The Initial Value Problem for Cohomogeneity One Einstein Metrics

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Introduction

A Riemannian manifold (M, g) is *Einstein* if its Ricci tensor satisfies $\mathrm{Ric}(g) = \lambda \cdot g$ for some constant λ. The terminology results from the fact that if (M, g) is a Lorentz 4-manifold, then the Einstein condition is precisely Einstein's field equation in vacuo. In this paper, we are interested in the situation where (M, g) admits a compact Lie group action by isometries with *cohomogeneity* one, i.e., an action whose principal orbits are hypersurfaces in M.

In the study of Einstein metrics, cohomogeneity one examples are of particular interest because the regular part of (M, g) corresponds in a natural way to a spatially homogeneous Lorentz Einstein manifold. Indeed, the first non-Kähler inhomogeneous compact Riemannian Einstein manifold was constructed by the physicist D. Page from the Taub–NUT solution [18]. His construction was generalized by Berard–Bergery in the unpublished preprint [3] and later independently by Page and Pope [19]. Subsequently, many authors have studied Einstein metrics of cohomogeneity one in various *specific* instances. While these works are too numerous to list, we mention especially the work of Sakane, for he constructed in [22] the first non-homogeneous examples of Kähler–Einstein

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metrics with positive anti-canonical class. These examples were later generalized by him and Koiso in [15] and [16].

Cohomogeneity one Einstein metrics are interesting from another point of view. As a partial differential equation, the Einstein equation is a very complicated non-linear system whose gauge group is the group of all diffeomorphisms of the manifold. A standard strategy in dealing with non-linear scalar partial differential equations is to study the rotationally symmetric case as a preliminary step. For the Einstein equation, the analogous step is the study of cohomogeneity one metrics, which includes the special case where the metric is rotationally symmetric about a point in the manifold. We shall see that having a singular orbit (i.e., one whose dimension is less than that of the principal orbits) which does not reduce to a point necessitates additional considerations and gives rise to new phenomena not encountered when the metric is rotationally symmetric.

The object of this paper is to undertake a general study of the problem of the local existence and uniqueness of a smooth invariant Einstein metric (of *any* sign of the Einstein constant) in the neighborhood of a singular orbit of the group action. If a compact Lie group G acts with cohomogeneity one on a manifold M with a singular orbit Q, then $Q = G/H$ and a tubular neighborhood of Q is equivariantly diffeomorphic to an open disk bundle of the normal bundle $E = G\times_H V$, where V is a normal slice of Q in M on which H acts orthogonally via the slice representation. The principal orbits are then the hypersurfaces which are the sphere bundles over Q of (small) positive radii. As the radius tends to zero, these principal orbits collapse onto the singular orbit in a smooth way. This basic geometry allows one to reformulate the Einstein condition in the tubular neighborhood as an initial value problem in which the initial data consist of a given invariant metric and a prescribed shape operator on Q. Our main result is the following.

Theorem. *Let G be a compact Lie group, H a closed subgroup with an orthogonal linear action on $V = \mathbb{R}^{k+1}$ which is transitive on the unit sphere S^k, and $E = G \times_H V$ be the vector bundle over $Q = G/H$ with fiber V. Denote by $\mathfrak{p}_-$ an ad-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Let $v_0 \in S^k$ have isotropy group K. Assume that as K-representations, V and $\mathfrak{p}_-$ have no irreducible sub-representations in common. Then, given any G-invariant metric g_Q on Q and any G-equivariant homomorphism $L_1 : E \longrightarrow S^2(T^*Q)$, there exists a G-invariant Einstein metric on some open disk bundle E' of E with any prescribed sign (positive, zero, or negative) of the Einstein constant.*

Some remarks about the theorem are in order. First, the assumption about the irreducible summands in V and $\mathfrak{p}_-$ may appear strange, but we shall see in Remark 2.7 that it is quite natural in the context of the Kaluza–Klein construction. Second, note that the initial metric and shape operator are actually not sufficient to ensure uniqueness of the local solution. Because of the requirement of smoothness, the distance spheres in V must become round to first order as the distance tends to 0. Therefore, one expects to be able to prescribe the initial second derivatives of the V-part of the metric in the initial value problem. However, we will find out in Section 5 that when the singular orbit G/H does not reduce to a point, still more initial conditions need to be prescribed before the solution of the initial value problem is unique. The number of parameters of local solutions with a given initial metric and shape operator can be computed explicitly using representation theory. In particular, in Section 5.5 we give an example of a sequence of manifolds (of increasing dimension) for which, in order to specify a solution of the initial value problem, one needs to prescribe higher and higher initial derivatives of the $\mathfrak{p}_-$-part of the metric.

Generally speaking, the Einstein condition for metrics of cohomogeneity one reduces to a non-linear second order system of ordinary differential equations. We include a derivation of this well-known reduction in Section 2. Our version is actually a reduction to a first order system that emphasizes the role of the shape operator of the orbits. For example, the existence of a smooth metric on E implies that Q must be a minimal submanifold. We also discuss an important observation of Back (Lemma 2.4) regarding the role of the second Bianchi identity (i.e., the infinitesimal action of the diffeomorphism group). It implies that when there is a singular orbit, any *smooth* invariant metric which satisfies the Einstein system in the principal orbit directions automatically satisfies the whole system.

The Einstein system is also complicated by the boundary conditions which are necessary to ensure that a solution actually corresponds to a smooth metric on M. While it is not difficult in specific situations to formulate the appropriate boundary conditions, it is not entirely trivial to do so in complete generality and in an *explicit* form. In Section 1, we give a necessary and sufficient condition for an invariant metric on $E \setminus Q$ to extend to a smooth metric on E. But, as can be seen from the statement of Lemma 1.1, a more explicit condition requires computations involving representation theory, which in turn requires more specific knowledge of the triple (G, H, K). Much of the difficulty in analyzing the Einstein system in generality stems from these "non-specific" boundary conditions, which make ordinary analytic techniques (such as the variational approach) hard to apply.

Another source of difficulty in treating the Einstein system in generality is the complexity of the expression of the Ricci tensor of a homogeneous space. Without specifying (G, H, K) explicitly, one must nevertheless extract enough geometrical information from the general expression of the Ricci tensor for use in solving the Einstein system. In the initial value problem, as the regular orbits collapse to the singular orbit, the Ricci tensor and shape operator blow up in the collapsing directions. In terms of the Einstein system, this is reflected in the singular point at the origin (which corresponds to the singular orbit). From a purely analytic point of view, one therefore cannot expect to have formal *power* series solutions, even when the system is linear.

We shall solve the initial value problem by the method of "asymptotic series." The first step is to show that there is in fact a formal power series solution. After that, one takes a high order truncation of the formal solution and applies Picard's iteration to it to get a real solution. Finally, one must ensure that the real solution actually gives a *smooth* Einstein metric. The bulk of the work of this paper is in establishing the first step and will be presented in Sections 4 and 5. Through studying the limiting behavior of the Ricci tensors of one-parameter families of homogeneous metrics satisfying the appropriate initial conditions, we show that a formal power series solution always exists, even though the linear operator ($\mathcal{L}_m$ in Section 5) in the recursion formula for the Taylor coefficients of the solution is not surjective in general. This is possible because the requirement for smoothness, when set up properly, ensures that we stay in the range of the operator $\mathcal{L}_m$. The non-uniqueness comes from the kernel of this operator.

We hope that an understanding of the initial value problem will eventually contribute towards the understanding of the general problem of constructing complete and compact cohomogeneity one solutions. We take the opportunity here to mention two very recent developments on this front. First, Christoph Böhm, in his Augsburg thesis [5] and [6], gave a *rigorous* construction of a cohomogeneity one Einstein metric on $\mathbb{H}P^2 \sharp (-\mathbb{H}P^2)$ and other low dimensional spaces. A numerical construction for such a metric was given in [20]. In a different direction, Jun Wang, in his Ph.D. thesis, has extended the work of Koiso and Sakane by showing that even when the existence of Kähler–Einstein metrics is obstructed, there exist Riemannian Einstein metrics on a large family of associated 2-sphere bundles over products of Kähler–Einstein manifolds with positive anti-canonical class.

1. Smoothness of tensor fields along the singular orbit

Let M be a connected (n+1)-dimensional manifold on which a compact Lie group G acts smoothly with *cohomogeneity one,* i.e., the codimension of the principal orbits is one. Let $Q = G \cdot q$

be a singular orbit with isotropy group $H = G_q$, and let $V = T_qM/T_qQ$ be its "normal space" at q on which H acts linearly with cohomogeneity one. The normal bundle of Q is $E = G\cdot V = G\times_H V$, and we may identify a neighborhood of Q in M with the total space of E. The principal orbits are $P = G\cdot v_0$ for any nonzero $v_0 \in V$; they are tubes around Q and can be identified with the sphere bundle of E, where we have chosen an H-invariant scalar product $\langle\ ,\ \rangle$ on V.

Abstractly, let G be a compact Lie group and H a closed subgroup which has an orthogonal representation (not necessarily faithful) on $V = \mathbb{R}^{k+1}$ with cohomogeneity one, i.e., it acts transitively on the unit sphere $S^k \subset V$. Let $E = G\times_H V$; this is a vector bundle over $Q = G/H$. We regard V as the fiber of E over $q = eH \in Q$ by the embedding

$$V \to E,\quad v \mapsto (e, v)\cdot H\ ,$$

where $e \in G$ denotes the unit element. The unit sphere bundle of E is $P = G\times_H S^k = G\cdot v_0 = G/K$ for some fixed $v_0 \in V$ with isotropy group $K = G_{v_0} = H_{v_0}$.

Let $a \in C^\infty(S^2TE)$ be a smooth G-invariant field of symmetric bilinear forms (quadratic forms) on the manifold E. Since $E = G\cdot V$, the tensor field a is determined by any H-invariant $a \in C^\infty(S^2(TE)|V)$. An $Ad(H)$-invariant (reductive) complement $\mathfrak{p}_-$ of $\mathfrak{h}$ in $\mathfrak{g}$ defines a G-invariant distribution $\mathcal{H}_g = dL_g\cdot\mathfrak{p}_-$ on G, i.e., a G-invariant connection on the principal bundle $G \to G/H = Q$, which in turn defines a G-invariant connection on the associated vector bundle E. So we obtain a G-invariant decomposition of TE into vertical and horizontal subbundles. These are canonically isomorphic to π^*E and π^*TQ, respectively, where $\pi : E \to Q$ is the projection. Thus,

$$TE = \pi^*E \oplus \pi^*TQ.$$

Over V, these pull-back bundles are trivial, i.e., we get an H-invariant trivialization

$$TE\,|V = V\times\left(V\oplus\mathfrak{p}_-\right)\ .$$

Hence, a G-invariant quadratic form is nothing but an H-equivariant smooth mapping

$$a : V \to S^2\left(V\oplus\mathfrak{p}_-\right)\ .$$

Since H acts transitively on the unit sphere $S^k \subset V$, it is convenient to describe the function a in polar coordinates

$$\phi : \mathbb{R}_+\times S^k \to V,\quad \phi(t, v) = \phi_t(v) = tv\ .$$

Then we obtain a 1-parameter family of H-equivariant mappings

$$a_t = a\circ\phi_t : S^k \to S^2\left(V\oplus\mathfrak{p}_-\right)$$

for $t \in \mathbb{R}_+ = [0,\infty)$. By equivariance, a_t is determined by the single value

$$a_t\left(v_0\right) \in S^2\left(V\oplus\mathfrak{p}_-\right)$$

for some arbitrary fixed $v_0 \in S^k$; this value $a_t(v_0)$ must be invariant under the isotropy group $K = H_{v_0}$, but is otherwise arbitrary.

