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Rank and symmetry of Riemannian manifolds

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Dedicated to Wilhelm Klingenberg on the occasion of his 70th birthday

Let M be a complete irreducible Riemannian manifold. A *k-flat* in M is a complete connected flat totally geodesic immersed submanifold of dimension k. The *rank* of M is the maximal dimension k such that every geodesic in M lies in a k-flat. Examples of manifolds of rank k are locally symmetric spaces of rank k. If M has sectional curvature $K \leq 0$, it is an open conjecture that these are the only examples for $k \geq 2$. This has been proved in many special cases (cf. [BBE], [BS], [BGS], [ES), [EH), [H]). In particular it is true if M is homogeneous [H]. On the other hand, if $K \geq 0$, there are homogeneous counterexamples [SS]. However, in these examples, the various k-flats are very different. What happens if we require that any two k-flats are isometric? More precisely, let us assume:

(I) A group G of isometries of M acts transitively on the set of pairs (p, F) where F is a k-flat and $p \in F$.

In the case $k = 1$, these are two-point-homogeneous spaces which are known to be rank-one symmetric (cf. [W], [Hg], [Sz]). Using the classification of strongly isotropy irreducible Riemannian manifolds, Heintze, Palais, Terng and Thorbergsson recently obtained the following result [HPTT]:

THEOREM A. *If M is compact of rank k and satisfies (I), the M is globally symmetric of rank k.*

The noncompact case is still open. The case $k = 2$ was recently solved by E. Samiou. In the present paper, we omit the dependence of the point in Condition (I) and discuss the following weaker condition which no longer implies homogeneity.

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(II) A group G of isometries of M acts transitively on the set of k-flats.

THEOREM B. *If M is compact of rank k satisfying (II), then M is locally symmetric of rank k. Moreover, if any k-flat is one-to-one immersed and intrinsically symmetric, then M is globally symmetric.*

(In fact we prove a slightly more general result, cf. Theorem B′, Ch. 2). The additional hypothesis for M being globally symmetric is necessary; if we omit it, the flat Kleinian bottle is a simple counterexample. For $k = 1$, following an idea of E. Heintze, we do not need compactness:

THEOREM C. *A complete Riemannian manifold M with the property that $I(M)$ acts transitively on the set of geodesics, is globally rank-one symmetric.*

In fact, we will prove a local statement which implies Theorem C (cf. Ch. 6).

We have started our work on this subject with a conceptual proof of Theorem A which is mainly as follows. We consider the action of the group $G = I(M)$ on the tangent bundle TM, equipped with the Sasaki metric $\langle\langle \, , \rangle\rangle$ which makes the vertical and the horizontal distributions (defined by the Levi-Civita connection) perpendicular. We noticed that the G-orbits on TM are perpendicular to the tangent spaces of any flat totally geodesic submanifold (cf. Ch. 1). This observation essentially goes back to R. Hermann [Hr]. For a rank-k manifold satisfying (I) this means that the horizontal distribution is tangent to the G-orbits. Thus, parallel vectors along a curve remain in the same G-orbit. In particular, the holonomy orbits are contained in the orbits of the isotropy group, and by the Berger-Simons holonomy theorem and our Theorem C, M is locally symmetric. An argument involving the deck group of the universal cover of M then finishes the proof (cf. Ch. 5).

If we only assume (II), the G-orbits are not large enough to contain all horizontal curves. Instead, we look for a collection of G-orbits, namely $G.P_v$ where P_v for some $v \in TM$ is the set of all vectors being tangent to the same k-flat as v and parallel to v. It turns out that in an open and sense subset of TM, these $G.P_v$ are manifolds and form a foliation which is perpendicular and transversal to the tangent spaces of the k-flats. Again, the horizontal distribution is tangent to these (larger) manifolds, and we get local symmetry by applying the Berger-Simons theorem to the local holonomy group; the local rank-one factors have to be treated separately (cf. Ch. 4, 6).

1. The Hermann Lemma on the tangent bundle

Let $(M, \langle \, , \rangle)$ be a Riemannian manifold. Then there exists a natural Riemannian metric $\langle\langle \, , \rangle\rangle$ on the tangent bundle TM of M. Let $\mathscr{V}(M)$ and $\mathscr{H}(M)$ be the vertical and horizontal distributions of TM (with rspect to the Levi-Civita connections on M). Then $\mathscr{V}(M) \perp \mathscr{H}(M)$ with respect to $\langle\langle\ ,\ \rangle\rangle$. Moreover $\langle\langle\ ,\ \rangle\rangle|_{\mathscr{V}(M)}$ is the usual metric on the fibers of $\pi : TM \to M$ and $\langle\langle\ ,\ \rangle\rangle|_{\mathscr{H}(M)}$ is such that π is a Riemannian submersion., So, if $v_1, v_2 : (-\epsilon, \epsilon) \to TM$ are C^1 with $v_1(0) = v_2(0)$,

$$\langle\langle v_1'(0), v_2'(0)\rangle\rangle = \langle (\pi v_1)'(0), (\pi v_2)'(0)\rangle + \left\langle \frac{D}{dt}\bigg|_0 v_1(0), \frac{D}{dt}\bigg|_0 v_2(0)\right\rangle$$

