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Endogenous Growth Cycles - Arrow´s Learning by Doing Reconsidered

von

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Abstract

A two-sector growth model is presented where human capital is acquired through learning by doing. It is shown that for both the competitive situation and the social optimum endogenous growth cycles, persistent or transitory, may be the outcome. Concerning the economic mechanisms possibly causing persistent oscillations we detect the degree of complementarity with respect to the stocks or positive cross effects between human capital and the capital stock or between investment and human capital. As to the economic policy recommendations it turns out that, as usual, any government intervention leading to a reallocation of resources from consumption to investment is welfare improving. But it must be emphasized that although taking these measures, policy makers cannot be sure to rule out growth cycles.

1 Introduction

With the publication of the papers by Romer (1986, 1990) and Lucas (1988) growth theory has received a new boost. Indeed, these papers present completely new ideas giving rise to the so-called "new growth theory". The traditional (neoclassical) growth theory postulated that the growth rate of the per-capita capital stock tends to a steady state in the sense of a rest point of the dynamic system (cf. Ramsey (1928), Solow (1956, 1957), Cass (1965)). Then, the growth rate of the population plays the significant role for the case of a given technology: it determines the growth rate of the total capital stock in the economy and is equal to it, giving the balanced growth path.

In the paper by Romer (1986), he presented a growth model in which this property does not hold any longer. In that paper he demonstrates that, under suitable assumptions, persistent per-capita growth of consumption and of the capital stock is feasible. The economic mechanism responsible for that phenomenon lies in the fact that the aggregate production function shows increasing returns to scale when varying the capital stock what, for its part, results from the fact that Romer does not focus on the physical capital stock but on technical knowledge. This good, technical knowledge, is now considered as being non-rival and partially excludable leading to spillovers on the macroeconomic level, a fact which gives rise to increasing returns.

In the paper by Lucas (1988) and Romer (1990) persistent per-capita growth is achieved through human capital which grows without an upper limit. This then leads to unbounded growth of the capital stock as well as consumption since the stock of human capital is used as an input factor in the aggregate production function. Given the unlimited growth of human capital, persistent per-capita growth is feasible even within a conventional aggregate production function revealing decreasing returns to scale.

A good deal of these models are essentially based on the learning by doing approach initiated by Arrow (1962). Thus, Arrow was one of the first to endogenize technical change in a formal macroeconomic model of economic growth. In his model, however, the economic variables converge to a rest point of the dynamic system, although there also exist spillovers. The reason for that is given by the fact that in his model increasing returns only occur when varying all input factors at the same time, whereas there are decreasing returns to scale when only one factor is increased (see also Sheshinski (1967) and Wan (1971)).

Arrow asserted that acquiring new knowledge is strongly related to experience. For examples, he refers to the airframe industry where a strong correlation between productivity growth and experience seems to exist. According to Arrow, a measure for the change in experience is given by investment and he maintains that cumulative investment represents a good index of experience.

Common to nearly all growth models with strong emphasis on intangible production factors like technical knowledge or human capital, is the fact that they only consider steady state solutions, i.e. dynamic paths with a constant positive per-capita growth rate (new growth theory) or a per capita growth rate equal to zero (traditional growth theory)[1]. Our goal with this paper, therefore, is to demonstrate that endogenous growth cycles (persistent or transitory) may be the outcome of a conventional two-sector growth model where learning by doing is present. That result holds both for the competitive situation as well as the social optimum, where the positive spillovers of investment are explicitly taken into consideration.

The rest of the paper is organized as follows. In section 2, we depict the model and formulate the representative individual's optimization problem. In section 3 we present analytical results concerning the optimal choice of investment for both the competitive as well as the socially optimal situation.

[1] To our knowledge the only paper taking account of cyclically oscillating growth rates is the one by Asada and Semmler (1993)

The dynamics of our model are then studied in section 4 and section 5 illustrates our results with the help of numerical examples. Section 6 finally concludes the paper.

2 The Two-Sector Model

We consider a two-sector economy which produces a consumption good $C(t)$ and an investment good denoted by $I(t)$ with two distinct production functions according to $C(t) = g^0(AL_0, K_0)$ and $I(t) = g^1(AL_1, K_1)$. L_0 and L_1 denote labor effort to produce one unit of the consumption good and one unit of investment good, respectively, with $L_0 + L_1 = L \equiv 1$, i.e. L which is normalized to one is the total amount of labor available to the economy. K_0 and K_1 represent the amounts of capital devoted to the production of consumption and investment, respectively, and $K_0 + K_1 = K$ denotes the total amount of capital in our economy. A represents human capital and acts as an efficiency index positively influencing the employed amount of labor.

As to the formation of A we assume that it reflects cumulated experience in the production of investment goods, that is, A is built up according to Arrow's learning by doing approach, Arrow (1962). In contrast to Arrow, however, we suppose that the contribution of gross investment to the formation of human capital further back in time is smaller than recent gross investment.

This assumption makes sense economically and can be formalized by defining the stock of human capital as a function of an integral of past gross investment with exponentially declining weights put on investment flows further back in time (cf. Ryder and Heal (1973) or Feichtinger and Sorger (1988)). $A(t)$, then is given by $A(t) = [\rho \int_{-\infty}^{t} e^{\rho(s-t)} I(s) ds]^{\alpha}$, $0 < \alpha$. In what follows, we will confine our investigations to the case $\alpha = 1$ since we thus do not restrict the economic content but can considerably simplify our subsequent analysis[2]. The parameter ρ represents the weight given to more recent levels of gross investment. The higher ρ, the larger is the contribution of more recent gross investment to the human capital stock in comparison to flows of investment dating back further in time.

As to the structure of our economy, we assume a closed economy with competitive markets and identical, rational agents. The production functions $g^0(\cdot)$ and $g^1(\cdot)$ are assumed to be C^2 and concave jointly in L_i, K_i, $i = 0, 1$, holding A fixed. Furthermore, they are strictly concave in each separate factor and linear homogenous in L_i, K_i, $i = 0, 1$ for a given A. Moreover, they are increasing in all arguments. Assuming efficient production the production possibility frontier (PPF), $T(A(t), K(t), I(t))$, then is obtained from the optimization problem $T(A(t), K(t), I(t)) = \max C(t) = g^0(AL_0(t), K_0(t)), s.t.\ I = g^1(AL_1(t), K_1(t)), K(t) = K_0(t) + K_1(t), 1 = L_0(t) + L_1(t); L_i(t), K_i(t) \geq 0; i = 1, 2.$

At that point it should be noted that the existence of a solution to this problem is guaranteed by standard assumptions. Moreover, given our assumptions above the chosen input levels $K_0(K, I), L_0(K, I), K_1(K, I)$ and $L_1(K, I)$ are continuous differentiable functions. The PPF, $T(A, K, I)$ is then equal to $g^0(AL_0(K, I), K_0(K, I))$. For given values of A it turns out to be also concave in K, I, increasing in K and decreasing in I. For our subsequent analysis we will need only the assumptions that the PPF, $T(A, K, I)$, is C^2 and has $T_I(\cdot) < 0$ and is strictly concave in I. Furthermore, it is increasing in the factors A and K with decreasing marginal productivities and has an upper bound for each marginal product.

For a survey of the results on PPF we refer to Boldrin (1989); a more thorough treatment can be found in Kuga (1972), Hirota and Kuga (1971) and Benhabib and Nishimura (1979).

Following standard economic procedure in growth theory, the represen-

[2]Sheshinski (1967) limited the range of α to $0 < \alpha < 1$. There are, however, also contributions to economic research which allow the more general case $0 < \alpha$, cf. Wan (1971), p. 226.

tative individual is supposed to maximize his discounted utility over an infinite time horizon,

$$\max_C \int_0^\infty e^{-rt}u(C(t))dt = \max_I \int_0^\infty e^{-rt}u(T(A(t), K(t), I(t))dt,$$

where $u(\cdot)$ is assumed to be increasing in C and concave.

