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LOCAL ROBUST STABILIZATION OF NONLINEAR OSCILLATORS UNDER PARAMETRIC EXCITATION

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ABSTRACT

The exponential growth behavior of nonlinear systems in the neighborhood of a fixed point is measured via the Lyapunov exponents of the linearized system at the fixed point. Hence Lyapunov exponents allow to obtain precise local stabilization results. This approach is utilized for robust stabilization of controlled oscillators with parametric excitation. We consider systems with arbitrary time varying perturbations in a given range, and analyze their local robust stabilizability at a fixed point using state feedback controls with values in a given set. The corresponding feedback results are compared to the situation where the controlled parameter is tuned to an optimal (but fixed) value. The van-der-Pol oscillator is treated in detail.

I. Introduction

Robust stabilization of nonlinear systems with parametric excitations has enjoyed considerable attention recently, as models with uncertain dynamics and with non negligable excitation have become common place. Various approaches to robust, nonlinear stabilization have been proposed, such as nonlinear H^∞-theory (see e.g. van der Schaft (1991) or Isidori et al. (1992)), or Lyapunov function techniques that utilize the fact that a known Lyapunov function may still work if perturbations of a certain size are included in the model. For one-dimensional systems, Lai and Lin (1994) have obtained precise results, by extending the concept of control sets (see Colonius et al. (1994a)) to uncertain control systems. They show in particular that robust stabilization via feedback controls does not imply robust stabilization using open loop controls, while the converse is always true. Therefore, we consider feedback systems in this paper.

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In this paper, we consider the problem of robust feedback stabilization for nonlinear oscillators of the form

(1)
$$\ddot{y}+(f_1(y,\dot{y})+uf_2(y,\dot{y}))\dot{y}+(g_1(y,\dot{y})+wg_2(y,\dot{y}))y=0,$$

where u is the control and w is the parametric excitation term. (Other combinations of control and excitation in the damping and/or restoring force can be treated in a similar way.) The problem addressed here is: Given a perturbation range $W\subset\mathbb{R}$ and a control range $U\subset\mathbb{R}$, does there exist a state feedback $U:\mathbb{R}^2\to U$ that stabilizes the system (1) at the fixed point $(y,\dot{y})=(0,0)$ for all possible (time varying) excitations with values in W? This includes in particular the situation, where W is a stochastic process with values in W. Stabilization in this context means local exponential stabilization at $(0,0)$.

More generally, we consider a family $W^\rho,\rho\geq 0$ of perturbation ranges and a family $U^\sigma,\sigma\geq 0$ of control ranges, and determine the exact borderline in the $\sigma-\rho$ plane that separates stabilizable from non-stabilizable systems. Specific attention is devoted to the question whether robust system performance can be enhanced by feedback controls with values in U^σ, as compared to simply tuning the parameter u to a constant (optimal) value in U^σ. The second strategy corresponds to using e.g. optimal material constants, while the use of a (non constant) feedback control requires active control elements in the system.

The method employed in this paper for robust local stability consists of linearizing the system around a fixed point, and computing the Lyapunov exponents of the linearized system in order to obtain the precise exponential growth behavior around the fixed point. (For a comparison of Lyapunov exponent and Lyapunov function design in perturbed linearized systems see e.g. Colonius et al. (1991).) If there exists a feedback control with values in U^σ such that the Lyapunov exponent of the linearized system is negative for all possible excitations with values in W^ρ, then this $\sigma - \rho$ pair corresponds to a robustly stabilizable linearized system. A theorem of Pinsky (1993) then guarantees that the nonlinear system is locally robust stabilizable at the fixed point.

Local stabilization is one part of the stabilization strategy for nonlinear systems at fixed points. The other part involves computing the domain of attraction of the fixed point (robust stability region) and a feedback control, again with values in some given range U^σ, that steers the system from any point in the domain of attraction to a neighborhood of the fixed point, independent of the perturbation. In this neighborhood the local strategy, computed via linearization as described above, can be used. For one-dimensional systems, this strategy is analyzed in Lai and Lin (1994). For nonlinear oscillators, global feedback analysis for perturbed control systems is currently under investigation.

This paper is organized as follows: In Section II. we describe the linearization of controlled oscillators with parametric excitation. Section III. analyzes the Lyapunov exponents of the linearized system and optimal robust stabilization techniques. In Section IV. we discuss the van-der-Pol oscillator in detail.

