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Existence and Uniqueness of Maps Into Affine Homogeneous Spaces.

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SUMMARY - We extend the usual existence and uniqueness theorem for immersions into spaces of constant curvature to smooth mappings into affine homogeneous spaces. We also obtain a result on reduction of codimension.

1. Statement of the results.

Let S be a smooth manifold with a connection D on its tangent bundle TS with parallel curvature and torsion tensors R and T . If S is simply connected and D is complete, such a space is precisely a reductive homogeneous space $S = G/H$ with its canonical connection (cf. [N], [K]). In this case, G can be chosen to be the group of affine diffeomorphisms; these are diffeomorphisms $g: S \rightarrow S$ with $g^*D = D$.

Let M be another smooth manifold and $f: M \rightarrow S$ a smooth mapping. Then its differential gives a vectorbundle homomorphism $F = df: TM \rightarrow E$ where E is the pull back bundle of TS :

$$E = f^*TS = \{(m, v); m \in M, v \in T_{f(m)}S\}.$$

The curvature and torsion tensors of S give bundle homomorphisms $T: \Lambda^2 E \rightarrow E$ and $R: \Lambda^2 E \rightarrow \text{End}(E)$ (the endomorphisms of E) satisfying the following structure equations (cf. [GKM]):

$$(1) \quad D_V F(W) - D_W F(V) - F([V, W]) = T(F(V), F(W)),$$

$$(2) \quad D_V D_W \xi - D_W D_V \xi - D_{[V, W]} \xi = R(F(V), F(W)) \xi,$$

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for any sections V, W of TM and ξ of E , where D denotes the pull-back connection on E . Abbreviating the left hand side of (1) by $dF(V, W)$ where d is the Cartan derivative of the E -valued 1-form F , and the left hand side of (2) by $R^E(V, W)\xi$ where $R^E: \Lambda^2 TM \rightarrow \text{End}(E)$ is the curvature tensor of the connection D on E , we can write these equations shortly as

$$(1) \quad dF = F^* T,$$

$$(2) \quad R^E = F^* R.$$

Since S has parallel torsion and curvature, the tensors R and T are parallel with respect to the connection D on E . More generally, let E be any vector bundle over M , equipped with a connection D . We say that E has the algebraic structure of S if there exist parallel bundle homomorphisms $T: \Lambda^2 E \rightarrow E$ and $R: \Lambda^2 E \rightarrow \text{End}(E)$ and a linear isomorphism $\Phi_0: E_p \rightarrow T_o S$ for some fixed $p \in M$, $o \in S$, which preserve R and T . Apparently, $E = f^* TS$ has the algebraic structure of S . We want to prove the following.

THEOREM 1. *Let S be a manifold with complete connection D with parallel torsion and curvature tensors. Let M be a simply connected manifold and E a vector bundle with connection D over M having the algebraic structure (R, T) of S . Let $F: TM \rightarrow E$ be a vector bundle homomorphism satisfying equations (1) and (2) above. Then there exists a smooth map $f: M \rightarrow S$ and a parallel bundle isomorphism $\Phi: E \rightarrow f^* TS$ preserving T and R such that*

$$df = \Phi \circ F.$$

If S is simply connected, then f is unique up to affine diffeomorphisms of S .

THEOREM 2 (Reduction of codimension). *Let S be as above and $f: M \rightarrow S$ a smooth map such that the image of df lies in a parallel subbundle $E' \subset f^* TS$ which is invariant under T and R . Then there is a totally geodesic subspace $S' \subset S$ with $f(M) \subset S'$.*

REMARKS. If $S = \mathbb{R}$, then the conditions (1), (2) are reduced to $dF = 0$. So Theorem 1 holds since $H^1(M) = 0$. If S is a Riemannian space form of constant curvature with its Levi-Civita connection and if F is injective and E is equipped with a parallel metric, then E can be identified with $TM \oplus \perp M$ where $\perp M = F(TM)^\perp$, and F is the embedding of the first factor. Now (1) is equivalent to $(dF)^\perp = 0$ which means that the second fundamental form is symmetric, and (2) contains pre-

cisely the Gauß, Codazzi and Ricci equations. So we receive the usual existence and uniqueness theorems for maps into space forms. In [EGT], a similar theorem for Kähler space forms was proved which is also covered by our result.

After finishing this work we learned that Theorem 1 was already proved in 1978 by a different method ([W], p. 36); unfortunately, this proof was never published in a Journal.

