Universität Augsburg

Institut für Mathematik

Peter Quast

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Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

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Peter Quast
Institut für Mathematik
Universität Augsburg
86135 Augsburg


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# REAL FORMS OF HERMITIAN SYMMETRIC SPACES, REVISITED

PETER QUAST

Abstract. In 1984 Masaru Takeuchi showed that every real form of a hermitian symmetric space of compact type is a symmetric $R$-space. In this paper we present a geometric proof of Takeuchi's result.

## 1. Introduction

Symmetric $R$-spaces are compact Riemannian symmetric spaces that are also $R$-spaces (generalized flag manifolds) in the sense of Takeuchi [T65], that is quotients of connected center-free semi-simple Lie groups by parabolic subgroups. Symmetric $R$-spaces can be realized are $s$-orbits of extrinsically symmetric elements (see [N65, Ko68, KT68, Ke71, Ke72, F74a] and Section 2). From this point of view they are the building blocks of extrinsically symmetric submanifolds in Euclidean space (see [F74b, F80, EH95]). Indecomposable symmetric $R$-spaces split into two types:

(i) irreducible hermitian symmetric spaces of compact type;
(ii) indecomposable symmetric $R$-spaces of non-hermitian type.

The (local) classification of indecomposable symmetric $R$-spaces is due to Kobayashi and Nagano [KN64, KN65] (see also [BCO03, p. 310f]). In this note we give a *geometric* proof of Takeuchi's result:

**Theorem 1** (Masaru Takeuchi [T84]). *Every symmetric $R$-space can be realized as a real form of a hermitian symmetric space of compact type. Vice-versa every real form of a hermitian symmetric space of compact type is a symmetric $R$-space.*

Our main tool is the extension of isometries of hermitian symmetric spaces of compact type presented in [EQT13].

A classification of real forms of hermitian symmetric spaces can also be found in [Le79].

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## 2. Preliminaries

2.1. **Symmetric $R$-space as $s$-orbits.** Every symmetric $R$-space arises in the following way (see [N65, Ko68, KT68, Ke71, Ke72, F74a] and also [BCO03, pp. 70-72]):

Let $S$ be a symmetric space of compact type (we always assume symmetric spaces to be connected) with a chosen base point $o \in S$, and $L$ the identity component of the isometry group of $S$. For the classical facts about symmetric spaces mentioned below we refer to the standard literature like Helgason's famous monograph [H78] or [W84, Part IV]. The geodesic symmetry $s_o$ of $S$ at the base point $o$ gives rise to an involutive automorphism

$$\sigma : L \to L, \quad l \mapsto s_o \circ l \circ s_o.$$

The differential $\sigma_*$ of $\sigma$ at the identity is therefore an involutive automorphism of the Lie algebra $\mathfrak{l}$ of $L$, called the *Cartan involution* of $(S, o)$. We denote by $\mathfrak{h}$ the fixed point set and by $\mathfrak{s}$ the $(-1)$-eigenspace of $\sigma_*$. The decomposition

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s},$$

called *Cartan decomposition* of $\mathfrak{l}$ corresponding to $(S, o)$, is orthogonal w.r.t. the Cartan-Killing form $B_\mathfrak{l}$ of $\mathfrak{l}$. This decomposition satisfies the *Cartan relations*, namely

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s} \text{ and } [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h}.$$

The Lie subalgebra $\mathfrak{h}$ is the Lie algebra of the identity component $H$ of the isotropy group of $o$ in $G$. Moreover, $\mathfrak{s}$ can be identified with the tangent space $T_oS$ by the restriction of the differential at the identity of the principal bundle $L \to S$, $l \mapsto l.o$. Here $l.o$ denotes the action of the isometry $l$ of $S$ on the point $o \in S$. Using the above identification $\mathfrak{s} \cong T_oS$, the lineare isotropy action of $H$ on $T_oS$, also known as *$s$-representation,* becomes the restriction of the adjoint action:

$$H \times \mathfrak{s} \to \mathfrak{s}, \quad (h, X) \mapsto \mathrm{Ad}_L(h)X.$$

A non-zero element $\xi \in \mathfrak{s}$ is called *extrinsically symmetric* (or *minuscule coweight*), if

$$\mathrm{ad}_\mathfrak{l}(\xi)^3 = -\mathrm{ad}_\mathfrak{l}(\xi),$$

or equivalently, if the eigenvalue spectrum of $\mathrm{ad}(\xi)$ equals $\{-i, 0, i\}$. For a description of extrinsically symmetric elements in terms of roots we refer to [MQ12, Lem. 2.1] and also to [KN64].