LEMMA 1. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold and let G be a Lie subgroup of $I(M)$ with bounded Killing fields. Let e be a parallel Jacobi field along some geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$, $\gamma'(0) = v$. Then the orbit Gv is perpendicular at v to the vertical submanifold $v + \mathbb{R}\cdot e(0) \subset T_pM \subset TM$.*

Proof. If $A \in L(G)$ and $g(t) = \exp tA$, we have to show that

$$\begin{aligned}\alpha &:= \left\langle\left\langle \frac{d}{dt}\bigg|_0 g(t)v, \frac{d}{dt}\bigg|_0 (v + te(0))\right\rangle\right\rangle \\ &= \left\langle \frac{D}{dt}\bigg|_0 f(t)_* v, e(0)\right\rangle\end{aligned}$$

vanishes. Recall that

$$\begin{aligned}\frac{D}{dt}\bigg|_0 g(t)_* v &= \frac{D}{\partial t}\frac{\partial}{\partial s}\bigg|_{0,0} g(t)\gamma(s) \\ &= \frac{D}{ds}\bigg|_0 A.\gamma(s),\end{aligned}$$

hence

$$\alpha = \frac{d}{ds}\bigg|_0 \langle A.\gamma(s), e(s)\rangle.$$

Since $x \mapsto A.x$ is Killing, $A.\gamma$ is a Jacobi field, and since e is parallel, we have

$$\frac{d^2}{ds^2}\langle A.\gamma(s), e(s)\rangle = 0.$$

Thus

$$\langle A.\gamma(s), e(s)\rangle = a + bs$$

for some $a, b \in \mathbb{R}$. But since A is bounded, we get $b = 0$ and therefore, $\alpha = 0$. □

COROLLARY. *Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold and G a Lie subgroup of $I(M)$ with bounded Killing fields. Let F be a complete flat totally geodesic immersed submanifold (called a "flat") in M and $w \in T_pF$. Then the orbit Gw is perpendicular to the vertical submanifold T_pF.* □

2. The set of regular vectors

From now on, let M be a complete irreducible Riemannian manifold with the following property.

(III) There is a k-dimensional flat (k-*flat*) F in M and a Lie subgroup G of $I(M)$ with bounded Killing fields such that $G.TF = TM$.

Clearly, (II) implies (III) if M is compact and of rank k. We will prove the following generalization of Theorem B:

THEOREM B′. *Let M be complete irreducible satisfying (III). Then M is locally symmetric of rank K. Moreover, if all k-flats in M are one-to-one immersed and intrinsically symmetric, then M is globally symmetric.*

The proof of this theorem will be finished in Ch. 5.

Consider the G-equivarient smooth map

$$\phi : G \times TF \to TM$$

$$\phi(g, v) = g.v = g_*v.$$

By (III), ϕ is onto. Sard's theorem says that the set of regular values of ϕ has full measure. Therefore, the subset

$$\mathscr{R} = \{w \in TM; \exists (g, v) \in \phi^{-1}(w) \colon d\phi\,|_{(g,v)} \text{ is onto}\}$$

(called the *set of regular vectors*) is open and dense in TM.

LEMMA 2. *For any $w \in \mathscr{R}$, there is exactly one k-flat F_w tangent to w and any parallel Jacobi field e along the geodesic γ_w is tangent to F_w. Moreover, $\mathscr{R} \cap TF$ is dense in TF.*

Proof. Let U be an open subset of $G \times TF$ where $d\phi$ has constant rank. By equivariance, we may assume $U = G' \times (TF)'$ where G' is a neighborhood of $1 \in G$

and $(TF)'$ an open subset of TF. Making U smaller if necessary, we may assume that $\phi(U)$ is a submanifold $\mathscr{S}$ of TM. Let $(g, v) \in U$ and $w = \phi(g, v)$. Then w is tangent to the flat $F_w := gF$, and

$$T_w\mathscr{S} = im\ d\phi\ |_{(g,v)} = T_w TF_w + T_w Gw. \tag{1}$$

Suppose that there is a parallel Jacobi field e along $\gamma = \gamma_w$ which is not tangent to F_w. Since F_w is totally geodesic and flat, we may assume $e \perp F_w$. It follows that the vertical vector $\hat{e}(0) = d/dt\ |_0\ (w + te(0)) \in T_w TM$ is perpendicular to the first term on the right hand side of (1). By Lemma 1, $\hat{e}(0)$ is also perpendicular to the second term. Hence, $\hat{e}(0) \perp T_w\mathscr{S}$.