II. Linearization of nonlinear oscillators

Consider the nonlinear oscillator with control and excitation term

(1)
$$\ddot{y} + (f_1(y,\dot{y}) + u f_2(y,\dot{y}))\dot{y} + (g_1(y,\dot{y}) + w g_2(y,\dot{y}))y = 0,$$

where w is a time varying perturbation, i.e. $w \in \mathcal{W} = \{w : \mathbb{R} \to W, \text{measurable}\}$, and the perturbation range W satisfies: $W \subset \mathbb{R}$ is a closed, bounded interval with 0 in its interior. The control u is a state feedback, i.e. $u \in \mathcal{F} = \{u : \mathbb{R}^2 \to U, \text{measurable}\}$, and the control range $U \subset \mathbb{R}$ is a closed, bounded interval with 0 in its interior. This means that we are looking at systems with given, bounded excitation and control range. The standing assumptions are:

(A) The functions f_1, f_2, g_1, g_2 are C^∞. For all initial conditions Equation (1) has a unique solution for all $w \in \mathcal{W}, u \in \mathcal{F}$, and $t \geq 0$. The fixed point $(y, \dot{y}) = (0,0)$ is an isolated fixed point.

Linearization of the nonlinear oscillator (1) with respect to the state variables $(y, \dot{y})$ around the fixed point $(0,0)$ yields

$$\ddot{z} + (f_1 + u f_2)\dot{z} + (g_1 + w g_2)z = 0, \tag{2}$$

where $f_1 = f_1(0,0), f_2 = f_2(0,0), g_1 = g_1(0,0), g_2 = g_2(0,0)$ are the values of the corresponding functions at the fixed point $(y, \dot{y}) = (0,0)$. Rewriting Equation (2) as a first order system with $(z, \dot{z}) = (x_1, x_2)$ gives

(3)
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g_1 & -f_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ 0 & -f_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w \begin{pmatrix} 0 & 0 \\ -g_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

For an initial value $x^0 = (x_1^0, x_2^0)$, and for $w \in \mathcal{W}, u \in \mathcal{F}$, we denote the solution of (3) at time $t \geq 0$ by $x(t, x^0, w, u)$. The Lyapunov exponent of a solution describes its exponential growth behavior (stability) with respect to the constant solution $x(t) \equiv 0$, and is defined as

$$\lambda(x^0, w, u) = \limsup_{t \to \infty} \frac{1}{t} \log |x(t, x^0, w, u)|. \tag{4}$$

Given a control $u \in \mathcal{F}$, maximal exponential growth of the solutions of (3) is described by

$$\lambda(u) = \sup_{w \in \mathcal{W}} \sup_{x^0 \neq 0} \lambda(x^0, w, u). \tag{5}$$

The question of robust exponential stabilization of (3), therefore, boils down to computing

$$\kappa = \min_{u \in \mathcal{F}} \lambda(u). \tag{6}$$

I.e. the linearized system (3) is exponentially stabilizable at 0 for all excitations $w \in \mathcal{W}$ if and only if there exists $u \in \mathcal{F}$ such that $\lambda(u) < 0$, i.e. if and only if $\kappa < 0$.

The following theorem is a special case of a result by Pinsky (1993), and it relates exponential stability of the linearized system (3) to local exponential stability of the nonlinear system (1).

Theorem 1. *If $\kappa < 0$, then there exists a feedback control $u \in \mathcal{F}$ and a neighborhood N of the fixed point $(0,0)$ of the nonlinear oscillator (1) such that the system (1) is exponentially stable in N. If $\kappa > 0$, then the nonlinear system (1) is exponentially unstable at $(0,0)$.*

This theorem reduces the problem of local robust stabilization of nonlinear systems to the computation of a $u \in \mathcal{F}$ with $\lambda(u) < 0$. In the next section we present some background material on Lyapunov exponents for two-dimensional linearized systems.

III. Lyapunov exponents and robust feedback stabilization

In this section we present some results on Lyapunov exponents for controlled systems with parametric excitation. These results allow us to extend some of the findings in Colonius et al. (1994b, 1994c) to robust feedback design.

