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Abstract

In previous work, we have presented a CCS-like process algebra PAFAS for comparing the efficiency of asynchronous systems. This note gives a complete axiomatization for a fragment of PAFAS.

1 Introduction

This note is an addendum to [JV98, JV99]; the present section gives a short introduction to the approach presented there, but to really understand this note the reader should read [JV98, JV99].

[JV98, JV99] presents a CCS-like process algebra, which we will henceforth call PAFAS (process algebra for fast asynchronous systems). Processes are regarded as asynchronous systems, and the intention is to consider their temporal efficiency. For this purpose it is assumed that actions happen within time 1, and time is assumed to be discrete. Thus, a process $a.P$ can either do a immediately and become P – as usual – or it can let time 1 pass and become $\underline{a}.P$; $\underline{a}$ is called *urgent* a, and $\underline{a}.P$ cannot let time pass, but can only do a to become P.

A testing scenario a la De Nicola/Hennessy [DNH84] is developed, where a test is only successfully passed if success is surely reached within a given time bound. As a consequence, the resulting testing preorder compares some P to Q if P serves all possible users or for all patterns of usage as efficiently as Q, i.e. P is *faster than* Q. This faster-than-relation is characterized using some sort of refusal traces, and it has to be refined slightly to become a precongruence $\leq$; the purpose of this note is to give a complete axiomatization for a fragment of PAFAS, consisting just of prefix and choice. Usually, such a fragment is trivial to axiomatize, but for PAFAS the completeness proof is not easy at all. It might be the case that the proof could be simplified if the process algebra were extended by operators that somehow model the characterising refusal traces directly; such an operator would presumably allow to model timeouts as the then-operator of [HR95], which of course makes sense but does not fit the setting of asynchronous processes. Furthermore, it seems that in the present

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setting such an operator would have to treat partial time steps (see below) explicitly, and use of such artificial operators in the axioms would make the axiomatization less pleasant in the opinion of the authors. Hence, it is an important feature that the axiomatization does (almost) not rely on an extension of the original process algebra.

The intention of [JV98, JV99] is to compare the efficiency of ordinary CCS-like processes; when considering the processes reachable from such ordinary processes, we will encounter urgent actions as above, but with two restrictions: in a term $a.P$ or $\underline{a}.P$, P will not contain urgent actions; in a sum $P + Q$ either both summands are urgent – as $\underline{a}.P$ – or both are ordinary. When defining the processes we consider here, we will keep the second restriction, but we will drop the first one – though we will almost apply it when giving our axiomatization. Thus, processes of the (prefix,choice)-fragment are defined by the following grammar, where I generates the *initial* or ordinary processes and U the *urgent* ones.

$$\begin{array}{rcl|c|c|c} I & ::= & \mathbf{0} & a.I & a.U & I+I \\ U & ::= & \mathbf{0} & \underline{a}.U & \underline{a}.I & U+U \end{array}$$

In this grammar, a is either the invisible action τ or a visible action from our base alphabet $\mathbb{A}$; since in this note we have no need for the special success action ω, we do not consider $\mathbb{A}_\omega$ as in [JV99].

The refusal traces $\mathsf{RT}(P)$ of a process P are generated by the following SOS-rules in the usual fashion; here, the occurrence of a (refusal) set $X \subseteq \mathbb{A}$ is called a (partial) *time step*: the actions listed in the set are not urgent, so P is justified in not performing them, but performing a time step instead. Other actions might be urgent, so as a stand-alone-process P might actually be unable to make a time step; but as a component of a larger systems, it might take part in a time step if the environment (e.g. a test) can refuse those other actions.

$$\frac{}{a.P \xrightarrow{a}_r P} \qquad \frac{}{\underline{a}.P \xrightarrow{a}_r P} \qquad \frac{P_1 \xrightarrow{a}_r P_1'}{P_1 + P_2 \xrightarrow{a}_r P_1'} \qquad \frac{P_2 \xrightarrow{a}_r P_2'}{P_1 + P_2 \xrightarrow{a}_r P_2'}$$

$$\frac{}{a.P \xrightarrow{X}_r \underline{a}.P} \qquad \frac{a \notin X \cup \{\tau\}}{\underline{a}.P \xrightarrow{X}_r \underline{a}.P} \qquad \frac{}{\mathbf{0} \xrightarrow{X}_r \mathbf{0}} \qquad \frac{\forall_{i=1,2}\ P_i \xrightarrow{X}_r P_i'}{P_1 + P_2 \xrightarrow{X}_r P_1' + P_2'}$$

Extending the transition relation to sequences w of visible actions and refusal sets as usual (i.e. suppressing τ's), we write $P \xRightarrow{w}_r P'$ and $P \xRightarrow{w}_r$; then, $\mathsf{RT}(P) = \{w \mid P \xRightarrow{w}_r\}$ is the set of *refusal traces* of P.

For a process P, $\mathcal{A}(P) = \{a \in \mathbb{A} \cup \{\tau\} \mid P \xrightarrow{a}_r\}$ is the set of immediately enabled actions; P is called *unstable* if $P \xrightarrow{\tau}_r$, *stable* otherwise.

The coarsest precongruence arising from the testing scenario in [JV98, JV99], i.e. the *efficiency precongruence* $\leq$ that we want to axiomatize, is now defined by: $P \leq Q$ if $(\mathsf{RT}(P) \subseteq \mathsf{RT}(Q))$ and (P stable iff Q stable) and (P initial iff Q initial). The inclusion of refusal traces characterizes the faster-than-relation arising from testing, and stability has to be considered (as usual) to obtain a precongruence for choice; remarkable is the fact that the stability requirement has to be an equivalence, although we work with a preorder where one might expect just one implication. The third requirement is necessary since for typing reasons we are not allowed to exchange e.g. an initial process for an urgent one in a sum of initial processes. We write $P = Q$ if $P \leq Q \wedge Q \leq P$. The precongruence results for choice and prefixing of [JV98, JV99] carry directly over to the additional processes we consider here, so $\leq$ is indeed a precongruence, which is of course essential when considering an axiomatization.

2 Axiomatization

Our aim is to give a complete axiomatization for initial processes in the fragment defined above; for this purpose, we also consider prefixing with urgent τ's, but not with other urgent actions. Thus, a process is in this section defined by the following grammar, where I generates again the *initial* and U the *urgent* processes, and $a \in \mathbb{A} \cup \{\tau\}$:

$$
\begin{array}{lcl|c|c|c}
I & ::= & \mathbf{0} & a.I & a.U & I+I \\
U & ::= & \mathbf{0} & \underline{\tau}.U & \underline{\tau}.I & U+U
\end{array}
$$

With this grammar, we deviate from the processes of [JV98, JV99] only slightly by allowing $\underline{\tau}$-prefixes also inside P in a process $a.P$ or $\underline{a}.P$. For the following reason, $\underline{\tau}$ is so important: τ combines the usual aspect of allowing an unobservable choice with the aspect of delaying a process by up to time 1. This combination poses a problem, and $\underline{\tau}$ helps to resolve this since $\underline{\tau}$ represents only the first aspect.