In particular we have shown that F_w is the only $\mathscr{S}$-flat tangent to w. (A k-flat F' is called *$\mathscr{S}$-flat* if an open subset of TF' belongs to $\mathscr{S}$.) It follows that for any $(g, v) \in (\phi \mid U)^{-1}(w)$,

$$\phi(\{g\} \times TF) = TF_w.$$

Thus the foliation of $U = G' \times (TF)'$ by the second factor induces the smooth foliation

$$\mathscr{T}\mathscr{F} : w \mapsto TF_w \cap \mathscr{S}$$

of $\mathscr{S}$, by applying the submersion $\phi \mid U : U \to \mathscr{S}$. Making U even smaller (if necessary), we may assume that $\mathscr{S} = \phi(U)$ is diffeomorphic to a product $S' \times (TF)'$ such that $\mathscr{T}\mathscr{F}$ becomes the foliation by the second factor. Since any $g \in G'$ preserves this foliation on $\mathscr{S} \cap g^{-1}(\mathscr{S})$, each Killing field $A \in L(G)$ induces a tangent vector field A_1 on the transversal manifold $\mathscr{S}'$, and we have $A_1(s') = 0$ for some $s' \in \mathscr{S}'$ if and only if $\exp tA$ leaves invariant the k-flat corresponding to $\{s'\} \times (TF)'$. Thus, for any $w \in \mathscr{S} \cap TF$, the connected component of the stabilizer G_w is contained in $G_F = \{g \in G; g(F) = F\}$ (by uniqueness of F_w) and

$$T_w TF \cap T_w Gw = T_w G_F w.$$

So by (1), we have

$$\begin{aligned} \dim \mathscr{S} &= \dim TF + \dim Gw - \dim G_F w \\ &= \dim TF + \dim (G/G_F). \end{aligned}$$

In particular, this holds if $\mathscr{S}$ is an open subset of $\mathscr{R}$ which shows that

$$\dim TF + \dim (G/G_F) = \dim TM.$$

Therefore, *any* such $\mathscr{S}$ has full dimension and must be an open subset of $\mathscr{R}$.

If $\mathscr{R} \cap TF$ were not dense, we could find an open subset $(TF)'$ of TF such that *rank*$(d\phi)$ is constant but not maximal on $\{1\} \times (TF)'$. By equivariance of ϕ, the same would be true on some open subset U of $G \times TF$ which we have excluded. □

Remark. The uniqueness of the flat F_w tangent to $w \in \mathscr{R}$ shows that $\mathscr{R}$ is in fact the set of regular values of ϕ.

3. The local holonomy

On the set $\mathscr{R}$ of regular vectors, we have introduced the smooth foliation

$$\mathscr{T}\mathscr{F} : w \mapsto TF_w$$

where F_w is the unique k-flat through w. (The smoothness of this foliation can also be seen using the arguments in [BBE], Lemma 2, where the Jacobi operator $R(\, , \gamma'_w)\gamma'_w$ has to be replaced by its square, for sake of definiteness.)

The tangent distribution $T\mathscr{T}\mathscr{F}$ of this foliation splits into its horizontal and vertical parts:

$$T\mathscr{T}\mathscr{F} = \mathscr{V}\mathscr{F} \oplus \mathscr{H}\mathscr{F}$$

where

$$(\mathscr{V}\mathscr{F})_w = \mathscr{V}_w \mathscr{F}_w = T_w T_{\pi(w)} F_w,$$
$$(\mathscr{H}\mathscr{F})_w = \mathscr{H}_w \mathscr{F}_w = T_w P_w$$

where $P_w \subset TF_w$ is the set of vectors which arise from w by parallel transport along curves in F_w. Now from Equation (1) in the previous section we get for all $w \in \mathscr{R}$:

$$T_w TM = T_w G w + (\mathscr{H}\mathscr{F})_w + (\mathscr{V}\mathscr{F})_w. \tag{2}$$

LEMMA 3. *The distribution $(\mathscr{V}\mathscr{F})^\perp$ on $\mathscr{R}$ is integrable with integral leaf $G.P'_v$ through $v \in \mathscr{R}$, where $P'_v = P_v \cap \mathscr{R}$.*

Proof. Let $g \in G$ and $v \in \mathscr{R} \cap TF$. Let $w \in P'_v$. Recall that G_F is the subgroup of G which leaves the flat F invariant. We have

$$\phi^{-1}(g.w) = \{(gh^{-1}, hw); h \in G_F\}.$$

Thus

$$T_{(g,w)}(\phi^{-1}(g.w)) = \{(-gA, Aw); A \in L(G_F)\}.$$

Since G has bounded Killing fields, $A \in L(G_F)$ induces an infinitesimal translation on F. Thus Aw is tangent to P'_v and

$$T_{(g,w)}(\phi^{-1}(g.w)) \subset T_{(g,w)}(G \times P'_v).$$

So $G \times P'_v$ is a submanifold of $\phi^{-1}(\mathscr{R})$ containing the fibres of the submersion $\phi : \phi^{-1}(\mathscr{R}) \to \mathscr{R}$. Therefore $G.P'_v = \phi(G \times P'_v)$ is a smooth submanifold of $\mathscr{R}$. Clearly

$$T_w(G.P'_v) = T_w Gw + (\mathscr{H}\mathscr{F})_w,$$

which shows that $T_w(G.P'_v)$ is the orthogonal complement of $(\mathscr{V}\mathscr{F})_w$ (cf. (2) and Lemma 1).

