Stochastic Resonance in Optical Bistable Systems: Amplification and Generation of Higher Harmonics

ROLAND BARTUSSEK,* PETER JUNG† and PETER HÄNGGI†

*Humboldt-University at Berlin, Department of Physics, Invalidenstr. 42, D-10115 Berlin, Germany; and †University of Augsburg, Department of Physics, Memmingerstr. 6, D-86135 Augsburg, Germany

Abstract—We investigate cooperative effects of noise and periodic forcing in an optical bistable system. It has been demonstrated in a recent experiment by Grohs *et al.* that noise induced switching between low and high output intensity can be synchronized via the stochastic resonance effect by a small periodic modulation of the input intensity. Here we present theoretical results for stochastic resonance in optical bistable systems.

1. MODEL AND BASIC EQUATIONS

A model for optical bistability was introduced by Bonifacio and Lugiato [1]. For the amplitude y of the input light and the transmitted amplitude x, they have derived the equation of motion

$$\dot{x} = y - x - 2c\frac{x}{1 + x^2} + \sqrt{D}\frac{x}{1 + x^2}\Gamma(t), \tag{1}$$

where Γ represents δ-correlated, Gaussian distributed noise with zero mean. A weak periodic modulation of the input intensity is taken into account by adding a periodic term to y, i.e. $y \rightarrow y(t) = y_0 + A\cos(\Omega t + \Psi)$. For the probability density of the transmitted amplitude, $P(x, t)$, we find the Fokker–Planck equation

$$\begin{aligned}\frac{\partial}{\partial t}P(x, t) = &-\frac{\partial}{\partial x}\left[y_0 - x - \frac{2cx}{1 + x^2} + D\frac{x(1 - x^2)}{(1 + x^2)^3} + A\cos(\Omega t + \Psi)\right]P(x, t)\\ &+ \frac{\partial^2}{\partial x^2}D\frac{x^2}{(1 + x^2)^2}P(x, t).\end{aligned} \tag{2}$$

The spectral density of the transmitted amplitude has δ-spikes at multiples $n\Omega$ of the driving frequency [2] with the corresponding weights w_n being a measure for the output power at the frequency $n\Omega$. They can be expressed in terms of the Fourier coefficients of the time periodic, asymptotic mean value [3]

$$\langle x(t)\rangle_{as} = \sum_{n=-\infty}^{\infty} |M_n| \exp[in(\Omega t + \Psi + \varphi_n)] \tag{3}$$

by

$$w_n = 2\pi|M_n|^2. \tag{4}$$

2. AMPLIFICATION OF THE OPTICAL SIGNAL

The amplification of the periodic signal is given by the ratio of the transmitted power at the driving frequency and the input power [3]

$$\eta_1(\Omega) = 4\frac{|M_1|^2}{A^2}. \tag{5}$$

We can solve the Fokker–Planck equation (2) numerically for the asymptotic mean value $\langle x(t)\rangle_{as}$ by using the method of matrix continued fractions [4]. In doing so we follow the reasoning put forward in refs [2 and 3] where the stochastic resonance in a symmetric double well has been investigated. The numerical results for η_1 are shown in Fig. 1 for various frequencies by the solid lines. Figure 1(a) corresponds to choosing y_0 such that $P(x, t)$ shows two peaks of nearly equal height in the limit $D \to 0$ what we call the 'symmetric case', and Fig. 1(b) corresponds to an 'asymmetric case', where the peaks of the stationary probability have different probabilistic weights in the limit $D \to 0$.

In the symmetric case we observe stochastic resonance [5] very much like in the quartic double well potential, i.e. a peak in the amplification of the modulation as a function of the noise intensity when the sum of the mean sojourn times in both stable states equals the period of the driving (these values of D are indicated as vertical dashed lines in Fig. 1).

In the asymmetric case, the peak of the amplification is *suppressed, because—in contrast to the symmetric case*—the corresponding contribution (i.e. the weight g_T in (8)) of hopping motion to the response of the system disappears exponentially for small noise [6]. The remaining maximum is only the tail of the amplification by synchronization at large noise.