The representative individual has as usual to decide what amount to consume and what to invest in the creation of physical capital thus increasing the consumption possibilities in the future. The evolution of the capital stock is constrained by the differential equation $\dot{K}(t) = I(t) - \delta K(t)$, with $K(0) = K_0 > 0$ given.

This, however, turns out to be non-optimal since it becomes intuitively clear that by only taking into account the effects of investment on the creation of capital neglects the positive effects of investment on the creation of human capital. In what follows we will call these positive externalities of investment spillovers and refer to the situation where those spillovers are neglected as the competitive situation. The social optimum will be called the solution where those positive externalities of investment are intentionally taken account of. Formally, this can be achieved by considering an additional constraint in the individual's optimization problem. From above we know that the stock of human capital is formed as a by-product of accumulated weighted gross investment, $A(t) = [\rho \int_{-\infty}^t e^{\rho(s-t)} I(s)ds]$, for $\alpha = 1$. The evolution of $A(t)$ is then given by the differential equation $\dot{A}(t) = \rho(I(t) - A(t))$ and the social optimal solution for the representative individual consists in maximizing his discounted stream of consumption subject to both the constraint giving the evolution of capital as well as the evolution of the stock of human capital. How these two problems differ will be seen in the next section where we analyze our model in detail.

3 Analytical Results

As shown in section 2 the competitive situation of our economy is described by a solution to the optimization problem (I):

$$\max_I \int_0^\infty e^{-rt} u(T(A(t), K(t), I(t)))dt,$$

subject to $\dot{K}(t) = I(t) - \delta K(t)$, $K(0) = K_0 > 0$.

The social optimum can be described by a solution to our problem (II):

$$\max_I \int_0^\infty e^{-rt} u(T(A(t), K(t), I(t)))dt,$$

subject to $\dot{K}(t) = I(t) - \delta K(t)$, $K(0) = K_0 > 0$ and $\dot{A}(t) = \rho(I(t) - A(t))$, $A(0) = A_0 > 0$.

As we do not require the PPF to be concave jointly in all its arguments we cannot be sure whether the necessary conditions do make sense as they are not sufficient for this case. Therefore, we first show that a solution to our optimization problems exists and then use the necessary conditions to describe this solution. This is done in Theorem 1.

Theorem 1 *Given the assumption of the strict concavity of $T(A, K, I)$ in I, there exists a unique path of investment that solves the optimal control problem (I) and the optimal control problem (II).*

The proof which is available on request in an appendix follows from a standard result in control theory (cf. Seierstad and Sydsaeter (1987), p. 237) and uses the fact that the domain of all possibly optimal values for the rate of investment is bounded. This, for its part, is a consequence of our assumption that the marginal product of each factor is bounded by above, i.e. there exists a constant M such that $T_i(A, K, I) \leq M$ for $i = A, K$. That property elucidates the fact that the existence of an upper bound for each marginal product in the social production function is sufficient to guarantee the boundedness of per-capita variables. Only if there does not exist an upper bound for the marginal products then persistent growth may be the outcome.

Given theorem 1 we can now characterize the solution to our optimization problems. First let us look at problem (I), the competitive situation. Here, the individual only takes into account the direct effects of investment expenditure, that is the building up of physical capital according to the differential equation $\dot{K}(t) = I(t) - \delta K(t)$, treating the evolution of human capital $A(t)$ as exogenously given.

In the following we will suppose a linear utility function. This assumption, however, does not change any of our results derived below. The Hamiltonian function for that problem then is given by the expression $H(\cdot) = \gamma_0 T(A, K, I) + \gamma_1(I - \delta K)$, with γ_1 denoting the current-value costate variable or shadow price of capital. The first order condition for $I(t)$ to yield a maximum for problem (I) then is $-T_I(\cdot) = \gamma_1$ (for $\gamma_0 = 1$). The evolution of γ_1 is given by $\dot{\gamma}_1 = (r + \delta)\gamma_1 - T_K(\cdot)$. Here, it should be noted that the stock of human capital also evolves over time, as a by-product of investment, and thus influences the evolution of physical capital, of its shadow price and of investment. But this property is not explicitely taken into account by our individual. Furthermore, the limiting transversality conditions are given by $\lim_{t\to\infty} e^{-rt}\gamma_1(t) = 0$. Note that the transversality conditions are necessary in this case (demonstrated in the appendix available on request). This result follows from Michel's corollary to his theorem (Michel (1982), pp. 977/979). Before analyzing the dynamic behaviour of our variables let us briefly turn to the social optimization problem, denoted by problem (II).

The only difference to problem (I) consists in the fact that in this problem the individual that may be termed a social planner takes account of the positive spillovers of investment. The Hamiltonian function now is written as $H(\cdot) = \gamma_0 T(A, K, I) + \gamma_1(I - \delta K) + \gamma_2 \rho(I - A)$, with γ_2 denoting the shadow price of human capital. Note that γ_2 only represents a shadow price, whereas γ_1, however, indeed represents the competitive price of the investment good as well (see e.g. Boldrin (1989), p. 235 or Otani and El-Hodiri (1987), p. 226). The rate of investment is now set according to $-T_I(\cdot) = \gamma_1 + \rho\gamma_2$.[3] It can already be seen that in problem (II) investment is always larger than in problem (I). The reason is that now investment is not only paid its competitive price γ_1 but also an additional (shadow) price γ_2, giving the value of an additional marginal unit of human capital. The dynamic behaviour of γ_2 is described by $\dot{\gamma_2} = (r+\rho)\gamma_2 - T_A(\cdot)$. The limiting transversality conditions for that case are given by $\lim_{t\to\infty} e^{-rt}(\gamma_1(t) + \gamma_2(t)) = 0$.

In the next section, we will investigate the dynamic behaviour and try to give economic conditions for a possibly cyclical behaviour of our variables.

4 The Dynamic Behaviour

4.1 The competitive economy

Let us first look at the competitive economy. We know that the evolution of physical capital and human capital is described by the differential equations $\dot{K}(t) = I(t) - \delta K(t)$ and $\dot{A}(t) = \rho(I(t) - A(t))$, respectively. Investment in these two equations is chosen such that $H(\cdot) = \gamma_0 T(A,K,I) + \gamma_1(I - \delta K)$ is maximized, giving $-T_I(A,K,I) = \gamma_1$, as already mentioned in the last section. Investment $I(t)$ is thus a function implicitly defined by A, K, γ_1, i.e. $I(t) = I(A(t), K(t), \gamma_1(t))$. As to the evolution of the shadow price $\gamma_1(t)$, we know from optimal control theory that it is given by the differential equation $\dot{\gamma_1}(t) = r\gamma_1(t) - \partial H(\cdot)/\partial K$, such that the system of differential equations can be written as

$$\dot{K}(t) = I(A(t), K(t), \gamma_1(t)) - \delta K(t), \quad (1)$$
$$\dot{\gamma_1}(t) = (r+\delta)\gamma_1(t) - T_K(\cdot), \quad (2)$$

[3] For interior solutions to problem (I) and (II) which may be justified by imposing Inada-type conditions on $T(\cdot)$ it can easily be seen that γ_0 can be set equal to 1.

$$\dot{A}(t) \;=\; \rho I(A(t), K(t), \gamma_1(t)) - \rho A(t). \tag{3}$$

To determine the dynamic behaviour of our system of differential equations, let us first investigate the question of the existence of a steady state in the sense of a rest point, that is a situation where the derivatives with respect to time equal zero. Here we can state Lemma 1.