Remark. There are no singular orbits on $\mathscr{R}$, i.e. the orbits of G define a smooth foliation on $\mathscr{R}$. In fact, let $w \in \mathscr{R}$ and G_w its stabilizer subgroup. If $X \in L(G_w)$ then $\exp tX$ preserves the k-flat F_w (by Lemma 2). Since G has bounded Killing fields, $\exp tX$ is a translation. But $(\exp tX).\pi(w) = \pi(w)$ and therefore $\exp tX_{|F_w} = id$ for all $t \in \mathbb{R}$. So $(G_w)_0 = (G^{F_w})_0$ where $G^{F_w} = \{g \in G : g \text{ fixes pointwise } F_w\}$. Thus $(G_{w'})_0$ is the same group for all $w' \in F_w$ which shows that all orbits have the same dimension.

COROLLARY. *If $w \in \mathscr{R}$ and $p = \pi(w)$ is its base point, then*

$$T_w(\Phi_p^{loc}.w) \perp T_w T_p F_w$$

where Φ_p^{loc} denotes the local holonomy group of M at p (cf. [KN], p. 94).

Proof. On $\mathscr{R}$, the horizontal distribution $\mathscr{H}M = (\mathscr{V}\mathscr{M})^{\perp}$ is contained in $(\mathscr{V}\mathscr{F})^{\perp}$. Thus, by Lemma 3, any $w \in \mathscr{R}$ has a neighborhood $\mathscr{U}_w$ in $\mathscr{R}$ such that any horizontal curve in $\mathscr{U}_w$ starting at w stays in $G.P'_w$. Let $B_\epsilon(w) \subset \mathscr{U}_w$. By the theorem in the appendix, there exists a neighborhood N of $1 \in \Phi_p^{loc}$ such that any $\varphi \in N$ is the parallel transport along a piecewise C^1-loop β starting and ending at p with length $L(\beta) < \epsilon$. Thus, the horizontal lift $\hat{\beta}$ of β starting at w has also length $<\epsilon$ which shows $\hat{\beta} \subset \mathscr{U}_w$. Hence $\hat{\beta} \subset G.P'_w$, and in particular, $\varphi.w \in G.P'_w$ since $\varphi.w$ is the end point of $\hat{\beta}$. Thus, if $t \mapsto \varphi_t$ is a smooth curve in N with $\varphi_0 = 1$, then

$$\frac{d}{dt}\Big|_0 \varphi_t.w \in T_w(G.P'_w) \perp (V\mathscr{F})_w = T_w T_p F_w$$

which finishes the proof. □

Now by the Berger-Simons holonomy theorem (cf. [Be], [S]), the curvature tensor on $\pi(\mathscr{R})$ is parallel unless we have locally a Riemannian product with a factor with transitive holonomy. This will be considered in the next section.

4. Local factors with transitive holonomy

Consider $w \in \mathscr{R}$ with base point $\pi(w) = p$. There exists a simply connected open neighborhood M' of p in M such that the holonomy group Φ of M' at p equals the local holonomy Φ_p^{loc} of M at p. By the de Rham theorem, perhaps making M' smaller if necessary, we may assume that M' splits metrically as

$$M' = M_0 \times M_1 \times \cdots \times M_k$$

with $p = (p_0, p_1, \ldots, p_k)$, and

$$\Phi = \Phi_1 \times \cdots \times \Phi_k$$

where Φ_j is the holonomy group of M_j at p_j, acting irreducibly on $T_{p_j}M_j$ for $j = 1, \ldots, k$, while M_0 is the flat factor.

By the Berger-Simons theorem ([Be], [S]), each M_j is locally symmetric unless Φ_j acts transitively on the unit sphere in $T_{p_j}M_j$. Call these latter factors *holonomy-transitive*. We will show that those are also locally symmetric:

LEMMA 4. *Each holonomy-transitive factor* M_i of M' *is locally symmetric of rank one.*

Proof. We will show that M_i satisfies the assumption of Lemma 6, Ch. 6. Let γ_i be a geodesic in M_i. We have $M' = M_i \times \bar{M}$ where $\bar{M}$ contains all other factors. So $\gamma = \gamma_i \times \{\bar{q}\}$ is a geodesic in M', for any $\bar{q} \in \bar{M}$. This geodesic lies in a k-flat $F' = g(F)$ for some $g \in G$. We may assume that $F' = F$. Since $\mathscr{R} \cap TF$ is dense in TF (Lemma 2), we may choose a regular vector $w \in TF$ with base point

$\pi(w) = p = (p_i, \bar{p}) \in F \cap M'$. Let $w = (w_i, \bar{w})$. We may assume $w_i \neq 0$. We saw in Lemma 3 that the holonomy orbit $\Phi.w$ is perpendicular to $T_w T_p F$. Thus $\Phi_i w_i$ is perpendicular to $\pi_i(T_w T_p F) = T_{w_i} T_{p_i}(\pi_i F)$ where π_i denotes the projection onto the ith factor on M', TM', TTM'; But $\Phi_i w_i$ is the sphere of radius $\|w_i\|$ in $T_{p_i} M_i$. Therefore, $T_{p_i}(\pi_i F)$ is one-dimensional, i.e. $F_i := \pi_i F$ is a geodesic in M'. Since $\gamma_i \subseteq \pi_i F$, we get $F_i = \gamma_i$, and w_i is a tangent vector of γ_i.

