Limit Behavior and Genericity for Nonlinear Control Systems*

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For nonlinear control systems of the form $\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x)$ with constrained control range $U \subset \mathbb{R}^m$ the limit behavior of the trajectories is analyzed. The limit sets are closely related to regions of the state space where the system is controllable. In particular, under a rank condition on the linear span of the derived vectorfields on a limit set, this limit set is contained in the interior of a control set. This result is a mathematical basis for the control of complicated behavior. It also allows the characterization of Morse sets of a differential equation as intersections of topologically transitive sets of control systems. Topological genericity theorems classify the possible limit behavior of a control system for open and dense sets in $\mathcal{U} \times M$, where $\mathcal{U}$ is the space of admissible control functions, and M is the state space. The methods are a combination of control theoretic arguments and chain transitivity for the control flow on $\mathcal{U} \times M$. The results are applied to the Lorenz equations, showing that its strange attractor is contained in a region of controllability, in which its dynamics can be altered, e.g., to yield periodic motions.

1. INTRODUCTION

In this paper we analyze the qualitative behavior of trajectories of nonlinear control systems and relate it to controllability properties, and we discuss genericity. This is accomplished by applying methods from global theory of dynamical systems to the control flow (defined in [CK(93)]) associated with the control system. The controls may also be interpreted as

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(deterministic, time-varying) disturbances. Thus our results may also be viewed as a contribution to global perturbation theory of nonlinear differential equations.

In order to describe recurrence properties of flows, a number of concepts of varying generality have been studied in the dynamical systems literature (cp. [Co(78), Ru(89), Ma(87), ER(85)]). In particular, we have the following relationship between various kinds of recurrent points for general dynamical systems: chain recurrent points $\supset$ ω-limit sets $\supset$ supports of ergodic measures. Limit sets are of particular importance in the study of attractors (i.e., sets A with a neighborhood U such that $\omega(U)=A$), the study of Morse decompositions associated with attractor–repeller sequences, and the study of complicated behavior and strange attractors. From the numerous results on the global behavior of dynamical systems we would like to mention in particular D. Ruelle's result [Ru(81)], which shows almost sure convergence to attractors for dynamical systems subject to small random perturbations.

In control theory, it is of particular interest to control complicated behavior; e.g., given a differential equation

$$\dot{x}=X_0(x)$$

with complicated limit behavior, can one simplify the behavior by applying suitable control functions (which act in an a priori given way on the system)? The papers [Hu(89), Fo(89), OGY(90)] contain approaches to this problem quite different from the one presented here.

In this paper we consider the following class of control–affine nonlinear systems with restricted control values,

$$\dot{x}(t)=X_0(x(t))+\sum_{i=1}^{m} u_i(t)\, X_i(x(t)), \qquad t\in\mathbb{R} \tag{1.1}$$

$$x(0)=x\in M \tag{1.2}$$

$$u\in\mathcal{U}:=\{u\colon \mathbb{R}\to\mathbb{R}^m : u(t)=(u_i(t))\in U \text{ for a.a. } t\in\mathbb{R}, \text{ locally integrable}\}, \tag{1.3}$$

where $X_0, ..., X_m$ are given C^∞ vectorfields on a smooth manifold M and $U\subset\mathbb{R}^m$ is convex and compact.

For simplicity, we assume that for all $u\in\mathcal{U}$, $x\in M$ there exists a unique global solution $\varphi(t, x, u)$, $t\in\mathbb{R}$, of (1.1), (1.2).

Given the control system (1.1)–(1.3) we characterize the limit sets

$$\pi_M\omega(u, x):=\{y\in M : \text{there are } t_k\to\infty \text{ with } \varphi(t_k, x, u)\to y\}. \tag{1.4}$$

This is done via the associated control flow Φ on $\mathcal{U} \times M$ (with an appropriate metric on $\mathcal{U}$ [CK(93)]):

$$\Phi: \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \qquad \Phi(t, u, x) := (u(t + \cdot), \varphi(t, x, u)).$$

We study the ω-limit sets of the control flow and its chain recurrent components, which induce "chain control sets" on M.

In particular, we characterize those pairs $(u, x) \in \mathcal{U} \times M$ for which $\pi_M \omega(u, x)$ lies in the interior of a region of complete controllability (i.e., a control set). Within such a region, H. Sussmann's theory of subanalytic feedbacks [Su(79)] can (in the analytic case) be applied. We show that, generically, limit sets of controlled trajectories lie in the interior of control sets. This relation between limit sets and controllability is remarkable, since it connects the asymptotic behavior for $t \to \infty$ with the finite time property of controllability. Here, genericity is understood in the topological sense of "holding on an open and dense set," not just on a residual set. Since there is no canonical shift invariant probability measure on the space $\mathcal{U}$ of admissible control functions, measure theoretic genericity, i.e., statements holding with probability one as in Ruelle's theorem [Ru(81)] or Oseledeč's theorem [Os(68)], is not adequate in this context.

The contents of this paper is as follows: In Section 2, we describe the setup and some basic concepts for control systems with bounded control range that are not completely controllable. We briefly recall necessary tools and results from [CK(93)], where control flows were introduced and their basic properties were discussed. Section 3 defines the notion of "inner pairs," which allows us to derive a result in Proposition 3.7 that may be considered as a control theoretic analogue of Bowen's [Bo(75)] shadowing lemma (showing how to obtain orbits close to pseudo-orbits). Section 4 contains the main results on the relation between limit behavior and controllability: Theorems 4.4 and 4.13 assure the existence of control sets that contain given limit sets in their interior under different sets of assumptions. The same situation is considered in Section 5 from a perturbation point of view: As the size of the control range varies with $\rho \geqslant 0$, the limit sets for ρ_0 are contained in the control sets for $\rho_1 > \rho_0$. In particular, the Morse sets of the dynamical system $\dot{x} = X_0(x)$ are the intersections of topologically transitive sets of a control system for small $\rho > 0$, if this system is sufficiently rich (Corollary 5.3). This result is a topological analogue of Ruelle's perturbation theorem in [Ru(81)]. In Section 6 the generic limit behavior of the trajectories of a control system is analyzed: Theorem 6.2 and the following propositions show that generically (on an open and dense set) in $\mathcal{U} \times M$ the closed (and the open) control sets contain all limit sets for $t \to \infty$ (and for $t \to -\infty$, respectively). This result can be rephrased in terms of stabilization theory as: There is no robust open loop stabilization outside of invariant (= closed) control sets (cp. also [CK(91)]). In Section 7 we apply the controllability results from Sections 4 and 5 to the Lorenz equation, showing that its "strange attractor" is contained in the closure of a control set with nonvoid interior. Consequences for the "control of chaos" in this and many other examples that can be treated with Theorems 4.4 and 4.13 are outlined in Remark 7.3.

Notation. We consider the state space M of (1.1), (1.2) as a Riemannian C^∞ manifold, and $d(\cdot, \cdot)$ denotes the Riemannian metric on M. The projections of $\mathscr{U} \times M$ onto its components are $\pi_{\mathscr{U}}$, and π_M, respectively. For a metric space (S, d) the interior and the closure of a set $A \subset S$ are denoted by int A and cl A; for $x \in S$, $B_\delta(x) := \{y \in S;\ d(x, y) < \delta\}$.

2. Preliminaries. The Control Flow of a Control System

In this section, we first specify our assumptions on the control system (1.1)–(1.3). Then we recall notions and results from [CK(93)] relating control systems to dynamical systems. We assume that the state space M of (1.1) is a paracompact Riemannian C^∞ manifold of dimension $d < \infty$ and $X_0, X_1, ..., X_m$ are C^∞ vectorfields on M. Furthermore we assume —unless otherwise specified—that the distribution $\Delta_{\mathscr{L}}$ generated by the Lie algebra $\mathscr{L} = \mathscr{L}\mathscr{A}\{X_0 + \sum u_i X_i;\ (u_i) \in U\}$ coincides with $T_x M$ at every $x \in M$, i.e.,

$$\dim \Delta_{\mathscr{L}}(x) = d = \dim M \qquad \text{for all} \quad x \in M. \tag{H}$$

Intuitively, the Lie algebra condition (H) means that at any point $x \in M$ the control system (1.1) can move in all directions of $T_x M$, i.e., M is the maximal integral manifold for the family of vectorfields $\{X_0 + \sum u_i X_i;\ (u_i) \in U\}$; see also Example 2.7 below. The positive orbits of (1.1) from $x \in M$ up to time $t \geq 0$ are defined as

$$\mathcal{O}^+_{\leq t}(x) := \{y \in M\text{: there exist } u \in \mathscr{U} \text{ and } 0 \leq \tau \leq t \text{ with } \varphi(\tau, x, u) = y\}.$$

Similarly, the negative orbits of x up to time $t \geq 0$ are

$$\mathcal{O}^-_{\leq t}(x) := \{y \in M\text{: there exist } u \in \mathscr{U} \text{ and } 0 \leq \tau \leq t \text{ with } \varphi(\tau, y, u) = x\},$$

and

$$\mathcal{O}^+(x) := \bigcup_{t \geq 0} \mathcal{O}^+_{\leq t}(x), \qquad \mathcal{O}^-(x) := \bigcup_{t \geq 0} \mathcal{O}^-_{\leq t}(x).$$

Then (H) implies (cp., e.g., [Is(89), NvS(90)]) that

$$\text{int}\, \mathcal{O}^+_{\leq t}(x) \neq \varnothing \qquad \text{and} \qquad \text{int}\, \mathcal{O}^-_{\leq t}(x) \neq \varnothing$$

for all $x \in M$ and all $t > 0$.

It is actually this property of the positive and negative orbits (often called local accessibility) which is crucial for many arguments in this paper. It says that the orbits are topologically "thick" in the state space, but, in general, they are strictly contained in M, i.e., the system is not completely controllable. Therefore, one must define the control structure of (1.1) on M. The following notion is central for control systems that are not completely controllable (cp. also [CK(93), Remark 3.2]).

2.1. Definition. A set $D \subset M$ is called a control set of (1.1)–(1.3) if for all $x \in D$, (i) there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$, (ii) $D \subset \text{cl}\, \mathcal{O}^+(x)$, and (iii) D is maximal (w.r.t. set inclusion) with these properties.

If (H) holds, exact controllability in int D follows. More precisely, define the first hitting time map $h: M \times M \to \mathbb{R} \cup \{+\infty\}$ by

$$h(x, y) := \inf_{u \in \mathcal{U}} \{t \geq 0: \varphi(t, x, u) = y\}.$$

Then [CK(89), Proposition 2.3] shows the following assertion.

2.2. Proposition. *Let $D \subset M$ be a control set and let $K_1 \subset D$ and $K_2 \subset \text{int}\, D$ be compact. Then there exists $T = T(K_1, K_2) < \infty$ such that $h(x, y) \leq T$ for all $x \in K_1$, $y \in K_2$. In particular,* int $D \subset \mathcal{O}^+(x)$ *for all* $x \in D$.

2.3. *Remark.* Actually, the proof of the result above shows that the same assertion is true if only piecewise constant controls $u \in \mathcal{U}$ are admitted.