Lemma 1 *The system of differential equations (1)-(3) has a unique optimal steady state* $K^\infty, \gamma_1^\infty, A^\infty$.

Proof: An optimal steady state is the solution to the equation system $\dot{K} = \dot{\gamma_1} = \dot{A} = 0$. This implies

$$\begin{aligned}
\gamma_1^\infty &= -T_I(K^\infty, \gamma_1^\infty, A^\infty), \\
\delta K^\infty &= I^\infty, \\
(r+\delta)\gamma_1^\infty &= T_K(K^\infty, \gamma_1^\infty, A^\infty), \\
A^\infty &= I^\infty.
\end{aligned}$$

Given that I^∞ is unique, which follows from the strict concavity of H in $I(t)$, the uniqueness and existence of a solution to the above equations can easily be checked, if Inada-type conditions are imposed on $T_K(\cdot)$ and $T_A(\cdot)$.

□

To derive the (local) dynamic behaviour of our economic variables we now compute the Jacobian of (1)-(3) and determine its eigenvalues. The derivatives of $I(t) = I(A(t), K(t), \gamma_1(t))$ are easily obtained by implicit differentiation as $I_A(\cdot) = -\frac{T_{IA}}{T_{II}}$, $I_K(\cdot) = -\frac{T_{IK}}{T_{II}}$, $I_{\gamma_1}(\cdot) = -\frac{1}{T_{II}} > 0$. Thus, the Jacobian can be seen to have the following form,

$$J = \begin{bmatrix} -\frac{T_{IK}}{T_{II}} - \delta & -1/T_{II} & -\frac{T_{IA}}{T_{II}} \\ -T_{KK} + \frac{T_{IK}^2}{T_{II}} & \frac{T_{IK}}{T_{II}} + r + \delta & -T_{KA} + \frac{T_{IK}T_{IA}}{T_{II}} \\ -\rho\frac{T_{IK}}{T_{II}} & -\rho/T_{II} & -\rho - \rho\frac{T_{IA}}{T_{II}} \end{bmatrix}.$$

The characteristic equation of our system is

$$\lambda^3 + (-\operatorname{trace} J)\lambda^2 + K_2\lambda + (-\det J) = 0,$$

with K_2 being defined as

$$K_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

with a_{ij} element of the i-th row and j-th column of J.

The dynamic behaviour of our variables may be quite different, depending on the parameter values. In particular we are interested in transitory or permanent oscillations stating that our economy may produce endogenous fluctuations, in contrast to conventional growth theory which only considers monotonic time paths with constant growth rates.

Especially the emergence of a Hopf bifurcation leading to persistent oscillations of economic variables has gained more and more attention in economic research. The technical prerequisite for that phenomenon lies in the presence of two eigenvalues crossing the imaginary axes, thus causing a change in the qualitative property of the solution of the system of differential equations. For our system we can use Lemma 2 in order to gain further insight in the properties of our economic model.

Lemma 2 *A necessary and sufficient condition for the characteristic equation* $\lambda^3 + (-\operatorname{trace} J)\lambda^2 + K_2\lambda + (-\det J) = 0$ *to possess a pair of two pure imaginary roots* $\pm\omega i, i = \sqrt{-1}, \omega \neq 0$ *is* $K_2 > 0$ *and* $K_2 \cdot (-\operatorname{trace} J) + \det J = 0$.

A proof of that Lemma can be found in Asada and Semmler (1992).

Applying this Lemma to our problem we have to find out that the second condition can be determined technically and becomes extremely complicated. Moreover, it is not apt to economic interpretation such that we will focus on the first one. Doing so, we calculate K_2 as $K_2 = -a +$

$A_1 + c + d$, with $a = \delta(r+\delta) > 0$, $A_1 = (-1/T_{II})[(r+2\delta)T_{IK} + T_{KK}]$, $c = -(1/T_{II})r\rho(T_{IA} + T_{II})$, $d = -\rho T_{KA}/T_{II}$.

Looking closer at the elements K_2 is composed of, we see that all may have positive or negative signs except a.

As to the economic meaning of those elements we can give a nice interpretation to A_1. A_1 is normally seen as a measure for complementarity over time. If $A_1 > 0$ we can speak of complementarity between adjacent dates (or briefly adjacent complementarity) with respect to the capital stock K. That means increasing investment at time t_3 implies a reallocation of resources from distant dates t_1 to nearby dates t_2. Correspondingly, if $A_1 < 0$ we speak of complementarity between distant dates (distant complementarity) with respect to the capital stock, meaning that an increase in investment at time t_3 leads to a reallocation of resources from nearby dates t_2 to distant ones t_1.[4]

Given these notions, we immediately see that adjacent complementarity is favourable for the emergence of persistent oscillations. In fact, if the cross derivatives T_{IA}, T_{KA} are non-positive, stable limit cycles are only possible if adjacent complementarity prevails.

If, however, positive cross derivatives can be observed, then persistent oscillations may occur even for distant complementarity with respect to the capital stock. The economic reason for that result seems to be clear: cyclical time paths require a sort of acceleration effect between the two stocks. If the cross derivative between physical capital and human capital is non-positive, however, it can be argued that an additional unit of human capital has a satiating effect on capital, implying convergence to the steady state. Moreover, it seems that a positive cross effect represents the more plausible case, since an additional unit of capital yields a higher return if the workers operating the machines dispose of higher skills, i.e. if the stock

[4] For a derivation of these concepts we refer to Dockner and Feichtinger (1991). See also Wan (1970) and Ryder and Heal (1973).

of human capital is higher.

The same holds if a higher stock of human capital lowers the opportunity cost of investment. In this case, a higher stock of human capital implies that less investment has to be given up in order to produce an additional unit of the consumption good.

We can summarize our results in the following Theorem.

Theorem 2 *A necessary condition for persistent oscillations of the economic variables consists in*
(i) adjacent complementarity with respect to the capital stock or
(ii) positive cross effects between the stock of human capital and physical capital or between human capital and investment.

The proof of that theorem follows from Lemma 2 together with the characterization of K_2.

It should be noted that this theorem only provides us with necessary conditions for persistent growth cycles. Because of those reasons, later on, we will present a numerical example to illustrate our results and depict the possible time paths.

Up to now we have derived results for our competitive economy. In particular we worked out some economic mechanisms possibly responsible for persistent growth cycles. It might be argued that our results change if we investigate the case of the social optimum and that then a steady state balanced growth path may be the outcome. This question will be investigated in the next subsection.

4.2 The Social Optimum and Policy Implications

As already mentioned in the previous section, for the social optimum maximization problem the rate of investment is at any point of time higher than in the competitive economy. The maximum principle now gives $-T_I(A,K,I) = \gamma_1 + \rho\gamma_2$ implicitly defining investment. The derivatives can again be calculated as $I_A(\cdot) = -\frac{T_{IA}}{T_{II}}$, $I_K(\cdot) = -\frac{T_{IK}}{T_{II}}$, $I_{\gamma_1}(\cdot) = -1/T_{II} > 0$ and $I_{\gamma_2}(\cdot) = -\rho/T_{II} > 0$. Substituting these relations in the other necessary conditions given by the differential equations describing the evolution of physical capital, the stock of human capital and its shadow prices, respectively, then gives the so-called Hamiltonian system, completely describing the dynamic behaviour of our economic variables. This system may be written as

$$\begin{aligned}
\dot{K}(t) &= I(A(t), K(t), \gamma_1(t), \gamma_2(t)) - \delta K(t), &(4)\\
\dot{A}(t) &= \rho I(A(t), K(t), \gamma_1(t), \gamma_2(t)) - \rho A(t), &(5)\\
\dot{\gamma_1}(t) &= (r+\delta)\gamma_1(t) - T_K(\cdot), &(6)\\
\dot{\gamma_2}(t) &= (r+\rho)\gamma_2(t) - T_A(\cdot). &(7)
\end{aligned}$$

Before going into details of our analysis we state Lemma 3.