Note first of all, that for an initial value $\alpha \cdot x^0$ with $\alpha \in \mathbb{R}, \alpha \neq 0$, we have $\lambda(\alpha x^0, w, u) = \lambda(x^0, w, u)$. Hence the Lyapunov exponents of (3) are determined by the angular behavior of the system, i.e. by its behavior on the projective space $\mathbb{P}$ in $\mathbb{R}^2$. Identifying the projective space with the angle space $[0,\pi)$, one obtains for the angular dynamics of (3)

$$\begin{aligned}\dot\varphi &= -\sin^2\varphi + (-f_1 - uf_2)\sin\varphi\cos\varphi \\ &\quad + (-g_1 - wg_2)\cos^2\varphi =: h(\varphi, w, u),\end{aligned} \tag{7}$$

and for the Lyapunov exponents

(8)

$$\lambda(x^0, w, u) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(\varphi(s,\varphi^0, w, u), w, u)\,ds$$

where $q(\varphi, w, u) = (1-g_1-wg_2)\sin\varphi\cos\varphi + (-f_1-uf_2)\sin^2\varphi$, and φ^0 is the projection of x^0 onto the projective space, compare e.g. Colonius et al. (1993). Equation (8) implies in particular that a stabilizing feedback, if it exists, can be chosen constant on the rays through the origin, i.e. as a function in $\mathcal{F}_{\mathbb{P}} = \{u : \mathbb{P} \to U$, measurable$\}$.

Fix $u \in \mathcal{F}$ for the moment, and consider $\lambda(u)$ as defined in (5). By the results in Joseph (1993) (compare Colonius et al. (1994b)), $\lambda(u)$ can be obtained in one of the following two ways:

Either (a) $\lambda(u)$ is a (maximal) real eigenvalue of (3) for some constant $w \in W$,

or (b) $\lambda(u)$ is obtained for a perturbation $w \in W$ such that the solution $\varphi(t, \varphi^0, w, u)$ of (7) is is periodic with period

$T = \min\{t > 0, \varphi(t, \varphi^0, w, u) = \varphi^0\}$, and $\varphi(t, \varphi^0, w, u)$ has no loops in $[0, \pi)$.

Computing the optimal κ (as defined in (6)) therefore means: For each $u \in U$ compute $\lambda^a(u) = \max_{w\in W} \mu(u, w)$, where $\mu(u, w)$ is the (maximal) real eigenvalue of the r.h.s. of (3) for u and w, and for each $u \in \mathcal{F}_{\mathbb{P}}$ compute $\lambda(u)$ as supremum over all $w \in W$ that satisfy (b), which we denote by $\lambda^b(u)$. Then $\kappa = \min_{u\in\mathcal{F}} (\max\{\lambda^a(u), \lambda^b(u)\})$.

As a consequence, we obtain the following partial result on robust stabilization:

Theorem 2. *Consider the linearized system (3) and its optimal Lyapunov exponent κ, given by (6).*

- (i) *If there exists $w \in W$ with $g_1 + wg_2 \leq 0$, then $\kappa \geq 0$.*
- (ii) *If $g_1 + wg_2 > 0$ for all $w \in W$ and*
  - *(ii1) if $(f_1 + uf_2)^2 \geq 4(g_1 + wg_2)$ for all $u \in U, w \in W$, then $\kappa < 0$ iff there exists $u_1 \in U$ with $f_1 + u_1 f_2 > 0$; in this case the system is robust stabilizable via constant feedback $F_1(\varphi) \equiv u_1$ for $\varphi \in [0, \pi)$,*
  - *(ii2) if there exists $u_2 \in U$ with $(f_1 + u_2 f_2)^2 \geq 4(g_1 + wg_2)$ and $f_1 + u_2 f_2 > 0$, then $\kappa < 0$; in this case a robust stabilizing feedback is given by $F_2(\varphi) \equiv u_2$ for $\varphi \in [0,\pi)$.*

Proof.