Conversely, if H-equivariant maps $a_t : S^k \to S^2(V\oplus\mathfrak{p}_-)$, or equivalently K-invariant symmetric 2-tensor $a_t(v_0) \in S^2(V\oplus\mathfrak{p}_-)^K$, are given for all $t \in \mathbb{R}_+$, what is the condition for the map

$$a : V\setminus\{0\} \to S^2\left(V\oplus\mathfrak{p}_-\right),\quad a(v) = a_{|v|}\left(\frac{v}{|v|}\right) \tag{1.1}$$

to have a smooth extension to the origin 0?

To answer this question, it is useful to introduce the vector space W of all smooth H-equivariant maps $L : S^k \to S^2(V \oplus \mathfrak{p}_-)$. Since H acts transitively on S^k, the evaluation map

$$\epsilon : W \to S^2\left(V \oplus \mathfrak{p}_-\right)^K , \quad \epsilon(L) = L\left(v_0\right)$$

is a linear isomorphism. In particular, W is finite-dimensional. Let $W_m \subset W$ be the subspace of all maps L which are restrictions to S^k of H-equivariant homogeneous polynomials $L : V \to S^2(V \oplus \mathfrak{p}_-)$ of degree m, and let $W^m = \sum_{p=0}^m W_p$. In fact, we have $W^m = W_m + W_{m-1}$ since we may alter the degree of any homogeneous polynomial by an even number without changing its restriction to S^k, just by multiplying it with powers of the H-invariant polynomial $v \mapsto \langle v, v\rangle$. Since by polynomial approximation $\cup_m W^m$ is dense in W, and W is finite-dimensional, we have $W = W^{m_0}$ for some positive integer m_0.

Lemma 1.1. *Let $t \mapsto a_t : \mathbb{R}_+ \to S^2(V \oplus \mathfrak{p}_-)^K$ be a smooth curve (i.e., at zero, the right-hand derivatives of all orders exist and are continuous from the right) with Taylor expansion at zero $a_t \sim \sum_p a_p t^p$. Then the map a on $V \setminus \{0\}$ defined by (1.1) has a smooth extension to 0 if and only if $a_p \in \epsilon(W_p)$ for all $p \geq 0$.*

Proof. If $a : V \to S^2(V \oplus \mathfrak{p}_-)$ is a smooth H-equivariant map, then a has a Taylor expansion $a \sim \sum_p L_p$ where $L_p : V \to S^2(V \oplus \mathfrak{p}_-)$ is a homogeneous H-equivariant polynomial of degree p. So the Taylor expansion of $t \mapsto a_t(v_0)$ is $\sum_p L_p(v_0)t^p$ as claimed.

Conversely, let $t \mapsto a_t \sim \sum_p a_p t^p : \mathbb{R}_+ \to S^2(V \oplus \mathfrak{p}_-)^K$ be a smooth map with Taylor coefficients $a_p = \epsilon(L_p) \in \epsilon(W_p)$. Let $L^1, \ldots , L^N$ be a basis of the vector space W, adapted to the filtration

$$W^0 \subset W^1 \subset \ldots \subset W^{m_0} = W^{m_0+1} = \ldots = W .$$

In other words, a linear combination of the L^i lies in $W^m \setminus W^{m-1}$ iff all L^i with nonzero coefficients lie in W^m and at least one of them does not lie in W^{m-1}. This means that no linear combination of the L^i as a polynomial is divisible by r^2 where $r : V \to \mathbb{R},\ r(v) = |v|$. We have

$$a_t = \sum_{i=1}^N f_i(t) L^i\left(v_0\right)$$

for smooth functions $f_i : \mathbb{R}_+ \to \mathbb{R}$. Let

$$f_i \sim \sum_p \alpha_{ip} t^p$$

be the corresponding Taylor expansion. Then

$$a_t \sim \sum_p \sum_{i=1}^N t^p \alpha_{ip} L^i\left(v_0\right) .$$

On the other hand, by assumption we have $a_t \sim \sum_p t^p L_p(v_0)$, hence

$$L_p\big|_{S^k} = \sum_{i=1}^N \alpha_{ip}\, L^i\Big|_{S^k} .$$

By homogeneity, we obtain on V the identity

$$L_p = \sum_{i=1}^{N} \alpha_{ip} r^{p-m_i} L^i \tag{$*$}$$

where m_i is the degree of L^i.

Now we claim that all the powers $p - m_i$ for nonzero α_{ip} are even and nonnegative. In fact, since the polynomial L_p contains only an even power of r as factor, all powers of r in $(*)$ must be even. Otherwise, we could rewrite $(*)$ in the form $A = r \cdot B$ for some polynomials A and B, but this is impossible (the left-hand side is C^∞ while the right-hand side is not). Furthermore, if some of the powers $p - m_i$ for nonzero α_{ip} were negative, dividing through by the highest negative power of r would give an identity of the form

$$r^{2k_0} L_p = \sum_i{}' \alpha_{ip} r^{2k_i} L^i + \sum_j{}'' \alpha_{jp} L^j ,$$

where k_0, k_i are positive and the second sum is nonzero. But then $\sum_j'' \alpha_{jp} L^j$ would be divisible by r^2, which was excluded by the choice of the L^i. This proves the claim.

Thus, from the Taylor expansion of f_i we get

$$f_i(t) = t^{m_i} \cdot g_i\left(t^2\right)$$

for some smooth function $g_i : \mathbb{R}_+ \to \mathbb{R}$. Hence,

$$a(v) = \sum_{i=1}^{N} g_i\left(|v|^2\right) \cdot L^i(v)$$

is smooth. □

Remark. There is obviously a C^k version of the above lemma. Also, all the above discussion holds if $S^2(V \oplus \mathfrak{p}_-)$ is replaced by End $(V \oplus \mathfrak{p}_-)$ or indeed by any finite-dimensional H-module.

We now make the following technical assumption:

(A) The representations of $K = G_{v_0}$ on $\mathfrak{p}_-$ and V have no equivalent irreducible factors.

In this case, we have a splitting

$$S^2\left(V \oplus \mathfrak{p}_-\right)^K = S^2(V)^K \oplus S^2\left(\mathfrak{p}_-\right)^K$$

and each W_m splits accordingly as $W_m^+ \oplus W_m^-$ where the polynomials in W_m^+ (resp., W_m^-) take values in $S^2(V)$ (resp., $S^2(\mathfrak{p}_-)$).

Remark. Assumption (A) is convenient for the following computations, but presumably it is not really necessary. On the other hand, we shall see in the next section that it is a natural assumption in the context of the Kaluza–Klein construction for Einstein metrics. Anyway, there are many cases in which it is satisfied. For example, (A) holds if H and K are locally products $H_1 \times H_2$ and $K_1 \times H_2$, respectively, such that $H/K = S^k$ and H_2 acts non-trivially on all irreducible summands in $\mathfrak{p}_-$. In many other cases, we can satisfy assumption (A) by passing to a finite extension of the group G. For example, let $Q = G/H$ be a symmetric space and $\sigma \in \mathrm{Aut}(G)$ the corresponding involution fixing H. Let $\hat{G}$ be the two-fold extension of G by σ. If we let σ act trivially on V, then we get a $\hat{G}$-action on E, and the new group element σ fixes the fiber $V \subset E$, so it lies in $\hat{K} = \hat{G}_{v_0}$ and it acts as id on V and as $-id$ on $\mathfrak{p}_-$. Thus, the representations of $\hat{K}$ on V and $\mathfrak{p}_-$ have no common factors.

Since V splits under the K-action as $V = \mathfrak{p}_+ \oplus \mathbb{R}v_0$ where $\mathfrak{p}_+ = v_0^\perp$, we obtain a K-invariant decomposition

$$S^2(V)^K \cong S^2\left(\mathfrak{p}_+\right)^K \oplus \mathfrak{p}_+^K \oplus \mathbb{R}\,. \tag{1.2}$$

In fact, any $a \in S^2(V)^K$ splits as $a = a_1 + a_2 + a_3$ with

$$a_2(x, y) = \langle x, v_0\rangle\, a(v_0, y) + \langle y, v_0\rangle\, a(v_0, x), \quad a_3(x, y) = \langle x, v_0\rangle\, \langle y, v_0\rangle\, a(v_0, v_0)\,.$$

Hence a_2 is determined by the K-invariant linear form $x \mapsto a(v_0, x)$ (or its dual vector) and a_3 by the constant $a(v_0, v_0)$. For any $v \in V$, we will denote by v^T the dual 1-form $\langle v, .\rangle$, and by $(v^T)^2$ the quadratic form $x \mapsto \langle v, x\rangle^2$ and its corresponding symmetric bilinear form.

Lemma 1.2. *Let $L \in W_m^+$ nonzero and $a = \epsilon(L)$. Then m is even and $a_i = \epsilon(L_i)$ for some $L_i \in W_2^+$, for $i = 1, 2, 3$.*

Proof. We extend $a_1 \in S^2(\mathfrak{p}_+)^K$ to an H-invariant symmetric 2-tensor field on the sphere S^k. We have to show that this is the restriction to S^k of a quadratic polynomial $L_1 \in W_2^+$. By examining the different transitive linear groups H on S^k, Ziller has described the space of H-invariant symmetric 2-tensors on S^k [25]. One may reformulate his description as follows: besides multiples of the constant curvature metric, which corresponds to the polynomial $v \mapsto (v^T)^2$, all other H-invariant symmetric 2-tensors are generated by projections along H-invariant Hopf fibrations associated to H-invariant complex or quaternionic or Cayley structures. More precisely, when there is an H-invariant complex structure J on V, we can form the quadratic polynomial $v \mapsto ((Jv)^T)^2$ which induces an H-invariant symmetric 2-tensor on S^k. Letting J vary over all H-invariant complex structures and taking linear combinations of the resulting polynomials, we obtain all the H-invariant symmetric 2-tensors with one exception: the case of $S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7)$. In this case, the Cayley division algebra gives rise to a Clifford family $J_1, \ldots, J_7$ of anticommuting complex structures on $\mathbb{R}^8$. This determines a Clifford system $C = \mathrm{Span}\,\{P_0, \ldots, P_8\}$ of anticommuting symmetric endomorphisms of order 2 (reflections) on $\mathbb{R}^{16}$ with the property that any $v \in \mathbb{R}^{16}$ is element of $Cv = \mathrm{Span}\,\{P_0v, \ldots, P_8v\}$; in fact

$$P_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_8 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad P_i = \begin{pmatrix} 0 & -J_i \\ J_i & 0 \end{pmatrix}$$

for $i = 1, \ldots, 7$. For any $v \in S^{15}$ there is exactly one $P_v \in C$ with $P_v v = v$, namely $P_v = \sum_{i=0}^{8} \langle P_i v, v\rangle\, P_i$ (note that $P_0v, \ldots, P_8v$ is an orthonormal basis of Cv), and the 8-dimensional fixed space E_v of P_v is spanned by v and the tangent space of the Hopf fiber through v. Thus, the corresponding symmetric tensor field on S^{15} is given by the projection on E_v which is $\frac{1}{2}(P_v + I) = \frac{1}{2}(\sum_{i=0}^{8} \langle P_i v, v\rangle\, P_i + \langle v, v\rangle\, I) \in W_2^+$.