Lemma 3 *The canonical system (4)-(7) possesses a unique optimal steady state $K^\infty, A^\infty, \gamma_1^\infty, \gamma_2^\infty$.*

The proof can easily be obtained by adopting the arguments to the proof of Lemma 1.

As in the preceding section we now calculate the Jacobian matrix near the steady state and determine its eigenvalues. The Jacobian is seen to be

$$J = \begin{bmatrix} -\frac{T_{IK}}{T_{II}} - \delta & -\frac{T_{IA}}{T_{II}} & -1/T_{II} & -\rho/T_{II} \\ -\rho\frac{T_{IK}}{T_{II}} & -\rho\frac{T_{IA}}{T_{II}} - \rho & -\rho/T_{II} & -\rho^2/T_{II} \\ -T_{KK} + \frac{T_{KI}^2}{T_{II}} & -T_{KA} + \frac{T_{KI}T_{AI}}{T_{II}} & r+\delta+\frac{T_{IK}}{T_{II}} & \rho T_{KI}/T_{II} \\ -T_{KA} + \frac{T_{KI}T_{AI}}{T_{II}} & -T_{AA} + \frac{T_{AI}^2}{T_{II}} & \frac{T_{IA}}{T_{II}} & r+\rho+\rho\frac{T_{IA}}{T_{II}} \end{bmatrix},$$

with the eigenvalues given by

$$\lambda_{1,2,3,4} = \frac{r}{2} \pm \sqrt{\left(\frac{r}{2}\right)^2 - \frac{K_1}{2} \pm \sqrt{\left(\frac{K_1}{2}\right)^2 - \det J}}.$$

K_1 is defined as

$$K_1 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + 2 \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix},$$

with a_{ij} again denoting the element of the i-th row and j-th column of J (see Dockner and Feichtinger (1991)).

Given the explicit characterization of the eigenvalues of our Jacobian matrix, we can use Lemma 4 in order to determine its stability properties.

Lemma 4 *(1) The conditions*

$$K_1 < 0,$$

$$0 < \det J \leq (K_1/2)^2$$

are necessary and sufficient for all eigenvalues to be real, two being positive two being negative.

(2) The conditions

$$\det J > (K_1/2)^2,$$

$$\det J > (K_1/2)^2 + r^2(K_1/2)$$

are necessary and sufficient for all eigenvalues to be complex, two having negative real parts and two having positive real parts.

(3) The conditions

$$\det J > (K_1/2)^2,$$

$$\det J - (K_1/2)^2 - r^2(K_1/2) = 0$$

are necesary and sufficient for all eigenvalues to be complex and two having zero real parts.

The proof of that Lemma can be found in Dockner and Feichtinger (1991).

For our system the constant K_1 and the determinant of the Jacobian can be written as follows,

$$\begin{aligned}
K_1 &= -a - b + A_1 + A_2 - 2\rho T_{AK}/T_{II}, \\
\det J &= ab - aA_2 - bA_1 + \rho T_{AK}/T_{II}[\delta(r+\rho) + \rho(r+\delta)], \\
\det J - (K_1/2)^2 &= 1/4[-(a-b)^2 - (A_1+A_2)^2 - 2(a-b)(A_2 - A_1)] \\
&+ \rho\frac{T_{AK}}{T_{II}}[\delta(r+\rho) + \rho(r+\delta)] + \rho\frac{T_{KA}}{T_{II}}K_1 + \left(\rho\frac{T_{KA}}{T_{II}}\right)^2,
\end{aligned}$$

where $a = \delta(r+\delta)$, $b = \rho(r+\rho)$, $A_1 = -(1/T_{II})[(r+2\delta)T_{IK} + T_{KK}]$, $A_2 = -(\rho/T_{II})[(r+2\rho)T_{IA} + T_{AA}]$.

It can already be seen that in the social optimum a steady state growth path is not necessarily the outcome. As in the competitive economy there may be persistent oscillations in the growth rate of the economy. To work out the economic mechanisms responsible for persistent or transitory oscillations we first consider our model for the case $T_{KA} = 0$. The constant K_1 and the determinant of the Jacobian are then given by

$$\begin{aligned}
K_1 &= -a - b + A_1 + A_2, \\
\det J &= ab - aA_2 - bA_1, \\
\det J - (K_1/2)^2 &= 1/4[-(a-b)^2 - (A_1+A_2)^2 - 2(a-b)(A_2 - A_1)].
\end{aligned}$$

Given Lemma 4 we can immediately see that our model is stable in the saddle point sense if both physical capital and human capital show distant complementarity. It should be noted, however, that the path to the steady state value may reveal cyclical (transitory) oscillations. If one of the stocks is characterized by adjacent complementarity whereas the other has distant complementarity, persistent oscillations may occur if the stock of human capital shows adjacent (distant) complementarity and physical capital has distant (adjacent) complementarity, if $\rho > \delta (\rho < \delta)$. If both stocks have adjacent complementarity, the degree of complementarity of the stock of human capital has to be larger (smaller) than the one of physical capital, if $\rho > \delta (\rho < \delta)$. That result can easily be seen to follow from the condition $\det J - (K_1/2)^2 > 0$.

Moreover, the degree of adjacent complementarity has to be sufficiently high, respectively, a fact which follows from the condition $K_1 > 0$.

It should be recalled that the above results were derived for the case $T_{KA} = 0$. If, however, $T_{KA} > 0$, i.e. if the marginal product of physical capital increases with an increase in human capital, then persistent oscillations may result even if both stocks show distant complementarity. Because then both K_1 and $\det J$ may become positive although $A_1 < 0$, $A_2 < 0$ holds. In analogy to the competitive equilibrium, we may again argue that these positive cross effects represent a sort of acceleration effect leading to stable limit cycles.

We have seen that the economic effects possibly leading to persistent growth cycles in the social optimum resemble those of the competitive economy. Again, intertemporal substitution effects or positive cross effects between the stock of human capital and physical capital may cause endogenous growth cycles. We can summarize our results in the following two Theorems.

Theorem 3 *Given zero cross derivatives with respect to human capital and physical capital for the social optimum maximization problem the following turns out to be true:*
(i) A necessary condition for persistent endogenous growth cycles is that human capital shows adjacent (distant) complementarity and physical capital has distant (adjacent) complementarity if $\rho > \delta (\rho < \delta)$ and the degree of adjacent complementarity has to be sufficiently high.
(ii) A necessary condition for persistent endogenous growth cycles is that the degree of adjacent complementarity of the stock of human capital is larger (smaller) than the one of physical capital, if $\rho > \delta (\rho < \delta)$, if both stocks show adjacent complementarity. Moreover, the degree of adjacent complementarity has to be sufficiently high.

As mentioned above, we may observe persistent endogenous oscillations despite the fact that both stocks show distant complementarity. This is the content of Theorem 4.

Theorem 4 *If both stocks have distant complementarity the following holds:*
(i) The model is stable in the saddle point sense, if there are non-positive cross derivatives.
(ii) A necessary condition for persistent endogenous growth cycles is a positive cross effect between the stock of human capital and physical capital, i.e. $T_{KA} > 0$.

The proof of these theorems follows immediately from our considerations above.

Before we go on and present numerical examples to illustrate our results, let us briefly summarize what we have done up to now. We have shown that for our competitive two-sector economy persistent endogenous growth cycles may be the outcome. The economic prerequisite for that result was seen to lie in intertemporal substitution effects or positive cross derivatives with respect to physical capital and human capital or investment and physical capital[5]. It is clear that the competitive solution does not represent the social optimum since in the former, the positive spillover effects caused by investment are not taken into account. But even for the social optimum where those positive externalities are taken into account, we have seen that non-monotonic time paths may represent the outcome.