For any Riemannian manifold X, let $\mathscr{K}(X)$ denote the space of Killing fields. A Killing field, being an infinitesimal isometry, can be applied to points of X as well as to vectors in TX. By Section 3, Equation (2), we have

$$\mathscr{K}(M).w + \mathscr{H}_w \mathscr{F} + \mathscr{V}_w \mathscr{F} = T_w TM'.$$

Applying the ith projection π_i (which commutes with the horizontal and vertical projections), we get

$$\pi_i \mathscr{K}(M).w + \mathscr{H}_{w_i} F_i + \mathscr{V}_{w_i} F_i = T_{w_i} TM_i.$$

Since Killing fields on M' project onto Killing fields on M_i, we have

$$\pi_i \mathscr{K}(M).w \subset \mathscr{K}(M_i).w_i,$$

thus

$$\mathscr{K}(M_i) w_i + \mathscr{H}_{w_i} \mathscr{F}_i + \mathscr{V}_{w_i} F_i = T_{w_i} TM_i.$$

Now we may apply Lemma 6, Ch. 6, to see that M_i is locally rank-one symmetric. This proves the first part of Theorem B′.

5. Global symmetry

For any symmetric space X, let $Tr(X)$ denote the transvection group, i.e. the subgroup of the isometry group $I(X)$ which is generated by the compositions of any two symmetries.

LEMMA 5. *Let M be complete, locally symmetric of rank k with universal cover $\pi : X \to M$. Suppose that there is a connected Lie subgroup $G \subset I(M)$ with the following properties*:

(1) *The connected component of the lift $\tilde{G}$ of G lies in $T := Tr(X)$.*
(2) *G acts transitively on the set of k-flats in M, and any k-flat is one-to-one immersed and intrinsically symmetric.*

Then M is globally symmetric.

Remark. If M satisfies condition (III) for a group G then the universal covering X of M satisfies also (III) for the lifting group $\tilde{G}$. Since X is symmetric, any k-flat has a regular element and it is easy to see that X satisfies also (II) for $\tilde{G}$. Since the set of k-flats in a (globally) symmetric space is connected, the connected component $\tilde{G}_0$ acts transitively on the set of k-flats of X and hence also $G_0 = \pi(\tilde{G}_0)$ acts transitively on the k-flats of M. Since $\tilde{G}_0$ acts with bounded Killing fields, it lies in $Tr(X)$ (which only says that it acts by translations on the euclidean factor). Thus, Lemma 5 proves the second part of Theorem B′.

Proof. X is globally symmetric, and $M = X/\Gamma$ where Γ is a discrete subgroup of $I(X)$. Let $\sigma \in I(X)$ be the symmetry at some point $x_0 \in X$. We have to show that σ descends to an isometry of M, i.e. that

$$\sigma(\Gamma) = \Gamma. \qquad (*)$$

(For any subgroup $\Gamma' \subset I(X)$ and $\tau \in I(X)$, we denote by $\tau(\Gamma')$ the conjugate subgroup $\tau\Gamma'\tau^{-1}$.) Let $\mathscr{F}$ denote the set of k-flats in X. For any k-flat $F \in \mathscr{F}$, let

$$\Gamma_F = \{g \in \Gamma; gF = F\}.$$

SUBLEMMA 1. *Each Γ_F acts as a translation subgroup on F, and*

$$\Gamma = \bigcup_{F \in \mathscr{F}} \Gamma_F.$$

Proof. Let $g \in \Gamma$, $x \in X$ and let γ be a geodesic connecting x and gx. There is a k-flat $F \in \mathscr{F}$ containing γ. By assumption, the k-flat $\pi(F) \subset M$ is symmetric without selfintersection. So the geodesic loop $\pi \circ \gamma$ in $\pi(F)$ is a closed geodesic, and $\pi(F) = \pi(gF)$. Thus $g \in \Gamma_F$. Moreover, any $g \in \Gamma_F$ translates the geodesic from y to gy for any $y \in F$, thus g is a translation on F. □

SUBLEMMA 2. *For any $t \in T$ and $F \in \mathscr{F}$,*

$$t(\Gamma_F) = \Gamma_{tF}.$$

Proof. $\tilde{G}$ normalizes Γ, so this holds for any $t \in \tilde{G}$. Since $\tilde{G}$ and hence $\tilde{G}_0$ act transitively on $\mathscr{F}$, we have $T = \tilde{G}_0 \cdot T_F$ where

$$T_F = \{t \in T; tF = F\}.$$

Recall that T_F is a finite extension (by the Weyl group) of its connected component $(T_F)_0$ which acts as translation group on F. Since T is connected, we get already $T = \tilde{G}_0 \cdot (T_F)_0$. Thus let $t = g \cdot \tau$ with $g \in \tilde{G}_0$ and $\tau \in (T_F)_0$. Then

$$t(\Gamma_F) = g\tau(\Gamma_F) = g(\Gamma_F) = \Gamma_{gF} = \Gamma_{tF}. \qquad \square$$