Associated with the control system (1.1)–(1.3) is a (continuous) dynamical system Φ on $\mathcal{U} \times M$, the induced control flow Φ (cp. [CK(93)] for the following results):

Define $\Phi: \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M$ by

$$\Phi(t, u, x) := (u(t + \cdot), \varphi(t, x, u)), \qquad t \in \mathbb{R}, \quad (u, x) \in \mathcal{U} \times M.$$

Here $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^*$ is equipped with the weak* topology; this makes $\mathcal{U}$ a completely metrizable compact space and we fix a corresponding metric d. Convergence of $u_n \to u$ in $\mathcal{U}$ implies uniform convergence of the solutions $\varphi(\cdot, x, u_n) \to \varphi(\cdot, x, u)$ of (1.1) on compact time intervals.

The ω-limit set of $(u, x) \in \mathcal{U} \times M$ with respect to the flow Φ is

$$\omega(u, x) := \{(v, y) \in \mathcal{U} \times M: \text{there are } t_k \to \infty \text{ with } (u(t_k + \cdot), \varphi(t_k, x, u)) \to (v, y)\}.$$

Note that the limit set in M, defined by (1.4), is the projection of this set onto the second component. The projection onto the first component (which is independent of x) is denoted by $\omega(u) = \pi_{\mathcal{U}}\omega(u, x)$. Recall that if the positive trajectory $\varphi^+(x, u) := \{\varphi(t, x, u) : t \geq 0\}$ is bounded, then $\omega(u, x)$ is a compact, connected, and invariant set for Φ. Furthermore, control sets D with nonvoid interior uniquely correspond to maximal topologically mixing sets $\mathcal{D}$ of $(\mathcal{U} \times M, \Phi)$ with $\operatorname{int} \pi_M \mathcal{D} \neq \emptyset$ via

$$\mathcal{D} = \operatorname{cl}\{(u, x) \in \mathcal{U} \times M : \varphi(t, x, u) \in \operatorname{int} D \text{ for all } t \in \mathbb{R}\}. \tag{2.1}$$

In general, the limit sets $\pi_M\omega(u, x)$ in M are not contained in control sets. They are contained in chain control sets, which satisfy a weaker controllability property, closely related to chain transitivity of the associated control flow as explained below.

2.4. Definition. Let $x, y \in M$ and $\varepsilon, T > 0$. A controlled (ε, T)-chain from x to y is given by $n \in \mathbb{N}$, $x_0, ..., x_n \in M$, $u_0, ..., u_{n-1} \in \mathcal{U}$, and $t_0, ..., t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) < \varepsilon \qquad \text{for} \quad j = 0, 1, ..., n-1.$$

2.5. Definition. A set $E \subset M$ is called a chain control set of (1.1)–(1.3) if

(i) for all $x \in E$ there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$;

(ii) for all $x, y \in E$ and all $\varepsilon, T > 0$ there is a controlled (ε, T)-chain from x to y; and

(iii) E is maximal with these properties.

2.6. *Remark.* The chain control sets $E \subset M$ correspond uniquely to the connected components $\mathcal{E} \subset \mathcal{U} \times M$ of the chain recurrent set of Φ via

$$\mathcal{E} = \operatorname{cl}\{(u, x) : \varphi(t, x, u) \in E \text{ for all } t \in \mathbb{R}\}.$$

In [CK(93)] it is shown that for all $(u, x) \in \mathcal{U} \times M$ with $\varphi^+(x, u)$ bounded

$$\pi_M\omega(u, x) \subset E \qquad \text{for some chain control set } E.$$

Below, we characterize the case where one actually has

$$\pi_M\omega(u, x) \subset \operatorname{int} D \qquad \text{for some control set } D.$$

The following example illustrates the concepts of this section and shows the difference between control sets and chain control sets.

2.7. EXAMPLE. Let $M = \mathbb{S}^1$, parametrized by the angle $\varphi \in [0, 2\pi)$. Consider the system

$$\dot{\varphi} = -u_1(t) \sin^2 \varphi + (u_2(t) - 1) \sin \varphi \cos \varphi + u_1(t) \cos^2 \varphi = f(\varphi, u) \qquad (2.2)$$

with $U = [0, \frac{1}{2}] \times [1, 2]$.

Since M is one-dimensional, the Lie algebra condition is satisfied iff for each $\varphi \in \mathbb{S}^1$ there exists $u \in U$ such that $f(\varphi, u) \neq 0$. (For systems on higher dimensional state spaces one must compute the Lie brackets, which give the directions in which the control system can move when switching between control values; see the example in Section 7.) The system (2.2) obviously satisfies Condition (H). Note that for $u^0 = (u_1, u_2) \equiv (0, 1)$, we have $\dot{\varphi} = 0$, and hence for this control value all points in $\mathbb{S}^1$ are fixed points.

The control sets on a one-dimensional state space are given by the zeros of the right hand side of the system equation; see [CK(92[a]), Sect. 3]. We obtain for this example: There are four control sets with nonvoid interior (see Fig. 1),

$$D_1 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \qquad D_2 = \left(\frac{3\pi}{4}, \pi\right), \qquad D_3 = D_1 + \pi, \qquad D_4 = D_2 + \pi,$$

and a continuum of one-point control sets

$$D_\alpha = \{\alpha\}, \qquad \alpha \in \mathbb{S}^1 \Big\backslash \left(\bigcup_{i=1}^{1} D_i \right) =: A.$$

The points in A are, of course, the fixed points of (2.2) corresponding to $u^0 \equiv (0, 1)$.

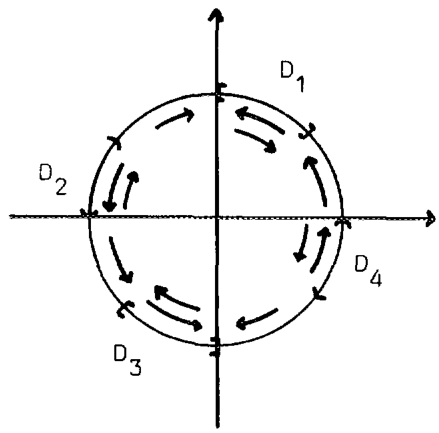

FIG. 1. Control sets and dynamics for the system (2.2).

The chain control sets on a one-dimensional state space contain (the closure of) the connected components of the union of all control sets; see [CK(92[a]), Theorem 3.16]. In this case we have a unique chain control set $E = \mathbb{S}^1$, since for any $T > 0$, $\varepsilon > 0$ and any $\varphi_1, \varphi_2 \in \mathbb{S}^1$ it suffices to use the control $u^0 \equiv (0, 1)$ in order to construct an (ε, T)-chain from φ_1 to φ_2. To describe the orbits of a point $\varphi \in \mathbb{S}^1$, take, e.g., $\alpha \in (0, \pi/4]$. Then

$$\mathcal{O}^+(\alpha) = \left[\alpha, \frac{\pi}{2}\right) \quad \text{and} \quad \mathcal{O}^-(\alpha) = \left(-\frac{\pi}{4}, \alpha\right].$$

Note that $\varphi(t, \alpha, u^0) \equiv \alpha$ stays at the boundary of $\mathcal{O}^+(\alpha)$ and of $\mathcal{O}^-(\alpha)$ for all $t \geqslant 0$ (and for all $t \leqslant 0$, respectively) and never enters a control set with nonvoid interior.

3. Chain Controllability and Inner Pairs

In this section we prove the main technical result that links chain controllability and controllability: If there exist controlled chains that link two points x and y and that are contained in the interior of the orbits from x (and from y, respectively), then x and y lie in the interior of the same control set; see Corollary 3.8 below. The appropriate concept which specifies the idea of "contained in the interior of an orbit," is that of an inner pair:

3.1. Definition. A pair $(u, x) \in \mathcal{U} \times M$ is called an inner pair, if there exist $T > 0$ and $S > 0$ such that

$$\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T+S}(x).$$

Using continuous dependence of the solutions of (1.1), (1.2) on the initial value, one easily sees that $\varphi(T_1, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T_1+S}(x)$ for some $T_1 > 0$ implies $\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T+S}(x)$ for all $T \geqslant T_1$. The following lemma shows a similar consequence for the negative orbits.

3.2. Lemma. *Let $(u, x) \in \mathcal{U} \times M$ be an inner pair. Then there exist $R_2 \geqslant R_1 > 0$ such that*

$$\varphi(R_1, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant R_2}(x) \quad \textit{and} \quad x \in \operatorname{int} \mathcal{O}^-_{\leqslant R_2}(\varphi(R_1, x, u)). \tag{3.1}$$

Proof. Let V be an open neighborhood of $x_T := \varphi(T, x, u)$ with $V \subset \mathcal{O}^+_{\leqslant T+S}(x)$, and define $y := \varphi(T + 2S, x, u)$. Pick a constant $u_1 \in U$ and a time $0 < t_0 < S$ such that $\hat{x} := \varphi(-t_0, x_T, u_1) \in V$. Then, using (H) and a standard construction in nonlinear control theory (cp., e.g., [Is(89), pp. 69] or [NvS(90), pp. 80]), one finds an open set $W \subset V$ such that $W \subset \mathcal{O}^-_{\leqslant 3S}(y)$. Since $W \subset V$, we have for $z \in W$ by continuous dependence on the initial value: $x \in \operatorname{int} \mathcal{O}^-_{\leqslant T+2S}(z) \subset \operatorname{int} \mathcal{O}^-_{\leqslant T+5S}(y)$. By definition of y we also have $y = \varphi(T+2S, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T+3S}(x)$, and with $R_1 = T+2S$, $R_2 = T+5S$ the statement (3.1) follows. ∎

The property (3.1), describing the existence of interior trajectories in positive and negative orbits, will be used frequently.

3.3. *Remark.* If (1.1)–(1.3) is a control system whose systems group is a (finite dimensional) Lie group, then one can easily characterize controls which lead to inner pairs. More precisely, suppose that the systems group G is a Lie group which acts transitively on M (this is guaranteed by (H)), and that the systems semigroup $\mathcal{S}$ has nonvoid interior in G. Under Assumption (H) we even know that $\operatorname{cl}(\operatorname{int} \mathcal{S}) = \operatorname{cl} \mathcal{S}$, and that for any neighborhood $U(e)$ of the identity e in G we have $\operatorname{int} \mathcal{S}_{\leqslant T} \cap U(e) \neq \varnothing$ for all $T > 0$; cp, e.g., [JS(72)]. This setup holds in particular for right invariant systems on Lie groups or homogeneous spaces, such as actions of linear systems groups on Grassmannians or flag manifolds.

CLAIM. For a piecewise constant control $u \in \mathcal{U}$ every pair $(u, x) \in \mathcal{U} \times M$ is an inner pair, iff there are $T > 0$, $S > 0$ such that the element $g_{u,T} \in \mathcal{S}_{\leqslant T}$ corresponding to $u|_{[0,T]}$ satisfies $g_{u,T} \in \operatorname{int} \mathcal{S}_{\leqslant T+S}$.