Further, a_2 determines an H-invariant vector field on S^k which is of type $X(v) = \alpha J v$ for some $\alpha \in \mathbb{R}$ and some H-invariant complex structure J on V. Hence $a_2 = \epsilon(L_2)$ with $L_2(v) = \alpha v^T (Jv)^T$. Moreover, $a_3 = \epsilon(L_3)$ with $L_3(v) = \beta(v^T)^2$ for some $\beta \in \mathbb{R}$. This finishes the proof. □

Now let M^{n+1} be as in the beginning of this section and let $\hat{g}$ be a G-invariant Riemannian metric on M. We will now identify E with the *metric* normal bundle νQ of the singular orbit Q. The metric $\hat{g}$ induces inner products on $\mathfrak{p}_-$, which can be identified with $T_q Q$, and on $V = \nu_q Q$.

These two inner products define a G-invariant "background" Riemannian metric $\hat{g}_0 = \langle\ ,\ \rangle$ on E (the connection metric for the connection on the principal bundle $G \to G/H$ given by the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-$): just take this inner product on the fiber of $TE|V = V \times (V \oplus \mathfrak{p}_-)$ and extend it by G to all of TE. We will identify T^*E and TE using $\hat{g}_0$, hence we also identify bilinear forms with endomorphism fields on E.

In particular, the metric $\hat{g}$ on a tubular neighborhood $M' = B_r(Q) \subset M$ is transplanted to $E' = \{v \in E;\ \|v\| < r\}$ by means of the normal exponential map $\exp|\nu Q$ of the metric $\hat{g}_0$. By the above lemma, $\hat{g}$ is given by a smooth curve

$$x = (\tilde{x}_+, x_-) : [0, r) \to S^2(V)^K \oplus S^2\left(\mathfrak{p}_-\right)^K$$

and the initial values are given by

$$\tilde{x}_+(0) = I_+, \quad x_-(0) = I_- \,. \tag{1.3}$$

Let us choose a reductive decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_+$, so that we may identify $T_{v_0}(S^k) = {v_0}^\perp$ with $\mathfrak{p}_+$. By construction, $\tilde{x}_+(t)v_0 = v_0$ and $\tilde{x}_+$ maps $\mathfrak{p}_+$ into itself for all $t \in [0, r)$. Hence, in the Taylor expansion $x(t) \sim \sum_p x_p t^p$, we have $v_0 \in \ker x_p$ for all $p \geq 1$. Furthermore, since $x_p = L_p(v_0)$ for some H-equivariant homogeneous polynomial $L_p : V \to S^2(V \oplus \mathfrak{p}_-)$ of degree p, we have $L_p(v)v = 0$ for all $v \in V$. In particular, for $p = 1$, it follows that the tri-linear map $(v, w, x) \mapsto \langle L_1(v)w, x\rangle$ on $V \times V \times V$ is anti-symmetric in the first two arguments and symmetric in the last two arguments; hence, it is zero. Thus, $(L_1)_+ = 0$, i.e., $(x_1)_+ = 0$, which is a well-known fact for exponential coordinates. On the other hand, $L_1 = (L_1)_-$ is the shape (Weingarten) operator of the singular orbit Q at q, and since the linear function $v \mapsto \operatorname{tr}(L_1(v))$ on V is H-invariant, it must vanish, i.e., Q is minimal. Thus, we obtain the initial derivatives

$$(\tilde{x}_+)'(0) = 0, \quad (x_-)'(0) = L_1(v_0) \tag{1.4}$$

with

$$\operatorname{tr}(x_-)'(0) = 0 \,. \tag{1.5}$$

Alternatively, the metric $\hat{g}$ can be described in "cylindrical coordinates" around Q by the map

$$\phi : [0, r) \times P \to E', \quad \phi(t, v) = tv \,,$$

where P is the unit normal bundle of Q. Then we have

$$\phi^*\left(\hat{g}\right) = dt^2 + g(t) \tag{1.6}$$

on $(0, r) \times P$, where $g(t)$ is the G-invariant metric on P given by

$$g(t) = t^2 x_+(t) \oplus x_-(t) \,,$$

and we have $\tilde{x}_+(t) \mid {v_0}^\perp = x_+(t)$.

2. The Einstein condition for metrics of cohomogeneity one

Let M be any manifold of dimension $n + 1$ with Riemannian metric $\hat{g} = \langle\ ,\ \rangle$. Denote by $\hat{\nabla}$ the Levi–Civita connection and by $\hat{R}$ the curvature tensor of $\hat{g}$. We consider an equi-distant hypersurface family in M, i.e., a diffeomorphism $\phi : I \times P \to M_0$ where $I \subset \mathbb{R}$ is some open interval, P an n-dimensional manifold, and $M_0 \subset M$ an open subset such that

$$\phi^*\left(\hat{g}\right) = dt^2 + g(t)$$

where $g(t)$ is a family of Riemannian metrics on P. We will think of $g(t)$ as a one-parameter family of isomorphisms $TP \to T^*P$. Let N be the unit normal field of the hypersurface family, i.e., $N = d\phi(\frac{\partial}{\partial t})$, which commutes with any vector field on M_0 induced by a vector field on P via $d\phi$.

Next, let $L(t)$ be the shape operator of the hypersurface $P_t = \phi(\{t\} \times P)$ given by

$$L(t)X = \hat{\nabla}_X N$$

for any $X \in TP_t$. We will consider $L(t)$ as a one-parameter family of endomorphisms on TP via the diffeomorphism ϕ. Then we have on TP

$$g' = 2g \circ L\,, \tag{2.1}$$

where $' = \frac{d}{dt}$ is the time derivative and $\circ$ denotes composition. We claim that $\hat{\nabla}_N L$ corresponds to L' under $d\phi$. In fact, for vector fields X, Y on P, considered also as vector fields on M_0 via $d\phi$, using covariant derivatives, we obtain

$$g(LX, Y)' = \hat{g}\left(\left(\hat{\nabla}_N L\right) X, Y\right) + 2g\left(L^2 X, Y\right)\,.$$

On the other hand, using the ordinary derivative and (2.1), we have

$$g(LX, Y)' = g'(LX, Y) + g\left(L'X, Y\right) = 2g\left(L^2X, Y\right) + g\left(L'X, Y\right)\,,$$

which proves our claim. Now, it is well known that L satisfies the Riccati equation $\nabla_N L + L^2 + \hat{R}_N = 0$, where $\hat{R}_N \cdot X = \hat{R}(X, N)N$ (cf [11]). Consequently, we have

$$L' + L^2 + \hat{R}_N = 0\,. \tag{2.2}$$

We will let R_t and Ric_t denote, respectively, the Riemann and Ricci tensors of the metrics $g(t)$. The Ricci endomorphism of TP, denoted by $r(t)$, is given by $g(r(X), Y) = \mathrm{Ric}(X, Y)$, where the t-dependence is suppressed; in other words, $r = g \circ \mathrm{Ric}$. Similarly, $\hat{r}$ and $\widehat{\mathrm{Ric}} = \hat{g} \circ \hat{r}$ will denote the Ricci endomorphism and the Ricci tensor of the metric $\hat{g}$ on M. In the following, we give a self-contained derivation of the Einstein condition for $\hat{g}$, which can also be found, in a slightly different notation, in [3] (4.2) or [2] p. 211. With an appropriate change of sign, the derivation also applies to Lorentzian metrics $\phi^*\hat{g} = -dt^2 + g(t)$.

First, we take the trace of (2.2). Notice that since endomorphisms of TP are studied, the trace does not depend on the metric. One obtains

$$\mathrm{tr}\left(L'\right) + \mathrm{tr}\left(L^2\right) + \widehat{\mathrm{Ric}}(N, N) = 0\,. \tag{2.3}$$

Using the Gauss equation for the hypersurface P_t and (2.2), we have, for $X, Y \in TP_t$ and an orthonormal basis $\{e_i\}$ of $\hat{g}$,

$$\begin{aligned} \widehat{\mathrm{Ric}}(X, Y) &= \sum_i \left\langle \hat{R}\left(X, e_i\right) e_i, Y\right\rangle + \left\langle \hat{R}_N \cdot X, Y\right\rangle \\ &= \mathrm{Ric}(X, Y) - \mathrm{tr}\,(L) g(LX, Y) - g(L'X, Y)\,. \end{aligned} \tag{2.4}$$

Furthermore, by the Codazzi equation,

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,N) &= \sum_i \left\langle \hat{R}\,(X,e_i)\,e_i, N\right\rangle \\ &= \sum_i \left\langle \left(\nabla_{e_i} L\right) X, e_i\right\rangle - \left\langle \left(\nabla_X L\right) e_i, e_i\right\rangle \\ &= -\operatorname{tr}\left(X \lrcorner\, d^{\nabla} L\right), \end{aligned} \tag{2.5}$$

where $d^{\nabla} L$ is the TP-valued 2-form on P which is the covariant exterior derivative of L regarded as TP-valued 1-form on P, and $\lrcorner$ denotes the interior product. Hence, in the case where $\hat{g}$ is an Einstein metric, i.e.,

$$\hat{r} = \lambda \cdot I, \quad \widehat{\mathrm{Ric}} = \lambda \cdot \hat{g}$$

for some fixed constant $\lambda \in \mathbb{R}$, we obtain:

Proposition 2.1. *Notation as above, the Einstein condition for the metric $\hat{g}$ on M_0 is given by*

$$g' = 2gL \tag{2.6a}$$

$$L' = -(\operatorname{tr} L)L + r - \lambda \cdot I \tag{2.6b}$$

$$\operatorname{tr} L' = -\operatorname{tr}\left(L^2\right) - \lambda \tag{2.6c}$$

$$\operatorname{tr}\left(X \lrcorner\, d^{\nabla} L\right) = 0 \tag{2.6d}$$

for all $X \in TP$.