Now fix a k-flat F_0 through x_0. Since $\pi(F_0)$ is symmetric, $\sigma \mid F_0$ descends to the symmetry of $\pi(F_0)$ at $\pi(x_0)$, and therefore

$$\sigma(\Gamma_{F_0}) = \Gamma_{F_0} = \Gamma_{\sigma F_0}. \qquad (*)$$

Next let F be an arbitrary k-flat and choose $g \in Tr(X)$ with

$$g\sigma F = F_0.$$

Since $\sigma g \sigma \in Tr(X)$, we have

$$\sigma g \sigma(\Gamma_F) = \Gamma_{\sigma g \sigma F} = \Gamma_{\sigma F_0} = \sigma(\Gamma_{F_0}),$$

using $(*)$. Hence we get

$$g\sigma(\Gamma_F) = \Gamma_{F_0}$$

and consequently

$$\sigma(\Gamma_F) = g^{-1}(\Gamma_{F_0}) = \Gamma_{g^{-1}F_0} = \Gamma_{\sigma F}$$

always applying Sublemma 2. Thus $\sigma(\Gamma_F) \subset \Gamma$ for every flat F, and by Sublemma 1 we get $\sigma(\Gamma) \subset \Gamma$ which shows that σ descends to M. This finishes the proof of Lemma 5 and Theorem B′.

6. The rank-one case

Let M be a Riemannian manifold, not necessarily complete. A geodesic γ in M will be considered as (part of a) 1-flat, so for any tangent vector v of γ we have the one-dimensional subspaces $\mathscr{H}_v\gamma$ (tangent to the curve $t \mapsto \gamma'(t)$) and $\mathscr{V}_v\gamma$ (tangent to $t \mapsto tv$) of T_vTM. Let $\mathscr{K}(M)$ denote the set of Killing fields (which we apply to points and to tangent vectors of M).

LEMMA 6. *Let M be any Riemannian manifold, possibly not complete, with the following property: There is an open subset $\mathscr{R}$ of TM such that any geodesic in M has a tangent vector in $\mathscr{R}$ and so that for any $v \in \mathscr{R}$,*

$$\mathscr{K}(M).v + \mathscr{H}_v\gamma + \mathscr{V}_v\gamma = T_vTM \tag{3}$$

Then M is locally rank-one symmetric.

Proof. The tangent vectors of the geodesics form the 2-dimensional foliation

$$\mathscr{T}\mathscr{F}(v) = \{s \cdot \gamma_v'(t);\ s \in (0, \infty),\ t \in Dom(\gamma_v')\}$$

of $TM \backslash \{\text{zero section}\}$ (notation as in section 3). By assumption, any geodesic γ has a tangent vector v such that the infinitesimal isometry orbit $\mathscr{K}(M).v$ is transversal to this foliation. In other words, for any $w \in TM$ which is sufficiently close to v, there is a local isometry g_w mapping w to a tangent vector of γ (note that tranversality is an open property).

For any $p \in M$ we consider

$$\mathscr{K}(M)_p = \{A \in \mathscr{K}(M);\ A.p = 0\}.$$

This is a Lie subalgebra of $L(O(T_pM))$, and there is a compact connected subgroup G_p of $O(T_pM)$ with $L(G_p) = \mathscr{K}(M)_p$; this is the connected component of the isometry group of a small ball in M centered at p. For any nonzero $x \in T_pM$, let $G_x \subset G_p$ denote the stabilizer of x, i.e. the group of local isometries fixing p and x.

Clearly, G_x does not change under the geodesic flow, more precisely, $G_{\Phi_t x}$ is isomorphic to G_x, where $\Phi_t x = \gamma_x'(t)$ denotes the action of the geodesic flow. This is because $g \in G_x$ fixes any tangent vector of γ_x in its domain. Moreover, the isomorphic type of G_x cannot change near a regular vector v since any x near v is mapped to some $\Phi_t v$ by a local isometry g_x which conjugates G_x and $G_{\Phi_t v} \cong G_v$. Thus, the isomorphic type of G_x is locally constant on the unit tangent bundle. In particular, for any $p \in M$ and any two unit vectors $x, y \in T_pM$, the subgroups G_x and G_y of G_p are isomorphic. Hence, all orbits of G on the unit sphere S_pM have the same dimension.