Proof. Note that for all $x \in M$, all $t > 0$ we have $(\operatorname{int} \mathcal{S}_{\leqslant t})x = \operatorname{int} \mathcal{O}^+_{\leqslant t}(x)$. Now, if $g_{u,T} \in \operatorname{int} \mathcal{S}_{\leqslant T+S}$, then obviously $g_{u,T}x \in (\operatorname{int} \mathcal{S}_{\leqslant T+S})x = \operatorname{int} \mathcal{O}^+_{\leqslant T+S}(x)$. ∎

The following lemma deals with the uniform size of balls in the interior of forward orbits, and it is an important technical tool for the subsequent analysis.

3.4. LEMMA. *Let $I \subset \mathcal{U} \times M$ be a compact Φ-invariant set consisting of inner pairs. Then there are $\varepsilon, T, S > 0$ such that for all $(u, x) \in I$*

$$B_\varepsilon(\varphi(T, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T+S}(x). \tag{3.2}$$

Proof. Let $(u, x) \in I$. Then there are $\delta_1 = \delta_1(u, x) > 0$, $T_1 = T_1(u, x) > 0$, and $S_1 = S_1(x, u) > 0$ with

$$B_{\delta_1}(x) \subset \operatorname{int} \mathcal{O}^-_{\leqslant T_1+S_1}(\varphi(T_1, x, u)).$$

Since I is invariant, $(u(T_1 + \cdot), \varphi(T_1, x, u)) \in I$ and hence it is an inner pair. Thus there are δ with $0 < \delta < \delta_1$ and $T_2 > 0$, $S_2 > 0$ with

$$B_\delta(\varphi(T_1 + T_2, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T_2+S_2}(\varphi(T_1, x, u)). \tag{3.3}$$

Abbreviate

$$V_x := \varphi(T_1 + T_2, \cdot, u)^{-1} [B_{\delta/2}(\varphi(T_1 + T_2, x, u))] \cap B_\delta(x) \subset M.$$

Clearly $x \in V_x$ and V_x is open. Let $y \in V_x$ be arbitrary. Then, since $\delta < \delta_1$,

$$y \in B_\delta(x) \subset \operatorname{int} \mathcal{O}^-_{\leqslant T_1+S_1}(\varphi(T_1, x, u)).$$

Thus

$$\operatorname{int} \mathcal{O}^+_{\leqslant T_2+S_2}(\varphi(T_1, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T_1+T_2+S_1+S_2}(y). \tag{3.4}$$

By definition of V_x

$$\varphi(T_1+T_2, y, u) \in B_{\delta/2}(\varphi(T_1+T_2, x, u)).$$

Note for later purpose that this implies

$$B_{\delta/2}(\varphi(T_1+T_2, y, u)) \subset B_\delta(\varphi(T_1+T_2, x, u)). \tag{3.5}$$

By (3.3) and (3.4) all $y \in V_x$ satisfy

$$\varphi(T_1+T_2, y, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T_1+T_2+S_1+S_2}(y). \tag{3.6}$$

By continuous dependence on the right hand side of the control equation, (3.6) remains valid for all v in an open neighborhood V_u of u. Now compactness implies that finitely many neighborhoods of the form $V_u \times V_x$ cover I.

One obtains: there exist $T_+ := \max T_{(u,x)} < \infty$ and $S_+ := \max S_{(u,x)} < \infty$ such that for all $(v, y) \in I$

$$\varphi(T_+, y, v) \in \operatorname{int} \mathcal{O}^+_{\leqslant T_+ + S_+}(y).$$

By time reversal and invariance of I one finds analogously $T_- > 0$ and $S_- > 0$ such that for all $(v, y) \in I$

$$y \in \operatorname{int} \mathcal{O}^-_{\leqslant T_- + S_-}(\varphi(T_-, y, v)).$$

Now repeat the part of the proof up to formula (3.6), but with T_- replacing T_1 and T_+ replacing T_2. Observe that, contrary to T_1 and T_2, the times $T := T_+ + T_-$ and $S := S_+ + S_-$ are independent of (u, x). We obtain using (3.5) that

$$B_{\delta/2}(\varphi(T, y, v)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T+S}(y)$$

for all (v, y) in an open neighborhood of (u, x). Now a compactness argument yields an $\varepsilon > 0$ independent of (u, x) in an open cover of I, with (3.2). ∎

3.5. *Remark.* Analogous arguments yield under the assumptions of Lemma 3.4 that there are ε', T', $S' > 0$ such that for $(u, x) \in I$

$$B_{\varepsilon'}(x) \subset \operatorname{int} \mathcal{O}^{-}_{\leqslant T'+S'}(\varphi(T', x, u)).$$

We need the following special class of controlled (ε, T)-chains to establish a relation between chain controllability and (the usual) controllability.

3.6. Definition. Let $I \subset \mathcal{U} \times M$ be a compact invariant set consisting of inner pairs. Then $x \in M$ is inner chain controllable to $y \in M$ (with respect to I) if for all $\varepsilon, T > 0$ there is a controlled (ε, T)-chain $(u_i, x_i) \in I$, $i = 0, ..., n-1$, from x to y.

The following proposition shows that inner chain controllability implies controllability. It may be viewed as a control theoretic analogue of Bowen's "Shadowing Lemma" [Bo(75)], without hyperbolicity assumption.

3.7. Proposition. *Let $x \in M$ be inner chain controllable to $y \in M$ w.r.t. $I \subset \mathcal{U} \times M$. Then there is $T > 0$ with $y \in \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(x)$.*

Proof. Choose ε, T', S according to the Uniformization Lemma 3.4. Take an (ε, T')-chain in I from x to y. Then for $i = 0, ..., n-1$

$$x_{i+1} \in B_{\varepsilon}(\varphi(T', x_i, u_i)) \subset \operatorname{int} \mathcal{O}^{+}_{\leqslant T'+S}(x_i).$$

Thus $x_n = y \in \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(x)$ with $T = n(T'+S)$. ∎

The result implies a connection between chain controllability and control sets.

3.8. Corollary. *Suppose that $I \subset \mathcal{U} \times M$ is compact and invariant for Φ and that for all $x, y \in \pi_M I \subset M$ the point x is inner chain controllable to y with respect to I. Then there is a control set D with $\pi_M I \subset \operatorname{int} D$.*

Proof. Pick $x, y \in \pi_M I$. By Proposition 3.7 there is $T > 0$ with

$$y \in \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(x) \quad \text{and} \quad x \in \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(y). \tag{3.7}$$

Hence there are neighborhoods V_x and V_y of x and y, respectively, with

$$V_y \subset \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(x) \quad \text{and} \quad V_x \subset \operatorname{int} \mathcal{O}^{+}_{\leqslant T}(y).$$

Since $\operatorname{int} \mathcal{O}^{-}_{\leqslant \tau}(y) \neq \varnothing$ for all $\tau > 0$, one finds $\tau > 0$ small enough, such that

$$\varnothing \neq \operatorname{int} \mathcal{O}^{-}_{\leqslant \tau}(y) \subset V_y.$$

Now every z in $\operatorname{int} \mathcal{O}^-_{\leqslant\tau}(y)$ can be reached from x and hence from y and therefore from any point in $\operatorname{int} \mathcal{O}^-_{\leqslant\tau}(y)$.

Thus $\operatorname{int} \mathcal{O}^-_{\leqslant\tau}(y) \subset \operatorname{int} D$ for some control set D. Furthermore, since $y \in D$ can be reached in finite time from $\operatorname{int} \mathcal{O}^-_{\leqslant\tau}(y)$, it follows that $y \in \operatorname{int} D$. ∎

4. Limit Sets and Control Sets

This section contains the main result of our paper: We characterize the ω-limit sets of a control system that are actually contained in the interior of some control set. Theorem 4.4 states the result under Hypothesis (H) and Theorem 4.13 is a variant under weaker assumptions. This last theorem will allow us to prove controllability in the Lorenz equations in Section 7. For complicated dynamics, where the control system does satisfy the Lie algebra condition (H), a similar result follows from Theorem 4.4. Example 4.7 shows a system that satisfies (H), but none of the equivalent statements in Theorem 4.4, hence this system has limit sets outside of control sets with nonvoid interior.

First, we recall some notions from the topological theory of dynamical systems (see in particular [Co(78)]). An (ε, T)-chain for a dynamical system Ψ on a metric space (S, d) from $x \in S$ to $y \in S$ consists of $x_0, x_1, ..., x_n \in S$ and $t_0, ..., t_{n-1} \geqslant T$ such that $x_0 = x$, $x_n = y$ and

$$d(\Psi(t_j, x_j), x_{j+1}) \leqslant \varepsilon \qquad \text{for } j = 0, 1, ..., n-1.$$

Furthermore, a set $S_0 \subset S$ is chain transitive, if for all $x, y \in S_0$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain in S from x to y.

The following result is a direct consequence of [Co(78), II.6.3.C]:

4.1. Lemma. *Suppose that for $(u, x) \in \mathcal{U} \times M$ the positive trajectory $\varphi^+(x, u) := \{\varphi(t, x, u): t > 0\}$ is bounded. Then $\omega(u, x)$ is chain transitive with respect to $\Phi|_{\omega(u,x)}$.*

This together with the results of preceding section implies:

4.2. Proposition. *Assume that for $(u, x) \in \mathcal{U} \times M$ the set $\varphi^+(x, u)$ is bounded, and $\omega(u, x)$ consists of inner pairs. Then for all $y, z \in \pi_M \omega(u, x)$, the point y is inner chain controllable to z with respect to $\omega(u, x)$ and $\pi_M \omega(u, x) \subset \operatorname{int} D$ for some control set D.*

Proof. By Lemma 4.1, $\omega(u, x)$ is chain transitive. Hence all (v, y), $(w, z) \in \omega(u, x)$ can be connected by chains consisting of inner pairs in the compact invariant set $I := \omega(u, x)$. Thus y is inner chain controllable to z w.r.t. I. Now Corollary 3.8 implies

$$\pi_M \omega(u, x) \subset \text{int } D \qquad \text{for some control set } D. \blacksquare$$

In order to prove a converse to this result, we note the following lemma.

4.3. LEMMA. *Let $I \subset \mathscr{U} \times M$ be compact and Φ-invariant with*

$$\pi_M I \subset \text{int } D \qquad \textit{for some control set } D.$$

Then I consists of inner pairs.

Proof. Let $(v, y) \in I$. There exists a compact set K with $\pi_M I \subset \text{int } K \subset \text{int } D$. Choose $T = T(K, K)$ according to Proposition 2.2. Then

$$\varphi(T, y, v) \in \pi_M I \subset \text{int } K.$$

Hence $\varphi(T, y, v) \in \text{int } \mathscr{O}^+_{\leqslant T}(y)$. ▮

The next theorem is the main result of this section.

4.4. THEOREM. *Assume that for $(u, x) \in \mathscr{U} \times M$ the positive trajectory $\varphi^+(x, u)$ is bounded. Then the following assertions are equivalent:*

(i) *$\pi_M \omega(u, x) \subset \text{int } D$ for some control set D.*

(ii) *$\omega(u, x)$ consists of inner pairs.*

(iii) *For all $y, z \in \pi_M \omega(u, x)$ the point y is inner chain controllable to z with respect to $\omega(u, x)$.*

Proof. Assertion (i) implies (ii) by Lemma 4.3. Assertion (ii) implies (iii) by Proposition 4.2. Assertion (iii) implies (i) by Corollary 3.8. ▮

4.5. *Remark.* The proof of Lemma 4.3 shows that $\omega(u, x)$ consists of inner pairs iff for all $(v, y) \in \omega(u, x)$ there is $T > 0$ with

$$\varphi(T, y, v) \in \text{int } \mathscr{O}^+_{\leqslant T}(y).$$

This, apparently, is a stronger statement than the one in Definition 3.1.