We will consider the following axioms, where $\epsilon \in \mathbb{A} \cup \{\tau, \underline{\tau}\}$:

$$
\begin{array}{llcll}
A00 & \underline{\tau}.\tau.\mathbf{0} & \leq & \underline{\tau}.\mathbf{0} & \text{Termination} \\
A10 & \epsilon.(\underline{\tau}.X + \underline{\tau}.Y) & = & \epsilon.X + \epsilon.Y & \text{Decision} \\
A11 & \underline{\tau}.(X+Y) & \leq & \underline{\tau}.X + \underline{\tau}.Y & \text{Nondeterminism} \\
A12 & \underline{\tau}.X & \leq & \underline{\tau}.X + Y & \text{Instability-0} \\
A13 & \tau.X & \leq & \tau.X + Y & \text{Instability-1} \\
A14 & X + \tau.X & \leq & \tau.X & \text{Subsumption} \\
A20 & \underline{\tau}.Y + \underline{\tau}.(X + \tau.Y) & \leq & \underline{\tau}.(X + \tau.Y) & \\
A30 & X + \mathbf{0} & = & X & \\
A31 & X + X & = & X & \\
A32 & X + Y & = & Y + X & \\
A33 & (X+Y)+Z & = & X + (Y+Z) &
\end{array}
$$

It is not hard to check that these axioms are sound for the efficiency precongruence $\leq$ defined in the preceding section in the sense that the (in)equalities are satisfied provided both terms under comparison are defined.

Note that two of the axioms would not hold in a synchronous setting: in such a setting, τ would necessarily take time 1; hence, in $A14$ and $A20$, the left-hand-side would have *additional* fast behaviour (due to X and $\underline{\tau}.Y$). In our setting, this fast behaviour can also be shown by the right-hand-side.

$A30 - A33$ will often be used implicitly. In particular, we will use $\stackrel{\sim}{\equiv}$ to denote $\equiv$ modulo $\mathbf{0}$-summands, i.e. up to applications of $A30$. Furthermore, in order to economize on derivations when reasoning about terms with similar structure, we will use square brackets [ ] around optional (sub)terms.

Also, as a consequence of $A30$, $A32$ and $A33$, we can write terms like $\sum_I X_i$, where i is understood to range over I and similarly for other letters, see e.g. $D15$ below. In accordance with $A30$, the empty sum is $\mathbf{0}$.

Proposition 1

The following laws for processes are derivable from the axioms, where $\epsilon \in \mathbb{A} \cup \{\tau, \underline{\tau}\}$ and $\mathcal{A} \subseteq \mathbb{A} \cup \{\tau\}$ and $I \neq \emptyset$ are finite:

$$
\begin{array}{lrcl}
D00 & \epsilon.\mathbf{0} & = & \epsilon.\tau.\mathbf{0} \\
D10 & \sum_I \epsilon.X_i & = & \epsilon.\sum_I \underline{\tau}.X_i \\
 & \text{in particular:} \quad \epsilon.X & = & \epsilon.\underline{\tau}.X \\
D11 & \epsilon.X & \leq & \epsilon.\tau.X \\
D12 & \underline{\tau}.X + \underline{\tau}.Y & = & \underline{\tau}.X + \underline{\tau}.Y + \underline{\tau}.(X+Y) \\
D13 & X + \tau.Y & \leq & \tau.(X + \tau.Y) \\
D14 & X + \tau.X & = & \tau.X \\
D15 & \underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a) & = & (\underline{\tau}.\sum_{\mathcal{A}} a.X_a) + (\underline{\tau}.\sum_{\mathcal{A}} a.Y_a) \\
D20 & \underline{\tau}.(X + \tau.Y) & = & \underline{\tau}.Y + \underline{\tau}.(X + \tau.Y) \\
D21 & \underline{\tau}.(U + \tau.V) + \underline{\tau}.(X + \tau.Y) & = & \underline{\tau}.(X + U + \tau.Y + \tau.V) \\
D22 & \underline{\tau}.Z + \underline{\tau}.(X + \tau.Y) & = & \underline{\tau}.Z + \underline{\tau}.(Z + X + \tau.Y)
\end{array}
$$

Proof: The following ordering respects the dependencies:

$D14$: by $A13$ and $A14$

$D10$: By induction on $|I|$:
$\epsilon.X = \epsilon.X + \epsilon.X = \epsilon.(\underline{\tau}.X + \underline{\tau}.X) = \epsilon.\underline{\tau}.X$ by $A31$, $A10$ and $A31$ again.
For larger I, we have w.l.o.g. that $0 \in I$; let $I' = I \setminus \{0\}$; then:
$\sum_I \epsilon.X_i = \epsilon.X_0 + \epsilon.\sum_{I'} \underline{\tau}.X_i = \epsilon.(\underline{\tau}.X_0 + \underline{\tau}.\sum_{I'} \underline{\tau}.X_i) = \epsilon.\sum_I \underline{\tau}.X_i$ by induction, $A10$ and induction again.

$D11$: $\underline{\tau}.X \leq \underline{\tau}.X + \underline{\tau}.(X + \tau.X)$ by $A12$, $\underline{\tau}.X + \underline{\tau}.(X + \tau.X) \leq \underline{\tau}.(X + \tau.X)$ by $A20$, $\underline{\tau}.(X + \tau.X) = \underline{\tau}.\tau.X$ by $D14$, hence $\epsilon.X = \epsilon.\underline{\tau}.X \leq \epsilon.\underline{\tau}.\tau.X = \epsilon.\tau.X$ by $D10$.

$D00$: $\leq$ by $D11$ and $\geq$ by $D10$, $A00$ and $D10$.

$D12$: $\underline{\tau}.X \leq \underline{\tau}.X + \underline{\tau}.(X+Y)$ by $A12$, hence $\underline{\tau}.X + \underline{\tau}.Y \leq \underline{\tau}.X + \underline{\tau}.Y + \underline{\tau}.(X+Y)$. On the other hand, $\underline{\tau}.(X+Y) \leq \underline{\tau}.X + \underline{\tau}.Y$ by $A11$, hence $\underline{\tau}.X + \underline{\tau}.Y + \underline{\tau}.(X+Y) \leq \underline{\tau}.X + \underline{\tau}.Y$.

$D13$: First, $\tau.Y \leq \tau.Y + \tau.(X + \tau.Y)$ by $A13$, hence $X + \tau.Y \leq (X + \tau.Y) + \tau.(X + \tau.Y) = \tau.(X + \tau.Y)$ by $D14$.

$D15$: First, for each $a \in \mathcal{A}$ we have
$\underline{\tau}.X_a \leq \underline{\tau}.X_a + \underline{\tau}.Y_a$ by $A12$, hence
$a.X_a = a.\underline{\tau}.X_a \leq a.(\underline{\tau}.X_a + \underline{\tau}.Y_a) = a.X_a + a.Y_a$ by $D10$, $A10$ resp., hence
$\underline{\tau}.\sum_{\mathcal{A}} a.X_a \leq \underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a)$;
by the same argument also
$\underline{\tau}.\sum_{\mathcal{A}} a.Y_a \leq \underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a)$ hence
$(\underline{\tau}.\sum_{\mathcal{A}} a.X_a) + (\underline{\tau}.\sum_{\mathcal{A}} a.Y_a) \leq \underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a)$.
On the other hand,
$\underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a) = \underline{\tau}.((\sum_{\mathcal{A}} a.X_a) + (\sum_{\mathcal{A}} a.Y_a))$ hence
$\underline{\tau}.\sum_{\mathcal{A}}(a.X_a + a.Y_a) \leq (\underline{\tau}.\sum_{\mathcal{A}} a.X_a) + (\underline{\tau}.\sum_{\mathcal{A}} a.Y_a)$ by $A11$.