Does this result have repercussions for economic policy recommendations? It can be stated that the amount of investment determined by the social optimum maximization problem yields a higher discounted stream of consumption than the competitive economy. This simply results from the optimality condition $\int_0^\infty e^{-rt}T(A^\star, K^\star, I^\star)dt > \int_0^\infty e^{-rt}T(A, K, I)dt$, where $\star$ denotes optimal values. Therefore, as usual, the government has to give

[5] Note that the intertemporal substitution effect also only works, if there is a positive cross effect between investment and physical capital.

incentives leading to higher investment. It can do this by imposing a lump-sum tax τ_1 on consumption and using this tax to pay a subsidy of σ_1 units of consumption goods for any unit of investment good our representative individual intends to produce. The optimal subsidy σ_1 then has to be calculated in a way such that the opportunity costs of investment for the competitive economy equal those for the social optimum. For our simple economy it can easily be seen that σ_1 has to be chosen such that $\sigma_1 = \rho\gamma_2$ for all $t \in [0, \infty)$. Then the path of investment for the competitive economy equals the one for the social optimum. The equality $\sigma_1 = \rho\gamma_2$ becomes intuitively clear since γ_2 denotes the shadow price of human capital the individual does not take account of in his decision problem and ρ determines when the positive spillovers of investment on human capital take effect. The same result can be achieved if, instead of a lump-sum tax and a subsidy, a proportional tax is imposed on consumption. The effect of this tax consists in lowering the opportunity costs of investment and in shifting the PPF inwards (in direction to the origin). If a balanced budget is supposed, the shift of the PPF in direction to the origin must then be compensated by a parallel outward shift leaving the economy with a change in the slope of the PPF (it becomes steeper) in comparison to the original competitive situation. Imposing a proportional tax, consumption is given by $(1 - \tau_2)T(\cdot)$. The tax rate then has to be fixed in a way such that $-\tau_2 T_I(\cdot) = \rho\gamma_2$ holds which can easily be derived from the Hamiltonian maximizing condition (also shown in the appendix).

But it should be noted that the government imposing the tax, cannot be sure to rule out fluctuations of the economic variables. As we have seen oscillations may represent the optimal solution even for the social optimum.

5 Numerical Examples

As mentioned above, in this section we will illustrate our analytical results by numerical examples. In analogy to the previous section we will again divide this section in two subsections, one for the competitive economy, the other for the social optimum.

5.1 A Simulation Run for the Competitive Economy

For our numerical example we suppose a PPF of the form $T(A, K, I) = a_1K + a_2A - aI^2/(A+K) + b_1IK + b_2IA$ (for a similar relationship see Kuga (1972), p. 735). The evolution of the capital stock is given by $\dot{K} = I - \delta K$. For the parameter values we choose $a = 1, a_1 = 0.35, a_2 = 1, b_2 = -0.25, r = 0.25$ and $\delta = 0.75$, $\rho = 0.5$. Forming the Hamiltonian for problem (I), maximizing with respect to I and substituting this value in the differential equations, then gives our dynamic system as

$$\begin{aligned}
\dot{K}(t) &= (b_1K(t) + b_2A(t) + \gamma_1(t))(A(t) + K(t))/2a - \delta K(t), \\
\dot{\gamma}_1(t) &= (r+\delta)\gamma_1(t) - \frac{1}{4a}(b_1K(t) + b_2A(t) + \gamma_1(t))^2 \\
&- b_1(b_1K(t) + b_2A(t) + \gamma_1(t))(A(t) + K(t))/2a - a_1, \\
\dot{A}(t) &= \rho(b_1K(t) + b_2A(t) + \gamma_1(t))(A(t) + K(t))/2a - \rho A(t).
\end{aligned}$$

Using the parameter values from above and taking b_1 as bifurcation parameter we see that for $b_{1,crit} = 0.16541$ two eigenvalues of the corresponding Jacobian matrix are pure imaginary. The steady states for this value of b_1 are given by $K^\infty = 3.17228$, $\gamma_1^\infty = 0.9272186$, $A^\infty = 2.37921$, $I^\infty = 2.37921$, $C^\infty = 2.3031218$, $GNP^\infty = 4.5091696$. $GNP(t)$ denotes gross national product and is given by $GNP(t) = C(t) + \gamma_1(t)I(t)$. Note that $\gamma_1(t)$ denotes the price of investment in terms of the consumption good which is used as numeraire. The derivative of the real part with respect to the bifurcation parameter of the pure imaginary eigenvalues at $b_1 = b_{1,crit}$, is $Re\,\lambda_1'(b_{1,crit}) = 4.8916$ indicating the emergence of a Hopf bifurcation[6] possibly leading to stable limit cycles.

As to the degree of adjacent complementarity we calculate for A_1, $A_1 = 1.36981$ whereas both c and d are negative, for $b_1 = b_{1,crit}$. By varying b_1 we determine the sign of A_1 giving the degree of complementarity of the capital stock with respect to time.

Taking $b_1 = 0.16$ a little smaller than $b_{1,crit}$ we calculate the eigenvalues of the Jacobian as $\lambda_{1,2} = -0.0291483 \pm 0.626392\,i$, $\lambda_3 = -0.359898$ indicating that for this case the dynamic behaviour of the variables is characterized by a stable focus.

In figure 1 the path for investment and human capital is depicted. It can be seen that both paths show cyclical oscillations with declining amplitude. Moreover, investment shows higher oscillations than human capital and the latter always lags behind investment.

Figure 1 about here

In figure 2 the time path for GNP(t) is depicted (denoted as BSP(t)).

Figure 2 about here

Figure 3 finally shows a three dimensional projection of the stable focus in the $(I(t) - K(t) - C(t))$ phase diagram.

Figure 3 about here

As mentioned above, our dynamical system undergoes a Hopf bifurcation for $b_1 = 0.16541$. For values smaller than this critical value, the system is stable. If we take b_1 a little larger than 0.16541 and take $b_1 = 0.16542$ we can observe that we now have stable limit cycles. In figure 4 again the time paths for investment and human capital are depicted. We see that the amplitude of the oscillations now remains constant.

Figure 4 about here

In figure 5 the time path for consumption is depicted which also shows

[6] For the numerical computations and the solution of the differential equations we used the computer software Mathematica (see Wolfram Research (1991)).

persistent oscillations.

Figure 5 about here

In figure 6 and 7 the limit cycle is depicted in the $(I(t)-A(t))$ phase diagram and in the three dimensional $(I(t) - BSP(t) - A(t))$ phase diagram. The orientation is counter-clockwise. In figure 7 it can clearly be seen how the trajectory approaches the limit cycle demonstrating that it is an attractor.

Figure 6/7 about here

Let us now present a numerical example demonstrating the possibility of endogenously generated growth cycles for the social optimum.