Of particular interest, e.g., for systems with complicated behavior are the limit sets for constant controls. We obtain the following result.

4.6. COROLLARY. *Let u^0 be a constant control with value in* int U, *and let $x \in M$ with $\varphi^+(x, u^0)$ bounded. Assume instead of* (H) *that for all $y \in \pi_M \omega(u^0, x)$ the following stronger condition is satisfied:*

With $X := X_0 + \sum_{i=1}^m u_i^0 X_i$, the linear span of

$$\{(\text{ad}_X^k X_i)(y),\ i = 1, ..., m,\ k = 0, 1, 2, ...\}$$

equals $T_y M$.

Then each element $(u^0, y) \in \omega(u^0, x)$ *is an inner pair and hence*

$$\pi_M \omega(u^0, x) \subset \operatorname{int} D \qquad \textit{for some control set } D.$$

Proof. Observe that the right hand side of the system equation (1.1) may be rewritten as

$$X_0 + \sum_{i=1}^{m} u_i X_i = X + \sum_{i=1}^{m} (u_i - u_i^0) X_i.$$

Thus we may assume that $u^0 = 0 \in \operatorname{int} U$. Now the result follows from a standard result in nonlinear control theory, guaranteeing that $\varphi(T, y, u) \in \operatorname{int} \mathcal{O}_T^+(y)$ for all $T > 0$, $y \in \pi_M \omega(u^0, x)$ (cp., e.g., [So(90), Lemma/Exercise 3.6.13]; observe that the condition required in the corollary holds in a neighborhood of $\pi_M \omega(u^0, x)$). ∎

4.7. EXAMPLE. Again consider Example 2.7, take a point $\alpha \in (0, \pi/4) \subset A$, and the constant control $u^0 \equiv (0, 1)$. Since α is a fixed point for this differential equation on $\mathbb{S}^1$, we have $\pi_{\mathbb{S}^1} \omega(u^0, \alpha) = \{\alpha\}$. But $\pi_{\mathbb{S}^1} \omega(u^0, \alpha)$ is not contained in a control set with nonvoid interior; see Example 2.7. The reason is, of course, that $\operatorname{int} \mathcal{O}^+(\alpha) = (\alpha, \pi/2)$ has the trajectory $\varphi(t, \alpha, u^0) \equiv \alpha$ at its boundary for all $t \geq 0$. Section 5 shines additional light on situations like this, when the control range U varies.

Next we relax Hypothesis (H) to obtain more general versions of Theorem 4.4. The first generalization concerns the behavior of a control system on its maximal integral manifolds, and the second one relaxes the assumption that the limit sets must be contained in one maximal integral manifold.

We start from a basic result due to H. J. Sussmann [Su(73)]: Recall that an (injectively immersed) submanifold J of M is said to be an integral manifold of a distribution Δ if for every $x \in J$ the tangent space J_x is exactly $\Delta(x)$. A distribution Δ has the maximal integral manifolds property if through every point $x \in M$ there passes a maximal integral manifold of Δ, i.e., an integral manifold J with the additional property that every connected integral manifold of Δ, which intersects J, is an open submanifold of J. Recall that $\mathcal{L}$ denotes the Lie algebra spanned by the vectorfields $X_0 + \sum u_i X_i$, $(u_i) \in U$. The following hypothesis is used below instead of (H).

The distribution $\Delta_{\mathcal{L}}$ spanned by the vectorfields in $\mathcal{L}$ has the maximal integral manifolds property. (I)

By [Su(73), Theorems 4.1, 4.2] it follows from (I) that the orbits (i.e., the equivalence classes of points x, x' in M for which there exists an element g in the group of diffeomorphisms generated by the vectorfields in $\mathscr{L}$ with $gx = x'$) coincide with the maximal integral manifolds of $\Delta_{\mathscr{L}}$ and that the dimension of $\Delta_{\mathscr{L}}$ on an orbit J is equal to the dimension of J. Thus the system restricted to a maximal integral manifold J (equipped with the unique differentiable structure such that J is a submanifold of M) satisfies Assumption (H). We obtain the following corollary to Theorem 4.4.

4.8. COROLLARY. *Assume that instead of* (H) *Hypothesis* (I) *is satisfied. Let* $(u, x) \in \mathscr{U} \times M$ *with bounded positive trajectory* $\varphi^+(x, u)$ *and assume that* $\pi_M \omega(u, x)$ *is contained in a maximal integral manifold* J *of the distribution* $\Delta_{\mathscr{L}}$ *generated by the vectorfields in* $\mathscr{L}$. *Then the following assertions are equivalent*:

(i) $\pi_M \omega(u, x) \subset \mathrm{int}_J D$ *for some control set* D.

(ii) *For all* $(v, y) \in \omega(u, x)$ *there is* $T > 0$ *with* $\varphi(T, y, v) \in \mathrm{int}_J \mathcal{O}^+_{\leqslant T+1}(y)$.

Here for a set $A \subset M$, $\mathrm{int}_J A$ *denotes the relative interior of* $A \cap J$ *with respect to* J.

Proof. By the preceding remarks, the system restricted to J satisfies Assumption (H). Now the assertion follows, as before, from Lemma 4.3, Proposition 4.2, and Corollary 3.8. Observe that every control set of the system restricted to J is contained in a control set of the system on M. ∎

4.9. *Remark.* Our proofs in Section 3 use local accessibility, i.e., $\mathrm{int}\, \mathcal{O}^+_{\leqslant T}(x) \neq \varnothing$ for all $T > 0$. Hence we cannot cope with the very general situation treated by [Su(73)]. In Sussmann's notation, we require $A_{D*} = P_D$, with $D = \{X_0 + \sum u_i X_i, (u_i) \in U\}$. This holds iff D is locally finitely generated. In particular, this is true in the analytic case [Su(73), p. 188].

4.10. *Remark.* Note that in the corollary above we do not require that $x \in J$.

For results in control theory, which involve only finite time properties, one can replace Hypothesis (H) by (I). Then, as noted above, (H) holds on the (immersed) maximal integral manifold J through x and $\mathcal{O}^+(x) \subset J$. However, in the analysis above, the limit behavior as $t \to \infty$ is relevant. The following example illustrates some of the difficulties which may occur here. Note that they vanish, if the integral manifold J is considered in its own topology (as an immersed submanifold); then trajectories with $\pi_M \omega(u, x) \not\subset J$ are unbounded in J.

4.11. EXAMPLE. Consider in $\mathbb{R}^3_+$ the control system

$$\dot{x}_1 = x_1[x_1 + ux_2 + \beta x_3 - M]$$
$$\dot{x}_2 = x_2[\beta x_1 + x_2 + ux_3 - M]$$
$$\dot{x}_3 = x_3[ux_1 + \beta x_2 + x_3 - M],$$

where

$$M = x_1[x_1 + ux_2 + \beta x_3] + x_2[\beta x_1 + x_2 + ux_3] + x_3[ux_1 + \beta x_2 + x_3]$$

and

$$u \in U := [\underline{\alpha}, \bar{\alpha}]$$

with

$$0 < \underline{\alpha} < \bar{\alpha} < 1 < \beta, \qquad \bar{\alpha} + \beta < 2.$$

For $u = \alpha$ such equations belong to the class of equations which is studied as a model for the self organization of macromolecules (cp. ([ES(77)]; cp. also [ML(75), SSW(79)] for a class of equations arising in competition analysis of three species, where a similar limiting behavior is observed). For the following assertions on the qualitative behavior see [SSW(79)].

For $u \equiv \alpha \in U$, the orbit of almost every point in the interior of $\mathbb{R}^3_+$ has as ω-limit set the boundary of the simplex $S_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$ formed by the unit vectors e_1, e_2, e_3 and the three segments (which are orbits) joining them. The exceptional points are the equilibrium $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ and its stable manifold, which consists of the two orbits on the line $x_1 = x_2 = x_3$ having x^* as ω-limit point.

Observe that Hypothesis (I) is satisfied since the involved vectorfields are analytic.

Since the planes $x_i = 0$, $i = 1, 2, 3$, are invariant, $\pi_{\mathbb{R}^3}\omega(\alpha, x)$ has nonvoid intersection with six maximal integral manifolds. The same argument shows that $\pi_{\mathbb{R}^3}\omega(\alpha, x)$ cannot be a subset of a control set.

The theory developed so far is not applicable to the Lorenz equations treated in Section 6. These equations have the feature that $\pi_M\omega(u, x)$ contains parts of more than one maximal integral manifold of the system. However, one can obtain the result of Theorem 4.4 and Corollary 4.8 on compact subsets of the maximal integral manifolds. Before we can prove a corresponding statement, we need a variant of Lemma 3.4, adapted to this situation.

4.12. LEMMA. *Let $u \in U$ and assume instead of* (H) *that $L \subset M$ is an open set such that for all $x \in L$ and all $T' > T > 0$*

$$\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T'}(x) \qquad \text{and} \qquad x \in \operatorname{int} \mathcal{O}^-_{\leqslant T'}(\varphi(T, x, u)).$$

Then for every compact $K \subset L$ and all $T' > T > 0$ there is $\varepsilon > 0$ such that for all $x \in K$

$$B_\varepsilon(\varphi(T, x, u)) \in \operatorname{int} \mathcal{O}^+_{\leqslant T'}(x).$$

Proof. There exists $T_1' > 0$ such that for every $x \in K$

$$\varphi(t, x, u) \in L \qquad \text{for all} \quad 0 \leqslant t \leqslant 2T_1'.$$

Fix $0 < T_1 < T_1'$ and $x \in K$. Then there is $\delta_1 = \delta_1(x) > 0$ such that

$$B_{\delta_1}(x) \subset \operatorname{int} \mathcal{O}^-_{\leqslant T_1'}(\varphi(T_1, x, u)).$$

Since $\varphi(T_1, x, u) \in L$ there is $\delta = \delta(x) > 0$ with $0 < \delta < \delta_1$ and

$$B_\delta(\varphi(2T_1, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T_1'}(\varphi(T_1, x, u)).$$

Abbreviate

$$V_x := \varphi(2T_1, \cdot, u)^{-1} [B_{\delta/2}(\varphi(2T_1, x, u))] \cap B_\delta(x).$$

Clearly, $x \in V_x$ and V_x is open. Let $y \in V_x$ be arbitrary. Then, since $\delta < \delta_1$,

$$y \in B_\delta(x) \subset \operatorname{int} \mathcal{O}^-_{\leqslant T_1'}(\varphi(T_1, x, u)).$$

Thus

$$\operatorname{int} \mathcal{O}^+_{\leqslant T_1'}(\varphi(T_1, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant 2T_1'}(y).$$

By definition of V_x

$$\varphi(2T_1, y, u) \in B_{\delta/2}(\varphi(2T_1, x, u))$$

and hence

$$\begin{aligned} B_{\delta/2}(\varphi(2T_1, y, u)) &\subset B_\delta(\varphi(2T_1, x, u)) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leqslant T_1'}(\varphi(T_1, x, u)) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leqslant 2T_1'}(y). \end{aligned}$$

By compactness, the set K can be covered by finitely many sets of the form V_x. Thus the assertion follows. ∎

4.13. THEOREM. *Let* $u \in U$ *and assume instead of* (H) *that there exists an open set* $L \subset M$ *such that for all* $x \in L$ *and all* $T' > T > 0$ *one has* $\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T'}(x)$ *and* $x \in \operatorname{int} \mathcal{O}^-_{\leqslant T'}(\varphi(T, x, u))$. *Let* $K \subset L$ *be compact. Then for every* $x \in M$ *with* $\varphi^+(x, u)$ *bounded there exists a control set D with*

$$K \cap \pi_M \omega(u, x) \subset \operatorname{int} D. \tag{4.1}$$

Observe that the assertion (4.1) *is void, if* $K \cap \pi_M \omega(u, x) = \emptyset$.