Remark 2.2. If we take the trace of (2.6b) and use (2.6c), we obtain the equation

$$s - (\operatorname{tr} L)^2 + \operatorname{tr}\left(L^2\right) = (n-1)\lambda\,, \tag{2.7}$$

where $s(t) = \operatorname{tr} r(t)$ denotes the scalar curvature of $g(t)$. This equation can be regarded as a conservation law or first integral of the system (2.6a) through (2.6d).

Remark 2.3. Note that L as a shape operator must be symmetric with respect to the metric g, which does not immediately follow from (2.6a) through (2.6d). But later we will replace L by $\frac{1}{2}g'g^{-1}$ which will settle this problem.

Back [1] has made the following useful observation about the system (2.6a) through (2.6d). Since [1] has not appeared in print to the best of our knowledge, we include a proof.

Lemma 2.4 (cf. [1]). *Let $\hat{g} = dt^2 + g(t)$ be a metric on $I \times P$ where $g(t)$ is a one-parameter family of metrics on P satisfying (2.6a), (2.6b) for some constant λ. Suppose further that the scalar curvature $\hat{s}$ of $\hat{g}$ is constant along $\{t\} \times P$ for all $t \in I$. Let $v(t)$ denote the volume distortion of $g(t)$ with respect to some fixed background metric on P. Then $\widehat{\mathrm{Ric}}(X, N) \cdot v$ is constant in time for any $X \in TP$. Furthermore, if (2.6d) is satisfied, then $(\widehat{\mathrm{Ric}}(N, N) - \lambda)v^2$ is also constant in time.*

Proof. We will apply the contracted second Bianchi identity $\mathrm{Div}\widehat{\mathrm{Ric}} = \frac{1}{2}d\hat{s}$. Now for any $X \in TP$,

$$\begin{aligned}
\frac{1}{2}d\hat{s}(X) &= \sum_i \left(\hat{\nabla}_{e_i}\widehat{\mathrm{Ric}}\right)(e_i, X) + \left(\hat{\nabla}_N\widehat{\mathrm{Ric}}\right)(N, X) \\
&= \sum_i \left\{e_i\left(\widehat{\mathrm{Ric}}\,(e_i, X)\right) - \widehat{\mathrm{Ric}}\left(\hat{\nabla}_{e_i}e_i, X\right) - \widehat{\mathrm{Ric}}\left(e_i, \hat{\nabla}_{e_i}X\right)\right\} \\
&\quad + N\left(\widehat{\mathrm{Ric}}(N, X)\right) - \widehat{\mathrm{Ric}}\left(N, \hat{\nabla}_N X\right) \\
&= \lambda\left\{\sum_i \left(\nabla_{e_i}g\right)(e_i, X)\right\} + \mathrm{tr}\,(L)\widehat{\mathrm{Ric}}(N, X) + \widehat{\mathrm{Ric}}(LX, N) \\
&\quad + N\left(\widehat{\mathrm{Ric}}(N, X) - \widehat{\mathrm{Ric}}(N, LX)\right) \\
&= \mathrm{tr}\,(L)\widehat{\mathrm{Ric}}(N, X) + N\left(\widehat{\mathrm{Ric}}(N, X)\right) = 0
\end{aligned}$$

since $d\hat{s}(X) = 0$. But we also have $v'(t) = \mathrm{tr}\,(L)v(t)$. Hence, one has $\frac{\partial}{\partial t}(v \cdot \widehat{\mathrm{Ric}}(N, X)) = 0$, which is the first assertion.

Furthermore, since $\widehat{\mathrm{Ric}}(e_i, e_i)$ is constant, we get for the scalar curvature $\hat{s} = \mathrm{tr}_{\hat{g}}\,\widehat{\mathrm{Ric}}$:

$$\frac{1}{2}d\hat{s}(N) = \frac{1}{2}N\left(\widehat{\mathrm{Ric}}(N, N)\right) .$$

On the other hand, the second Bianchi identity gives

$$\frac{1}{2}d\hat{s}(N) = \sum_i \left(\hat{\nabla}_{e_i}\widehat{\mathrm{Ric}}\right)(e_i, N) + \left(\hat{\nabla}_N\widehat{\mathrm{Ric}}\right)(N, N) .$$

If (2.6d) is satisfied, i.e., $\widehat{\mathrm{Ric}}(X, N) = 0$ for all $X \in TP$, we get from these two equations

$$\begin{aligned}
\frac{1}{2}N\left(\widehat{\mathrm{Ric}}(N, N)\right) &= \sum_i \left\{\widehat{\mathrm{Ric}}\left(\hat{\nabla}_{e_i}e_i, N\right) + \widehat{\mathrm{Ric}}\,(e_i, Le_i)\right\} \\
&= \lambda \cdot \mathrm{tr}\,(L) - \mathrm{tr}\,(L)\widehat{\mathrm{Ric}}(N, N) .
\end{aligned}$$

This gives

$$-\frac{1}{2}N\left(\widehat{\mathrm{Ric}}(N, N) - \lambda\right) = \mathrm{tr}\,(L)\left(\widehat{\mathrm{Ric}}(N, N) - \lambda\right) ,$$

which integrates to give the second assertion (using $\mathrm{tr}\ L = v'/v$). □

Corollary 2.5. *Let G be a compact Lie group acting transitively on P. For every choice of a G-invariant metric g_0 on P and a g_0-symmetric G-equivariant endomorphism field L_0 of TP satisfying $\mathrm{tr}\,(X \lrcorner\, d^\nabla L_0) = 0$ for all $X \in TP$, there exists a unique Einstein metric $\hat{g}$ of the form $dt^2 + g(t)$ on $(-a, a) \times P$ for some $a > 0$ with initial conditions $g(0) = g_0$, $L(0) = L_0$.*

Proof. For a homogeneous metric g on P, the Ricci endomorphism $r(g)$ is a rational function of g, so (2.6a), (2.6b) is an ODE system, depending on the parameter λ. Using (2.7), we choose λ so that

$$(n-1)\lambda = s_0 - (\mathrm{tr}\ L_0)^2 + \mathrm{tr}\left(L_0^2\right) \qquad (*)$$

where s_0 is the scalar curvature of g_0. Let $g(t)$, $L(t)$ be the unique solution determined by the initial conditions on an interval $(-a, a)$ with $a > 0$. From (2.5) we have $\widehat{\mathrm{Ric}}(X, N) = 0$ for any $X \in TP$

at the time $t = 0$, hence by Lemma 2.4, $\widehat{\mathrm{Ric}}(X, N) = 0$ holds for all t. Further, taking the trace of (2.6b) and using $(*)$ we get (2.6c) for $t = 0$. This means $\widehat{\mathrm{Ric}}(N, N) = \lambda$ at $t = 0$ and hence at any $t \in (-a, a)$, due to Lemma 2.4. □

Corollary 2.6. *Let $\hat{g}$ be a G-invariant Riemannian C^3-metric on $E = G \times_H V$ as in Section 1. Suppose that $k \geq 1$, i.e., $\dim Q < \dim P$. Then, if (2.6a) and (2.6b) are satisfied for $g(t)$ [which is defined as in Section 1, Equation (1.6)], $\hat{g}$ is Einstein and hence real analytic.*

Proof. Since there is a singular orbit ($k \geq 1$), we get that $\lim_{t \to 0} v(t) = 0$, while the amount of smoothness implies that $\hat{\mathrm{Ric}}(N, X)$ and $\hat{\mathrm{Ric}}(N, N)$ are finite. Applying Lemma 2.4, we see that g is Einstein and finally, by Theorem 5.2 in [9], $\hat{g}$ must be real analytic. □

Remark 2.7. Although we are primarily interested in Riemannian metrics of cohomogeneity one in this paper, the above analysis holds quite generally for an equi-distant hypersurface family, which includes in particular the following situation. Let Z be a principal H-bundle over a smooth manifold B such that there is a smooth H-action of cohomogeneity one on a manifold F. Let M be the associated fiber bundle $Z \times_H F$ over B. We choose a principal connection ω on Z, a metric g_B with constant scalar curvature on B, and an H-invariant metric g_F of the form $dt^2 + h(t)$ on F. Then the above choices determine a unique metric $\hat{g}$ on M such that the projection onto B is a Riemannian submersion with totally geodesic fibers. This is the Kaluza–Klein construction, in which one seeks choices such that $\hat{g}$ becomes Einstein. More generally, let M_0 denote the open submanifold of M obtained by removing the singular orbits of F in the above construction. Then we may replace the metric g_B by a one-parameter family $g_B(t)$ of smooth metrics each having constant scalar curvature. As a result, we obtain on M_0 a metric precisely of the form considered in this section. Of course, as in Section 1, there would be further conditions that are necessary to guarantee that $\hat{g}$ is smooth on all of M. Furthermore, $(M, \hat{g})$ in general will have very little symmetry.

The cohomogeneity one case we are concerned with in this paper becomes the special case of the above where $B = G/H = Q$, $Z = G$, and $F = V$ or S^k. The hypersurfaces P are given by $Z \times_H (H/K)$, where K is the principal isotropy group of the H-action on F. Finally, the principal connection in this case is given by the reductive splitting in Section 1.

Notice that the Einstein condition (2.6b) in Proposition 2.1 can be satisfied by a metric of the above type only if its Ricci transformation r preserves the splitting induced by the principal connection chosen above. The technical condition (A) introduced in Section 1 is a natural and verifiable condition on the triple (G, H, K) which guarantees this property of r.

Now we can extend Corollary 2.6 as follows. Let Z be a smooth principal H-bundle over a manifold B with H compact, and ω be a principal connection on Z. Let $g_B(t)$ be a one-parameter family of metrics on B such that for each t the scalar curvature is constant and the norm of the curvature form of ω is a constant function on Z. Let (F, g_F), $g_F = dt^2 + h(t)$, be a Riemannian manifold with a cohomogeneity one isometric action of H. Suppose that $\hat{g}$ is a Riemannian metric of class C^3 constructed using ω on $M = Z \times_H F$ of the form $dt^2 + h(t) + \pi^* g_B(t)$. If F has a singular orbit, i.e., an orbit of dimension strictly smaller than that of the principal orbit, and if (2.6a) and (2.6b) are satisfied, then $\hat{g}$ is necessarily Einstein, and hence is real analytic.

To see this, we first remark that the norm of the curvature of ω is defined using a fixed bi-invariant metric on H and $g_B(t)$. Since for a fixed t this norm is constant and $g_B(t)$ has constant scalar curvature, it follows that the corresponding metric on $Z \times_H (H/K)$ has constant scalar curvature for each t (cf. [4], (9.70d), p. 253). The rest of the proof is the same as above.

3. Setting up the initial value problem

In this section we will set up the initial value problem and make some general remarks regarding its solution.