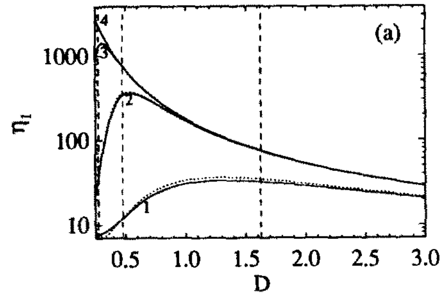
The numerical results are compared in Fig. 1 with those obtained within linear response approximation [3] (dotted lines). In this approximation we find in terms of the response function $R(t)$

$$\langle x(t)\rangle_{as} - \langle x\rangle_{st} = \int_{-\infty}^{\infty} R(t-t')A\cos(\Omega t' + \Psi)\,\mathrm{d}t' - \int_0^{\infty} xP_{st}(x)\,\mathrm{d}x, \tag{6}$$

with the stationary solution $P_{st}(x)$ of the undriven system. The response function $R(t)$ is expressed via a fluctuation theorem by a correlation function $K(t)$ of the undriven system

$$R(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)h(x(0))\rangle \equiv \frac{\mathrm{d}}{\mathrm{d}t}K(t) \tag{7}$$

with $h(x) = (1/D)(-x^{-1} + 2x + \frac{1}{3}x^3)$. $K(t)$ is approximated by a sum of exponentials with



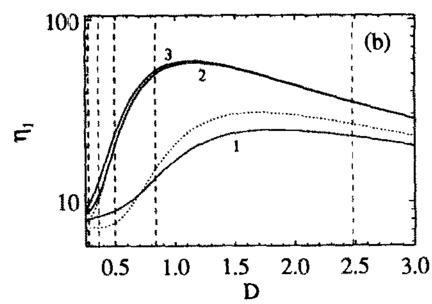


Fig. 1. Spectral amplification η_1 at $c = 6$, $A = 10^{-4}$ for $y_0 = 6.72584$ (a) and $y_0 = 6.8$ (b). Curves with label n correspond to $\Omega = 10^{-n}$. The dotted lines correspond to results within linear response approximation ((6)–(8)).

the typical time scales of the system λ_T and $\lambda_{1,2}$—stemming from hopping and local motion in the potential wells respectively, i.e. [3, 6]

$$K(t) \simeq \sum_{i=1,2,T} g_i e^{\lambda_i t}. \tag{8}$$

The weights g_i are determined by the correlation function $K(t)$ and its derivatives at $t = 0$.

3. GENERATION OF HIGHER HARMONICS

The generation of the nth harmonic in the output due to the nonlinearities is characterized by the ratio

$$\eta_n(\Omega) = 4\frac{|M_n|^2}{A^2}. \tag{9}$$

The second harmonic depends on the noise strength as shown for the symmetric and asymmetric case in Fig. 2. In the symmetric case (Fig. 2(a)) a 'dip' appears which becomes sharper with decreasing frequencies. In the asymmetric case (Fig. 2(b)) we do not observe such behaviour.

For the third harmonic, η_3, we find a smooth curve in the symmetric case and a dip in the asymmetric case.

We have confirmed the results for the higher harmonics within an adiabatic approximation, valid for small driving frequencies.

4. PHASE SHIFT OF THE OUTPUT SIGNAL

We note that the periodic asymptotic mean value in (3) involves complex-valued Fourier coefficients M_n. It is of interest to investigate the behaviour of the corresponding phases, $\{\varphi_n\}$, of (3)—which induce a characteristic lag of the deterministic phase $(\Omega t + \Psi)$—as a function of the parameters characterizing the stochastic resonance. In the following we have numerically studied the behaviour of the phases φ_1 and φ_2 as a function of increasing noise intensity D, with all other parameters kept fixed at values denoted in Fig. 1. In Fig. 3, the phase shifts φ_n, defined in (3), are shown for the first and second harmonic of the asymptotic mean value $\langle x(t)\rangle_{as}$. We again distinguish between the symmetric (Fig. 3(a), (c)) and asymmetric case (Fig. 3(b), (d)). The results within linear response theory are shown by dotted lines. The phase shift φ_1 in the symmetric case looks like in the quartic model: the extremum results from the competition between internal motion and

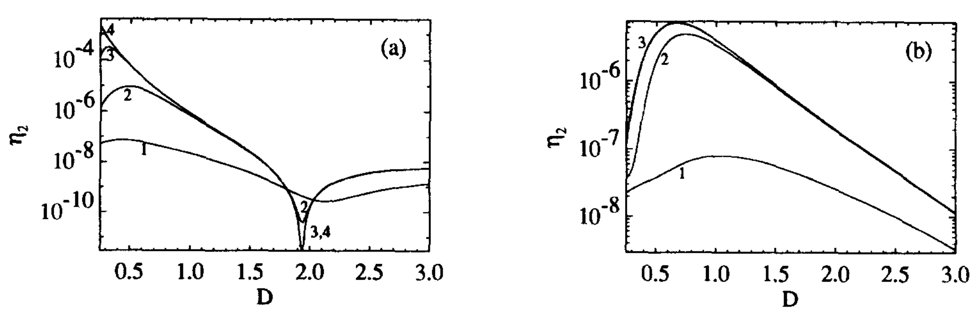


Fig. 2. Higher harmonic η_2, parameters as in Fig. 1.