A *symmetric $R$-space* is an isotropy orbit ($s$-orbit)

$$M = \mathrm{Ad}_L(H)\xi \subset \mathfrak{s},$$

of $S$ where $\xi \in \mathfrak{s}$ is an extrinsically symmetric element.

If $S$ is an irreducible symmetric space of compact type, we call $M$ *indecomposable.* If $S$ is an irreducible symmetric space of compact type, but not a compact simple Lie group, $M$ is an indecomposable symmetric $R$-space of non-hermitian type (see e.g. [BCO03, p. 310f.]).

2.2. **Hermitian symmetric spaces of compact type as $R$-spaces.** If $S$ is a compact connected semi-simple center-free Lie group $G$, then $L$ is isomorphic to $G \times G$. (see [H78, Chap. IV, §6]). The linear isotropy representation on the tangent space $T_eG$ is isomorphic to the adjoint representation of $G$ on $\mathfrak{g}$.

Let $\xi \in \mathfrak{g}$ be extrinsically symmetric and assume that the projections of $\xi$ onto the simple factors of $\mathfrak{g}$ never vanish. It is well-known (see [Li58, pp. 165 ff.] and [Hi70]) that $P := \mathrm{Ad}(G)\xi \subset \mathfrak{g}$, endowed with the Riemannian metric induced by the scalar product $-B_{\mathfrak{g}}$ on $\mathfrak{g}$, is a hermitian symmetric space of compact type. Let $X \in P$, then $\mathrm{Ad}\,(\exp{(\pi/2 \cdot X)})$ and $\mathrm{ad}(X)$ coincide on $T_XP \subset \mathfrak{g}$ and they define a Kähler structure $J_X$ of $P$ at the point $X$, that is

$$J_X = \mathrm{Ad}\,(\exp{(\pi/2 \cdot X)})\,|_{T_XP} = \mathrm{ad}(X)|_{T_XP}, \tag{1}$$

which turns $P$ into a hermitian symmetric space.

The geodesic symmetry $s_X$ of $P$ at the point $X$ extends to the reflection $\rho_X$ of $\mathfrak{g}$ along the normal space $N_XP = \{Y \in \mathfrak{g} : \ \mathrm{ad}(X)Y = 0\}$ given by the involutive automorphism

$$\rho_X := \mathrm{Ad}(\exp(\pi X)) \tag{2}$$

of $\mathfrak{g}$. Finally, $G$ can be identified with the identity component of the isometry group of $P$.

Conversely every hermitian symmetric space $P$ of compact type can be realized as such an orbit ([Li58, pp. 165 ff.] and [Hi70]) in the Lie algebra of its infinitesimal isometries. If we endow this Lie algebra with a scalar product that coincides on each irreducible factor with the Cartan-Killing form up to a suitable negative constant, this embedding is isometric. We call this the *standard embedding* of a hermitian symmetric space of compact type.

2.3. **Real forms of hermitian symmetric spaces.** Following Takeuchi [T84], a *real form* of a hermitian symmetric space $P$ is a connected component of the fixed point set of some involutive and antiholomorphic isometry $f$ of $P$. Therefore real forms are in particular totally geodesic real submanifolds of $P$.

## 3. The proof

In this section we present a geometric proof of Takeuchi's result, Theorem 1 (see [T84]). As a major tool we use in Paragraph 3.2 the results of Eschenburg, Tanaka and the author on the extension of isometries of hermitian symmetric spaces published in [EQT13].

3.1. **The proof of the first implication.** The arguments presented in this paragraph are classical and straight forward. They may also be adapted to more general situations.

Let $S$ be a symmetric space of compact type, $o \in S$ a base point, $\sigma_*$ the corresponding Cartan involution and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s}$ the induced Cartan decomposition of the semi-simple Lie algebra $\mathfrak{l}$ of infinitesimal isometries of $S$. Let $\xi \in \mathfrak{s}$ be an extrinsic symmetric element and $M := \mathrm{Ad}_L(H)\xi$ a symmetric $R$-space. We may assume that no projection of $\xi$ onto a simple factor of $\mathfrak{l}$ vanishes.