- (i) Computing the eigenvalues for the r.h.s. of (3) yields $\mu(u, w) = -\frac{1}{2}(f_1 + uf_2) \pm \sqrt{\frac{1}{4}(f_1 + uf_2)^2 - (g_1 + ug_2)}$. Let $w_0 \in W$ with $g_1 + w_0 g_2 \leq 0$, then for all $u \in U$ there exist two real eigenvalues, hence two eigendirections, and we are in case (a) above, see Colonius et al. (1994b). The larger of the two eigenvalues is ≥ 0 for all $u \in U$, hence $\kappa \geq 0$.
- (ii) The projected system (7) has a proper invariant subset on $[0,\pi)$ iff $R^+ \neq \emptyset$ and $R^- \neq \emptyset$ (see Lai and Lin (1994)), where $R^+ = \{\varphi \in [0,\pi)$; there exists $u \in U$ with $h(\varphi, w, u) > 0$ for all $w \in W\}$, and $R^- = \{\varphi \in [0,\pi)$; there exists $u \in U$ with $h(\varphi, w, u) < 0$ for all $w \in W\}$. Note that $R^- = \emptyset$ in any case, because $h(\frac{\pi}{2}, w, u) < 0$ for all $w \in W$, all $u \in U$. And $R^+ \neq \emptyset$ iff there exists $u \in U$ with $(f_1 + uf_2)^2 \geq 4(g_1 + wg_2)$ for all $w \in W$. (ii1) guarantees that there exists a proper invariant subset of $[0,\pi)$ for all $u \in U$, hence we are in case (a) above and $\kappa < 0$ iff there exists $u \in U$ with $f_1 + u_1 f_2 > 0$, compare Colonius et al. (1994b). In situation (ii2), choosing $F_2(\varphi) \equiv u_2$ results in a proper

invariant subset, and its maximal eigenvalue is negative, hence $\kappa < 0$. □

Remark. Some local stabilization strategies for homogeneous systems (they include systems of the form (3) as special cases) are based on the idea that one first chooses a feedback to obtain a 1-point invariant set of the projected system on the projective space, and then stabilizes the system in this direction. Note that for perturbed systems of the form (3) is strategy is not applicable, because 1-point invariant sets for (7) exist only if $g_2 = 0$, i.e. if the perturbation in (1) does not show up in the linearization. In this case, optimal stabilizing feedbacks can be computed as in Colonius et al. (1993).

Theorem 2. covers only the situations, in which the existence of a robust stabilizing feedback can be decided by considering the real eigenvalues of the r.h.s. of the linearized system (3). In these cases, if the system is stabilizable, then a constant feedback suffices. I.e. optimal robust stabilization even in the presence of time varying excitations boils down to the appropriate tuning of the system parameter $u \in U$. In all other cases, time varying perturbations and non- constant feedback laws have to be considered. In general, this can only be done numerically by exploring the characterizations (a) and (b) above. The next section discusses an example along these lines.

IV. The Van-der-Pol oscillator

In this section we discuss the van-der-Pol oscillator with controlled damping and uncertain restoring force. Other nonlinear oscillators can be treated in a similar fashion.

Consider the equation

$$\ddot{y} + u(y^2+1)\dot{y} + (1+w)y = 0. \tag{9}$$

Linearizing this equation around the fixed point $(y, \dot{y}) = (0,0)$ yields

(10)
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + w\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

For the control, and the excitation range, respectively, we use the families of sets $U^\sigma = [-\sigma, \sigma], \sigma \geq 0$, and $W^\rho = [-\rho, \rho], \rho \geq 0$.

First we consider this system with constant $u \in U^\sigma$, i.e. treating the damping as a tuning parameter, but time varying excitations $w \in W^\rho$. Figure 1. shows the level curves of the optimal κ for this case in the $\sigma - \rho$ plane, for $\sigma \in [0,3]$ and $\rho \in [0,2]$. Note that the line $\kappa = 0$ separates unstable systems from those that are exponentially stable for the corresponding $\sigma - \rho$ combination, compare Colonius et al. (1993).

Next, we discuss the consequences of Theorem 2. for the van-der-Pol oscillator:

(i) If $\rho \geq 1$, then the system is not stabilizable by any feedback $u \in \mathcal{F}$.

(ii) If $\rho < 1$ and $\sigma^2 \geq 4(1+\rho)$, then the system is stabilizable via the constant feedback $F \equiv \sigma$.

Figure 2. shows the corresponding stability and instability regions in the $\sigma = \rho$ plane.

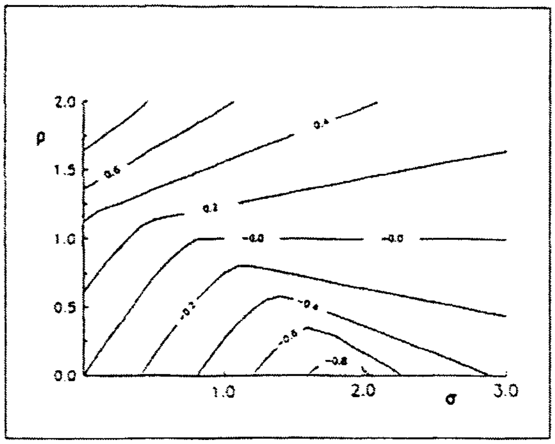

Figure 1. Level curves of the optimal Lyapunov exponent κ for the system (10).