$D20$: We have $\leq$ by $A12$ and $\geq$ is $A20$.
$D21$: On the one hand, we have $\geq$ by $A11$. On the other hand, we have $\tau.Y \leq \tau.Y + U + \tau.V$ by $A13$ and, hence, $\underline{\tau}.(X + \tau.Y) \leq \underline{\tau}.(X + U + \tau.Y + \tau.V)$; similarly, $\underline{\tau}.(U + \tau.V) \leq \underline{\tau}.(X + U + \tau.Y + \tau.V)$ and $\leq$ follows with $A31$.
$D22$: $\leq$ follows from $\tau.Y \leq \tau.Y + Z$ by $A13$ and $\geq$ follows from $\underline{\tau}.Z + \underline{\tau}.(X + \tau.Y) \geq \underline{\tau}.(Z + X + \tau.Y)$ by $A11$ and with $A31$. ■ 2.1

Definition 2

Henceforth, we will write $\leq_e$ or $=_e$ if the (in)equality is *derivable* from the axioms above, where we can also use the derived laws listed in the preceding proposition. In derivations, one often has the situation that a subterm is transformed, then an axiom or law is applied and the transformation is reversed; we will indicate this with the words *'by reversal'*.

We now define normal forms that are also typed as i- and u-normal-forms or i-nf, u-nf resp. for short.

1. Let $\mathcal{A} \subseteq \mathbb{A} \cup \{\tau\}$ be finite. Then $\sum_{\mathcal{A}} a.P_a$ is an i-nf if all P_a are u-nf.
2. Let I be finite. Then $\sum_I \underline{\tau}.P_i$ is a u-nf if all P_i are i-nf.

Note that $\mathcal{A} = \emptyset$ yields i-nf $\mathbf{0}$ and $I = \emptyset$ yields u-nf $\mathbf{0}$.

We will use $\mathcal{A}, \mathcal{B}, \ldots$ for subsets of $\mathbb{A} \cup \{\tau\}$ and $A, B, \ldots$ for subsets of $\mathbb{A}$; if both, $\mathcal{A}$ and A, appear, then A denotes $\mathcal{A} \setminus \{\tau\}$ etc. ■ 2.2

We want to show that each process can be brought into the appropriate normal form using our axioms; some of the proofs involved apply induction on the depths of processes defined as follows.

Definition 3

The *depth* $|P|$ of a process P is a natural number defined structurally as

Nil: $|\mathbf{0}| = 0$
Pref: $|a.P| = 1 + |P|$ for $a \in \mathbb{A} \cup \{\tau\}$ and $|\underline{\tau}.P| = |P|$
Sum: $|P + Q| = \max(|P|, |Q|)$ ■ 2.3

The intended result given in Proposition 6 below uses two lemmata, which deal with $\underline{\tau}$-prefixing and choice.

Lemma 4

For each normal form P there is a normal form $N =_e \underline{\tau}.P$ with $|N| \leq |\underline{\tau}.P|$.

Proof:
If P is an i-nf, then $N \equiv \underline{\tau}.P$ with $|N| = |P|$ is a u-nf according to Definition 2.2, and we are done; this also treats the case $P \equiv \mathbf{0}$. If $P \equiv \sum_I \underline{\tau}.P_i$ is a u-nf with $I \neq \emptyset$, then $\underline{\tau}.P =_e P$ by induction on $|I|$ with $D10$; hence we can choose $N \equiv P$ in this case. ■ 2.4

Lemma 5

Let P and Q be both i-nf or both u-nf.

1. Then there is a normal form $N =_e P + Q$ with $|N| \leq |P + Q|$ and
2. if $P \equiv \sum_{\mathcal{A}} a.P_a$ and $Q \equiv \sum_{\mathcal{B}} a.Q_a$ are i-nf, then $N \equiv \sum_{\mathcal{A}\cup\mathcal{B}} a.N_a$; moreover,
3. if $\mathcal{B} \subseteq \mathcal{A}$, then $\underline{\tau}.P + \underline{\tau}.N =_e \underline{\tau}.N$ and
4. if $\mathcal{B} = \mathcal{A}$, then $\underline{\tau}.P + \underline{\tau}.Q =_e \underline{\tau}.N$.

Proof:

We perform NOETHERian induction on $|P + Q|$ to show 1.; assume it to hold for all normal forms P, Q with $|P+Q| < n$ for some $n \in \mathbb{N}_0$ and consider P, Q with $|P+Q| = n$. We can assume $P \not\equiv \mathbf{0} \not\equiv Q$ by $A30$. Furthermore, if P and Q are u-nf, then already $N \equiv P + Q$ is a u-nf by Definition 2.2.

Hence, let $P \equiv \sum_{\mathcal{A}} a.P_a$ and $Q \equiv \sum_{\mathcal{B}} a.Q_a$ be i-nf and let $P' \equiv \sum_{\mathcal{A}\setminus\mathcal{B}} a.P_a$ and $Q' \equiv \sum_{\mathcal{B}\setminus\mathcal{A}} a.Q_a$. Then $P+Q =_e P'+Q'+\sum_{\mathcal{A}\cap\mathcal{B}}(a.P_a+a.Q_a) =_e P'+Q'+\sum_{\mathcal{A}\cap\mathcal{B}} a.(\underline{\tau}.P_a+\underline{\tau}.Q_a)$ by $A10$. For each $a \in \mathcal{A}\cap\mathcal{B}$ there is u-nf $R_a =_e \underline{\tau}.P_a + \underline{\tau}.Q_a$ with $|R_a| \leq |\underline{\tau}.P_a + \underline{\tau}.Q_a|$ by induction: there are u-nf $P''_a =_e \underline{\tau}.P_a$ and $Q''_a =_e \underline{\tau}.Q_a$ by Lemma 4, such that $|P''_a| \leq |\underline{\tau}.P_a| = |P_a| < |a.P_a| \leq |P|$ and analogously for $|Q''_a|$, hence $|P''_a+Q''_a| = \max(|P''_a|, |Q''_a|) < \max(|P|, |Q|) = |P + Q|$. Thus, $P + Q =_e P' + Q' + \sum_{\mathcal{A}\cap\mathcal{B}} a.R_a =_e N$, where N is of the form $N \equiv \sum_{\mathcal{A}\cup\mathcal{B}} a.N_a$ with $|N| \leq |P + Q|$. These constructions also prove 2.

For 3. we have $\underline{\tau}.P + \underline{\tau}.N$

$=_e \underline{\tau}.\sum_{\mathcal{A}} a.P_a + \underline{\tau}.\sum_{\mathcal{A}} a.N_a$	by 2.
$=_e \underline{\tau}.\sum_{\mathcal{A}}(a.P_a + a.N_a)$	by $D15$
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.(\underline{\tau}.P_a + \underline{\tau}.N_a)$	by $A10$
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.(\underline{\tau}.P_a + \underline{\tau}.(\underline{\tau}.P_a[+\underline{\tau}.Q_a]))$	by the above [if $a \in \mathcal{B}$]
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.(\underline{\tau}.P_a + \underline{\tau}.P_a[+\underline{\tau}.Q_a])$	by $A10$ [if $a \in \mathcal{B}$]
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.(\underline{\tau}.P_a[+\underline{\tau}.Q_a])$	by $A31$ [if $a \in \mathcal{B}$]
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.N_a$	by the above
$=_e \underline{\tau}.N$	

For 4. we have $\underline{\tau}.P + \underline{\tau}.Q \equiv \underline{\tau}.\sum_{\mathcal{A}} a.P_a + \underline{\tau}.\sum_{\mathcal{A}} a.Q_a$

$=_e \underline{\tau}.\sum_{\mathcal{A}}(a.P_a + a.Q_a)$	by $D15$
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.(\underline{\tau}.P_a + \underline{\tau}.Q_a)$	by $A10$
$=_e \underline{\tau}.\sum_{\mathcal{A}} a.N_a$	by the above
$\equiv \underline{\tau}.N$	■ 2.5

Proposition 6

For each process P there is a normal form $N =_e P$ with $|N| \leq |P|$.