5.2 Fluctuations in the Social Optimum

The PPF for the social optimization problem is again assumed to be given by $T(A,K,I) = a_1K + a_2A - aI^2/(A+K) + b_1IK + b_2IA$. The evolution of the capital stock is given by $\dot{K} = I - \delta K$ and human capital follows $\dot{A} = \rho(I - A)$. For the parameter values we now choose $a = 0.15, a_1 = 3.2, a_2 = 2.175, b_2 = 3.0855, r = 0.25$ and $\delta = 0.035$, $\rho = 0.15$.[7] Again, b_1 is selected as bifurcation parameter. Forming the Hamiltonian (taking explicitly the constraint $\dot{A} = \rho(I-A)$ into consideration), maximizing with respect to I and substituting this value in the differential equations then yields the modified Hamiltonian system as

$$\begin{aligned}
\dot{K}(t) &= (b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))(A(t) + K(t))/2a - \delta K(t), \\
\dot{A}(t) &= \rho(b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))(A(t) + K(t))/2a - \rho A(t), \\
\dot{\gamma_1}(t) &= (r+\delta)\gamma_1(t) - \frac{1}{4a}(b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))^2 \\
&- b_1(b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))(A(t) + K(t))/2a - a_1, \\
\dot{\gamma_2}(t) &= (r+\rho)\gamma_1(t) - \frac{1}{4a}(b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))^2 \\
&- b_2(b_1K(t) + b_2A(t) + \gamma_1(t) + \rho\gamma_2(t))(A(t) + K(t))/2a - a_2.
\end{aligned}$$

[7] Note that we can vary these parameter values by a time transformation.

To investigate this dynamic system we used the code BIFDD[8]. It turns out that this system has two pure imaginary eigenvalues for $b_1 = -0.712938$. The derivative of the real part of the pure imaginary eigenvalues with respect to b_1 at $b_1 = b_{1,crit}$ is given by $Re\lambda_1'(b_{1,crit}) = 24.73895$. BIFDD also calculates the coefficient β_2 determining the stability of the limit cycles which is given by $\beta_2 = -12.99039$. As $\beta_2 < 0$ the limit cycles are stable. The steady state values for these parameters are now seen to be $K^\infty = 18.46036$, $A^\infty = 0.6461126$, $\gamma_1^\infty = 9.612614$, $\gamma_2^\infty = 10.42188$, $I^\infty = 0.6461126$, $C^\infty = 53.2597$, $GNP^\infty = 59.470531$.

Given these informations we can solve our system of differential equations. For slightly larger values of b_1 than $b_{1,crit}$ we again can observe stable limit cycles.

Taking $b_1 = -0.7129$ the limit cycles are depicted in figure 8 and 9. Figure 8 shows how the trajectory approaches the limit cycle in the $(I(t) - C(t))$ phase diagram. In figure 9, the three dimensional limit cycle in the $(I(t) - A(t) - K(t))$ diagram is shown.

Figure 8/9 about here

For $b_1 = -0.71345 < b_{1,crit}$ the model is stable in the saddle point sense with the stable manifold representing the optimal solution. As for the competitive economy, we can observe that the dynamic path shows oscillations with declining amplitude. This case is illustrated in figure 10 and 11 in a two dimensional and three dimensional phase diagram, respectively.

Figure 10/11 about here

As to the economic mechanisms for our numerical example we see that there is distant complementarity with respect to physical capital, with $A_1 = -14.5202$, and adjacent complementarity with respect to human capital, with $A_2 = 16.2148$ for $b_1 = b_{1,crit}$ and the corresponding parameter values. Note that b_1 again influences the value of A_1 thus determining the degree of

[8]For a description of the related code BIFOR2 we refer to Hassard, Kazarinoff and Wan (1981).

complementarity of the stock of capital over time. As to the cross derivative T_{KA} we see that it is negative in the steady state, but extremely small, namely $T_{KA} = -1.79554 \cdot 10^{-5}$ such that this effect can be neglected.

6 Conclusion

The goal of this paper was to show that in a usual two-sector growth model where human capital is built according to Arrow's learning by doing, endogenously caused fluctuations may be the outcome. It should be mentioned that all of our results remain valid for a one-sector economy if a strictly concave utility function $u(c(t))$ is supposed. This can easily be seen if we substitute $T(A, K, I)$ by $u(F(A, K) - I)$, with $F(A, K)$ denoting the macroeconomic production function. A learning by doing framework was used as this model serves as a basis for a good deal of models belonging to the new growth theory.

In contrast to growth models with pure conventional goods, where certain factor intensities are the cause for persistent cycles (see e.g. Benhabib and Nishimura (1979), or Nishimura and Takahashi (1993)), this result does not hold for that sort of human capital model presented by us because human capital does not have a competitive price. Here, intertemporal substitution effects or positive cross effects may cause cyclical behaviour of the economic variables. Moreover, it cannot be ruled out that even for the social optimum, cycles may turn out to be optimal. But, of course, we do not assert that cycles are inherently good or bad and it is clear that governments can influence the steady state values of the variables by economic policy influencing the discount rate for example (the complete effect of such a policy could be calculated by comparative dynamics). But the fact we would like to emphasize with this paper is that growth cycles may be caused endogenously and cyclical fluctuations observed in reality cannot simply be dismissed as stochastic deviations from a balanced growth path or merely as a result from exogenously given shocks. Therefore policy makers always have to reckon with break-ins in the growth rates of capitalist economies and cannot rely on a competitive economy guaranteeing a steady state growth path. Even if the economy converges to the rest point in the long run, these oscillations do matter since this stability property is an asymptotic one, meaning that the steady state cannot be reached within finite time.

As to future research we believe that models revealing positive per-capita growth rates should be investigated concerning the possible appearence of endogenously caused cycles. Assuming that firms intentionally invest in the creation of new technical knowledge or their workers' skills (what they certainly do) more realistic models could be built. But then a competitive framework cannot be maintained if positive spillovers are postulated (cf. Schumpeter (1947), Shell (1967) or Romer (1990)). The dynamics of these sort of models is certainly worth investigating and especially the economic mechanisms leading to cyclical dynamics should be worked out.

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Captions of the Figures

Figure 1: Optimal paths of investment and human capital for the competitive economy (stable focus).

Figure 2: Optimal path of gross national product for the competitive economy (stable focus).

Figure 3: Stabel focus in the investment-capital-consumption space for the competitive economy.

Figure 4: Optimal paths of investment and human capital for the competitive economy (limit cycles).

Figure 5: Optimal path of consumption for the competitive economy (limit cycle).

Figure 6: Limit cycle in the investment-human capital plane for the competitive economy.

Figure 7: Limit cyle in the investment-GNP-human capital space for the competitive economy.

Figure 8: Limit cycle in the investment-consumption plane for the social optimum.

Figure 9: Limit cycle in the investment-human capital-capital space for the social optimum.

Figure 10: Stable focus in the consumption-human capital plane for the social optimum.

Figure 11: Stable focus in the consumption-capital-GNP space for the social optimum.