The inclusion $H \hookrightarrow L$ of the identity component $H$ of the isotropy group of $o$ into the identity component $L$ of the full isometry group of $S$ provides a natural inclusion

$$\mathfrak{s} \supset M = \mathrm{Ad}_L(H)\xi \longrightarrow \mathrm{Ad}_L(L)\xi =: P \subset \mathfrak{l}$$

of the symmetric $R$-space $M$ into the hermitian symmetric space $P$.

The linear endomorphism $F := -\sigma_*$ of $\mathfrak{l}$ preserves the scalar product on $\mathfrak{l}$ and maps adjoint orbits onto adjoint orbits. Since $\xi$ lies in $\mathfrak{s}$, the $(-1)$-eigenspace of $\sigma_*$, $\xi$ is a fixed point of $F$. Thus $F$ leaves $P$ invariant and $f := F|_P$ is an involutive isometry of $P$. Let $f_*$ denote the differential of $f$ at the fixed point $\xi$. To show that $f$ is anti-holomorphic, it is sufficient to verify that $f_*(J_\xi X) = -J_\xi f_*(X)$ for all $X \in T_\xi P$, because the complex structure $J$ of $P$ is parallel. Equation 1 implies

$$\begin{aligned} & f_*(J_\xi X) = F[\xi, X] = -\sigma_*[\xi, X] = -[\sigma_*\xi, \sigma_* X] \\ = \; & [\xi, \sigma_* X] = -[\xi, FX] = -J_\xi f_*(X). \end{aligned}$$

Recall that $T_\xi P$ is the orthogonal complement in $\mathfrak{l}$ of $\{X \in \mathfrak{l} : [X, \xi] = 0\}$ and $T_\xi M$ is the orthogonal complement in $\mathfrak{s}$ of $\{X \in \mathfrak{s} : [X, \xi] = 0\}$ (see e.g. [BCO03, p. 71]). Thus $T_\xi M = T_\xi P \cap \mathfrak{s}$ and $M$ is a connected component of the fixed point set of $f$. This shows that $M$ is a real form of $P$.

3.2. **The proof of the converse implication.** We now show the converse, namely that every real form of a hermitian symmetric space $P$ of compact type is a symmetric $R$-space.

Since $P$ is simply connected (see e.g. [H78, Ch. VIII, Thm. 4.6]), $P$ is a product of its irreducible de Rham factors

$$P = P_1 \times \ldots \times P_k,$$

where each factor is an irreducible hermitian symmetric space of compact type (see [W84, Cor. 8.7.11]). An involutive anti-holomorphic isometry $f$ either respects a de Rham factor or permutes isometric de Rham factors pairwise. Thus it is sufficient to only consider the following two cases:

(I) $P$ is the Riemannian product of two equal irreducible hermitian symmetric spaces $Q$ of compact type, that is $P = Q \times Q$, and $f$ interchanges both factors.

(II) $P$ is irreducible.

We start by investigating the first case. Let $\iota$ denote the isometry of $P = Q \times Q$ that just interchanges both factors, that is $\iota(x, y) = (y, x)$ for all $x, y \in Q$. Then $f$ has the form $f = (f_1 \times f_2) \circ \tau$ where $f_{1,2}$ are isometries of $Q$. Since $f$ is involutive we get $f_2 = f_1^{-1}$, that is $f = (f_1 \times f_1^{-1}) \circ \tau$. The fixed point set of $f$,

$$\{(x, y) \in P :\ f(x, y) = (x, y)\} = \{(x, f_1^{-1}(x)) :\ x \in Q\},$$

is isomorphic to $Q$ and hence a symmetric $R$-space.

To treat the second case we actually prove the following sharpened formulation of Theorem 1:

**Proposition 2.** *Every real form of an irreducible hermitian symmetric space $P$ of compact type is an indecomposable symmetric $R$-space of non-hermitian type.*

If $P = \mathrm{Ad}(G)\xi \subset \mathfrak{g}$ is a standardly embedded irreducible hermitian symmetric space of compact type, then the Lie algebra $\mathfrak{g}$ of its infinitesimal isometries is simple. We consider $\mathfrak{g}$ endowed with the scalar product that coincides with $B_{\mathfrak{g}}$ up to a negative factor. The Cartan involution corresponding to $(P, \xi)$ is $\rho_\xi$ given in Equation 2. The induced Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the fixed point set of $\rho_\xi$.