Proof:

We perform induction on the structure of P: the base case $\mathbf{0}$ is clear, hence by induction and Lemma 4 and Lemma 5.1 we can restrict attention to the case $a.P$ with $a \in \mathbb{A}_{\omega\tau}$, assuming P to be a normal form: then there is u-nf $P' =_e \underline{\tau}.P$ with $|P'| \leq |\underline{\tau}.P| = |P|$ by Lemma 4, hence – according to Definition 2.1 – we can choose normal form $N \equiv a.P' =_e a.P$ by $D10$ with $|N| = 1 + |P'| \leq 1 + |P| = |a.P|$. ■ 2.6

Normal forms are only a first step; to achieve our final goal – proving completeness of our axioms –, we have to consider saturated normal forms. First, we define saturated i-normal forms. In this definition we use $\stackrel{\sim}{\equiv}$; since the form of P is given first, one can see that the only reason for writing $P \stackrel{\sim}{\equiv} \ldots$ is that the first sum on the right-hand-side might be empty and is actually missing in P. Similarly, Q_i has the given form except that the sum over B_i and/or Q^i_τ might be $\mathbf{0}$. Other cases below are similar, e.g. P in Definition 9 is syntactically equal to $P_s + P_\tau$ except that one or both summands might be $\mathbf{0}$ and actually missing.

Definition 7

Let $P \equiv \sum_{\mathcal{A}} a.P_a$ be an i-nf. Then P is *saturated* if P is stable or

$$P \stackrel{\sim}{\equiv} \sum_{A} a.P_a + \tau.\sum_{I} \underline{\tau}.Q_i \quad \text{where, for each } i \in I, \quad Q_i \stackrel{\sim}{\equiv} \sum_{B_i} b.Q^i_b + Q^i_\tau$$

is saturated (such that Q^i_τ is $\mathbf{0}$ or of the form $\tau.Q$), $B_i \subseteq A$ and $a.Q^i_a + a.P_a =_e a.P_a$ for all $a \in B_i$. ■ 2.7

Examples of saturated i-nf are $\mathbf{0}$, $\tau.\mathbf{0}$, $a.\mathbf{0} + \tau.\mathbf{0}$, $\tau.\underline{\tau}.\tau.\mathbf{0}$ and $a.\mathbf{0} + \tau.\underline{\tau}.(a.\mathbf{0} + \tau.\mathbf{0})$, but not $\tau.\underline{\tau}.a.\mathbf{0}$, since visible behaviour after $\tau\underline{\tau}$ must also be present on the 'top level', i.e. in $\sum_A a.P_a$. The proof of the following proposition consists of a construction to make this true. This construction is not as complicated as it might be, since for P_a or Q^i_b saturation is not required. (But note that, due to the recursive definition, Q^i_τ has to be saturated.)

Proposition 8

For each i-nf P there is a saturated i-nf $N =_e P$ with $|N| \leq |P|$.

Proof:

We perform induction on $|P|$ and assume the property to hold for all normal forms with depth $< |P|$. Let $P \equiv \sum_{\mathcal{A}} a.P_a$. If $\tau \notin \mathcal{A}$ we can choose $N \equiv P$, hence let $P_\tau \equiv \sum_I \underline{\tau}.Q_i \equiv \sum_I \underline{\tau}.(\sum_{B_i} b.Q^i_b + Q^i_\tau)$ where we can assume Q_i to be saturated for each $i \in I$ by induction. Then $P \equiv \sum_A a.P_a + \tau.P_\tau$

$\equiv \sum_A a.P_a + \tau.\sum_I \underline{\tau}.(\sum_{B_i} b.Q^i_b + Q^i_\tau)$

$=_e \sum_A a.P_a + \sum_I \tau.(\sum_{B_i} b.Q^i_b + Q^i_\tau)$ by $A10$

$=_e \sum_A a.P_a + \sum_I \tau.(\sum_{B_i} b.Q^i_b + Q^i_\tau) + (\sum_I(\sum_{B_i} b.Q^i_b + Q^i_\tau))$ by $D14$

$=_e \sum_A a.P_a + \sum_I \tau.(\sum_{B_i} b.Q^i_b + Q^i_\tau) + (\sum_I(\sum_{B_i} b.Q^i_b + Q^i_\tau)) + \sum_I \sum_{B_i} b.Q^i_b$ by $A31$

$=_e \sum_A a.P_a + \tau.P_\tau + \sum_I \sum_{B_i} b.Q^i_b$ by reversal

$=_e \sum_{A \setminus B} a.P_a + \sum_{A \cap B}(a.P_a + \sum_{I_a} a.Q^i_a) + \sum_{B \setminus A} \sum_{I_b} b.Q^i_b + \tau.P_\tau$

where $B = \bigcup_{i \in I} B_i$ and $I_a = \{i \in I \mid a \in B_i\}$

$=_e \sum_{A \setminus B} a.P_a + \sum_{A \cap B} a.P'_a + \sum_{B \setminus A} b.Q'_b + \tau.P_\tau \stackrel{\sim}{\equiv} N$

where N is without $\mathbf{0}$-summands and $P'_a =_e \underline{\tau}.P_a + \sum_{I_a} \underline{\tau}.Q_a$ is a u-nf with $|P'_a| \leq |\underline{\tau}.P_a + \sum_{I_a} \underline{\tau}.Q_a|$ for each $a \in A \cap B$, and $Q'_b =_e \sum_{I_b} \underline{\tau}.Q^i_b$ is a u-nf with $|Q'_b| \leq |\sum_{I_b} \underline{\tau}.Q^i_b|$ for each $b \in B \setminus A$. These u-nf exist by Proposition 6.

Now $N =_e P$ is an i-nf with $|N| \leq |P|$, and it is saturated since:

- for all $j \in I$ with $a \in B_j \cap A$ we have
  $a.Q^j_a + a.P'_a =_e a.Q^j_a + a.(\underline{\tau}.P_a + \sum_{I_a} \underline{\tau}.Q^i_a) =_e a.Q^j_a + a.P_a + \sum_{I_a} a.Q^i_a =_e a.P'_a$
  (by $D10$ and, with $j \in I_a$, by $A31$)

– for all $j \in I$ with $b \in B_j \setminus A$ we have
$b.Q_b^j + b.Q'_b =_e b.Q_b^j + b.\sum_{I_b} \underline{\tau}.Q_b^i =_e b.Q_b^j + \sum_{I_b} b.Q_b^i =_e b.Q'_b$
(by $D10$ and, with $j \in I_b$, by $A31$) ■ 2.8

Now we define when a u-nf is saturated.

Definition 9

Let $P \equiv \sum_I \underline{\tau}.P_i$ be a u-nf and let $I = I_s \dot{\cup} I_\tau$ be partitioned, s.t. P_i stable iff $i \in I_s$, i.e.:

$$P \,\hat{\equiv}\, P_s + P_\tau \quad \text{with} \quad P_s \equiv \sum\nolimits_{I_s} \underline{\tau}.\sum\nolimits_{A_i} a.P_a^i \quad \text{and} \quad P_\tau \equiv \sum\nolimits_{I_\tau} \underline{\tau}.P_i$$

where $A_i \subseteq \mathbb{A}$ for all $i \in I_s$ and P_i unstable for all $i \in I_\tau$. Then P is *saturated* if $P \equiv \mathbf{0}$ or $P \equiv \underline{\tau}.\mathbf{0}$ or it satisfies the following conditions:

1. $|I_\tau| \leq 1$ and $P_i \not\equiv \mathbf{0}$ for all $i \in I_s$.
2. If $P_\tau \,\hat{\equiv}\, \underline{\tau}.(P_\tau^s + \tau.Q)$, then $Q \,\hat{\equiv}\, \sum_{J_s} \underline{\tau}.Q_j + Q_\tau$ is a saturated u-nf and:
   i) if $Q_j \equiv \sum_{B_j} b.Q_b^j \not\equiv \mathbf{0}$ for $j \in J_s$, then $A_i = B_j$ and $Q_j + P_i =_e P_i$ for some $i \in I_s$.
   ii) $P_i + P_\tau^s =_e P_\tau^s$ for all $i \in I_s$.
3. If $i \neq j \in I_s$, then $A_i \neq A_j$ and $A_i \cup A_j = A_k$ and $P_i + P_j + P_k =_e P_k$ for a $k \in I_s$.