2. Proof of the theorems.

Let M be a manifold, E a vector bundle over M with connection D and $F: TM \rightarrow E$ a bundle homomorphism. We need to generalize the Cartan structure equations of the tangent bundle to this situation. Let $b = (b_1, \dots, b_n)$ be a local frame on some open subset $U \subset M$. Then there are 1-forms $\theta = (\theta_i)$, $\omega = ((\omega^i_j))$ on U (where $i, j = 1, \dots, n$) such that

$$F = b \cdot \theta := \sum \theta^i b_i, \quad Db = b \cdot \omega$$

where the last expression means $Db_j = \sum \omega^i_j b_i$. Then

$$(3) \quad dF = Db \wedge \theta + b \cdot d\theta = b(\omega \wedge \theta + d\theta),$$

$$(4) \quad dDb = Db \wedge \omega + b \cdot d\omega = b(\omega \wedge \omega + d\omega),$$

where $dDb = (dDb_1, \dots, dDb_n) = (R^E b_1, \dots, R^E b_n)$.

Now let there be parallel homomorphisms $T: \Lambda^2 E \rightarrow E$ and $R: \Lambda^2 E \rightarrow \text{End}(E)$. Using a fixed frame at some point $p \in M$, we identify E_p with \mathbb{R}^n and get linear maps $T_0: \Lambda^2 \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $R_0: \Lambda^2 \mathbb{R}^n \rightarrow \text{End}(\mathbb{R}^n)$. Let $H \subset \text{Gl}(n, \mathbb{R})$ be group of linear automorphisms of \mathbb{R}^n preserving T_0 and R_0 . The vector bundle E is associated to a principal H -bundle as follows. For any $m \in M$, a frame (b_1, \dots, b_n) of E_m can be considered as a linear isomorphism $b: \mathbb{R}^n \rightarrow E_m$ with $b(e_i) = b_i$. Let PE be the bundle of (T, R) -frames, i.e. those frames which map T_0 onto T and R_0 onto R . Clearly, PE is a principal H -bundle, where the group H acts from the right on PE . The advantage is that the coefficients of T and R are the same for any $b \in PE$:

$$T(b_i, b_j) = \sum t_{ij}^k b_k, \quad R(b_i, b_j) b_k = r_{ijk}^l b_l$$

where t_{ij}^k and r_{ijk}^l are the coefficients of T_0 and R_0 .

Now let us assume equations (1) and (2). Choose a local (T, R) -

frame, i.e. a local section $b: U \rightarrow PE|U$. Then

$$dF(v, w) = T(F(v), F(w)) = \sum T(\theta^i(v) b_i, \theta^j(w) b_j) = \sum \theta^i(v) \theta^j(w) t_{ij}^k b_k,$$

hence

$$dF = \frac{1}{2} \sum \theta^i \wedge \theta^j t_{ij}^k b_k.$$

Likewise,

$$dDb_k = \frac{1}{2} \sum \theta^i \wedge \theta^j r_{ijk}^l b_l.$$

Together with (3) and (4) we get the structure equations of Cartan type

$$(5) \quad \begin{aligned} d\theta &= -\omega \wedge \theta + \sum t_{ij} \theta^i \wedge \theta^j, \\ d\omega &= -\omega \wedge \omega + \sum r_{ij} \theta^i \wedge \theta^j, \end{aligned}$$

where $t_{ij} = (t_{ij}^1, \dots, t_{ij}^n)$ and r_{ij} is the matrix $((r_{ijk}^l))$, i.e. $r_{ij}(e_k) = \sum r_{ijk}^l e_l$.

Now recall that the forms θ and ω on U are just the pull backs by $b: U \rightarrow PE$ of global forms on PE which we also call θ and ω . Namely, the forms $\theta \in \Omega^1(PE) \otimes \mathbb{R}^n$ and $\omega \in \Omega^1(PE) \otimes \underline{h}$ (where $\underline{h} \subset \text{End}(\mathbb{R}^n)$ is the Lie algebra of H) are defined as follows. If $b \in PE$ and $X \in T_b PE$, then

$$(6) \quad b \cdot \theta(X) = F(d\pi_b(X))$$

where $\pi: PE \rightarrow M$ is the projection, and

$$(7) \quad b \cdot \omega(X) = \pi_v(X)$$

where $\pi_v: TPE \rightarrow VE$ is the vertical projection determined by the connection; here, $VE \subset TPE$ is the vertical distribution $(VE)_b = T_b(bH)$. Clearly, these forms on PE also satisfy (5).