Let $E = G \times_H V \to G/H = Q$ as in Section 1 and let $\hat{g}$ be a smooth G-invariant metric on E. Then, on $M_0 = E \setminus Q$, we have $\phi^*\hat{g} = dt^2 + g(t)$ where $g(t)$ is a one-parameter family of homogeneous metrics on P. This metric is Einstein iff $(g(t), L(t))$ satisfies Equations (2.6a) and (2.6b) in Section 2, where $L(t)$ denotes the Weingarten map of $P_t = \phi(\{t\} \times P)$. We remind the reader that in Section 1 we have chosen a background metric $\hat{g}_0$ with which we identify TP and T^*P, so that $g(t)$ becomes an endomorphism of TP. This background metric is determined by the restriction of $\hat{g}$ to Q, the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-$, and the restriction of $\hat{g}$ to V.

Conversely, in constructing an Einstein metric $\hat{g}$, we will use the background metric determined by the desired initial G-invariant metric on Q and any H-invariant inner product on V, which of course induces the constant curvature 1 metric on $S^k \cong H/K$. By assumption (A) in Section 1, $g(t)$, and hence $L(t)$ as well as $r(t)$ split into $+$ and $-$ parts, and keeping in mind the results of Section 1, we choose $x(t),\ \eta(t) \in \mathrm{End}\,(\mathfrak{p})^K$ which preserve the splitting of $\mathfrak{p}$ such that

$$\begin{aligned} g_+(t) &= t^2 x_+(t) \\ g_-(t) &= x_-(t) \\ L_+(t) &= \frac{1}{t} I + \eta_+(t) \\ L_-(t) &= \eta_-(t)\,. \end{aligned}$$

By Equations (1.3) and (1.4) in Section 1, in order that $\hat{g}$ admits a smooth extension to (an open neighborhood of Q in) E, we must impose the initial conditions

$$x(0) = I, \quad \eta_+(0) = 0, \quad \eta_-(0) = L_1\,(v_0)\ ,$$

where L_1 is the given shape operator of the singular orbit Q, which must be an H-equivariant linear map $V \to S^2(\mathfrak{p}_-)$. Notice that the initial metric on Q has been encoded in the background metric through the identification of TP and T^*P. Also, the H-equivariance of L_1 implies that its trace must be 0, i.e., Q must be a minimal submanifold in M (cf. (1.5) in Section 1). In these new variables, Equations (2.6a) and (2.6b) of Section 2 become

$$\begin{aligned} x' &= 2x\eta\,, \\ \eta' &= -(\mathrm{tr}\ \eta)\eta - \frac{k}{t}\eta - \left(\frac{k-1}{t^2} + \frac{\mathrm{tr}\ \eta}{t}\right) I_+ + r - \hat{r}\,, \end{aligned}$$

where I_+ denotes the projection onto $\mathfrak{p}_+$ and $\hat{r}$ is the Ricci endomorphism for the metric $\hat{g}$, which, in most cases, will be $\lambda \cdot I$. (Actually, to be precise, in the second equation above $\hat{r}$ denotes the $\mathfrak{p}$-component of the restriction of $\hat{r}$ to $\mathfrak{p}$.)

However, in order to get a metric, we must impose the further condition that x and $x\eta$ are *symmetric* endomorphisms with respect to the background metric. To deal with this unpleasant quadratic condition, we change variables again and put

$$y = x\eta\,.$$

In the variables x, y the equations become

$$\begin{aligned} x' &= 2y\,, \\ y' &= 2yx^{-1}y - \operatorname{tr}\left(x^{-1}y\right)y - \frac{k}{t}y - \left(\frac{k-1}{t^2} + \frac{\operatorname{tr}\left(x^{-1}y\right)}{t}\right)x_+ + xr - x\hat{r}\,, \\ x(0) &= I,\ \ y_+(0) = 0,\ \ y_-(0) = L_1(v_0) \end{aligned} \tag{3.1}$$

These equations will be considered as an ODE system with values in $(S^2\mathfrak{p})^K$. Note that

$$\begin{aligned} x_+r_+ &= \frac{1}{t^2}g_+r_+ = \frac{1}{t^2}\operatorname{Ric}_+\,, \\ x_-r_- &= g_-r_- = \operatorname{Ric}_- \end{aligned}$$

are symmetric, and similarly, $x\hat{r}$ is symmetric.

Next we need the formula for the Ricci tensor of a homogeneous metric g on $P = G/K$ (see [4], p. 185 for a derivation). For any basis $X_1, \ldots, X_n$ of $\mathfrak{p}$ and any $X, Y \in \mathfrak{p}$ we have

$$\begin{aligned} \operatorname{Ric}(X, Y) &= -\frac{1}{2}\operatorname{tr}_{\mathfrak{g}}(ad(X)ad(Y)) - \frac{1}{2}\sum_{ij} g\left([X, X_i]_{\mathfrak{p}}, [Y, Y_j]_{\mathfrak{p}}\right)g^{ij} \\ &\quad + \frac{1}{4}\sum_{ijpq} g\left(X, [X_i, X_p]\right)g\left(Y, [X_j, X_q]\right)g^{ij}g^{pq}\,, \end{aligned}$$

where $g_{ij} = g(X_i, X_j)$ and (g^{ij}) denotes the inverse matrix of (g_{ij}). Here, we choose our basis so that $X_\alpha := U_\alpha$ for $\alpha = 1, \ldots, k$ and $X_i := Z_i$ for $i = k+1, \ldots, n$ are, respectively, bases of $\mathfrak{p}_+$ and $\mathfrak{p}_-$ which are orthonormal with respect to the above chosen background metric $\hat{g}_0$. We have

$$g^{\alpha\beta} = \frac{1}{t^2}\cdot x_+^{\alpha\beta},\quad g^{ij} = x_-^{ij}\,,$$

and hence the Ricci endomorphism $r_\pm(t)$ splits as follows into a regular part and a singular part having a double pole at $t = 0$ (where we have left the routine verification to the reader):

Lemma 3.1.

$$r = \frac{1}{t^2}r_{\text{sing}} + r_{\text{reg}}$$

with

$$\begin{aligned} xr_{\text{sing}}(U, V) &= \operatorname{Ric}\left(S^k, x_+\right)(U, V) - \frac{1}{2}\Big\{x_-\left([U, Z_i], [V, Z_j]\right)x_-^{ij} \\ &\quad + \operatorname{tr}\left(ad(U)ad(V)|_{\mathfrak{p}_-}\right)\Big\}\,, \\ xr_{\text{sing}}(X, Y) &= \frac{1}{2}\Big\{x_-\left(X, [Z_i, U_\alpha]\right)x_-\left(Y, [Z_j, U_\beta]\right)x^{ij} \\ &\quad - x_-\left([X, U_\alpha], [Y, U_\beta]\right)\Big\}x_+^{\alpha\beta} \end{aligned}$$

(summation over repeated indices) for all $U, V \in \mathfrak{p}_+$ and $X, Y \in \mathfrak{p}_-$.

Hence,

$$y' = \frac{1}{t^2}A(x) + \frac{1}{t}B(x, y) + C(x, y, t),$$

where

$$
\begin{aligned}
A(x) &= x r_{\text{sing}} - (k-1)x_+ \,, \\
B(x,y) &= -ky - \operatorname{tr}\left(x^{-1}y\right)x_+ \,, \\
C(x,y,t) &= 2yx^{-1}y - \operatorname{tr}\left(x^{-1}y\right)y + x r_{\text{reg}} - x\hat{r} \,.
\end{aligned}
$$

Notice that C depends explicitly on t since r_{reg} does.

We have therefore reduced the initial value problem for a cohomogeneity one Einstein metric to a corresponding initial value problem for a non-linear first order system of ordinary differential equations with a singular point at the origin. Since Einstein metrics are real analytic, we will solve the initial value problem using the method of "asymptotic series," which was first used by Poincaré in the linear case, see [21]. A more modern account can be found in [24], Chapter 9.

The general method consists of showing that there always exists formal series solutions of an appropriate type and then using Picard iteration on sufficiently high truncations of the formal solutions to produce real solutions. In the linear case, it turns out that the initial value problem can always be solved, see [8], Chapter 5. It is interesting to compute the linearization of our equations. One finds that it is of the form $z' = t^{-2}\mathcal{A}(t)z$ where $\mathcal{A}(0)$ is lower triangular. This falls under the most complicated case of the linear theory. In the non-linear case, formal solutions do not always exist, let alone formal *power* series solutions. However, the *geometric* origin of the equation brings with it nice properties which we exploit in Sections 4 and 5 to give us formal power series solutions. Once one has formal power series solutions, one obtains a real solution by invoking a general theorem of Malgrange (Theorem 7.1 in [17]) or by carrying out the Picard iteration directly, as indicated in Section 6.

4. Equivariant variation of the Ricci tensor

In view of Section 5, we have to study families (variations) $g(t) = t^2 x_+(t) + x_-(t)$ of G-invariant metrics on P with initial condition

$$x_+(0) = I_+, \quad x_-(0) = I_- \,.$$

Lemma 4.1. *Let $\rho : H \to O(k+1)$ be a homomorphism by which H acts transitively on the unit sphere $S^k \subset V = \mathbb{R}^{k+1}$. Let $v_0 \in S^k$ have isotropy group $K \subset H$. Suppose that $(\ ,\ )$ is an $Ad(H)$-invariant inner product on $\mathfrak{h}$ and $\mathfrak{p}_+$ is the orthogonal complement of $\mathfrak{k}$. Finally, let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{p}_+$ induced by the constant curvature 1 metric on $S^k = H/K$. Then for any orthonormal basis $U_1, \cdots, U_k$ of $(\mathfrak{p}_+, \langle\ ,\ \rangle)$, we have*

$$-\sum_\alpha \rho_*(U_\alpha)^2 v_0 = k v_0$$

Proof. We first show that $-\sum_\alpha \rho_*(U_\alpha)^2 v_0$ is independent of the choice of the orthonormal basis $\{U_1, \cdots, U_k\}$. Let $v \in \mathbb{R}^{k+1}$ and consider

$$\beta_v(X,Y) = \frac{1}{2}\left\langle(\rho_*(X)\rho_*(Y) + \rho_*(Y)\rho_*(X))\, v_0, v\right\rangle \,.$$

This bilinear form on $\mathfrak{p}_+$ has trace equal to $\langle\sum \rho_*(U_\alpha)^2 v_0, v\rangle$, which of course does not depend on the choice of $\{U_1, \cdots, U_k\}$ as long as it is orthonormal with respect to $\langle\ ,\ \rangle$.