■ 2.9

E.g. $\mathbf{0}$, $\underline{\tau}.\mathbf{0}$, $\underline{\tau}.\tau.\mathbf{0}$ are saturated, but $\underline{\tau}.\mathbf{0} + \underline{\tau}.\tau.\mathbf{0}$ is not. We show how to saturate a u-nf.

Proposition 10

For each u-nf P there is a saturated u-nf $N =_e P$ with $|N| \leq |P|$.

Proof:

We perform induction on $|P|$ and assume the property to hold for all normal forms with depth $< |P|$. If $P \equiv \sum_I \underline{\tau}.P_i$ is not saturated, then $\mathbf{0} \not\equiv P \not\equiv \underline{\tau}.\mathbf{0}$ and some of the conditions in Definition 9 must be violated, and we show how to saturate P:

1.
First assume $P_i \equiv \mathbf{0}$ for some $i \in I$; if $P_j \equiv \mathbf{0}$ for all $j \in I$, then $P =_e \underline{\tau}.\mathbf{0}$ by $A31$, hence assume $P_j \not\equiv \mathbf{0}$ for some $j \in I$; then $|P| \geq |P_j| \geq 1$ since P_j is an i-nf; now $P =_e \underline{\tau}.\mathbf{0} + \sum_{I\setminus\{i\}} \underline{\tau}.P_i =_e \underline{\tau}.\tau.\mathbf{0} + \sum_{I\setminus\{i\}} \underline{\tau}.P_i$ by $D00$, hence we may assume P_i to be replaced by $\underline{\tau}.\tau.\mathbf{0}$ without increasing the depth of P (since $|\underline{\tau}.\tau.\mathbf{0}| = 1 \leq |P|$), thus $P_i \not\equiv \mathbf{0}$ for all $i \in I_s$. Now let $|I_\tau| > 1$. Then $P \,\hat{\equiv}\, P_s + P_\tau$
$\hat{\equiv}\, P_s + \sum_{I_\tau} \underline{\tau}.(P_i^s + \tau.Q_i)$
$=_e P_s + \underline{\tau}.\sum_{I_\tau}(P_i^s + \tau.Q_i)$ by $D21$
$=_e P_s + \underline{\tau}.(P_\tau^s + \tau.Q)$
where $P_\tau^s + \tau.Q =_e \sum_{I_\tau}(P_i^s + \tau.Q_i)$ is an i-nf with $|P_\tau^s + \tau.Q| \leq |\sum_{I_\tau}(P_i^s + \tau.Q_i)|$ which exists by Lemma 5.1. Hence, we can replace P_τ by $\underline{\tau}.(P_\tau^s + \tau.Q)$, yielding $|I_\tau| = 1$, without increasing the depth of P.

2.i)
Let $P_\tau \,\hat{\equiv}\, \underline{\tau}.(P_\tau^s + \tau.Q)$. If $Q \equiv \mathbf{0}$ or $Q \equiv \underline{\tau}.\mathbf{0}$, then P vacuously satisfies 9.2.i). Hence

assume $\mathbf{0} \not\equiv Q \not\equiv \underline{\tau}.\mathbf{0}$ now; then $\underline{\tau}.Q =_e Q$ by $D10$, thus P_τ

$=_e \underline{\tau}.(P^s_\tau + \tau.Q)$

$=_e \underline{\tau}.Q + \underline{\tau}.(P^s_\tau + \tau.Q)$ by $D20$

$=_e \underline{\tau}.Q + P_\tau$

$=_e Q + P_\tau$ $(*)$

Moreover, Q is a u-nf with $|Q| < |P_\tau| \leq |P|$, hence we may assume Q to be saturated by induction and of the form $Q \hat{\equiv} Q_s + Q_\tau \hat{\equiv} (\sum_{J_s} \underline{\tau}. \sum_{B_j} b.Q^j_b)[+\underline{\tau}.(Q^s_\tau + \tau.R)]$. In the case that the optional summand exists we have $\tau.Q + \tau.R$

$=_e \tau.(\underline{\tau}.Q + \underline{\tau}.R)$ by $A10$

$=_e \tau.(Q + \underline{\tau}.R)$ by $\underline{\tau}.Q =_e Q$

$=_e \tau.(Q_s + \underline{\tau}.(Q^s_\tau + \tau.R) + \underline{\tau}.R)$

$=_e \tau.(Q_s + \underline{\tau}.(Q^s_\tau + \tau.R))$ by $D20$

$=_e \tau.Q$

Hence – in any case – we have $\tau.Q[+\tau.R] =_e \tau.Q$ $(**)$. Now P_τ

$=_e Q + \underline{\tau}.(P^s_\tau + \tau.Q)$ by $(*)$

$=_e \sum_{J_s} \underline{\tau}.Q_j[+\underline{\tau}.(Q^s_\tau + \tau.R)] + \underline{\tau}.(P^s_\tau + \tau.Q)$ see above

$=_e \sum_{J_s} \underline{\tau}.Q_j + \underline{\tau}.((P^s_\tau[+Q^s_\tau]) + (\tau.Q[+\tau.R]))$ [by $D21$]

$=_e \sum_{J_s} \underline{\tau}.Q_j + \underline{\tau}.(P' + \tau.Q)$ by $(**)$

where $P' =_e P^s_\tau[+Q^s_\tau]$ is an i-nf with $|P'| \leq |P^s_\tau[+Q^s_\tau]|$ by Lemma 5.1. Also noting that $|Q_j| < |P|$ for all $j \in J_s$, we conclude that replacing P_τ according to $P_\tau =_e \sum_{J_s} \underline{\tau}.Q_j + \underline{\tau}.(P' + \tau.Q)$ does not increase the depth of P and – assuming J_s and I_s disjoint – just adds J_s to I_s; 9.1 is still satisfied, since for $j \in J_s$ $B_j \neq \emptyset$ by induction and saturation of Q, and we can choose $i = j$ in the enlarged set $J_s \cup I_s$ and have $A_i = B_j$ and $Q_j + P_i =_e Q_j \equiv P_i$. Note that saturated u-nf Q is not changed by these transformations. Thus, the transformed P now satisfies the conditions of 9.1 and 9.2.i); clause 9.2.ii) is treated below.

3.

Let I_s be partitioned into $I_s = \bigcup_{l \in \mathbb{N}} I_l$ where $I_l = \{i \in I_s \,|\, l = |A_i|\}$. Assume the conditions of 9.3 to hold for all $i \neq j \in \bigcup_{0 \leq l < m} I_l$ for some $m \in \mathbb{N}_0$; we show that we can transform P such that they also hold for all $i \neq j \in \bigcup_{0 \leq l \leq m} I_l$.