On the other hand,

$$\mathcal{K}(M)_p v = \mathcal{K}(M)v \cap T_v T_p M.$$

Hence, if $v \in \mathcal{R}$ with $\pi(v) = p$, then by (3), $\mathcal{K}(M)_p v$ has codimension 2 or less in $T_v T_p M$. Thus the principal orbits of G_p in the unit sphere $S_p M$ are hypersurfaces unless G_p acts transitively on $S_p M$. In the first case, the G_p-orbits form a family of homogeneous isoparametric hypersurfaces. These have focal manifolds, i.e. orbits of lower dimension, but this was excluded. So G_p acts transitively on $S_p M$. Now by Szabo's local theorem on 2-point homogeneous spaces [Sz], M is locally rank-one symmetric. □

Now Theorem C follows from Lemma 6 and Lemma 5. Lemma 5 can be used since any geodesic in M must be a one-to-one immersed circle or line. Namely, if there exists a geodesic loop, then any geodesic is a loop, so the cut locus distance is finite and M must be compact. Hence any Killing field X translates a geodesic (namely the geodesic γ_v for $v = X(p)$ where $\|X\|$ takes its maximum at p), so the geodesic loop must be a closed geodesic.

Appendix

The goal of this appendix is to prove the fact that parallel translations along short loops form a neighborhood of the identity in the local holonomy group. This was used in the proof of the corollary in section 3.

Let G be a Lie group of dimension n and let $F \subset G$ be such that

(i) F generates (algebraically) G.
(ii) $F^{-1} = F$.
(iii) For each $f \in F$ there exists $\gamma : [0, \epsilon_\gamma) \to F$ of class C^1 such that $\gamma(0) = e$ and $\gamma(t_0) = f$, for some $t_0 \in [0, \epsilon_\gamma)$.

For $k \in \mathbb{N}$, let

$$F^k = \{f_1 \cdot \ldots \cdot f_k ; f_1, \ldots, f_k \in F\},$$

$$\Gamma^k = \{\gamma : [0, \epsilon_\gamma) \to F^k; \gamma \text{ is } C^1, \gamma(0) = e\}.$$

Define

$$S_k = \{\gamma'(0) : \gamma \in \Gamma^k\} \subset \mathcal{G} = L(G).$$

Observe:

(i) $\gamma \in \Gamma^k \Rightarrow \gamma^{-1} \in \Gamma^k$ and $(\gamma^{-1})'(0) = -\gamma'(0)$.
(ii) $\gamma_1 \in \Gamma^k, \gamma_2 \in \Gamma^j \Rightarrow \gamma_1 . \gamma_2 \in \Gamma^{k+j}$.
(iii) If $\gamma \in \Gamma^k, r > 0$, then $\bar{\gamma} = \{t \mapsto \gamma(r \cdot t), t \in [0, \epsilon_\gamma . r^{-1})\} \in \Gamma^k$ and $\bar{\gamma}'(0) = r\gamma'(0)$.
(iv) $S_k \subset S_{k+1}$.

If $\gamma_1, \gamma_2 \in \Gamma^k$, then

$$\gamma_1(t)\gamma_2(t)\gamma_1^{-1}(t)\gamma_2^{-1}(t) = w(t^2)$$

for some $w \in \Gamma^{4k}$, and

$$w'(0) = [\gamma_1'(0), \gamma_2'(0)]$$

(see [KN], Appendix 4). On the other hand, if $\gamma_1, \ldots, \gamma_n \in \Gamma^k$, then $\gamma_1 \cdot \ldots \cdot \gamma_n \in \Gamma^{kn}$ and $(\gamma_1 \cdot \ldots \cdot \gamma_n)'(0) = \gamma_1'(0) + \cdots + \gamma_n'(0)$. Hence,

$$S_1 + \langle [S_1, S_1] \rangle \subset S_{4n},$$

$$S_{4n} + \langle [S_{4n}, S_{4n}] \rangle \subset S_{(4n)^2},$$

$$S_{(4n)^2} + \langle [S_{(4n)^2}, S_{(4n)^2}] \rangle \subset S_{(4n)^3},$$

and so on. Since $\dim(\mathcal{G}) = n$ we get that after n steps, that $S_{(4n)^n}$ must be a Lie subalgebra of $\mathcal{G}$.

LEMMA A1. $S_{(4n)^n} = \mathcal{G}$.

Proof. Let $f \in F$ and let $\gamma \in \Gamma^1$ with $\gamma(t_0) = f$ for some $t_0 \in [0, \epsilon_\gamma)$. Then, for each $s_0 \in [0, \epsilon_\gamma)$,

$$\bar{\gamma} := (t \mapsto \gamma^{-1}(s_0)\gamma(t + s_0)) \in \Gamma^2$$

(cf. [KN], Appendix 4) and therefore $\bar{\gamma}'(0) \in S_2 \subset S_{(4n)^n}$. Hence, $\bar{\gamma}$ is a C^1 curve which lies in the left invariant distributions of G defined by $S_{(4n)^n}$. Since $\bar{\gamma}(0) = e$ we get that $\gamma(t_0) \in H$, where H is the Lie subgroup of G corresponding to $S_{(4n)^n}$. Since F generates G we get $H = G$ and hence $S_{(4n)^n} = \mathcal{G}$. □