Next, let S be the $\mathrm{Ad}(K)$-invariant endomorphism on $\mathfrak{p}_+$ such that $\langle X, Y\rangle = (S(X), Y)$ for all $X, Y \in \mathfrak{p}_+$. Then there is an $\langle\ ,\ \rangle$-orthonormal basis $U_1, \cdots, U_k$ of eigenvectors of S. For $X \in \mathfrak{p}_+$, we have

$$\begin{aligned}\left\langle \rho_*(U_\alpha)^2 v_0, \rho_*(X)v_0\right\rangle &= -\langle \rho_*(U_\alpha) v_0, \rho_*(U_\alpha)\rho_*(X)v_0\rangle \\ &= -\langle \rho_*(U_\alpha) v_0, \rho_*(X)\rho_*(U_\alpha) v_0\rangle + \langle \rho_*(U_\alpha) v_0, \rho_*([X, U_\alpha]) v_0\rangle\ .\end{aligned}$$

The first term is 0 since $\rho_*(X)$ is skew-symmetric and the second term is by definition $\langle U_\alpha, [X, U_\alpha]_{\mathfrak{p}_+}\rangle = (S(U_\alpha), [X, U_\alpha]_{\mathfrak{p}_+}) = \lambda(U_\alpha, [X, U_\alpha]) = -\lambda([U_\alpha, U_\alpha], X) = 0$, by the $\mathrm{Ad}(H)$-invariance of $(\ ,\ )$. So $\rho_*(U_\alpha)^2 v_0$ is perpendicular to the H-orbit through v_0, i.e., $\rho_*(U_\alpha)^2 v_0$ is a multiple of v_0. But $\langle \rho_*(U_\alpha)^2 v_0, v_0\rangle = -\langle \rho_*(U_\alpha)v_0, \rho_*(U_\alpha)v_0\rangle = -\langle U_\alpha, U_\alpha\rangle = -1$. Summing over α gives the desired result. □

Lemma 4.2. *For any variation $x_\pm(t)$, we have*

$$\begin{aligned}(r_-)_{\mathrm{sing}}(0) &= 0\,, \\ \delta\,(r_-)_{\mathrm{sing}} &= \frac{1}{2}\mathcal{C}\,(\delta x_-)\end{aligned}$$

with $\mathcal{C} = -\sum_\alpha ad(U_\alpha)^2$ (where the adjoint representation of $\mathfrak{h}$ on $\mathfrak{p}_-$ is canonically extended to End $(\mathfrak{p}_-)$ *).*

Proof. Let (Z_i) and (U_α) be orthonormal bases for the background metrics on $\mathfrak{p}_-$ and $\mathfrak{p}_+$ and $X \in \mathfrak{p}_-$. Using Lemma 3.1 and the facts that $x_\pm(0) = I_\pm$ and that $x_-(0)$ is H-invariant, we get the first equality. Furthermore, by summing over repeated indices, we have

$$\begin{aligned}2\delta\,(r_-)_{\mathrm{sing}}(X) &= \big\{-\delta x_-\,(X, [Z_i, U_\alpha])\big\langle[Z_p, U_\alpha], Z_i\big\rangle - \langle[X, U_\alpha], Z_i\rangle\,\delta x_-\,\big(Z_p, [Z_i, U_\alpha]\big) \\ &\quad - \delta x_-\,\big([X, U_\alpha], [Z_p, U_\alpha]\big) \\ &\quad - \langle[X, U_\alpha], Z_i\rangle\big\langle[Z_p, U_\alpha], Z_j\big\rangle\delta x_-\,(Z_i, Z_j)\big\}\cdot Z_p \\ &\quad - \big\{\langle X, [Z_i, U_\alpha]\rangle\big\langle[Z_p, U_\alpha], Z_i\big\rangle - \big\langle[X, U_\alpha], [Z_p, U_\alpha]\big\rangle\big\}\,\delta x_-\,\big(Z_p, Z_q\big)\,Z_q \\ &= -\Big\{\delta x_-\,\Big(X, ad\,(U_\alpha)^2\,Z_p\Big) + \delta x_-\,\Big(Z_p, ad\,(U_\alpha)^2\,X\Big) \\ &\quad + 2\delta x_-\,\big(ad\,(U_\alpha)\,X, ad\,(U_\alpha)\,Z_p\big)\Big\}\,Z_p \\ &= \mathcal{C}\,(\delta x_-)\,\big(X, Z_p\big)\,Z_p \\ &= \mathcal{C}\,(\delta x_-)\,X\,.\end{aligned}$$

□

Remark. If the standard metric on S^k is normal homogeneous with respect to H, then we can extend the orthonormal basis (U_α) of $\mathfrak{p}_+ \subset \mathfrak{h}$ to an orthonormal basis (V_a) of a biinvariant metric on $\mathfrak{h}$, and $\mathcal{C} = -\sum_a ad(V_a)^2$ is a Casimir operator for the adjoint representation of H on $\mathfrak{p}_-$ and End $(\mathfrak{p}_-)$. However, if the standard metric on S^k is not normal homogeneous with respect to H, then $\mathcal{C}$ is not a Casimir.

The space $(S^2\mathfrak{p}_-)^K$ of K-invariant endomorphisms is isomorphic to the space W_- of all H-equivariant maps $L : S^k \to S^2(\mathfrak{p}_-)$, where the isomorphism is given by evaluation at v_0, namely $\epsilon : L \mapsto L(v_0)$. Consider the subspace $W_-^m \subset W_-$ of all maps L which are restrictions to S^k of H-equivariant polynomials $V \to S^2(\mathfrak{p}_-)$ of degree $\leq m$.

Lemma 4.3. *$\mathcal{C}$ stabilizes $\epsilon(W_-^m)$ and its eigenvalues on $\epsilon(W_-^m)$ are $\lambda_p := p(p-1+k)$ for $p = 0, \cdots, m$.*

Proof. Let $L : V \to S^2(\mathfrak{p}_-)$ be an H-equivariant homogeneous polynomial of degree $p \leq m$. We consider L as an H-homomorphism $L : S^p(V) \to S^2(\mathfrak{p}_-)$, where $S^p(V)$ denotes the pth symmetric tensor power of V. Let $v^p \in S^p(V)$ denote the pth symmetric tensor power of the vector $v \in V$. Then from the H-equivariance of L we obtain

$$\mathcal{C}(\epsilon(L)) = \mathcal{C}\left(L\left(v^p\right)\right) = L\left(\mathcal{C}v^p\right) ,$$

where now $\mathcal{C} := -\sum_\alpha \rho_*(U_\alpha)^2$ for *any* representation ρ of H. Here, the representation in question is the pth symmetric power of the given representation of H on V. Thus, for any $h \in H$ we have $h(v^p) = (hv)^p$, and by differentiation we get $U(v^p) = p \cdot U(v) \cdot v^{p-1}$ for any $U \in \mathfrak{h}$. Hence

$$U^2\left(v^p\right) = p \cdot U\left(U(v) \cdot v^{p-1}\right) = p \cdot \left\{U^2(v) \cdot v^{p-1} + (p-1) \cdot U(v)^2 \cdot v^{p-2}\right\} .$$

Recall that $\{v_0, U_1(v_0), \ldots , U_k(v_0)\}$ is an orthonormal basis of V. There is an H-equivariant embedding

$$\tau : S^{p-2}V \to S^pV, \quad v^{p-2} \mapsto \sum_{i=0}^{k} (e_i)^2 \cdot v^{p-2}$$

where $e_0, \ldots , e_k$ is any orthonormal basis of V, and τ does not depend on the choice of this basis. Using Lemma 4.1 we get from the equation above

$$\begin{aligned} \mathcal{C}v_0{}^p &= p\left\{k \cdot v_0{}^p - (p-1) \cdot \tau\left(v_0{}^{p-2}\right) + (p-1) \cdot v_0{}^p\right\} \\ &= p(k+p-1) \cdot v_0{}^p - p(p-1) \cdot \tau\left(v_0{}^{p-2}\right) . \end{aligned}$$

The adjoint $\tau^* : S^m(V)^* \to S^{m-2}(V)^*$ extends to a linear map $\tau^* : W_-^m \to W_-^{m-2}$, and we have

$$\mathcal{C}(\epsilon(L)) = \lambda_p \epsilon(L) - p(p-1)\epsilon\left(\tau^* L\right) . \tag{$*$}$$

Thus, the subspace $\langle L\rangle$ of $\epsilon(W_-^m)$ spanned by

$$\epsilon(L), \epsilon\left(\tau^* L\right), \epsilon\left(\left(\tau^*\right)^2 L\right), \cdots$$

is invariant under $\mathcal{C}$, and $\mathcal{C}$ acts on this space as a lower triangular matrix with (possible) diagonal entries $\lambda_p, \lambda_{p-2}, \lambda_{p-4}, \cdots$. Thus, any element of $\epsilon(W_-^m)$ lies in a $\mathcal{C}$-invariant subspace where $\mathcal{C}$ has the above eigenvalues. This finishes the proof. $\square$

Remark. Of course, the generators of $\langle L\rangle$ can be linearly dependent so that the two terms on the right-hand side of $(*)$ may cancel each other. In other words, not all of $\lambda_0, \cdots, \lambda_p$ arise as eigenvalues on $\langle L\rangle$, but if λ_q is an eigenvalue on $\langle L\rangle$ for some $q \leq p$, then λ_{q-2i} are also eigenvalues on $\langle L\rangle$ for all $i \leq q/2$. The full eigenspace for λ_m is a complement of W_-^{m-1} in W_-^m. As an example for the cancellation, take the polynomial $L(v) = \langle v, v\rangle^q \cdot I_-$ of degree $p = 2q$. By polarization,

$$L\left(e_i^2 v^{q-2}\right) = \frac{1}{2q-1}\left(\langle e_i, e_i\rangle \langle v, v\rangle^{q-1} + (2q-2)\langle e_i, v\rangle^2 \langle v, v\rangle^{q-2}\right) \cdot I_- .$$

If v is a unit vector, then

$$\left(\tau^* L\right)\left(v^{2q-2}\right) = \left(\frac{1}{2q-1}((k+1) + (2q-2))\right) I_- = \left(\frac{k+p-1}{p-1}\right) I_- ,$$

showing that the right-hand side of $(*)$ is zero (which corresponds to the fact that $L \in W_-^0 \subset W_-^p$).

Lemma 4.4. *For any variation $x_\pm(t)$, we have*

$$\begin{aligned} (r_+)_{\text{sing}}(0) &= r\left(S^k, x_+\right)(0) = (k-1)I_+ , \\ \delta(r_+)_{\text{sing}} &= \delta r\left(S^k, x_+\right) . \end{aligned}$$

Proof. By Lemma 3.1 we have for any $U \in \mathfrak{p}_+$

$$-2\left((r_+)_{\text{sing}} - r\left(S^k, x_+\right)\right)(U) = q\left(U, U_\alpha\right) x^{\alpha\beta} U_\beta ,$$

where

$$q\left(U, U_\alpha\right) := x_-\left(\left[U, Z_i\right], \left[U_\alpha, Z_j\right]\right) x^{ij} + \operatorname{tr}\left(ad(U)ad\left(U_\alpha\right)|_{\mathfrak{p}_-}\right) .$$

Now $q(U, U_\alpha)$ vanishes at $t = 0$, by the H-invariance of the inner product $x_-(0)$. This proves the first equation (recall that $x_+(0)$ induces the curvature-one metric on S^k).