Proof. We show that for all $y, z \in K \cap \pi_M \omega(u, x)$ there is $S > 0$ with $z \in \operatorname{int} \mathcal{O}^+_{\leqslant S}(y)$. Then the assertion follows as in the proof of Corollary 3.8.

Let $K_1 \subset L$ be compact with $K \subset \operatorname{int} K_1$. Then there is $T' > 0$ with

$$\varphi(t, x_1, u) \in K_1 \qquad \text{for all} \quad x_1 \in K \text{ and all } 0 < t \leqslant T'.$$

Denote $T := T'/2$. Then by Lemma 4.12, applied to K_1, we find $\varepsilon > 0$ depending only on T, T', and K_1, such that for all $x_1 \in K_1$ we have

$$B_\varepsilon(\varphi(T, x_1, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T'}(x_1).$$

Now pick $y, z \in K \cap \pi_M \omega(u, x)$. There exist $t_1 > t_0 > 0$ with

(i) $\varphi(t_0, x, u) \in B_\varepsilon(\varphi(T, y, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T'}(y)$, because $\varphi(T, y, u) \in K_1 \cap \pi_M \omega(u, x)$, and

(ii) $z \in B_\varepsilon(\varphi(t_1 + T, x, u)) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T'}(\varphi(t_1, x, u))$, where the first statement holds, because $z \in \pi_M \omega(u, x)$ and $\varepsilon > 0$ is fixed, and for the inclusion we can assume w.l.o.g. (by using the minimum of ε and the distance between ∂K_1 and ∂K) that $\varphi(t_1, x, u) \in K_1$.

Now observe that

$$\begin{aligned} \operatorname{int} \mathcal{O}^+_{\leqslant T'}(\varphi(t_1, x, u)) &\subset \operatorname{int} \mathcal{O}^+_{\leqslant t_1 - t_0 + T'}(\varphi(t_0, x, u)) && (\text{because } t_1 > t_0 > 0) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leqslant t_1 - t_0 + 2T'}(y) && \text{by (i).} \end{aligned}$$

With $S = t_1 - t_0 + 2T'$ we obtain $z \in \operatorname{int} \mathcal{O}^+_{\leqslant S}(y)$. ∎

5. PERTURBATIONS OF THE CONTROL RANGE

So far we have considered the second term $\sum u_i X_i$ on the right hand side of (1.1) as a bounded control term with given control range U. In this section we analyze the change of the limit and the control structure, depending on the size of the control range $U_\rho = \rho \cdot U$, $\rho \geqslant 0$. For small ρ this can be considered as a result on the time varying perturbations of ordinary differential equations on the entire time axis, since we study the ρ-dependence of limit sets. In a nutshell it turns out that the Morse sets of an ordinary differential equation (i.e., $\rho = 0$) expand to control sets for the control system (i.e., $\rho > 0$). For $\rho > 0$ we prove upper semicontinuity of the chain control sets with respect to ρ and show—under additional assumptions—that chain control sets for ρ are contained in control sets for $\rho' > \rho$. We embed the differential equation

$$\dot{x} = X_0(x) \tag{5.1}$$

into the following family of differential equations,

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t)\, X_i(x), \tag{5.2$^\rho$}$$

$u = (u_i) \in \mathcal{U}^\rho := \{u\colon \mathbb{R} \to \mathbb{R}^m;\ u(t) \in U^\rho = \rho \cdot U\}$ with $\rho \geqslant 0$ and $U \subset \mathbb{R}^m$ compact and convex and $0 \in \operatorname{int} U$. In particular, $U = \{u \in \mathbb{R}^m;\ |u| \leqslant \rho\}$ for any norm $|\cdot|$ on $\mathbb{R}^m$ is a possible choice.

For $\rho = 0$, we obtain an interpretation of (5.1) as a control system, with the one-point control range $U^0 = \{0\}$. The connected components of the chain recurrent set for (5.1) are the chain control sets of (5.2$^\rho$) with $\rho = 0$.

The following theorem does not need Hypothesis (H). It establishes the announced upper semicontinuity property of chain control sets with respect to ρ.

5.1. THEOREM. *Let M be compact and fix $\rho \geqslant 0$. Assume that the system (5.2$^\rho$) has only finitely many chain control sets E_i^ρ, $i = 1, ..., n$. Then for each $i \in \{1, ..., n\}$ there is an increasing sequence of chain control sets $E_i^{\rho'}$ of (5.2$^{\rho'}$) such that $E_i^\rho = \bigcap_{\rho' > \rho} E_i^{\rho'}$. Vice versa, if for a sequence $\rho_k \searrow \rho$ and chain control sets E^{ρ_k} of (5.2$^{\rho_k}$) the set of limit points $L := \{y \in M;$ there are $x^k \in E^{\rho_k}$ with $x^k \to y\}$ is nonvoid, then there is $i \in \{1, ..., n\}$ such that $L = E_i^\rho$.*

Proof. First of all, note that for $0 \leqslant \alpha < \beta$ we have $\mathcal{U}^\alpha \subset \mathcal{U}^\beta$ and $\Phi^\beta|_{\mathcal{U}^\alpha \times M} = \Phi^\alpha$. Thus if E^α is a chain control set of (5.2$^\alpha$), then there is a chain control set E^β of (5.2$^\beta$) with $E^\alpha \subset E^\beta$. This implies that for all $i = 1, ..., n$ and all $\rho' > \rho$ there is a chain control set $E_i^{\rho'}$ of (5.2$^{\rho'}$) with $E_i^\rho \subset E_i^{\rho'}$. Hence there is an increasing sequence of chain control sets $E_i^{\rho'}$ such that $E_i^\rho \subset \bigcap_{\rho' > \rho} E_i^{\rho'}$. The converse inclusion in an obvious consequence of the last assertion, which we prove now.

Pick $y^1, y^2 \in L = \{y \in M;$ there are $x^k \in E^{\rho_k}$ with $x^k \to y\}$. We must show that y^1 and y^2 are in the same chain control set of (5.2$^\rho$).

Let ε, $T > 0$. We construct a controlled (ε, T)-chain from y^1 to y^2. For $i = 1, 2$, $y^i = \lim_{k \to \infty} x_k^i$ with $x_k^i \in E^{\rho_k}$, $k \in \mathbb{N}$. For all $k \in \mathbb{N}$ there are controlled ($\varepsilon/3$, T)-chains from x_k^1 to x_k^2; i.e., there are $n_k \in \mathbb{N}$, $z_0^k, ..., z_{n_k}^k \in M$, $u_0^k, ..., u_{n_k - 1}^k \in \mathcal{U}^{\rho_k}$, and $t_0^k, ..., t_{n_k - 1}^k \geqslant T$ with $z_0^k = x_k^1$, $z_{n_k}^k = x_k^2$, and

$$d(\varphi(t_j^k, z_j^k, u_j^k), z_{j+1}^k) < \frac{\varepsilon}{3} \qquad \text{for } j = 0, 1, ..., n_k - 1. \tag{5.3}$$

Clearly,

$$\sup_{u \in U^{\rho_k}} \inf_{v \in U^{\rho}} |v-u| \to 0 \qquad \text{as} \quad k \to \infty.$$

Hence using compactness of M one finds $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ and all $j=0, 1, ..., n_k - 1$ there are $v_j^k \in \mathscr{U}^\rho$ such that for all $\tau \in \mathbb{R}$, all $z \in M$, and all $0 < t \leqslant 2T$

$$d(\varphi(t, z, v_j^k(\tau + \cdot)), \varphi(t, z, u_j^k(\tau + \cdot))) < \frac{\varepsilon}{3}. \tag{5.4}$$

We may choose k_0 so large that also

$$d(\varphi(T, y^1, v_0^k), \varphi(T, x_k^1, u_0^k)) < \frac{\varepsilon}{3} \tag{5.5}$$

and

$$d(x_k^2, y^2) < \frac{\varepsilon}{3}. \tag{5.6}$$

Fix $k \in \mathbb{N}$ satisfying (5.4)–(5.6). In the following we drop the index k. Define an (ε, T)-chain with controls in $\mathscr{U}^\rho$ from y^1 to y^2 in the following way,

$$\begin{aligned} &y_{0,0} = y^1, \qquad y_{0,1} = \varphi(T, x^1, u_0), \\ &y_{j,i} = \varphi(iT, z_j, u_j), \qquad j = 1, 2, ..., n-1, \quad i = 0, 1, ..., i_j, \\ &y_n = y^2, \end{aligned} \tag{5.7}$$

where $i_j \in \mathbb{N}$ is such that $(i_j + 1)T \leqslant t_j < (i_j + 2)T$,

$$\begin{aligned} &t_{0,0} := T, \\ &t_{j,i} := T \qquad \text{for} \quad j = 1, 2, ..., n-1, \quad i = 0, 1, ..., i_j - 1, \\ &t_{j,i_j} := t_j - i_j T, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &v_{0,0} := v_0, \\ &v_{j,i} := v_j(iT + \cdot), \qquad j = 1, 2, ..., n-1, \quad i = 0, 1, ..., i_j. \end{aligned} \tag{5.9}$$

In fact, this is an (ε, T)-chain from y^1 to y^2 with controls in $\mathscr{U}^\rho$ and $1 + \sum_{j=1}^n i_j$ jumps of length $< \varepsilon$:

$$\begin{aligned} &d(\varphi(t_{0,0}, y_{0,0}, v_{0,0}), y_{0,1}) \\ &\quad = d(\varphi(T, y^1, v_0), \varphi(T, x^1, u_0)) \\ &\quad < \varepsilon \qquad \text{by (5.5);} \end{aligned}$$

for $j=1, ..., n-1$, $i=0, 1, ..., i_j-1$,

$$\begin{aligned} &d(\varphi(t_{j,i}, y_{j,i}, v_{j,i}), y_{j,i+1}) \\ &\quad = d(\varphi(T, \varphi(iT, z_j, u_j), v_j(iT+\cdot)), \varphi((i+1)\, T, z_j, u_j)) \\ &\quad = d(\varphi(T, \varphi(iT, z_j, u_j), v_j(iT+\cdot)), \varphi(T, \varphi(iT, z_j, u_j), u_j(iT+\cdot))) \\ &\quad < \varepsilon \qquad \text{by (5.4);} \end{aligned}$$

for $j=0, 1, ..., n-1$, $\mathrm{i}=i_j$,

$$\begin{aligned} &d(\varphi(t_{j,i_j}, y_{j,i_j}, v_{j,i_j}), y_{j+1,0}) \\ &\quad = d(\varphi(t_j - i_j T, \varphi(i_j T, z_j, u_j), v_j(i_j T+\cdot)), z_{j+1}) \\ &\quad = d(\varphi(t_j - i_j T, \varphi(i_j T, z_j, u_j), v_j(i_j T+\cdot)), \\ &\qquad\quad \varphi(t_j - i_j T, \varphi(i_j T, z_j, u_j), u_j(i_j T+\cdot))) \\ &\qquad + d(\varphi(t_j, z_j, u_j), z_{j+1}) \\ &\quad < \varepsilon \qquad \text{by (5.4) and (5.3).} \end{aligned}$$