LEMMA A2. *$F^{(4n)^n \cdot n}$ contains a neighborhood of e in G.*

Proof. (cf. [KN], Appendix 4). Let $\gamma_1, \dots, \gamma_n \in \Gamma^{(4n)^n}$ be such that $\gamma_1'(0), \dots, \gamma_n'(0)$ is a basis of $\mathscr{G}$. We may assume that $\gamma_i : [0, \epsilon) \to F^{(4n)^n}$ and define $\bar{\gamma}_i : (-\epsilon, \epsilon) \to F^{(4n)^n}$ by

$$\bar{\gamma}_i(t) = \begin{cases} \gamma_i(t), & \text{if } t \in [0, \epsilon) \\ \gamma_i^{-1}(-t), & \text{if } t \in (-\epsilon, 0]. \end{cases}$$

Clearly, $\bar{\gamma}_i$ is C^1 ($i = 1, \dots, n$) and $(s_1, \dots, s_n) \mapsto \bar{\gamma}_1(s_1) \dots \bar{\gamma}_n(s_n)$ defines a coordinate system of G near e. Since $\bar{\gamma}_1(s_1) \dots \bar{\gamma}_n(s_n) \in F^{(4n)^n \cdot n}$ we get that $F^{(4n)^n \cdot n}$ contains a neighborhood of e in G. ❑

Now let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, $p \in M$, and let Φ^{loc} denote the local holonomy group at p. Let ϵ be such that $exp_p : B^E_{4\epsilon}(0) \to B_{4\epsilon}(p)$ is a diffeomorphism, where B^E denotes euclidean balls in T_pM and B balls in M. We may assume that the holonomy group of $B_{4\epsilon}(p)$ is Φ^{loc}. In $B_{3\epsilon}(p)$ we have also the Euclidean metric $\langle\ ,\ \rangle_E$ induced by $exp_p : B^E_{3\epsilon(0)} \to B_{3\epsilon}(p)$.

We have that there exist $\alpha, \beta \in \mathbb{R}$, $0 < \alpha, \beta \leq 1$ such that

$$\beta \text{ length}(c) \leq \text{length}_E(c) \leq \alpha^{-1} \text{ length}(c) \tag{I}$$

for all piecewise C^1-curves in $B_{3\epsilon}(p)$.

For each $0 < r < \epsilon$ let A_r (resp. A^E_r) be the set of piecewise C^1 loops through p of length (resp. Euclidean length) less than r. Let P_r (resp. P^E_r) denote the set of all parallel transports along loops in A_r (resp. A^E_r).

From (I) we get

$$A_{\alpha r} \subset A^E_r, \qquad A^E_{\beta r} \subset A_r \tag{II}$$

and hence

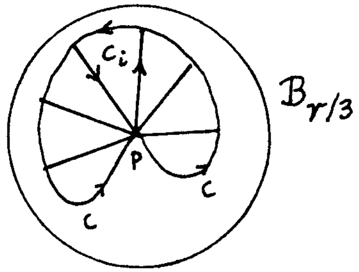
$$P_{\alpha r} \subset P^E_r, \qquad P^E_{\beta r} \subset P_r. \tag{III}$$

LEMMA A3. *For all $r \in (0, \epsilon)$, P_r (resp. P^E_r) generates algebraically Φ^{loc}.*

Proof. Let $c : [0, 1] \to B_{r/3}$ be piecewise C^1 with $c(0) = p = c(1)$. Then there exist piecewise C^1-curves $c_1, \dots, c_k : [0, 1] \to B_{r/3}$ with $c_1(0) = c_1(1) = \dots = c_k(0) = c_k(1) = p$ and such that

(i) $\tau_c = \tau_{c_k} \circ \tau_{c_{k-1}} \circ \dots \circ \tau_{c_1}$, where τ denotes parallel transport.

(ii) $\text{length}(c_i) \leq r$ for all $i = 1, \dots, k$.



Since the holonomy group of $B_{r/3}$ is Φ^{loc} we get that P_r generates Φ^{loc}. From (III) we get now that P_r^E also generates Φ^{loc}. □

THEOREM. *For all $r \in (0, \epsilon)$, P_r contains a neighborhood of e in Φ^{loc}.*

Proof. By (III) it suffices to show that P_r^E contains a neighborhhod of e in Φ^{loc}. Let $m = (4n)^n \cdot n$, where $n = \dim(\Phi^{loc})$ (cf. Lemma A2). Then, by Lemma A3, $P_{r/m}^E$ generates Φ^{loc}. Let $g \in P_{r/m}^E$ and let $c : [0, 1] \to M$ belong to $A_{r/m}^E$ such that $g = \tau_c$. Let, for $s \in [0, 1]$, $c_s : [0, 1] \to M$ be defined by $c_s(t) = s \cdot c(t)$ (we identify $B_{r/m}(p)$ with $B_{r/m}^E(0)$). Then $\text{length}_E(c_s) \leq \text{length}(c)$.

If $g_s = \tau_{c_s}$ then $(s \mapsto g_s)$ defines a C^1 curve in $P_{r/m}^E$. We can now apply Lemma A2 to $F = P_{r/m}^E$ in order to conclude that $(P_{r/m}^E)^m$ contains a neighborhood of e in Φ^{loc}. But $(P_{r/m}^E)^m \subset P_r^E$. Hence P_r^E contains a neighborhood of e in Φ^{loc}. □

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