# Spacetimes with geodesic light rays

J-H Eschenburg

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Federal Republic of Germany

**Abstract.** In a spacetime fibred by spacelike hypersurfaces (called 'space') such that light travels on geodesics in space, i.e. light geodesics have geodesic space projections, the space geometry must be fixed for all times up to rescaling, and the time scale is spatially independent.

Let $(M, g)$ be a Lorentzian $(n + 1)$-manifold $(n \geq 2)$ which is diffeomorphic to $I \times \Sigma$ where $I$ is a real interval and $\Sigma$ an $n$-dimensional manifold, such that the tangent vector field $\partial/\partial t$ of the factor $I$ is timelike with respect to the Lorentzian metric $g$ and perpendicular to $\Sigma_t := \{t\} \times \Sigma$. The projection onto $I$ will be called the *time function* $t$, and we consider the spacelike hypersurfaces $\Sigma_t$ as the spatial universe at time $t$.

*Theorem.* Suppose that the projection to $\Sigma$ of any null geodesic in $M$ is a geodesic (up to parametrization) with respect to the metric $h_t$ on $\Sigma$ induced by the embedding $\Sigma \to \Sigma_t$, for any $t \in I$. Then $g$ is a warped product metric

$$g = -f(t)^2 \cdot \mathrm{d}t^2 + k(t)^2 \cdot h$$

where $h$ is some Riemannian metric on $\Sigma$.

*Comment.* The converse statement is easy and well known; e.g. cf [1]. Our result improves a theorem of Tzanakis [2] who assumes that *all* causal geodesics of $M$ project onto $h_t$-geodesics of $\Sigma$. The motivation for this improvement was the question whether light travels on geodesics in space. So our theorem says: this is true only if the space geometry is fixed for all time up to scaling, and the timescale is spatially independent.

*Proof.* In the following, *geodesics* are always understood to have arbitrary parametrizations.

*Step 1.* Let $\gamma: J \to \Sigma$ be an $h_t$-geodesic in $\Sigma$ for some $t \in I$. We want to show that the surface $f: J \times I \to M$, $f(s, t) = (t, \gamma(s))$ is totally geodesic. Let $S$ be the spacelike unit vector field in the $s$ direction and $T$ the timelike unit vector field in the $t$ direction along $f$. Then $Z = T + S$ is a null vector, and the null geodesic $\mu$ in $M$ starting at $f(s, t)$ with initial vector $Z(s, t)$ by assumption projects onto some $h_t$-geodesics on $\Sigma$ which (by unicity) must agree with $\gamma$ up to parametrization. This has two consequences: (1) $\gamma$ is $h_t$-geodesic for *any* $t \in I$, (2) $\mu$ stays forever in the surface $f$ and $Z(\mu(u))$ is tangent to

$\mu$ for any $u$. The same is true for the other null vector field $W = T - S$: its integral curves are null geodesics which lie on $f$ and project onto $\gamma$ with reversed orientation. Since $Z$ and $W$ are tangent vector fields of geodesics in $M$,

$$\nabla_Z Z = a \cdot Z \qquad \nabla_W W = b \cdot W$$

for some real functions $a$ and $b$, where $\nabla$ denotes the covariant derivative on $M$. Moreover, since $s \to (t, \gamma(s))$ is a geodesic in $\Sigma_t$, we have

$$\nabla_S S = c \cdot S + d \cdot T$$

since $T(s, t)$ is the unit normal vector of $\Sigma_t$ at $f(s, t)$. Therefore the second fundamental form $\alpha$ of the surface $f$ (recall the definition $\alpha(U, V)$ = normal component of $\nabla_U V$ for any tangent vector fields $U$, $V$ of $f$) satisfies

$$\alpha(Z, Z) = \alpha(W, W) = \alpha(S, S) = 0$$

which shows that $\alpha = 0$; hence $f$ is totally geodesic.

*Step 2.* Let $T$ be the unit tangent field of the factor $I$.