Now let S be as above. Replacing (M, E, F) with (S, TS, Id) , we get also forms θ', ω' on PTS satisfying equations (5) which are now the usual Cartan structure equations of TS . We will consider $\theta, \omega, \theta', \omega'$ as forms on the product $PE \times PTS$ by pulling back via the projections pr_1, pr_2 onto the two factors. Since both (θ, ω) and (θ', ω') satisfy (5), we get that $d(\theta^i - \theta'^i)$ and $d(\omega_j^i - \omega'^i_j)$ lie in the ideal generated by $\theta^i - \theta'^i$ and $\omega_j^i - \omega'^i_j$; note that in any ring we have the identity

$$ab - a'b' = (a - a')b + a'(b - b').$$

Therefore the distribution

$$\underline{D} = \{(X, X') \in T(PE \times PTS); \theta(X) = \theta'(X'), \omega(X) = \omega'(X')\}$$

on $PE \times PTS$ is integrable.

Let $L \subset PE \times PTS$ be a maximal integral leaf of this distribution. We have $\dim L = \dim PE$ since the number of equations determining L is $n + \dim \mathfrak{h} = \dim PTS$. Moreover, L intersects the second factor $\{b\} \times PTS$ everywhere transversally. Namely, if some vector $(0, X')$ lies in TL , then $\theta'(X') = 0$ and $\omega'(X') = 0$, hence $X' = 0$ since the forms θ^i, ω^j span T^*PTS . Moreover, L is invariant under H acting diagonally on $PE \times PTS$. Namely, if $(b, b') \in L$ and $\alpha = (\alpha^i_j) \in \mathfrak{h}$, then $(b\alpha, b'\alpha) \in T_{(b, b')}L$ because $b\alpha$ and $b'\alpha$ are vertical vectors (so θ and θ' vanish) and $\omega(b\alpha) = \alpha = \omega'(b'\alpha)$. Thus the map $p_L := pr_1 | L: L \rightarrow PE$ is an H -equivariant local diffeomorphism.

Let us assume from now on that S is simply connected (which is no restriction since we may always pass to the universal cover). Then there is a group G which acts transitively on S by affine diffeomorphisms and also transitively on PTS (from the left) via differentials (cf. [K], Thm. I.17). Then also gL is an integral leaf for any $g \in G$, where we let G act only on the second factor of $PE \times PTS$. This is because θ' and ω' are invariant under affine diffeomorphisms of S since their differential preserves the horizontal and vertical distribution on PTS . (In fact, if we identify PTS by the action with G/kernel , then θ' and ω' are the components of the Maurer-Cartan form with respect to the $Ad(H)$ -invariant decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$.)

Now we claim that the mapping $p_L = pr_1 | L: L \rightarrow PE$ is onto. Since it is a local homeomorphism, its image is open. Since M is connected and p_L maps H -orbits diffeomorphically onto H -orbits, it is sufficient to show that the image is closed. So let $(b_k, b'_k)_{k \geq 0}$ be a sequence in L such that $b_k \rightarrow b$ in PE . We will show that also $b \in pr_1(L)$. Since G acts transitively on PTS , there exists $g_k \in G$ such that $g_k b'_k = b'_0$. Then the maximal integral leaves $g_k L$ contain the points (b_k, b'_0) . So they converge to the maximal integral leaf L' through (b, b'_0) . Hence $pr_1(L')$ contains a neighborhood of b in PE , and for big enough k , there exists $b' \in PTS$ with $(b_k, b') \in L'$. Therefore $L' = gL$ where $g \in G$ is such that $b' = gb'_k$, and in particular, $b \in pr_1(L)$ since $pr_1(L) = pr_1(gL)$.

It follows that p_L is a covering map. If U is a neighborhood of some $(b, b') \in L$ where $p_L | U$ is a diffeomorphism, then $p_L^{-1}(p_L(U))$ is a disjoint union of copies gU of U , where $g \in G$ leaves L invariant. Since M is simply connected, any element of the fundamental group $\pi_1(PE)$ can be represented by a closed curve in some fibre (H -orbit), and since p_L maps any H -orbit in L diffeomorphically onto an H -orbit in PE , it in-

duces a surjective homomorphism of the fundamental groups. Therefore, the covering map p_L is actually a global diffeomorphism which means that L is a graph over PE . So there exists a smooth H -equivariant map $Pf: PE \rightarrow PTS$ with $\text{Graph}(Pf) = L$, and by uniqueness, any other integral leaf is the graph of $g \circ Pf$ for some $g \in G$. The fact that $\text{Graph}(Pf)$ is an integral leaf means

$$(8) \quad Pf^* \theta' = \theta, \quad Pf^* \omega' = \omega.$$

Since Pf maps fibres onto fibres, it is a bundle map, i.e. it determines a smooth mapping of the base spaces $f: M \rightarrow S$ such that the following diagram commutes:

$$\begin{array}{ccc} PE & \xrightarrow{Pf} & PTS \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & S. \end{array}$$

