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Spacetimes with geodesic light rays

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Abstract. In a spacetime fibred by spacelike hypersurfaces (called 'space') such that light travels on geodesics in space, i.e. light geodesics have geodesic space projections, the space geometry must be fixed for all times up to rescaling, and the time scale is spatially independent.

Let (M, g) be a Lorentzian $(n + 1)$ -manifold ($n \geq 2$) which is diffeomorphic to $I \times \Sigma$ where I is a real interval and Σ an n -dimensional manifold, such that the tangent vector field $\partial/\partial t$ of the factor I is timelike with respect to the Lorentzian metric g and perpendicular to $\Sigma_t := \{t\} \times \Sigma$. The projection onto I will be called the *time function* t , and we consider the spacelike hypersurfaces Σ_t as the spatial universe at time t .

Theorem. Suppose that the projection to Σ of any null geodesic in M is a geodesic (up to parametrization) with respect to the metric h_t on Σ induced by the embedding $\Sigma \rightarrow \Sigma_t$, for any $t \in I$. Then g is a warped product metric

$$g = -f(t)^2 \cdot dt^2 + k(t)^2 \cdot h$$

where h is some Riemannian metric on Σ .

Comment. The converse statement is easy and well known; e.g. cf [1]. Our result improves a theorem of Tzanakis [2] who assumes that *all* causal geodesics of M project onto h_t -geodesics of Σ . The motivation for this improvement was the question whether light travels on geodesics in space. So our theorem says: this is true only if the space geometry is fixed for all time up to scaling, and the timescale is spatially independent.

Proof. In the following, *geodesics* are always understood to have arbitrary parametrizations.

Step 1. Let $\gamma: J \rightarrow \Sigma$ be an h_t -geodesic in Σ for some $t \in I$. We want to show that the surface $f: J \times I \rightarrow M$, $f(s, t) = (t, \gamma(s))$ is totally geodesic. Let S be the spacelike unit vector field in the s direction and T the timelike unit vector field in the t direction along f . Then $Z = T + S$ is a null vector, and the null geodesic μ in M starting at $f(s, t)$ with initial vector $Z(s, t)$ by assumption projects onto some h_t -geodesics on Σ which (by unicity) must agree with γ up to parametrization. This has two consequences: (1) γ is h_t -geodesic for any $t \in I$, (2) μ stays forever in the surface f and $Z(\mu(u))$ is tangent to

μ for any u . The same is true for the other null vector field $W = T - S$: its integral curves are null geodesics which lie on f and project onto γ with reversed orientation. Since Z and W are tangent vector fields of geodesics in M ,

$$\nabla_Z Z = a \cdot Z \quad \nabla_W W = b \cdot W$$

for some real functions a and b , where ∇ denotes the covariant derivative on M . Moreover, since $s \rightarrow (t, \gamma(s))$ is a geodesic in Σ_t , we have

$$\nabla_S S = c \cdot S + d \cdot T$$

since $T(s, t)$ is the unit normal vector of Σ_t at $f(s, t)$. Therefore the second fundamental form α of the surface f (recall the definition $\alpha(U, V) = \text{normal component of } \nabla_U V$ for any tangent vector fields U, V of f) satisfies

$$\alpha(Z, Z) = \alpha(W, W) = \alpha(S, S) = 0$$

which shows that $\alpha = 0$; hence f is totally geodesic.

Step 2. Let T be the unit tangent field of the factor I .

Claim. The integral curves of T are geodesics. Fix some $x \in \Sigma$ and consider two different h_t -geodesics β, γ in Σ starting at x . The corresponding surfaces $f(s, t) = (t, \beta(s))$ and $g(s, t) = (t, \gamma(s))$ are totally geodesic and intersect precisely in the curve $t \rightarrow (t, x)$. This must be a geodesic since the intersection of totally geodesic submanifolds is totally geodesic.

Step 3. Let T' be the gradient of the function t , i.e. $g(T', X) = \partial_X t$ for any vector field X . Then T' is perpendicular to the level hypersurfaces Σ_t and therefore $T' = u \cdot T$ for some real function u .

Claim. $u = \text{const}$ along each Σ_t . The integral curves of T' and T agree up to parametrization, hence they are geodesics by step 2 and therefore,

$$\nabla_{T'} T' = e \cdot T'.$$

So for any tangent vector field X of Σ_t we have

$$\partial_X g(T', T') = 2g(\nabla_X T', T') = 2g(\nabla_{T'} T', X) = 0$$

(note that $\nabla T'$ is self adjoint with respect to g since T' is a gradient). Hence $u = |g(T', T')|^{1/2}$ is constant along Σ_t . Rescaling the time function t we may assume from now on that T' is a unit vector, so $T' = T$.

Step 4. Now we have to compute how h_t depends on t . Let X be a vector field on Σ . We consider X also as a vector field $(0, X)$ on $M = I \times \Sigma$ which is everywhere tangent to Σ_t . Then

$$\frac{d}{dt} (h_t(X, X)) = \partial_T g(X, X) = 2g(\nabla_T X, X) = 2g(\nabla_X T, X).$$

Recall that T is a unit normal vector on Σ_t , so ∇T is the second fundamental tensor of Σ_t . By step 1, the vectors X and T at any point (t, x) span the tangent plane of a totally geodesic surface. Therefore $\nabla_X T \in \text{Span}\{X, T\}$, but $\nabla_X T \perp T$, so $\nabla_X T$ is a multiple of X

for any X . Hence $\nabla T = H \cdot Id$ for some real function H , i.e. Σ_t is an umbilic hypersurface. So

$$\frac{d}{dt} h_t = H \cdot h_t \quad (*)$$

Step 5. It remains to show that H does not depend on the space variable. We proceed as in the case of umbilic hypersurfaces in Euclidean space. Let $A = \nabla T$. For tangent vectors X, Y of Σ_t we have the Codazzi equation

$$R(X, Y)T = (\nabla_X A)Y - (\nabla_Y A)X = (\partial_X H) \cdot Y - (\partial_Y H) \cdot X.$$

where R is the curvature tensor of g . Consequently,

$$(\partial_Y H) \cdot g(X, X) = (\partial_X H) \cdot g(Y, X) + g(R(X, Y)T, X) = 0$$

for $X \perp Y$ since $R(T, X)X \in \text{Span}\{T, X\}$ (recall that T, X span the tangent plane of a totally geodesic surface) and therefore

$$g(R(X, Y)T, X) = g(R(T, X)X, Y) = 0.$$

So we have shown that $\partial_Y H = 0$ for any tangent vector Y of Σ_t , and therefore H is constant along Σ_t , i.e. $H = H(t)$. Integrating (*) we get $g = -dt^2 + k(t) \cdot h$ where $h = h_0$ and k is the solution of $k' = H \cdot k$, $k(0) = 1$.

References

- [1] O'Neill B 1984 *Semi-Riemannian Geometry* (New York: Academic)
- [2] Tzanakis C 1991 *Class. Quantum Grav.* **8** 1913–37