Let $f$ be an involutive anti-holomorphic isometry of $P$ and assume that $f$ fixes $\xi \in P$. We denote by $M$ the connected component of the fixed point set of $f$ that contains $\xi$. Then $M$ is a real form of $P$ and we must only show that $M$ is an indecomposable symmetric $R$-space of non-hermitian type.

The differential $f_*$ of $f$ at $\xi$ is an involutive linear automorphism of $\mathfrak{p} \cong T_\xi P$. The fixed point set $\mathfrak{m}$ of $f_*$ is canonically identified with the tangent space $T_\xi M$.

Following the reasoning in [EQT13, Sect. 3] we consider the Lie group automorphism

$$\phi : G \to G, \quad g \mapsto f \circ g \circ f$$

of the identity component $G$ of the full isometry group of $P$. Since $\phi$ leaves the stabilizer $K$ of $\xi$ in $G$ invariant, its differential $\phi_*$ at the identity induces an automorphism of $\mathfrak{k}$. We conclude (see [EQT13, Lemma 3.1]) that

$$\phi_*(\xi) \in \{\pm\xi\}.$$

**Lemma 3.** $\phi_*(\xi) = -\xi$.

*Proof.* Assume that $\phi_*(\xi) = \xi$. Then the derivative of the one-parameter family

$$\mathbb{R} \to G,\ s \mapsto \phi(\exp(s \cdot \xi)) = f \circ \exp(s \cdot \xi) \circ f$$

at $s = 0$ is $\phi_*(\xi) = \xi$. Hence

$$\exp(s \cdot \xi) = f \circ \exp(s \cdot \xi) \circ f \quad \text{for all } s \in \mathbb{R}.$$

The geodesic $\gamma$ in $M$ that starts at $\xi \in M$ in direction $X \in \mathfrak{m} \setminus \{0\}$ is given by $\gamma(t) = \mathrm{Ad}(\exp(tX))\xi = \exp(tX).\xi$. Taking $s = \frac{\pi}{2}$ we get

$$\begin{aligned} \exp\left(\frac{\pi}{2} \cdot \xi\right).\gamma(t) &= \left(f \circ \exp\left(\frac{\pi}{2} \cdot \xi\right) \circ f\right).\gamma(t) \\ &= \left(f \circ \exp\left(\frac{\pi}{2} \cdot \xi\right)\right).\gamma(t). \end{aligned}$$

The derivative at $t = 0$ yields

$$\begin{aligned} &\left(f_* \circ d\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right)_\xi\right) X = f_*\left(\mathrm{Ad}\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right) X\right) \\ = \;& f_*(J_\xi X) = d\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right)_\xi X = \mathrm{Ad}\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right) X = J_\xi X \end{aligned}$$

(see Equation 1). The equation $f_*(J_\xi X) = J_\xi X = J_\xi f_*(X)$ for a nonzero $X \in \mathfrak{m} \cong T_\xi M$ contradicts the fact that $f$ is anti-holomorphic. □

The proof of Lemma 3.1 in [EQT13] shows that

$$F := -\phi_* \circ \rho_\xi : \mathfrak{g} \to \mathfrak{g}$$

extends the isometry $f$ of $P \subset \mathfrak{g}$ to a linear isometry of the ambient space $\mathfrak{g}$.

**Lemma 4.** *$\tau := -F = \phi_* \circ \rho_\xi$ is an involutive automorphism of $\mathfrak{g}$ that commutes with $\rho_\xi$.*

*Proof.* As a composition of two automorphisms, $\tau$ is obviously an automorphism of $\mathfrak{g}$. Recall that $\phi_*$ preserves $\mathfrak{k}$ and therefore also $\mathfrak{p}$. Notice further that $\rho_\xi = \mathrm{Ad}(\exp(\pi\xi))$ is the identity on $\mathfrak{k}$ and $-\mathrm{Id}$ on $\mathfrak{p}$. This implies

$$\phi_* \circ \rho_\xi = \rho_\xi \circ \phi_*.$$

Hence $\tau^2 = \phi_*^2 \circ \rho_\xi^2 = \mathrm{Id}$ and $\tau \circ \rho_\xi = \phi_* = \rho_\xi \circ \tau$. □