Finally, for $j=n-1$ and $i=i_j=i_{n-1}$

$$\begin{aligned} &d(\varphi(t_{j,i}, y_{j,i}, v_{j,i}), y_{j+1}) \\ &\quad = d(\varphi(t_{n-1} - i_{n-1} T, y_{n-1,i_{n-1}}, v_{n-1,i_{n-1}}), y_n) \\ &\quad = d(\varphi(t_{n-1} - i_{n-1} T, \varphi(i_{n-1} T, z_{n-1}, u_{n-1}), v_{n-1}(i_{n-1} T+\cdot)), y^2) \\ &\quad \leqslant d(\varphi(t_{n-1} - i_{n-1} T, \varphi(i_{n-1} T, z_{n-1}, u_{n-1}), v_{n-1}(i_{n-1} T+\cdot)), \\ &\qquad\quad \varphi(t_{n-1} - i_{n-1} T, \varphi(i_{n-1} T, z_{n-1}, u_{n-1}), u_{n-1}(i_{n-1} T+\cdot))) \\ &\qquad + d(\varphi(t_{n-1}, z_{n-1}, u_{n-1}), x^2) + d(x^2, y^2) \\ &\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{by (5.4), (5.3) (with } z_n = x^2\text{), and (5.6).} \quad \blacksquare \end{aligned}$$

Under additional assumptions one also finds that chain control sets can be obtained as intersections of control sets. In the following we assume (H) for the system (5.2^ρ) for all $\rho > 0$. Recall the definition of $\mathscr{E}$ in Remark 2.6.

5.2. COROLLARY. *Let M be compact and fix $\rho \geqslant 0$. Assume that the number n of chain control sets $E_1^\rho, ..., E_n^\rho$ of (5.2^ρ) is finite and that for all $\rho' > \rho$ all $(u, x) \in \bigcup_{i=1}^n \mathscr{E}_i^\rho$ are inner pairs of ($5.2^{\rho'}$). Then for each $i \in \{1, ..., n\}$ there is an increasing sequence of control sets $D_i^{\rho'}$, $\rho' > \rho$, such that $E_i^\rho \subset \operatorname{int} D_i^{\rho'}$ and $E_i^\rho = \bigcap_{\rho' > \rho} D_i^{\rho'}$.*

Proof. By Corollary 3.8 there are for $\rho' > \rho$ control sets $D_i^{\rho'}$ with $E_i^{\rho} \subset \text{int } D_i^{\rho'}$ proving one inclusion. The other one follows from Theorem 5.1 and the fact that every control set is contained in a chain control set. ∎

Setting $\rho = 0$ in the preceding results one obtains a relation between the limit structure of the differential equation (5.1) and the perturbed systems (5.2^{ρ}). Suppose that (5.1) possesses a finest Morse decomposition $\mathcal{M} = \{M_0, ..., M_n\}$, i.e., the M_i, $i = 1, ..., n$, are the connected components of the chain recurrent set of the flow associated with (5.1); cp., e.g., [Co(78)].

5.3. COROLLARY. *Let M be compact. Assume that the chain recurrent set of* (5.1) *has finitely many components M_i, $i = 1, ..., n$, and that for all $\rho > 0$, all $x \in \bigcup_{i=1}^{n} M_i$, the point $(0, x)$ is an inner pair of (5.2^{ρ}). Then for each $i \in \{1, ..., n\}$, there is a decreasing sequence of control sets D_i^{ρ} of (5.2^{ρ}) with $M_i \subset \text{int } D_i^{\rho}$ such that $M_i = \bigcap_{\rho > 0} D_i^{\rho}$. Vice versa if for a sequence $\rho_k \to 0$ and control sets D^{ρ_k} of (5.2^{ρ_k}) the set of limit points $L := \{y \in M$: there are $x^k \in D^{\rho_k}$ with $x_k \to y\}$ is nonvoid then there is $i \in \{1, ..., n\}$ such that $L = M_i$.*

5.4. COROLLARY. *Let the assumptions of Corollary 5.3 be satisfied and assume that $\rho > 0$ is small enough. Then the system (5.2^{ρ}) is controllable iff the flow of* (5.1) *is chain recurrent.*

Compare [CTE(75), Lo(74), JQ(78)] for related results on controllability with arbitrarily small controls.

Recall from Section 2 that the lifts $\mathcal{D} \subset \mathcal{U} \times M$ of the control sets $D \subset M$ with $\text{int } D \neq \emptyset$ uniquely correspond to the maximal topologically mixing (and transitive) components of the control flow Φ. Hence Corollary 5.3 says that the chain recurrent components of (5.1) "blow up" to topologically transitive components of a control system, if the embedding of (5.1) into the family (5.2^{ρ}) is rich enough, i.e., if (H) holds and if the points $x \in M$ in the Morse sets for $\dot{x} = X_0(x)$ are inner pairs in $\mathcal{U} \times M$. One can view this result as a topological analogue of [Ru(81)], where a differential equation is embedded into a stochastic system with small random perturbations, which satisfy a nondegeneracy condition. (The analogy is even more striking if one takes into account the genericity results from Section 6, where it is shown that generically on an open and dense set in $\mathcal{U} \times M$ only the invariant control sets determine the limit behavior.) Also note the similarities to the Wentzell–Freidlin theory concerning random perturbations of dynamical systems.

In view of Corollary 5.3, it may be tempting to conjecture that, for $\rho > 0$ small enough, the number of control sets with nonvoid interior is equal to the number n of chain connected components of (5.1), i.e., that $D_1^{\rho}, ..., D_n^{\rho}$ are the only control sets with nonvoid interior of (5.2^{ρ}). The following example on $M=S^1$ shows that this conjecture is false; in fact, here the number of control sets with nonvoid interior is unbounded as ρ tends to zero. This is also remarkable since for fixed control range the number of control sets with nonvoid interior on $M=S^1$ is always finite (under quite weak assumptions: cp. Section 3 of [CK(92[a])]).

5.5. EXAMPLE. We consider a system on the unit circle $M=S^1$ with $U=[-1,1]\times[-1,1]$ of the form

$$\dot{x}(t)=f_0(x)-3u_1+6u_2=:f(x,u_1,u_2),$$

where x denotes the angle $x\in[0,2\pi)$.

We construct the C^∞-function f_0 on $[0,2\pi)$: For $n\in\mathbb{N}$

$$x_n:=\pi+\frac{1}{n}$$

$$I_n:=\left(x_n-\frac{1}{n^2},x_n+\frac{1}{n^2}\right)$$

$$J_n:=\left(\frac{x_n+x_{n+1}}{2},\frac{x_n+x_{n-1}}{2}\right).$$

For n_0 large enough and all $n\geqslant n_0$, $x_n\in I_n\subset J_n\subset(0,2\pi)$ and $J_n\cap J_{n+1}=\varnothing$. Choose a C^∞-function f_0 on $[0,2\pi)$ with

$$-\frac{2}{n}<f_0(x)<-\frac{1}{n}\qquad\text{for}\quad x\in I_n$$

$$f_0(x)\leqslant-\frac{2}{n}\qquad\text{for}\quad x\in J_n\backslash I_n$$

such that for some $y_n\in J_n\backslash I_n$, $y_n<x_n$,

$$f_0(y_n)=-\frac{10}{n}.$$

Clearly $f_0(\pi)=0$; we also require that for some $z_0\in(0,\pi)$ one have

$$f_0(z_0)-3-6>0.$$

(Hence one cannot steer the system from $x=\pi$ to $x=0=2\pi$.) Then for $x\in(\pi,2\pi)$, $u_1\geqslant0$,

$$f(x,u_1,0)=f_0(x)-3u_1<0.$$

Consider $\rho_N = 1/N$, i.e., $U^{\rho_N} = [-1/N, 1/N] \times [-1/N, 1/N]$. Then for all $n \in \mathbb{N}$ with $N/2 \leqslant n \leqslant N$ and $u_1 = -1/N$, $u_2 \geqslant 0$, $x \in I_n$:

$$\begin{aligned} f\left(x, -\frac{1}{N}, u_2\right) &= f_0(x) - 3u_1 + 6u_2 \\ &> -\frac{2}{n} + \frac{3}{N} + 6u_2 \\ &\geqslant \frac{1}{N} + 6u_2 > 0. \end{aligned}$$

Hence I_n is contained in the interior of some control set D_n^N. Furthermore π is also contained in the interior of some control set D_0^N:

$$f\left(\pi, \frac{1}{N}, 0\right) = f_0(\pi) - \frac{3}{N} = -\frac{3}{N} < 0$$

$$f\left(\pi, 0, \frac{1}{N}\right) = f_0(\pi) + \frac{6}{N} = \frac{6}{N} > 0.$$

On the other hand, for $N > 2n_0$ and $j, n \in \mathbb{N}$ with $N/2 \leqslant j < n \leqslant N$, one cannot steer the system from D_j^N to D_n^N (i.e., $D_j^N \neq D_n^N$): The point y_n lies between D_j^N and D_n^N and for all $(u_1, u_2) \in U^{\rho_N} := [-1/N, 1/N] \times [-1/N, 1/N]$

$$f(y_n, u_1, u_2) = f_0(y_n) - 3u_1 + 6u_2 \leqslant -\frac{10}{n} + \frac{3}{N} + \frac{6}{N} \leqslant -\frac{1}{N} < 0.$$

Similarly one sees that $D_0^N \neq D_n^N$, $N/2 \leqslant n \leqslant N$. Thus for even $N \in \mathbb{N}$ we have at least $N/2 + 2 - n_0$ control sets with nonvoid interior D_0^N, $D_{N/2-n_0}^N, ..., D_N^N$, while f_0 may be chosen such that $\dot{x} = f_0(x)$ has only two chain connected components.

Clearly the number of control sets tends to infinity as $\rho_N = 1/N$ tends to zero.