Moreover, Pf defines a vector bundle isomorphism $\Phi: E \rightarrow f^* TS$ as follows. If $\xi = \sum x^i b_i = bx \in E_m$ for some $b = (b_1, \dots, b_n) \in (PE)_m$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, we put

$$\Phi(\xi) = (m, Pf(b) \cdot x) \in (f^* TS)_m = \{m\} \times T_{f(m)} S.$$

The map Φ is well defined, by the H -invariance of Pf , and it is clearly a bundle isomorphism preserving T and R . Moreover, if $\xi(t)$ is a parallel section of E along some curve in M , then $\xi(t) = b(t)x$ for some horizontal curve $b(t)$ in PE , i.e. $\omega\left(\frac{d}{dt} b(t)\right) = 0$. Since $Pf^* \omega' = \omega$, the curve $Pf(b(t))$ in PTS is horizontal again, so $\Phi(\xi(t))$ is also parallel. This shows that Φ is parallel.

Now let $v \in T_m M$ and $V \in T_b PE$ any lift, i.e. $\pi(b) = m$ and $d\pi_b(V) = v$. Then $df(v) = d\pi'(dPf(V))$. Recall that by (5) for any $b' \in PTS$, $V' \in T_{b'} PTS$, $v' = d\pi'(V')$ we have

$$v' = b' \cdot \theta(V').$$

Using the basis $b' = Pf(b)$ of $T_{f(m)} S$ to represent the vector $v' = df(v)$, we get (omitting the base points)

$$(9) \quad df(v) = Pf(b) \cdot \theta'(dPf(V)).$$

On the other hand, $F(v) = F(d\pi(V)) = b \cdot \theta(V)$ hence

$$(10) \quad \Phi(F(v)) = Pf(b) \cdot \theta(V).$$

Since $Pf^*(\theta') = \theta$, we get $df = \Phi \circ f$.

It remains to show the uniqueness of f . So let $f: M \rightarrow S$ be any smooth map with $df = \Phi \circ f$ for some parallel bundle isomorphism $\Phi: E \rightarrow f^*TS$ preserving T and R . Then we define a bundle map $Pf: PE \rightarrow PTS$ covering $f: M \rightarrow S$ by

$$Pf(b) = \Phi(b)$$

where $\Phi(b) = (\Phi(b_1), \dots, \Phi(b_n))$ for $b = (b_1, \dots, b_n) \in PE$. As above, Pf satisfies (9) and (10), and thus $df = \Phi \circ f$ implies that $Pf^*\theta' = \theta$. Moreover, since Φ is parallel, Pf maps horizontal curves in PE onto horizontal curves in PTS , and therefore $Pf^*\omega' = \omega$. This shows that $\text{Graph}(Pf)$ is an integral leaf of the distribution \underline{D} . But we have shown that there is only one integral leaf up to the action of G , so f is uniquely determined up to composition with $g \in G$. This finishes the proof of Theorem 1.

Now we prove Theorem 2. Fix $p \in M$ and let $o = f(p)$. Then $V' := E'_p$ is a linear subspace of $V = (f^*TS)_p = T_oS$ which is invariant under \bar{R} and T . We may assume that S is simply connected, hence an affine homogeneous space G/H . Then there is a totally geodesic homogeneous subspace $S' = G'/H'$ of S through 0 with $T_oS' = E'_p$ (e.g. cf. the Proof of Thm. I.17 in [K]; we put $\underline{h}' = \{A \in \underline{h}; A(V') \subset V'\}$, $\underline{g}' = \underline{h}' \oplus V'$), and E' has the algebraic structure of S' . By Theorem 1, there exists a smooth map $f': M \rightarrow S'$ with $f'(p) = 0$ and $df' = \Phi' \circ df$ for some parallel (R, T) preserving isomorphism $\Phi': E' \rightarrow f'^*TS'$. But f'^*TS' is a parallel subbundle of f^*TS as well as E' , and their fibres agree at the point p , so these subbundles are the same, and $\Phi = id$ since Φ is parallel and $\Phi = id$ at the point p . So we see from the unicity part of Theorem 1 that $f' = f$.

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