First, assume $A_i = A_j$ for some $i, j \in I_m$; then with Lemma 5.1 there is i-nf $P_k =_e P_i + P_j$ with $|P_k| \leq \max(|P_i|, |P_j|)$ and $A_k = A_i = A_j$ by Lemma 5.2. Furthermore, $\underline{\tau}.P_i + \underline{\tau}.P_j =_e \underline{\tau}.P_k$ by Lemma 5.4, thus we can assume i and j in I_m to be replaced by k, yielding a smaller I_m. This preserves the conditions for $i' \neq j' \in \bigcup_{0 \leq l < m} I_l$: $P_{i'}$ and $P_{j'}$ are untouched, hence still $A_{i'} \neq A_{j'}$, and if w.l.o.g. $P_{i'} + P_{j'} + P_i =_e P_i$, then $P_{i'} + P_{j'} + P_k =_e P_{i'} + P_{j'} + P_i + P_j =_e P_i + P_j =_e P_k$ now, thus k replaces i – and analogously j – for this property. Similarly, the properties of 9.1 and 9.2.i) are untouched or preserved. Hence, we can assume $A_i \neq A_j$ for all $j \in \bigcup_{0 \leq l \leq m} I_l$ in the following.

Now assume $A_j \subset A_i$ for some $i \in I_m$ and $j \in \bigcup_{0 \leq l < m} I_l$; then with Lemma 5.1 again there is i-nf $P_k =_e P_i + P_j$ with $|P_k| \leq \max(|P_i|, |P_j|)$ and $A_k = A_i$ by Lemma 5.2. Furthermore, $\underline{\tau}.P_i + \underline{\tau}.P_k =_e \underline{\tau}.P_k$ by Lemma 5.3; with this and $D12$ we get $\underline{\tau}.P_i + \underline{\tau}.P_j =_e \underline{\tau}.P_i + \underline{\tau}.P_j + \underline{\tau}.(P_i + P_j) =_e \underline{\tau}.P_i + \underline{\tau}.P_j + \underline{\tau}.P_k =_e \underline{\tau}.P_j + \underline{\tau}.P_k$ and thus we can replace i by k in I_m. This removes the problematic pair (j, i), since now $P_i + P_k + P_k =_e P_k$; also, it preserves the conditions for $i' \neq j' \in \bigcup_{0 \leq l < m} I_l$ and those of 9.1 and 9.2.i) just as in the previous case, and keeps $A_{i'} \neq A_{j'}$ also for $i', j' \in I_m$. Even more, if $P_{j'} + P_{i'} + P_{k'} = P_{k'}$

for some $i' \in I_m$ and $A_{j'} \subseteq A_{i'}$ and some k' before, then also now: for $i = k'$ this follows as above and for $i = i'$ we must have $i = k'$.

Finally, if $A_j \not\subseteq A_i$ for some $i \in I_m$ and $j \in \bigcup_{0<l\leq m} I_l$, then with Lemma 5.1 again there is an i-nf $P_k =_e P_i + P_j$ with $|P_k| \leq \max(|P_i|, |P_j|)$ and $|A_k| > |A_i|$, hence $k \notin \bigcup_{0\leq l\leq m} I_l$ and it suffices to add k to I_s using $D12$, keeping all previously ensured properties untouched.

We see that all transformations keep the properties of Definition 9.1 and 9.2.i) and do not increase the depth of P; since $I_m = \emptyset$ for all $m > |\bigcup_{i\in I_s} A_i|$, we are done.

2.ii)
Let $P_\tau \tilde{\equiv} \underline{\tau}.(P^s_\tau + \tau.Q)$; then P
$=_e \sum_{I_s} \underline{\tau}.P_i + \underline{\tau}.(P^s_\tau + \tau.Q)$
$=_e \sum_{I_s} \underline{\tau}.P_i + \underline{\tau}.(\sum_{I_s} P_i + P^s_\tau + \tau.Q)$ by $D22$
$=_e \sum_{I_s} \underline{\tau}.P_i + \underline{\tau}.(P'_\tau + \tau.Q)$
where $P'_\tau =_e \sum_{I_s} P_i + P^s_\tau$ is an i-nf with $|P'_\tau| \leq |\sum_{I_s} P_i + P^s_\tau|$ which exists by Lemma 5.1 and is stable by Lemma 5.2. Hence we can transform P to satisfy 9.2.ii) without increasing the depth or touching any of the other conditions of Definition 9. ■ 2.10

We will now give a series of lemmata, leading to our theorem; these lemmata study the semantic properties of saturated normal forms, i.e. their RT-semantics or specific properties for pairs that are comparable under $\leq$.

Lemma 11

Let $P \equiv \sum_{\mathcal{A}} a.P_a$ be a saturated i-nf, $c \in \mathbb{A}$, $\Sigma \subseteq \mathbb{A}$ and $w \in (\mathbb{A} \cup 2^{\mathbb{A}})^*$. Then

1. $cw \in \mathsf{RT}(P)$ if and only if $c \in \mathcal{A}$ and $w \in \mathsf{RT}(P_c)$.
2. Let $\tau \in \mathcal{A}$; then $\emptyset\mathbb{A}w \in \mathsf{RT}(P)$ iff $\Sigma w \in \mathsf{RT}(P_\tau)$ and $\emptyset\mathbb{A}cw \in \mathsf{RT}(P)$ iff $cw \in \mathsf{RT}(P_\tau)$.

Proof:

1.
We perform induction on $|P|$ and assume the property to hold for all saturated i-nf with depth $< |P|$. The 'if'-direction is clear and the 'only-if'-direction is straightforward for stable P. Hence, let unstable $P \tilde{\equiv} \sum_A a.P_a + \tau.\sum_I \underline{\tau}.Q_i$ where $Q_i \tilde{\equiv} \sum_{B_i} b.Q^i_b + Q^i_\tau$ for all $i \in I$ and assume $cw \in \mathsf{RT}(\tau.P_\tau)$. Then P saturated implies Q_i for all $i \in I$ saturated, thus $cw \in \mathsf{RT}(Q_i)$ implies $c \in B_i$ and $w \in \mathsf{RT}(Q^i_c)$ by induction. Now by Definition 7 $c \in A$ and $c.Q^i_c + c.P_c =_e c.P_c$, thus $w \in \mathsf{RT}(P_c)$.

2.
Let $\emptyset\mathbb{A}w \in \mathsf{RT}(P)$; then we have $P \overset{\emptyset}{\to}_r P' \overset{\mathbb{A}}{\Rightarrow}_r \overset{w}{\Rightarrow}_r$ or $P \overset{\tau}{\to}_r P_\tau \overset{\emptyset}{\Rightarrow}_r \overset{\mathbb{A}}{\Rightarrow}_r \overset{w}{\Rightarrow}_r$; since τ is urgent in P' in the first case and since time steps can be omitted by [JV99, Prop. 4.10 ii)] in the second case, we conclude $\mathbb{A}w \in \mathsf{RT}(P_\tau)$; hence $\Sigma w \in \mathsf{RT}(P_\tau)$ for all $\Sigma \subseteq \mathbb{A}$ by [JV99, Prop. 4.2.1]. On the other hand, $\Sigma w \in \mathsf{RT}(P_\tau)$ implies $\mathbb{A}w \in \mathsf{RT}(P_\tau)$ since Σ must be performed by an initial subterm of P_τ, thus $\emptyset\mathbb{A}w \in \mathsf{RT}(P)$.

Now let $\emptyset\mathbb{A}cw \in \mathsf{RT}(P)$; then $\mathbb{A}cw \in \mathsf{RT}(P_\tau)$ as we have just shown, hence $cw \in \mathsf{RT}(P_\tau)$, since time steps can be omitted. On the other hand, $cw \in \mathsf{RT}(P_\tau)$ implies $\mathbb{A}cw \in \mathsf{RT}(P_\tau)$, since c must be performed by an initial subterm of P_τ, hence $\emptyset\mathbb{A}cw \in \mathsf{RT}(P)$ by the above.
■ 2.11

Lemma 12

Let $P \equiv \sum_{\mathcal{A}} a.P_a$ and $Q \equiv \sum_{\mathcal{B}} b.Q_b$ be saturated i-nf with $P \leq Q$. Then

1. we have $\mathcal{A} \subseteq \mathcal{B}$ and $\underline{\tau}.P_a \leq \underline{\tau}.Q_a$ for all $a \in \mathcal{A}$ and
2. if P and Q are stable, then $\mathcal{A} = \mathcal{B}$; in particular, $P \equiv \mathbf{0}$ iff $Q \equiv \mathbf{0}$.