Figure 1

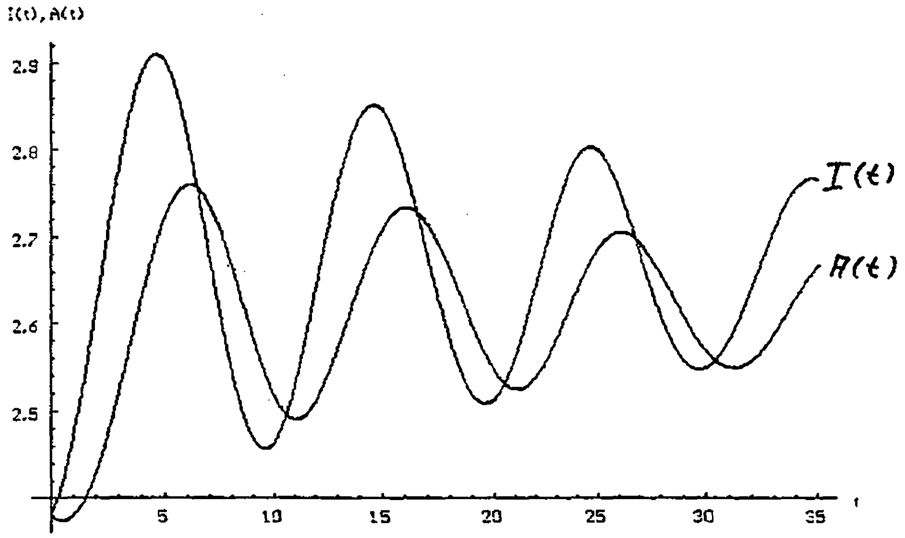


Figure 2

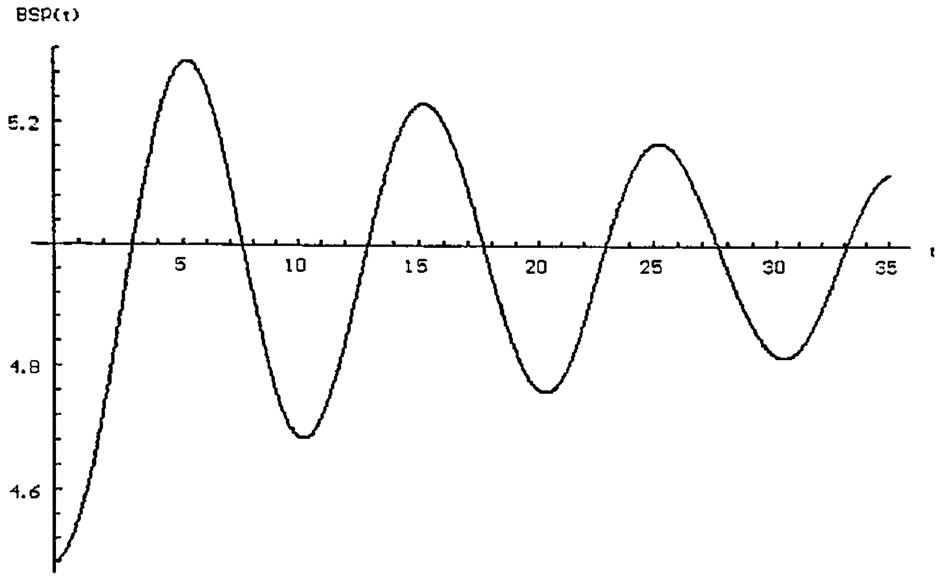


Figure 3

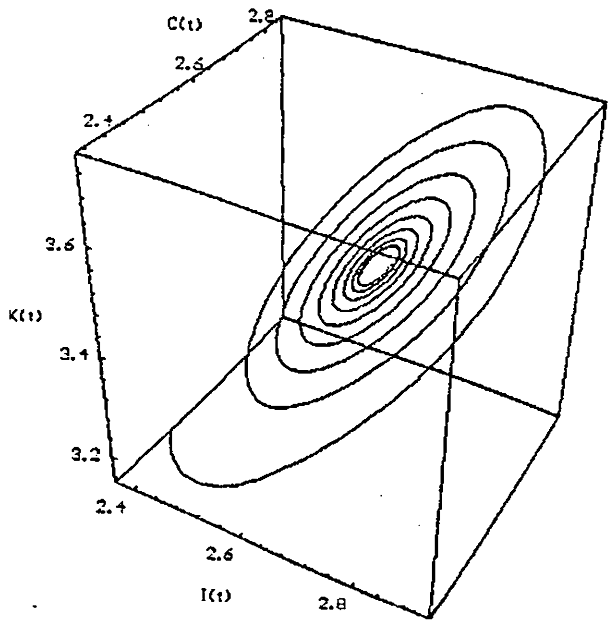


Figure 4

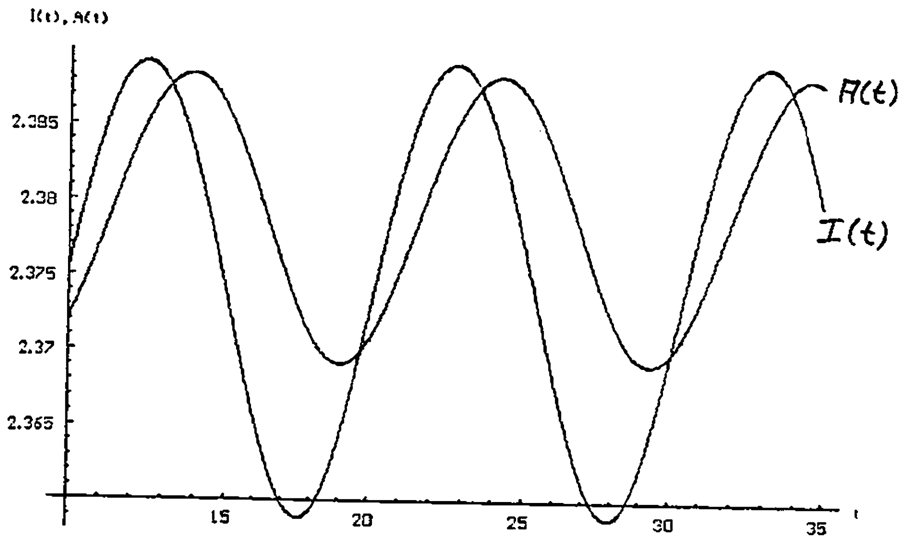


Figure 5

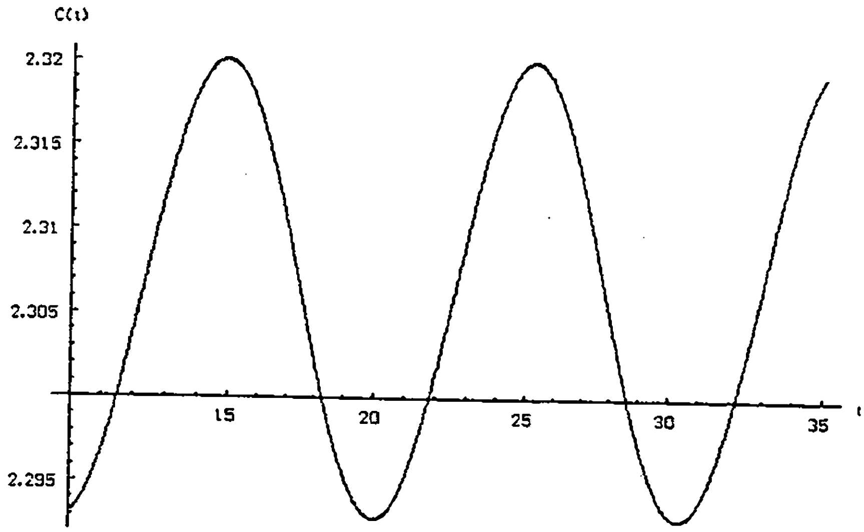


Figure 6

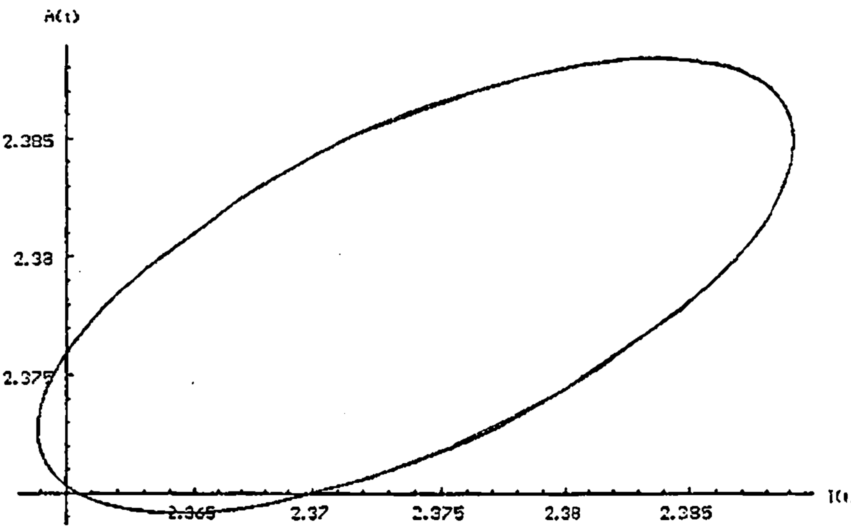
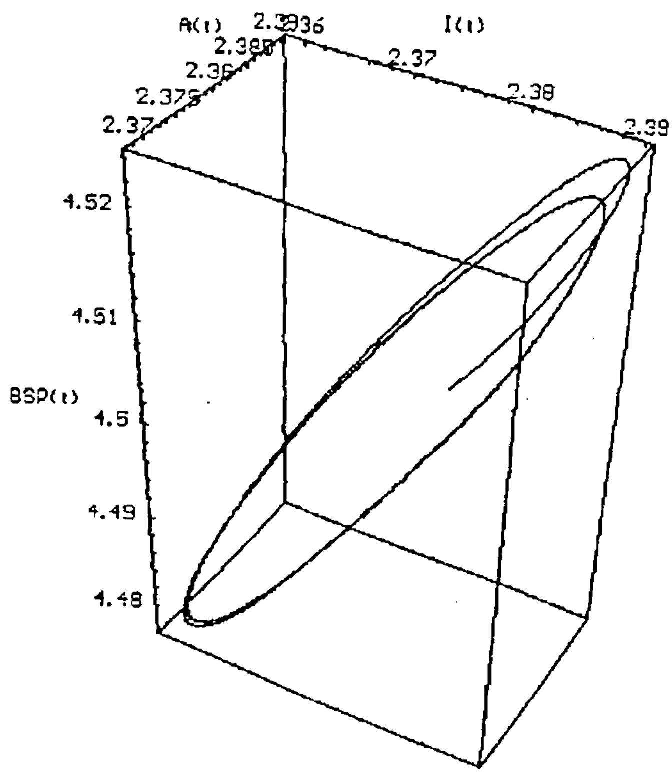