Moreover, for the second equation we only have to consider the variation of the factor $q(U, U_\alpha)$. Note that the second term is independent of t. So we get (summing over repeated indices)

$$\begin{aligned} \delta q\left(U, U_\alpha\right) &= \delta x_-\left(\left[U, Z_i\right], \left[U_\alpha, Z_i\right]\right) \\ &\quad - \left\langle\left[U, Z_i\right], \left[U_\alpha, Z_j\right]\right\rangle \delta x_-\left(Z_i, Z_j\right) \\ &= \left\langle\left[U, Z_i\right], Z_p\right\rangle\left\langle\left[U_\alpha, Z_i\right], Z_q\right\rangle \delta x_-\left(Z_p, Z_q\right) \\ &\quad - \left\langle\left[U, Z_i\right], Z_r\right\rangle\left\langle\left[U_\alpha, Z_j\right], Z_r\right\rangle \delta x_-\left(Z_i, Z_j\right) \\ &= 0 , \end{aligned}$$

which finishes the proof. □

Next we consider the *Hopf fibrations* on S^k,

(a) $S^1 \subset S^k \to \mathbb{C}P^m$, $k = 2m+1$,
(b) $S^3 \subset S^k \to \mathbb{H}P^q$, $k = 4q+3$,
(c) $S^7 \subset S^k \to S^8$, $k = 15$.

These are Riemannian submersions with totally geodesic fibers. Let $x_0 = \langle\, , \,\rangle$ be the standard curvature-one metric on S^k and consider the *canonical variation* (cf [4])

$$x_t = x_0|_{\mathcal{H}} + (1+t) \cdot x_0|_{\mathcal{V}} ,$$

where $\mathcal{H}$ and $\mathcal{V}$ denote the horizontal and vertical distributions. Let

$$d_V = \dim \mathcal{V}, \quad d_H = \dim \mathcal{H}$$

be, respectively, the dimensions of the fibers and the base.

Lemma 4.5. *The Ricci tensor of (S^k, x_t) is given by*

$$r_t = \rho_H(t) \cdot I|_{\mathcal{H}} + \rho_V(t) \cdot I|_{\mathcal{V}}$$

where

$$\rho_H(t) = d_H + 3d_V - 1 - 2d_V(1+t), \quad \rho_V(t) = (d_V - 1)\frac{1}{1+t} + d_H(t+1)$$

Proof. Recall that the Ricci tensor as well as the metric is invariant under the automorphism group of the Hopf fibration, which consists of the fiber-preserving isometries of S^k. Hence, we get from Proposition 9.70, p. 253, in [4] for the Ricci tensor Ric_t of x_t that $\mathrm{Ric}_t(\mathcal{H}, \mathcal{V}) = 0$ and

$$\begin{aligned} \mathrm{Ric}_t|_{\mathcal{H}} &= (\bar{\rho} - (1+t)\mu)\, x_0|_{\mathcal{H}} \\ \mathrm{Ric}_t|_{\mathcal{V}} &= \left(\hat{\rho} + (1+t)^2\nu\right) x_0|_{\mathcal{V}} \end{aligned}$$

where $\bar{\rho}$ and $\hat{\rho}$ are, respectively, the Ricci curvatures of the base manifold and the fiber, and μ and ν are real constants. These can be computed by taking $t = 0$:

$$\mu = \bar{\rho} - \rho, \quad \nu = \rho - \hat{\rho}\,,$$

where $\rho = k - 1$ denotes the Ricci curvature of S^k. Since $r_t = x_t^{-1} \cdot \mathrm{Ric}_t$, we obtain

$$\begin{aligned} \rho_H(t) &= \bar{\rho} - (\bar{\rho} - \rho)(1+t)\,, \\ \rho_V(t) &= \hat{\rho} \cdot \frac{1}{1+t} + \left(\rho - \hat{\rho}\right)(1+t)\,. \end{aligned}$$

The values of the Ricci curvatures are

$$\rho = d_H + d_V - 1, \quad \hat{\rho} = d_V - 1, \quad \bar{\rho} = d_H + 3d_V - 1\,.$$

In fact, on the base space, each vector lies in d_V 2-planes of curvature 4 and $d_H - d_V - 1$ 2-planes of curvature 1; hence, $\bar{\rho} = 4d_V + d_H + d_V - 1 = d_H + 3d_V - 1$. Plugging in these values, we get the result. □

Lemma 4.6. *Let H be a compact Lie group acting orthogonally and transitively on the unit sphere $S^k \subset V$. Let $x_+(t)$ be a smooth one-parameter family of H-invariant metrics on S^k such that $x_+(0)$ is the standard curvature-one metric. Then*

$$\delta r\left(S^k, x_+\right) = (k+1)\delta x_+ - 2\,\mathrm{tr}\,(\delta x_+) \cdot I_+\,.$$

Proof. In the following we will drop the subscript "+". In computing the desired formula, we clearly may assume that H is connected and acts almost effectively on S^k. Since the desired formula is *a priori* linear in δx_+, we claim that it is sufficient to consider only the following two types of one-parameter variations of $x(0)$:

(1) the homotheties $x(t) = (1+t)x(0)$,
(2) the canonical variation of all possible Hopf fibrations

$$x(t) = (1+t)x(0)|_{\mathcal{V}} + x(0)|_{\mathcal{H}}\,.$$

This is because the only groups H with more than a two-parameter family of H-invariant metrics are $Sp(m)$ and $Sp(m) \times U(1)$ with $k = 4m+3$ (cf. [25]) and in these cases the variations are generated by Hopf fibrations by circles corresponding to various complex structures on $V = \mathbb{R}^{4m+4}$.

For variations of type (1) we get $r(t) = \rho(t) \cdot I$ with

$$\rho(t) = \frac{k-1}{1+t}\,.$$

Thus, $\delta\rho = -(k-1)$. On the other hand, $\delta x = I$ and $\operatorname{tr}\,\delta x = k$, so the right-hand side of the desired equation is $(k+1-2k)I$ which shows the equality.

For variations of type (2) we have $r(t)$ as in Lemma 4.5. Thus,

$$\delta\rho_H = -2d_V,\quad \delta\rho_V = d_H - d_V + 1$$

On the other hand, $\delta x_V = 1$ while $\delta x_H = 0$. Thus, $\operatorname{tr}\,\delta x = d_V$ and

$$\begin{aligned}(k+1)\delta x_H - 2\operatorname{tr}(\delta x) &= -2d_V\,,\\ (k+1)\delta x_V - 2\operatorname{tr}(\delta x) &= d_H + d_V + 1 - 2d_V = d_H - d_V + 1\,.\end{aligned}$$

This finishes the proof. □

5. Formal power series solutions

5.1. The initial value problem (3.1) in Section 3 takes the following general form:

$$\begin{aligned}x' &= 2y && (5.1a)\\ y' &= \frac{1}{t^2}A(x) + \frac{1}{t}B(x,y) + C(t,x,y) && (5.1b)\\ x(0) &= a && (5.1c)\\ y(0) &= b\,, && (5.1d)\end{aligned}$$

where $x(t), y(t), a, b \in (S^2\mathfrak{p})^K \subset S^2(V \oplus \mathfrak{p}_-)^K$ are symmetric endomorphisms of a Euclidean vector space $\mathfrak{p}$ on which K acts linearly and orthogonally, and A, B, C are analytic mappings. Suppose that

$$x(t) = \sum_m \frac{x_m}{m!}\,t^m,\quad y(t) = \sum_m \frac{y_m}{m!}\,t^m$$

with $y_m = \frac{1}{2}x_{m+1}$ give a formal power series solution of (5.1a) through (5.1d) so that $x_0 = a,\ y_0 = b$. Since the left-hand side of (5.1b) has neither $1/t$ nor $1/t^2$ terms, the initial values a, b must satisfy

$$\begin{aligned}A(a) &= 0 && (5.2a)\\ 2\,dA_a \cdot b + B(a,b) &= 0\,. && (5.2b)\end{aligned}$$

We also let

$$A(x(t)) = \sum_m \frac{A_m}{m!}\,t^m,\quad B(x(t),y(t)) = \sum_m \frac{B_m}{m!}\,t^m,\quad C(t,x(t),y(t)) = \sum_m \frac{C_m}{m!}\,t^m\,.$$

We substitute these expressions into (5.1b) and obtain

$$y'(t) = \sum_m \left(\frac{1}{(m+1)!}\left(\frac{A_{m+2}}{m+2} + B_{m+1}\right) + \frac{C_m}{m!}\right)t^m\,.$$

Hence,

$$\frac{1}{2}x_{m+2} = y_{m+1} = C_m + \frac{1}{m+1}\left(\frac{A_{m+2}}{m+2} + B_{m+1}\right) . \tag{5.3}$$

Now

$$\begin{aligned} A_{m+2} &= \frac{d^{m+2}}{dt^{m+2}} A(x(t))|_{t=0} \\ &= \frac{d^{m+1}}{dt^{m+1}} \left((dA)_{x(t)} \cdot x'(t)\right)\Big|_{t=0} \\ &\equiv (dA)_a \cdot x_{m+2} \qquad (\text{mod } x_1, \dots , x_{m+1}) . \end{aligned}$$

Similarly,

$$B_{m+1} \equiv \frac{1}{2} \left(\partial_y B\right)_{(a,b)} (x_{m+2}) \ (\text{mod } x_1, \dots , x_{m+1})$$

where $\partial_y B$ denotes the partial derivative of B with respect to the variable y. Moreover, $C_m \equiv 0 \ (\text{mod } x_1, \dots , x_{m+1})$. Consequently,

$$x_{m+2} = \frac{1}{m+1}\left(\frac{2}{m+2}(dA)_a \cdot x_{m+2} + \left(\partial_y B\right)_{(a,b)} \cdot x_{m+2} + D_m\right) \tag{5.4}$$

for some function $D_m = D_m(x_1, \dots , x_{m+1})$. This can be written as

$$\mathcal{L}_m x_{m+2} = D_m \tag{5.5}$$

where $\mathcal{L}_m$ is the linear endomorphism on $(S^2\mathfrak{p})^K$ given by

$$\mathcal{L}_m := (m+1)I - \frac{2}{m+2}(dA)_a - \left(\partial_y B\right)_{(a,b)} .$$

Equation (5.5) gives the necessary and sufficient condition for the existence of a formal power series solution, namely

$$D_m \in \text{im } (\mathcal{L}_m) \tag{5.6}$$

for all $m \geq 0$. Since $(dA)_a$ and $(\partial_y B)_{(a,b)}$ are bounded, $\mathcal{L}_m$ will be invertible for all $m \geq m_0$ for some m_0. Thus, if (5.6) is satisfied for $m < m_0$, we can prescribe further "initial conditions" $x_2, \dots , x_{m_0}$ satisfying (5.4), and then the formal power series solution is uniquely determined.