5.6. *Remark.* Actually, the correspondence between Morse sets and control sets extends to the (partial) orders that can be defined on these structures: Let $\mathcal{M} = \{M_1, ..., M_n\}$ be the finest Morse decomposition of (5.1). Define

$$M_i \prec M_j \quad \text{if there are Morse sets } M_{i_1} = M_i, ..., M_{i_k} = M_j \text{ and points } x_1, ..., x_k \in M \text{ with } \omega^*(x_l) \subset M_{i_l} \text{ and } \omega(x_l) \subset M_{i_{l+1}} \text{ for } l = 1, ..., k-1; \tag{5.10}$$

compare [Co(78)]. For control sets we write $D_i \prec D_j$ if there exists $x \in D_j$ with $\mathcal{O}^+(x) \cap D_j \neq \emptyset$; see [(CK(93)]. Under the assumptions of Corollary 5.3 let $M_i = \bigcap_{\rho > 0} D_i^\rho$ for $i = 1, ..., n$. Then we have:

(ii) If $M_i \prec M_j$, then $D_i^\rho \prec D_j^\rho$ for all $\rho > 0$.

(ii) If there is $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$: $D_i^\rho \prec D_j^\rho$, then $M_i \prec M_j$.

(iii) The invariant control sets C^ρ of (5.2^ρ) correspond, for ρ small enough, exactly to the maximal Morse sets of (5.1), i.e., to the Morse sets, which are attractors. Similarly there is a 1–1 correspondence between open control sets and minimal Morse sets.

For a proof of these statements see Theorem 9 in [CK(92[b])]. Since each invariant control set C^ρ of (5.2^ρ) contains a maximal Morse set of (5.1), this result implies in particular that "additional" control sets, like those in Example 5.5, are neither open nor closed, and, in particular, they are variant control sets.

6. GENERICITY RESULTS

In this section we analyze the generic behavior of the control flow as $t \to \pm\infty$. While the limit and the control structure of the system (1.1) can be very rich, generically all trajectories in a compact, invariant set follow a simple dichotomy: As $t \to \infty$, the trajectories enter invariant control sets (and stay there), while for $t \to -\infty$ they end up in open control sets. Recall from Remark 5.6 that invariant control sets form around the attractors of the (uncontrolled) system (5.1), and minimal control sets develop around the minimal Morse sets (under the conditions of Theorem 5.1). Hence the limit structure of (5.1) is still visible in the generic behavior of (1.1). (Here genericity is understood in the strong (topological) sense that this statement holds in an open and dense set.)

These results are the control theoretic analogues (with topological genericity) of results in stochastic differential equations, where on a compact manifold—with probability one—every trajectory ends up in an invariant control set, and where the support of invariant measures coincides with invariant control sets (cp. [Kl(87)]). Compare in this context also [Ru(81)], where it is shown that a dynamical system with small random perturbations asymptotically "lives" on attractors. If attention is restricted to trajectories in $\mathscr{D}$ that stay for all times in the interior of some control set $\mathscr{D}$, then a standard result from dynamical systems theory shows that for a residual set in $\mathscr{D}$ the control set $\mathscr{D}$ will be filled out densely by trajectories as $t \to \infty$ or $t \to -\infty$.

Our analysis is based on the following lemma:

6.1. LEMMA. *Suppose that M is compact. Then for each $x \in M$ there exist an invariant control set C, a control $u \in \mathcal{U}$, and a time $T > 0$ such that $\varphi(t, x, u) \in \operatorname{int} C$ for all $t \geqslant T$.*

For a proof cp. [K1(87)].

If the state space M is compact, we obtain the following genericity result.

6.2. THEOREM. *Suppose that M is compact. Then*

$$\{(u, x) \in \mathcal{U} \times M : \text{there is } T > 0 \text{ such that for all } t \leqslant -T,\ \varphi(t, x, u) \in \operatorname{int} C^* \text{ for some open control set } C^* \text{ and for all } t \geqslant T,\ \varphi(t, x, u) \in \operatorname{int} C \text{ for some closed control set } C\}$$

in open and dense in $\mathcal{U} \times M$.

Proof. Suppose that there is $T > 0$ with $\varphi(t, x, u) \in \operatorname{int} C$ for all $t \geqslant T$ and some invariant control set C.

Choose a neighborhood N of $(u, x) \in \mathcal{U} \times M$ such that for all $(v, y) \in N$

$$\varphi(T, y, v) \in \operatorname{int} C.$$

Then invariance of int C implies openness.

Concerning density, let $(u, x) \in \mathcal{U} \times M$ be arbitrary. Then it suffices (cp., e.g., [CK(93)]) to consider neighborhoods N of (u, x) of the form

$$N = \left\{ (v, y) \in \mathcal{U} \times M : d(x, y) < \varepsilon \text{ and } \int_{\mathbb{R}} [u(t) - v(t)]\, f_i(t)\, dt < \varepsilon \text{ for } i = 1, \ldots, n \right\}, \tag{6.1}$$

where $\varepsilon > 0$, $n \in \mathbb{N}$, and $f_1, \ldots, f_n \in L^1(\mathbb{R}, \mathbb{R}^m)$.

Choose T large enough such that for $i = 1, \ldots, n$

$$\int_T^\infty |f_i(t)|\, dt < \frac{\varepsilon}{2 \cdot \operatorname{diam}(U)}.$$

Pick $(v, y) \in N$. Then there exist $\tilde{T} > 0$ and $\tilde{u} \in \mathcal{U}$ with

$$\varphi(\tilde{T}, \varphi(T, y, v), \tilde{u}) \in \operatorname{int} C$$

for some invariant control set C by Lemma 6.1. Then $(\tilde{v}, y) \in N$, where

$$\tilde{v}(t) := \begin{cases} v(t), & t \in (-\infty, T] \\ \tilde{u}(t - T), & t > T; \end{cases}$$

furthermore $\varphi(T + \tilde{T}, y, \tilde{v}) \in \operatorname{int} C$. This proves density. The rest of the assertion follows by time reversal. ∎

6.3. *Remark.* The compactness assumption on M can be weakened, if only the limit behavior for $t \to +\infty$ is considered. It suffices, e.g., to require that there exist a compact set $A \subset M$ such that for all $(u, x) \in \mathcal{U} \times M$ there is $T = T(u, x) > 0$ with $\varphi(t, x, u) \in A$ for all $t \geqslant T$. Then for all (u, x) in an open dense subset of $\mathcal{U} \times M$ it follows that $\varphi(t, x, u) \in \operatorname{int} C$ for some invariant control set C and all t large enough.

The next results concern the generic behavior of the trajectories in case where the state space M is not compact. We first show that for not completely controllable systems generically the trajectories leave any control set forward or backward in time:

6.4. PROPOSITION. *Suppose that D is a control set with nonvoid interior such that* cl D *is not a connected component of M. Then the set*

$$A := \{(u, x) \in \mathcal{U} \times M : \text{there is } T \in \mathbb{R} \text{ such that } \varphi(t, x, u) \notin \operatorname{cl} D \text{ for all } t \geqslant T \text{ or } \varphi(t, x, u) \notin \operatorname{cl} D \text{ for all } t \leqslant T\}$$

is open and dense in $\mathcal{U} \times D$.

Proof. Consider first an invariant control set C. Openness of A is clear. Concerning density, it suffices to consider $(u, x) \in \mathcal{U} \times \operatorname{int} C$. Take a basic neighborhood N of (u, x) of the form (5.1).

There is $T > 0$ large enough such that for $i = 1, ..., n$

$$\int_{-\infty}^{T} |f_i(t)| \, dt < \frac{\varepsilon}{2 \cdot \operatorname{diam}(U)}.$$

If $\varphi(-T, x, u) \notin C$ we are done since then $(u, x) \in A$. Otherwise we may assume (using (H)) that $\varphi(-T, x, u) \in \operatorname{int} C$.

Since by assumption C is not a connected component of M, there is $x_0 \in C \cap \operatorname{cl}(M \backslash C)$. There are $\tau > 0$ and $u_0 \in \mathcal{U}$ with $\varphi(\tau, x_0, u_0) \in \operatorname{int} C$. Thus, by continuous dependence on the initial value, there is $x_1 \in M \backslash C$ with $\varphi(\tau, x_1, u_0) \in \operatorname{int} C$. By Proposition 2.2, the point $\varphi(\tau, x_1, u_0)$ can be steered to $\varphi(-T, x, u)$ by, say, a control u_1 in time t_1. Define the composition control

$$v(t) = \begin{cases} u(t), & t \geqslant -T \\ u_1(T + t_1 - t), & t \in [-T - t_1, -T) \\ u_0(T + t_1 + \tau - t), & t \in [-T - t_1 - \tau, -T - t_1) \\ \in U, & t < -T - t_1 - \tau. \end{cases}$$

Then $(v, x) \in N$ and $\varphi(t, x, v) \notin C$ for all $t < -T - t_1 - \tau$, by maximality of control sets. This proves the assertion for closed control sets. Again, the other cases are similar. ∎

Note that we have actually proved the stronger statement: For all $(u, x) \in \mathscr{U} \times D$ and all neighborhoods $N(u) \subset \mathscr{U}$ there exists $v \in N(u)$ with $\varphi(t, x, u) \notin \operatorname{cl} D$ for all $t \geqslant T$, or for all $t \leqslant T$.

Next we analyze the generic behavior of trajectories which remain for all times in the interior of a control set. Let D be a control set and recall from Section 2 that

$$\mathscr{D} = \operatorname{cl}\{(u, x) \in \mathscr{U} \times M\colon \varphi(t, x, u) \in \operatorname{int} D \text{ for all } t \in \mathbb{R}\} \tag{6.2}$$

is topologically transitive.

6.5. PROPOSITION. *Let D be a bounded control set in M. Then*

$$\{(u, x) \in \mathscr{D}\colon \pi_M \omega(u, x) = D\}$$

is residual in $\mathscr{D}$, i.e., contains a countable intersection of open and dense sets.

Proof. This follows from [Ma(87), Proposition I.11.4], since $\mathscr{D}$ is topologically transitive. Thus

$$\{(u, x) \in \mathscr{D}\colon \omega(u, x) = \mathscr{D}\}$$

is residual in $\mathscr{D}$. ∎

The genericity statement above is formulated with respect to $\mathscr{D}$. The following results show that—except in trivial cases—the set $\mathscr{D}$ is very "thin" in $\mathscr{U} \times M$. Hence genericity with respect to $\mathscr{D}$ is rather weak.

6.6. PROPOSITION. *Suppose that D is a control set with nonvoid interior such that* cl D *is not a connected component of M. We have*

(i) *if D is closed, then for all $x \in D$ there exist $u \in U$ and $T < 0$ such that $\varphi(t, x, u) \notin D$ for all $t < T$;*

(ii) *if D is open, then for all $x \in D$ there are $u \in U$ and $T > 0$ such that $\varphi(t, x, u) \notin \operatorname{cl} D$;*

(iii) *if D is neither open nor closed, then for all $x \in D$ there are $u \in U$ and $T > 0$ such that $\varphi(t, x, u) \notin \operatorname{cl} D$ for all $t < -T$ and all $t > T$.*

Proof. (i) Let $x \in \partial D$; then there is a control $u\colon [0, S] \to U$ with $\varphi(S, x, u) \in \operatorname{int} D$. Hence there is a neighborhood $V(x)$ with $\varphi(S, y, u) \in \operatorname{int} D$ for all $y \in V(x)$, and thus there exists a point $y \in \operatorname{int} D$ with $\mathcal{O}^-(y) \cap D^c \neq \varnothing$. By Proposition 2.2 this means that $\mathcal{O}^-(z) \cap D^c \neq \varnothing$ for all $z \in D$. The assertion now follows from the maximality of control sets. (Here D^c denotes the complement of D.)