Figure 7



A(t)
2.3936
2.385
2.38
2.375
2.37
I(t)
2.37
2.38
2.39
4.52
4.51
BSP(t)
4.5
4.49
4.48

Figure 8

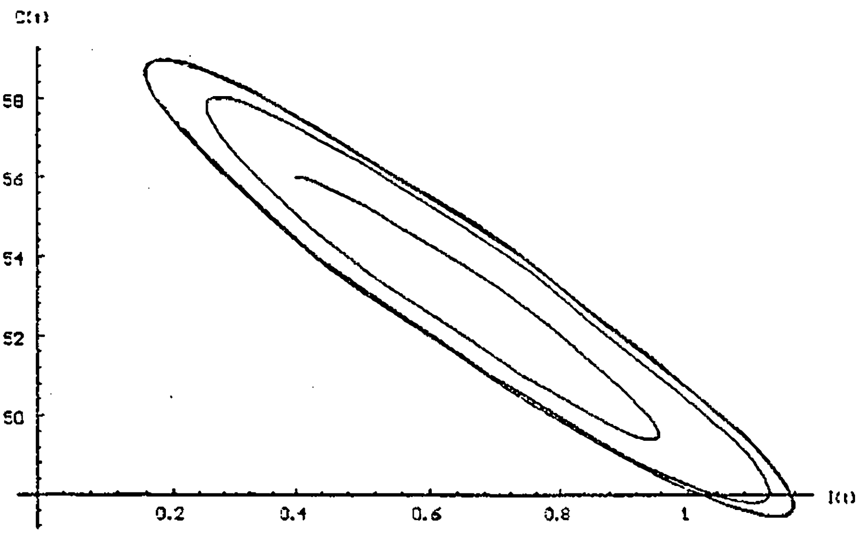


Figure 9

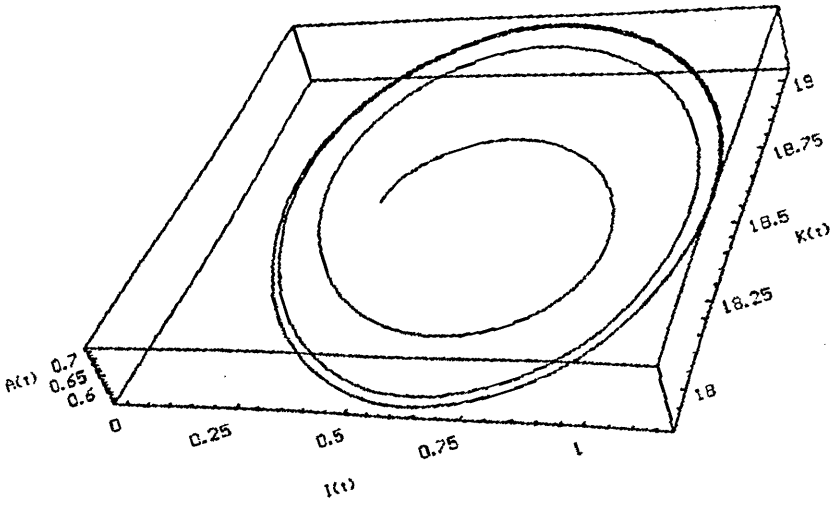


Figure 10

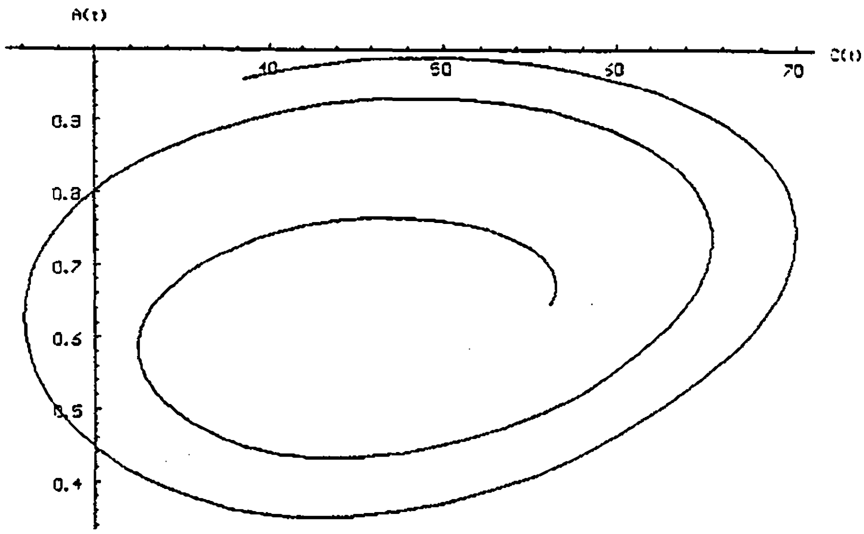
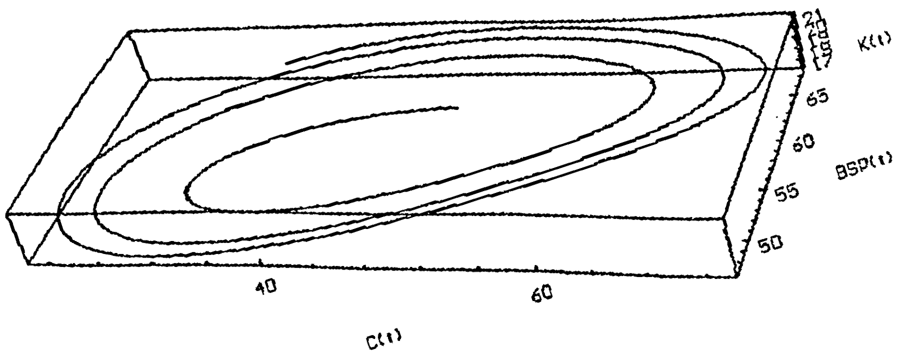


Figure 11



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