Thus $(\mathfrak{g}, \tau)$ is an orthogonal involutive Lie algebra (see e.g. [W84, Ch. 8]). Let $\mathfrak{h}$ be the fixed point set of $\tau$ and $\mathfrak{s}$ the fixed point set of $F$. Then the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$$

is the Cartan decomposition of some irreducible pointed symmetric space $S$ of compact type (see e.g. [W84, Sect. 8.3]), which is not a compact Lie group (see e.g. [H78, p. 379]).

Moreover, since $\tau$ and $\rho_\xi$ commute, we get a common eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k}_+ \oplus \mathfrak{k}_- \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+,$$

where $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$, $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+$, $\mathfrak{h} = \mathfrak{k}_+ \oplus \mathfrak{p}_+$ and $\mathfrak{s} = \mathfrak{k}_- \oplus \mathfrak{p}_-$. Notice that $\xi \in \mathfrak{k}_- \subset \mathfrak{s}$ and that $\mathfrak{m} = \mathfrak{p} \cap \mathfrak{s} = \mathfrak{p}_-$.

We observe that $M$ is the connected component of $P \cap \mathfrak{s}$ that contains $\xi$. Let $H$ be the identity component of the closed subgroup of $G$ formed by all elements $g \in G$ enjoying the property $\mathrm{Ad}_G(g)\mathfrak{s} = \mathfrak{s}$. Since the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ is orthogonal, we get $\mathrm{Ad}_G(h)\mathfrak{h} = \mathfrak{h}$ for all $h \in H$. One easily checks that $\mathfrak{h}$ is the Lie algebra of $H$.

Since the representation $\mathrm{Ad}_G(H)|_{\mathfrak{s}}$ is isomorphic to the $s$-representation of some irreducible symmetric space of compact type, which is not a compact Lie group, the following Lemma implies Theorem 2:

**Lemma 5.** $M = \mathrm{Ad}_G(H)\xi$.

*Proof.* The inclusion $\mathrm{Ad}_G(H)\xi \subset M$ is evident. Since both $M$ and $\mathrm{Ad}_G(H)\xi$ are compact submanifolds of $P$ without boundary, it now suffices to show that the dimensions of $M$ and $\mathrm{Ad}_G(H)\xi$ coincide.

The Lie algebra of the stabilizer of $\xi$ in $H$ is $\mathfrak{k}_+ = \{X \in \mathfrak{h} :\ \mathrm{ad}(X)\xi = 0\}$ and therefore $\dim(\mathrm{Ad}_G(H)\xi) = \dim(\mathfrak{p}_+)$. On the other hand we have $\dim(M) = \dim(\mathfrak{m}) = \dim(\mathfrak{p}_-)$. The automorphism $\mathrm{Ad}(\exp(\pi/2 \cdot \xi))$ of $\mathfrak{g}$, which coincides on $\mathfrak{p}$ with $J_\xi$ (see Equation 1), identifies $\mathfrak{p}_-$ with $\mathfrak{p}_+$. Indeed for $X \in \mathfrak{p}_\pm$ we get:

$$\begin{aligned}
\tau\left(\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\cdot\xi\right)\right)X\right) &= \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\cdot\tau(\xi)\right)\right)\tau(X)\\
&= \pm\mathrm{Ad}\left(\exp\left(-\frac{\pi}{2}\cdot\xi\right)\right)X\\
&= \pm\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\cdot\xi\right)\right)(\mathrm{Ad}(\exp(-\pi\cdot\xi))X)\\
&= \pm\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\cdot\xi\right)\right)(\mathrm{Ad}(\exp(\pi\cdot\xi))X)\\
&= \mp\mathrm{Ad}\left(\exp\left(\frac{\pi}{2}\cdot\xi\right)\right)X.
\end{aligned}$$

In the last equality used Equation 2. □

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(Peter Quast) Institut of Mathematics, University of Augsburg, 86135 Augsburg, Germany

*E-mail address*: `peter.quast@math.uni-augsburg.de`