Proof:
1. $P \leq Q$ implies $\mathcal{A} \cap \{\tau\} = \mathcal{B} \cap \{\tau\}$, and with Lemma 11.1 and .2 and $\mathsf{RT}(P) \subseteq \mathsf{RT}(Q)$ we conclude $\mathcal{A} \subseteq \mathcal{B}$ and $\mathsf{RT}(P_a) \subseteq \mathsf{RT}(Q_a)$ for all $a \in \mathcal{A}$, hence $\underline{\tau}.P_a \leq \underline{\tau}.Q_a$ for all $a \in \mathcal{A}$ by definition of $\leq$.

2. By 1., it suffices to show $\mathcal{B} \subseteq \mathcal{A}$: for stable P we have $\mathbb{A}\{a\} \in \mathsf{RT}(P)$ if and only if $a \notin \mathcal{A}$, hence: $a \notin \mathcal{A}$ implies $\mathbb{A}\{a\} \in \mathsf{RT}(P) \subseteq \mathsf{RT}(Q)$, which implies $a \notin \mathcal{B}$, thus $\mathcal{B} \subseteq \mathcal{A}$. ■ 2.12

Definition 13

For a saturated u-nf $P \,\hat{\equiv}\, P_s + P_\tau$ let $|P|_\tau$ be the τ-*depth* defined inductively as $|P|_\tau = 0$ if $P_\tau \equiv \mathbf{0}$ and $|P|_\tau = 1 + |P'|_\tau$ if $P_\tau \equiv \underline{\tau}.([P^s_\tau +]\tau.P')$. ■ 2.13

This τ-depth is not really needed, but it sharpens the next lemma, where $\circ$ denotes string composition extended to sets.

Lemma 14

Let $P \,\hat{\equiv}\, (\sum_{I_s} \underline{\tau}. \sum_{A_i} a.P^i_a) + P_\tau$ be a saturated u-nf, let $d > |P|_\tau$, let $\emptyset \neq A \subseteq \mathbb{A}$, let $\overline{A} = \mathbb{A} \setminus A$ and let $w \in (\mathbb{A} \cup 2^{\mathbb{A}})^*$. Then

1. If $P \overset{\emptyset^d}{\Rightarrow}_r P'$, then P' is stable and urgent.
2. $\{\emptyset^d \overline{A}\} \circ A \subseteq \mathsf{RT}(P)$ if and only if $A = A_i$ for some $i \in I_s$.
3. Let $A = A_i$ for some $i \in I_s$ and let w start with an action; then $\emptyset^d \overline{A} w \in \mathsf{RT}(P)$ if and only if $w \in \mathsf{RT}(P_i)$.
4. If $A = A_i$ for some $i \in I_s$, then $\mathsf{RT}(P_i) = (2^{\mathbb{A}} \cup \{\lambda\}) \circ (2^{\overline{A_i}})^* \circ \{w \in \mathsf{RT}(P_i) \mid w \text{ starts with an action or } w = \lambda\}$
5. Let $\Sigma, \Sigma' \subseteq \mathbb{A}$ and $P_\tau \not\equiv \mathbf{0}$. Then $\Sigma\Sigma' w \in \mathsf{RT}(P_\tau)$ implies $\mathbb{A}\mathbb{A} w \in \mathsf{RT}(P_\tau)$, and $\mathbb{A}\mathbb{A} w \in \mathsf{RT}(P)$ implies $\mathbb{A}\mathbb{A} w, \Sigma\Sigma' w \in \mathsf{RT}(P_\tau)$.

Proof:
1. Induction on $|P|_\tau$, where the base case $|P|_\tau = 0$ is clear. If $P \,\hat{\equiv}\, \sum_{I_s} \underline{\tau}.P_i + \underline{\tau}.(P^s_\tau + \tau.P')$ with $|P'|_\tau < d - 1$ and $P \overset{\emptyset}{\Rightarrow}_r P''$, then either P'' is some P_i after a time step with $i \in I_s$, hence stable and urgent, or we are done by induction.

2. The 'if'-direction is straightforward. For the 'only-if'-direction, observe that whenever $P \overset{\emptyset^d \overline{A}}{\Rightarrow}_r P'$, then P' is stable and urgent by 1. and we have $\mathcal{A}(P') \subseteq A$; hence, if $P' \overset{a}{\rightarrow}_r$ for some $a \in A$, then P' is a stable subterm R_a of P after a time step; R_a could e.g. be some Q_j with $j \in J_s$ in Definition 9. Now iterated application of 9.2.i) shows that for each $a \in A$ there is some $i \in I_s$ with $a \in A_i \subseteq A$. Hence, we can conclude $A_i = A$ for some $i \in I_s$ by repeated application of 9.3.

3. The 'if'-direction is straightforward. For the 'only-if'-direction, consider P' as in the proof of 2.: we see that w starts with some $a \in A$ and that there is some $j \in I_s$ with $A_j \subseteq A$ and $w \in \mathsf{RT}(P_j)$; for $j \neq i$ we apply 9.3, where we must have $k = i$, and conclude $w \in \mathsf{RT}(P_i)$.

4. The crucial observation is that P_i reaches the same process after performing an action, no matter whether it performs time steps before that action or not.

5. When P_τ performs $\Sigma\Sigma' w$, it reaches Q along the way; hence, we have $P_\tau \xrightarrow{\tau}_r \xrightarrow{\mathbb{A}}_r \xrightarrow{\tau}_r Q \overset{\Sigma' w}{\Rightarrow}_r$, omitting the time step Σ (see above) in case it is performed by Q. With a similar argument, we conclude $\mathbb{A}\mathbb{A}w \in \mathsf{RT}(P_\tau)$. Since $P_\tau \not\equiv \mathbf{0}$, none of the A_i, $i \in I_s$, is empty, hence $\mathbb{A}\mathbb{A}w \in \mathsf{RT}(P)$ must stem from P_τ, and this also gives $\Sigma\Sigma' w \in \mathsf{RT}(P_\tau)$ by [JV99, Prop. 4.2.1]. ∎ 2.14

Lemma 15

Let P be a saturated u-nf.

1. If $P \leq \underline{\tau}.\mathbf{0}$, then $P \leq_e \underline{\tau}.\mathbf{0}$.
2. If $\underline{\tau}.\mathbf{0} \leq P$, then $\underline{\tau}.\mathbf{0} \leq_e P$.

Proof:

1. We perform induction on $|P|$ and assume the property to hold for all saturated u-nf with depth $< |P|$. We have $P \not\equiv \mathbf{0}$ due to instability and we only have to consider the case $P \not\equiv \underline{\tau}.\mathbf{0}$; furthermore, $P \leq \underline{\tau}.\mathbf{0}$ implies $\mathsf{RT}(P) \subseteq (2^{\mathbb{A}})^*$, hence $\mathcal{A}(P) \subseteq \{\tau\}$; thus by Definition 9.1 we can assume $P \mathrel{\hat{\equiv}} P_s + P_\tau$ where $P_s \equiv \mathbf{0}$, thus $P \equiv \underline{\tau}.\tau.Q$ with saturated u-nf Q. Now also $\mathsf{RT}(Q) \subseteq (2^{\mathbb{A}})^*$, hence either $Q \equiv \mathbf{0}$ or $Q \leq_e \underline{\tau}.\mathbf{0}$ by induction, since $|Q| = |P| - 1$. In the first case we are done by $D00$ alone, in the latter case by $D10$ and then $D00$.