*Claim.* The integral curves of $T$ are geodesics. Fix some $x \in \Sigma$ and consider two different $h_t$-geodesics $\beta$, $\gamma$ in $\Sigma$ starting at $x$. The corresponding surfaces $f(s, t) = (t, \beta(s))$ and $g(s, t) = (t, \gamma(s))$ are totally geodesic and intersect precisely in the curve $t \to (t, x)$. This must be a geodesic since the intersection of totally geodesic submanifolds is totally geodesic.

*Step 3.* Let $T'$ be the gradient of the function $t$, i.e. $g(T', X) = \partial_X t$ for any vector field $X$. Then $T'$ is perpendicular to the level hypersurfaces $\Sigma_t$ and therefore $T' = u \cdot T$ for some real function $u$.

*Claim.* $u = \text{const}$ along each $\Sigma_t$. The integral curves of $T'$ and $T$ agree up to parametrization, hence they are geodesics by step 2 and therefore,

$$\nabla_{T'} T' = e \cdot T'.$$

So for any tangent vector field $X$ of $\Sigma_t$ we have

$$\partial_X g(T', T') = 2g(\nabla_X T', T') = 2g(\nabla_{T'} T', X) = 0$$

(note that $\nabla T'$ is self adjoint with respect to $g$ since $T'$ is a gradient). Hence $u = |g(T', T')|^{1/2}$ is constant along $\Sigma_t$. Rescaling the time function $t$ we may assume from now on that $T'$ is a unit vector, so $T' = T$.

*Step 4.* Now we have to compute how $h_t$ depends on $t$. Let $X$ be a vector field on $\Sigma$. We consider $X$ also as a vector field $(0, X)$ on $M = I \times \Sigma$ which is everywhere tangent to $\Sigma_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(h_t(X, X)) = \partial_T g(X, X) = 2g(\nabla_T X, X) = 2g(\nabla_X T, X).$$

Recall that $T$ is a unit normal vector on $\Sigma_t$, so $\nabla T$ is the second fundamental tensor of $\Sigma_t$. By step 1, the vectors $X$ and $T$ at any point $(t, x)$ span the tangent plane of a totally geodesic surface. Therefore $\nabla_X T \in \text{Span}\{X, T\}$, but $\nabla_X T \perp T$, so $\nabla_X T$ is a multiple of $X$

for any $X$. Hence $\nabla T = H \cdot Id$ for some real function $H$, i.e. $\Sigma_t$ is an umbilic hypersurface. So

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t = H \cdot h_t \tag{*}$$

*Step 5.* It remains to show that $H$ does not depend on the space variable. We proceed as in the case of umbilic hypersurfaces in Euclidean space. Let $A = \nabla T$. For tangent vectors $X$, $Y$ of $\Sigma_t$ we have the Codazzi equation

$$R(X, Y)T = (\nabla_X A)Y - (\nabla_Y A)X = (\partial_X H) \cdot Y - (\partial_Y H) \cdot X.$$

where $R$ is the curvature tensor of $g$. Consequently,

$$(\partial_Y H) \cdot g(X, X) = (\partial_X H) \cdot g(Y, X) + g(R(X, Y)T, X) = 0$$

for $X \perp Y$ since $R(T, X)X \in \mathrm{Span}\{T, X\}$ (recall that $T, X$ span the tangent plane of a totally geodesic surface) and therefore

$$g(R(X, Y)T, X) = g(R(T, X)X, Y) = 0.$$

So we have shown that $\partial_Y H = 0$ for any tangent vector $Y$ of $\Sigma_t$, and therefore $H$ is constant along $\Sigma_t$, i.e. $H = H(t)$. Integrating (*) we get $g = -\mathrm{d}t^2 + k(t) \cdot h$ where $h = h_0$ and $k$ is the solution of $k' = H \cdot k$, $k(0) = 1$.

## References

[1] O'Neill B 1984 *Semi-Riemannian Geometry* (New York: Academic)
[2] Tzanakis C 1991 *Class. Quantum Grav.* **8** 1913–37