Part (ii) is proved in the same way as (i) with time reversed.

Part (iii) follows immediately from the construction in (i) and (ii). ∎

7. Control of Complex Systems: The Lorenz Equation

Control and stabilization are two of the major design goals in control theory of nonlinear systems. Much attention has been devoted to criteria for complete controllability of systems; i.e., for any two points $x, y \in M$ there exist a control function u and a time $t \geq 0$ such that $\varphi(t, x, u) = y$. If the control range U is bounded, one cannot expect complete controllability in general, as simple examples, such as Example 2.7, show. The question then is, to what extent can the behavior of the differential equation $\dot{x} = X_0(x)$ be altered by the control system $\dot{x} = X_0(x) + \sum_{i=1}^m u_i(t) X_i(x)$ for a given control range U? Of particular interest are situations where $\dot{x} = X_0(x)$ has complicated limit sets or Morse sets, and one tries to "simplify" this behavior to obtain, e.g., fixed points or periodic trajectories. Theorems 4.4 and 4.13 provide an answer to this question under different sets of assumptions. Corollary 5.3 states that under the inner point assumption any Morse set M_i of $\dot{x} = X_0(x)$ can be embedded into a control set with nonvoid interior (and, in fact, M_i is the limit of such control sets as the control range tends to zero).

The problem of stabilization (e.g., with respect to fixed points or periodic orbits) requires controllability around these orbits, and convergence towards them. The results in Sections 4 and 5 clarify the assumptions under which we have controllability around a reference trajectory, namely if this trajectory lies in the interior of a control set. The genericity statements in Section 6 show that open loop controls will not result in robust stabilization with respect to reference trajectories outside of invariant control sets. Hence feedback controls are needed for this task. Their design, based on additional dynamical systems concepts for control systems, such as invariant manifolds, is currently under investigation.

Here we concentrate on the controllability problem. We discuss the controlled Lorenz equation as an example which illustrates the relation between "chaotic behavior" and control sets as derived in Section 4. We have chosen this example, on the one hand, because the standard control theoretic assumption (H) is not satisfied, and the treatment via Theorem 4.13 needs some additional arguments. (This should also show how to obtain results for systems that do satisfy (H).) On the other hand, the Lorenz equation is often quoted as an example for a strange attractor, and any "control of chaos theory" should be able to handle this case. Recall, however, that up to now there is no proof of the existence of a "strange" attractor for the Lorenz equation, only numerical evidence. For our approach this is not important, because we show how to embed the existing attractor of the Lorenz equation into some control set.

We consider the control system

$$\begin{aligned} \dot{x}_1(t) &= p[x_2(t) - x_1(t)] \\ \dot{x}_2(t) &= -x_1(t)\,x_3(t) + u(t)\,x_1(t) - x_2(t) \\ \dot{x}_3(t) &= x_1(t)\,x_2(t) - bx_3(t) \\ u(t) &\in U := [r^0 - \rho, r^0 + \rho], \end{aligned} \tag{7.1$^\rho$}$$

where $p > 0$, $b > 0$, $r^0 > 1$, and ρ is some positive constant. This equation serves as a finite dimensional model of the Rayleigh–Bénard convection. We take the coefficient r corresponding to the Rayleigh coefficient, i.e., to the applied temperature difference at the boundary, as the control input (cp. [BPV(84), Lo(63)]). A numerically observed "strange attractor" occurs for $p = 10$, $b = \frac{8}{3}$, $r^0 = 28$, and $\rho = 0$, which is approximated for $t \to \infty$, e.g., by the trajectory $\varphi(\cdot, x^0, r^0)$ starting in $x^0 := (-8, 8, 27)$. We show that there exists a control set D with nonvoid interior such that the attractor is contained in the closure of D, i.e., in a region of complete controllability.

For the following discussion cp., e.g., [HNW(87), p. 117] or [Sp(82)]. For a constant control $u(t) \equiv r$, the ball

$$R_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - p - r)^2 \leqslant c^2\}$$

is positively invariant for c large enough. Hence for $x^0 \in R_0$, the positive trajectory $\varphi^+(x, u)$ is bounded. For $r > 1$, the equation possesses three (hyperbolic) equilibria, the origin $0 \in \mathbb{R}^3$ and

$$\begin{aligned} x^* &= (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \\ x^{**} &= (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1). \end{aligned}$$

Note that the x_3-axis is invariant under all controls, but for constant controls $u \in U$ the fixed points x^* and x^{**} vary with u.

Denote by $\omega(r^0, x^0)$ the ω-limit set of (7.1) for the constant control value $u = r^0$ (i.e., $\rho = 0$) and for $x^0 \in \mathbb{R}^3$.

7.1. THEOREM. *Consider the control system* (7.1$^\rho$) *and assume* $r_0 - \rho > 1$. *Let* $K \subset \mathbb{R}^3$ *be compact with* $K \cap \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\} = \varnothing$. *Then for each initial point* $x_0 \in \mathbb{R}^3 \backslash \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ *there exists a control set D with*

$$K \cap \pi_{\mathbb{R}^3}\omega(r^0, x^0) \subset \operatorname{int} D, \tag{7.2}$$

Proof. We must verify the assumptions of Theorem 4.13 with $L = \mathbb{R}^3 \backslash \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. We make use of the strong accessibility criterion of Corollary 4.6.

For the system (7.1) and $u \equiv r^0$, one has

$$Y(x) = X_1(x)\, x_1 \frac{\partial}{\partial x_2},$$

$$X(x) = \{p(x_2 - x_1)\} \frac{\partial}{\partial x_1} + \{-x_1 x_3 + u x_1 - x_2\} \frac{\partial}{\partial x_2} + \{x_1 x_2 - b x_3\} \frac{\partial}{\partial x_3}.$$

One computes

$$(\mathrm{ad}_X Y)(x) = \{p(x_2 - x_1)\} \frac{\partial}{\partial x_1} + \{x_1 - x_1 p - x_1^2\} \frac{\partial}{\partial x_2},$$

$$(\mathrm{ad}_X^2 Y)(x) = \{p(x_2 - x_1)(1 - p - 2x_1 + x_3 - u - x_2)\} \frac{\partial}{\partial x_1}$$

$$+ \{(-x_1 x_3 + u x_1 - x_2) p + (x_1 - x_1 p - x_1^2)(1 - p - x_1)\} \frac{\partial}{\partial x_2},$$

$$(\mathrm{ad}_X^3 Y)(x) = f_1(x) \frac{\partial}{\partial x_1} + f_2(x) \frac{\partial}{\partial x_2} + p(x_1 x_2 - b x_3)(x_2 - 2x_1) \frac{\partial}{\partial x_3}.$$

Assume

(i) $x_1 \neq 0$ and

(ii) $x_1 \neq x_2$.

Then $Y(x)$ and $\mathrm{ad}_X Y(x)$ are linearly independent. If, additionally,

(iii) $x_1 x_2 - b x_3 \neq 0$, and

(iv) $x_2 - 2x_1 \neq 0$,

then linear span $\{Y(x), \mathrm{ad}_X Y(x), \mathrm{ad}_X^3 Y(x)\} = T_x \mathbb{R}^3$.

Thus in points x satisfying (i)–(iv), the strong accessibility condition is satisfied and hence for all $T' > T > 0$

$$\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^+_{\leqslant T'}(x) \qquad \text{and} \qquad x \in \operatorname{int} \mathcal{O}^-_{\leqslant T'}(\varphi(T, x, u)). \tag{7.3}$$

We show that for all $x \in K$ violating (i), (ii), (iii), or (iv) the trajectory $\varphi(t, x, u)$ satisfies (i)–(iv) for $|t| > 0$ small enough. Hence also for these points (7.3) holds, proving Theorem 7.1.

We may assume that $x \notin \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Then if $x_1 = 0$ one has $x_2 \neq 0$ and hence

$$X(x) = (p x_2, -x_2, -b x_3)$$

has a nonvanishing first component. Thus for all $|t| > 0$ small enough, $\varphi(t, x, u)$ has a nonvanishing first component, i.e., (i) holds.

If $x_1 = x_2$, then $x_1 \neq 0$ and

$$X(x) = (0, (-x_3 + u - 1)x_1, x_1^2 - bx_3).$$

Thus for $x_3 \neq r_0 - 1$, and $|t| > 0$ small enough, $\varphi(t, x, u)$ has two unequal first components, i.e., (ii) holds.

If $x_3 = r_0 - 1$, then the last component can be taken to be $\neq 0$ (since x^* and x^{**} vary with u), and hence for $|t|$ small enough, $\varphi(t, x, u)$ has a third component $\neq r_0 - 1$. Therefore we can use the same argument as before.

If $x_1 x_2 - bx_3 = 0$, then

$$X(x) = \left(p(x_2 - x_1), -\frac{1}{b} x_1^2 x_2 + ux_1 - x_2, 0 \right).$$

Hence $X(x)$ is not tangential to the surface $x_3 = (1/b) x_1 x_2$ for any $x \in L$. Thus for all $|t| > 0$ small enough, condition (iii) is satisfied by $\varphi(t, x, u)$.

Finally, if $x_2 - 2x_1 = 0$, then

$$X(x) = (px_1, -x_1 x_3 + ux_1 - 2x_1, 2x_1^2 - bx_3).$$

Hence for $x \in L$ either $X(x)$ is not tangential to the plane $x_2 - 2x_1 = 0$ or the argument for (iii) can be applied; then for all $|t| > 0$ condition (iv) holds for $\varphi(t, x, u)$.

In conclusion we find that each of the (finitely many) conditions (i)–(iv) holds for $\varphi(t, x, u)$ and all $|t| > 0$ small enough. This proves the theorem for initial points $x_0 \in \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ and shows that every limit set of Eq. (7.1) is contained in the interior of a control set, when intersected with any compact set K as specified above. ▮

7.2. *Remark.* Assume that there are $r^0 > 1$ and an initial point $x^1 \in \mathbb{R}^3$ such that $\pi_{\mathbb{R}^3} \omega(r^0, x^1)$ coincides with the chain recurrent set of the Lorenz equation with constant parameter value r^0 (this would hold if it has, as conjectured, a strange attractor). Then for $\rho > 0$ small enough, the control system (7.1^ρ) has a unique control set D satisfying (7.2).

7.3. *Remark.* The result above shows a close connection with recent results [OGY(90)] on the control of chaos: Since $K \cap \pi_{\mathbb{R}^3} \omega(r^0, x^0) \subset \text{int } D$, any trajectory of the uncontrolled equation with initial value outside the x_3-axis will eventually enter int D. Through all points in int D there passes a variety of periodic trajectories of the controlled equation that stay within int D, and because of complete controllability in int D, the system can be steered (by an open loop control) such that it follows any predetermined periodic motion in int D. The idea therefore is to embed a strange attractor (of "odd" dimension) in the interior of a control set D (with dim(int D) =

dim M), where it can be controlled to have regular behavior, e.g., follow a periodic motion γ (with dim $\gamma = 1$). The question of feedback stabilization to regular motion will be studied elsewhere.

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