2. We again perform induction on $|P|$. Again, $P \not\equiv \mathbf{0}$, and the case $P \equiv \underline{\tau}.\mathbf{0}$ is clear. Otherwise, $\mathsf{RT}(\underline{\tau}.\mathbf{0}) = (2^{\mathbb{A}})^* \subseteq \mathsf{RT}(P)$. Since $P_i \not\equiv \mathbf{0}$ and hence $\mathbb{A}\mathbb{A} \notin \mathsf{RT}(P_i)$ for all $i \in I_s$, we have $P_\tau \equiv \underline{\tau}.([P^s_\tau +]\tau.Q)$; whenever P performs some $\mathbb{A}\mathbb{A}w$, this must involve a transition to Q, thus $\mathsf{RT}(\underline{\tau}.\mathbf{0}) \subseteq \mathsf{RT}(Q)$. If $Q \equiv \mathbf{0}$, then $\underline{\tau}.\mathbf{0} \leq_e \underline{\tau}.\tau.\mathbf{0}$ by $D00$; otherwise, Q is an unstable u-nf, $\underline{\tau}.\mathbf{0} \leq_e Q$ by induction and hence $\underline{\tau}.\mathbf{0} \leq_e \underline{\tau}.\tau.\underline{\tau}.\mathbf{0} \leq_e \underline{\tau}.\tau.Q$ by $D00$ and $D10$. In any case, we get $\underline{\tau}.\mathbf{0} \leq_e P$ by $A13$ and $A12$. ∎ 2.15

Lemma 16

Let $P \mathrel{\hat{\equiv}} \sum_{I_s} \underline{\tau}.P_i + P_\tau$ and $Q \mathrel{\hat{\equiv}} \sum_{J_s} \underline{\tau}.Q_j + Q_\tau$ be saturated u-nf with $P \leq Q$. Then

1. $P \equiv \underline{\tau}.\mathbf{0}$ or, for each $i \in I_s$, there is some $j \in J_s$ with $P_i \leq Q_j$, and
2. if $P_\tau \not\equiv \mathbf{0}$, then either $P \leq_e Q \equiv \underline{\tau}.\mathbf{0}$, or $Q_\tau \not\equiv \mathbf{0}$ and $P_\tau \leq Q_\tau$.

Proof:

1. Let $P \not\equiv \underline{\tau}.\mathbf{0}$; it suffices to show that for each $i \in I_s$ there is some $j \in J_s$ with $\mathsf{RT}(P_i) \subseteq \mathsf{RT}(Q_j)$. If $P \equiv \mathbf{0}$ we are done vacuously. Otherwise, $A_i \neq \emptyset$ for $i \in I_s$, and by Lemma 14.2 there is some $j \in J_s$ with $\mathcal{A}(P_i) = A_i = \mathcal{A}(Q_j)$. 14.3 shows that each $w \in \mathsf{RT}(P_i)$ that starts with an action is in $\mathsf{RT}(Q_j)$ and now we are done with 14.4.

2. Let $P_\tau \not\equiv \mathbf{0}$, i.e. $Q \not\equiv \mathbf{0}$ by instability; now $\mathbb{A}\mathbb{A} \in \mathsf{RT}(P) \subseteq \mathsf{RT}(Q)$, hence either $Q \equiv \underline{\tau}.\mathbf{0}$ and we are done by Lemma 15.1 or $Q_\tau \not\equiv \mathbf{0}$ by Definition 9. In the latter case, take some $w \in \mathsf{RT}(P_\tau) \subseteq \mathsf{RT}(Q)$; if $w = aw'$, then $w \in \mathsf{RT}(Q_\tau)$ by 9.2.ii) applied to Q, and the same applies for $w = \Sigma aw'$.

Finally, let $w = \Sigma\Sigma' w' \in \mathsf{RT}(P_\tau)$; then $\mathbb{A}\mathbb{A}w' \in \mathsf{RT}(P_\tau) \subseteq \mathsf{RT}(Q)$ by 14.5, and this gives $\mathbb{A}\mathbb{A}w', \Sigma\Sigma' w' \in \mathsf{RT}(Q_\tau)$ again by 14.5. We conclude $\mathsf{RT}(P_\tau) \subseteq \mathsf{RT}(Q_\tau)$, hence $P_\tau \leq Q_\tau$ due to instability and urgency. ■ 2.16

Theorem 17

Let P and Q be processes with $P \leq Q$. Then $P \leq_e Q$.

Proof:

We perform induction on $|P+Q|$. By Propositions 6, 8 and 10 we may assume P and Q to be in saturated normal form; it is important that these transformations do not increase the depth in order to apply induction. $P \leq Q$ implies that either both are i-nf or both are u-nf.

First let P and Q be saturated i-nf. If P and Q are stable, then $\underline{\tau}.P_a \leq \underline{\tau}.Q_a$ for all $a \in \mathcal{A} = \mathcal{B}$ by Lemma 12.1 and .2, hence $\underline{\tau}.P_a \leq_e \underline{\tau}.Q_a$ by induction; thus, $a.P_a \leq_e a.Q_a$ by $D10$ for all $a \in \mathcal{A} = \mathcal{B}$, and we can directly conclude $P \leq_e Q$. If P and Q are unstable, then $a.P_a \leq_e a.Q_a$ for all $a \in \mathcal{A} \subseteq \mathcal{B}$ as above using 12.1, and since $\tau \in \mathcal{A} \cap \mathcal{B}$ we can conclude $P \leq_e Q$ with $A13$.

Now let P and Q be saturated u-nf. If $P \equiv \mathbf{0}$, then $Q \equiv \mathbf{0}$ due to stability; if $P \equiv \underline{\tau}.\mathbf{0}$, we are done by Lemma 15.2. Otherwise, for each $i \in I_s$, there is some $j \in J_s$ with $P_i \leq Q_j$ by Lemma 16.1, hence $P_i \leq_e Q_j$ by the above or induction, since $|P_i + Q_j| \leq |P+Q|$. If $P_\tau \equiv \mathbf{0}$, we have $P \leq_e Q$ by $A12$; if $P_\tau \not\equiv \mathbf{0}$, then by 16.2 we have $Q \equiv \underline{\tau}.\mathbf{0}$ and are done or $Q_\tau \not\equiv \mathbf{0}$ and $P_\tau \leq Q_\tau$. For $P_\tau \equiv \underline{\tau}.P'_\tau$ and $Q_\tau \equiv \underline{\tau}.Q'_\tau$, we have $\mathsf{RT}(P'_\tau) \subseteq \mathsf{RT}(Q'_\tau)$ and P'_τ and Q'_τ are initial and unstable. Thus, $P'_\tau \leq Q'_\tau$ and $P'_\tau \leq_e Q'_\tau$ again by the above or induction, and we are done by $A12$. ■ 2.17

Corollary 18

The axioms $A00$ – $A33$ are sound and complete for $\leq$ on the (prefix,choice)-fragment of PAFAS as defined in this section. ■ 2.18

Since general relabelling $[\Phi]$ as defined in [JV98, JV99] (comprising ordinary relabelling and hiding) clearly distributes over choice and satisfies $(a.P)[\Phi] = \Phi(a).P[\Phi]$ and $(\underline{\tau}.P)[\Phi] = \underline{\tau}.P[\Phi]$, this result can be extended to this operation. Besides recursion, it remains to treat parallel composition; here it seems that a complete axiomatization requires a significant extension of the algebra by a variant of the then-operator of [HR95]; in contrast to this then-operator, it seems that one actually needs a family of operators indexed by the possible refusal sets.

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