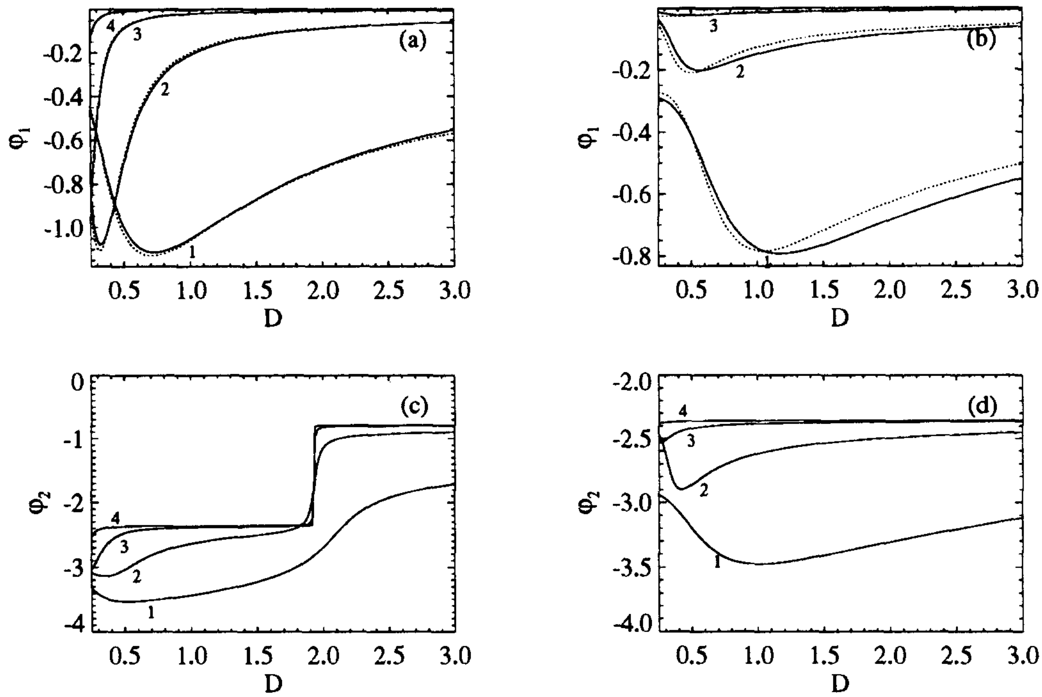


Fig. 3. Phase shifts, parameters as in Fig. 1. In (b), line 4 ($\Omega = 10^{-4}$) coincides with the axes $\varphi_1 = 0$.

hopping processes. In the asymmetric case the extremum is suppressed for small frequencies because the hopping disappears at small noise strength.

At values of D, for which a dip in a higher harmonic appears, the corresponding phase shift approaches a step function for small driving frequencies Ω. This characteristic behaviour, which cannot be explained on a pure deterministic level (i.e. $D = 0$), still awaits a simple physical intuitive explanation. We hope to be able to shed light onto this open problem in future work.

REFERENCES

1. R. Bonifacio and L. A. Lugiato, *Phys. Rev. A* **18**, 1192 (1978).
2. P. Jung and P. Hänggi, *Europhys. Lett.* **8**, 905 (1989).
3. P. Jung and P. Hänggi, *Phys. Rev. A* **44**, 8032 (1991); P. Jung, Periodically driven stochastic systems, *Phys. Rep.* **234**, 175 (1993).
4. H. Risken, *The Fokker-Planck-Equation*. Springer, Berlin (1984).
5. For a review: F. Moss, *Ber. Bunsenges. Phys. Chem.* **95**, 303 (1991).
6. P. Hänggi and H. Thomas, *Phys. Rep.* **88**, 207 (1982); see section 6.3, equations (6.3.46) and (6.3.63).