Comparing Figures 1. and 2. we see that an improvement of stability via nonstant feedbacks can only be expected in the $\sigma - \rho$ parameter region $[0, 0.81] \times [0,1)$, where $\sigma \sim 0.81$ corresponds to the point, where the $\kappa = 0$ level curve in Figure 1. meets the $\rho = 1$ line. In this region, the optimal Lyapunov exponent κ and the corresponding excitation $w \in \mathcal{W}$ and feedback $u \in \mathcal{F}_{\mathbb{P}}$ have to be computed numerically.

This is done using the alternatives (a) and (b) in Section III, where in case (b) the formula (8) is used for periodic solutions on $[0, \pi)$ in the following way:

$$\lambda(w,u) = \frac{1}{T}\int_0^\pi \frac{q(\varphi, w, u)}{|h(\varphi, w, u)|} d\varphi \tag{11}$$

with $T = \int_0^\pi \frac{1}{|h(\varphi,w,u)|} d\varphi$, i.e. by converting the time into the space variable on the projective space $\mathbb{P}$.

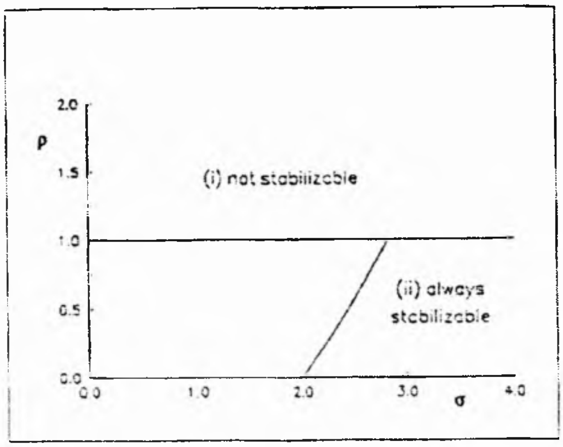

Figure 2. Stabilizable and not stabilizable regions of the oscillator (10) according to Theorem 2.

The optimization problem (5,6) to obtain κ, and the corresponding excitation and control, was solved by discretizing $[0,\pi), U^\sigma = [-\sigma,\sigma]$ for $\sigma \in [0,0.8]$, and $W^\rho = [-\rho,\rho]$ for $\rho \in [0,1]$. For each σ and ρ ten partition intervals were used in U^σ and in W^ρ, and 63 partition intervals were used in $[0,\pi)$. For each σ and ρ, the corresponding program runs between 17 and 21 h on a DEC station 3000 with the ALPHA chip.

The results of these computations are:

(1) The worst excitation leading to $\lambda(u)$ as defined in (5), is attained either by a constant value $w \in W^\rho$, or by a function $w(t) \in \mathcal{W}^\rho$ that takes on two values.

(2) The optimal control $u \in \mathcal{F}_\mathbf{P}$ that realizes κ as defined in (6), is constant for all $\sigma - \rho$ combinations that were analyzed, namely the σ-values 0.25, 0.5, and 0.65, and the ρ-values 0.25, 0.5, 0.75, and 1.0. These $\sigma - \rho$ combinations were chosen, because they represent, for fixed controls $u \equiv \sigma$, a variety of different forms for the perturbation $w \in \mathcal{W}^\rho$ leading to the maximal exponent $\lambda(u)$. In each case the constant $u(\varphi) \equiv \sigma$ provided the smallest exponential growth rate among the $\lambda(u)$.

As a consequence of our studies we conjecture that robust feedback stabilization of the linearized system (10), and hence robust local exponential stabilization of the van-der-Pol oscillator (9) at the fixed point $(y,\dot{y}) = (0,0)$ can be achieved exactly in the stability region indicated in Figure 1., and stabilization, if possible at all, can always be achieved using a constant sible at all, can always be achieved using a constant feedback. Hence active control does not show any performance improvement over optimal parameter tuning in this case. It is interesting to note, however, that the Lypanov exponent design featured in this paper, improves upon Lyapunov function techniques for the stabilization of uncertain systems, compare e.g. Figure 1. in Colonius et al. (1991).

The results presented here do not imply that constant feedbacks yield optimal performance for local stabilization in all types of nonlinear oscillators. Further investigations in this direction are currently under way.

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