5.2. In our case, all endomorphisms preserve the splitting $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, and hence decompose into a $(+)$ and a $(-)$ part. The initial values are

$$a = I, \ \ b_+ = 0, \ \ b_- = L_1(v_0) ,$$

with $L_1 \in W_1$, so that in particular $\text{tr } L_1 = 0$. Furthermore, using the expressions in Section 3, and Lemmas 4.2 and 4.4, we have

$$\begin{aligned} dA_a \cdot \xi &= d\left(r_{\text{sing}}\right)_I \cdot \xi && (5.7a) \\ B(a,b) &= -kb , && (5.7b) \\ \partial_y B_{(a,b)} \cdot \xi &= -k\xi - (\text{tr } \xi) I_+ && (5.7c) \\ C(x,y,t) &= 2yx^{-1}y - \text{tr}\left(x^{-1}y\right) y + x r_{\text{reg}} - x\hat{r} , && (5.7d) \end{aligned}$$

where $d(r_{\text{sing}})_I$ is obtained from Lemmas 4.2, 4.4, and 4.6:

$$
\begin{aligned}
d\left(r_{\text{sing}}\right)_I \cdot \xi_+ &= (k+1)\xi_+ - 2\left(\operatorname{tr}\xi_+\right) I_+ && (5.8a)\\
d\left(r_{\text{sing}}\right)_I \cdot \xi_- &= \frac{1}{2}\mathcal{C}\xi_- . && (5.8b)
\end{aligned}
$$

We now have two problems to solve:

Problem (i). We have to show that a formal power series solution exists, i.e., we must verify Equations (5.2a)–(5.2b) and (5.6).

Problem (ii). If $x(t) = \sum \frac{x_m}{m!} t^m$ is such a formal power series solution, we have to show that

$$
x^q(t) := \sum_{m=0}^{q} \frac{x_m}{m!} t^m
$$

defines a smooth G-invariant metric on E' for all q. By Lemma 1.1 this means that for all $m \geq 0$ we have to show that $x_m \in \epsilon(W_m)$.

5.3. Let us first verify Equations (5.2a)–(5.2b). In fact, (5.2a) is true by the first equations in Lemmas 4.2 and 4.4. The (+) part of (5.2b) follows from $b_+ = 0$ and (5.7b). Moreover, since $b_- \in \epsilon W_1$, we have $dA_a \cdot b_- = \frac{1}{2}kb_-$, by (5.7a), (5.8b), and Lemma (4.3), while $B(a,b)_- = -kb_-$. This proves (5.2b).

Next we have to verify Equation (5.6). From Lemmas 4.2, 4.4, and 4.6 we get

$$
\begin{aligned}
(\mathcal{L}_m\xi)_+ &= m\left(1+\frac{k+1}{m+2}\right)\xi_+ + \left(\frac{4}{m+2}\operatorname{tr}\xi_+ + \operatorname{tr}\xi\right) I_+ && (5.9a)\\
(\mathcal{L}_m\xi)_- &= (m+1+k)\xi_- - \frac{1}{m+2}\,\mathcal{C}\left(\xi_-\right) && (5.9b)
\end{aligned}
$$

for any $\xi = (\xi_+, \xi_-) \in S^2(\mathfrak{p})^K$. We will compute the eigenvalues of $\mathcal{L}_m$. If $\xi_+ = I_+$, $\xi_- = 0$, then $\operatorname{tr}\xi = \operatorname{tr}\xi_+ = k$ and hence $\mathcal{L}_m(\xi) = \mu_1\xi$ with

$$
\mu_1 = \frac{(2k+m)(m+3)}{m+2} > 0 .
$$

On the other hand, if $\xi_+ = 0$ and $\xi_- = I_-$, then $\operatorname{tr}\xi = \operatorname{tr}\xi_- = n-k$, so $(\mathcal{L}_m\xi)_+ = (n-k)I_+$ and $(\mathcal{L}_m\xi)_- = (m+1-k)I_-$. Note that $I_- \in \epsilon W_0$, so that $\mathcal{C}I_- = 0$ by Lemma 4.3. Hence $\mathcal{L}_m$ is a triangular matrix on Span $\{(I_+,0),(0,I_-)\} \subset \epsilon(W_2)$ with diagonal entries (eigenvalues) μ_1 and

$$
\mu_2 = m+1+k > 0 .
$$

We next consider $\mathcal{L}_m$ on the trace-free elements in $S^2(\mathfrak{p})^K$. If $\xi_- = 0$ and $\operatorname{tr}(\xi_+) = 0$, then we have an eigenvector of $\mathcal{L}_m$ with eigenvalue

$$
\mu_3 = m\left(1+\frac{k+1}{m+2}\right) \geq 0 ,
$$

which is zero precisely when $m = 0$. Finally, let ξ be such that $\xi_+ = 0$ and tr $\xi_- = 0$. Suppose further that $\xi_- \in \epsilon W^-_{m+2}$ (this will be true for $\xi = x_{m+2}$, cf. 5.4). By Lemma 4.3, the eigenvalues of $\mathcal{L}_m$ on this subspace are

$$\nu_p = m + 1 + k - \frac{p(p-1+k)}{m+2} \geq 0$$

for $p = 0, \ldots, m+2$. We have $\nu_p = 0$ precisely for $p = m+2$. Now we have decomposed $\epsilon(W_{m+2})$ into eigenspaces of $\mathcal{L}_m$. The eigenspaces corresponding to nonzero eigenvalues all lie in the subspace $\epsilon(W_m) \subset \epsilon(W_{m+2})$, hence $\mathcal{L}_m(\epsilon W_{m+2}) \subset \epsilon W_m$, and $\mathcal{L}_m$ maps this subspace bijectively onto itself if $m > 0$. Therefore, a solution of (5.5) exists iff $D_m \in \epsilon W_m$, and the kernel of $\mathcal{L}_m$ is isomorphic to W^-_{m+2}/W^-_m.

5.4. It remains to show that $x_p \in \epsilon(W_p)$ for all p (Problem (ii)) and that $D_m \in \epsilon W_m$. We proceed by induction over m. By assumption, $x_0 = a = I \in \epsilon W_0$ and $x_1 = 2b \in \epsilon(W_1)$. Suppose that we already have $x_p \in \epsilon(W_p)$ for $p = 1, \ldots, m+1$. Let

$$\tilde{x}(t) = x^{m+1}(t) = \sum_{p=0}^{m+1} \frac{x_p}{p!} t^p .$$

By the discussion in Section 1, $\tilde{x}$ defines a smooth G-invariant metric on E. Let

$$\tilde{r}(t) = \sum_{m=0}^{\infty} \frac{\tilde{r}_m}{m!} t^m$$

be the Ricci endomorphism of $\tilde{x}(t)$ on E at the point tv_0. Since the Ricci tensor is smooth and G-invariant, we have $(\tilde{x}\tilde{r})_m \in \epsilon W_m$ for all m. Put $\tilde{y} = \frac{1}{2}\tilde{x}'$. Then $(\tilde{x}, \tilde{y})$ solve (5.1a) through (5.1d) with $\hat{r}$ replaced by $\tilde{r}$ (cf. (3.1) in Section 3). Since $\tilde{x}_{m+2} = 0$, we get from (5.5) that $\tilde{D}_m = 0$, and (5.3), (5.4), and (5.7d) show that

$$\frac{1}{2(m+1)} D_m = \frac{1}{2(m+1)} \left(D_m - \tilde{D}_m\right) = C_m - \tilde{C}_m = -\left((x\hat{r})_m - (\tilde{x}\tilde{r})_m\right)\big|_{\mathfrak{p}} \in \epsilon W_m$$

by Lemma 1.2. By Section 5.3 there exists a solution x_{m+2} of (5.5) in ϵW_{m+2}, (in fact even in ϵW_m), but we can add an arbitrary element of the zero eigenspace of $\mathcal{L}_m$. This finishes the induction and shows the existence of a formal power series solution for our initial value problem (5.1a) through (5.1d).

5.5. Finally, we will describe the indeterminacy in the formal solution above more precisely. By the discussion in Section 5.3, notice that once the $(-)$ part of (5.5) is solved for, provided that $m > 0$, there is a unique solution of the $(+)$ part of (5.5). When $m = 0$, the trace-free part of x_{2+} is arbitrary. This is to be expected because $x_+(0)$ and $y_+(0)$ as well as the radial part of the metric are fixed by the geometry and so the usual freedom in the initial value problem falls upon the trace-free part of x_{2+}.

We saw in Section 5.3 that ker $(\mathcal{L}_m) \cong W^-_{m+2}/W^-_m$. On the other hand, by the discussion before Lemma 1.1 in Section 1 we see that the spaces W^-_m eventually stablize. Suppose that $W^-_{2m} = W^-_{2m_0}$ for all $m > m_0$ and $W^-_{2m+1} = W^-_{2m_1+1}$ for all $m > m_1$. Then the indeterminacy of the initial value problem is given by elements in

$$\left(W^-_{2m_0}/W^-_0\right) \oplus \left(W^-_{2m_1+1}/W^-_1\right) .$$

We note that $W_0 = S^2(V)^H \oplus S^2(\mathfrak{p}_-)^H$, where the first summand is one-dimensional (generated by the background metric) since V is irreducible as an H-representation.

Thus, once the G-manifold is given, the indeterminacy in the initial value problem is completely determined by representation theory. Below we will give some examples of the kind of computation necessary to determine the indeterminacy explicitly.

First let H be an effective compact linear group acting transitively on the unit sphere in V. Using the proof of Lemma 1.2 in Section 1, it is not hard to compute the dimension of $\mathrm{Hom}(S^2V, S^2V)^H$, which gives the indeterminacy in x_{2+}. (Indeed, this lemma also shows that $\mathrm{Hom}(S^mV, S^2V)^H \cong 0$ for odd m and that they are all isomorphic when m is a positive even integer.)

Next, we consider $\mathrm{Hom}(S^mV, S^2(\mathfrak{p}_-))^H \cong W_m^-$.

Example 5.1. Let $G = SO(p+n)$, $H = SO(p) \times SO(n)$, and $K = SO(p) \times SO(n-1)$.

In this case, $\mathfrak{p}_-$ as an H-representation is just the tensor product of the usual representations ρ_p of $SO(p)$ and ρ_n of $SO(n)$. V, on the other hand, is the tensor product of the trivial representation of $SO(p)$ and the usual representation of $SO(n)$. As is well known (see, e.g., [13] Exercise 19.21, p. 296),

$$S^mV = \sigma_m \oplus \sigma_{m-2} \oplus \cdots \oplus \sigma_{m-2[m/2]} ,$$

where σ_k is the irreducible representation of $SO(n)$ whose dominant weight is k times that of ρ_n. On the other hand, since $\mathfrak{p}_-$ is just $\rho_p \otimes \rho_n$, we have

$$S^2\left(\mathfrak{p}_-\right) = S^2\left(\rho_p\right) \otimes S^2\left(\rho_n\right) \oplus \Lambda^2\left(\rho_p\right) \otimes \Lambda^2\left(\rho_n\right) ,$$

which decomposes as

$$(\sigma_2 \otimes \sigma_2) \oplus (\mathbf{1} \otimes \sigma_2) \oplus (\sigma_2 \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}) \oplus \left(ad_p \otimes ad_n\right) .$$

Therefore, by Schur's lemma, $\dim W_m^-$ is 2 when $m \geq 2$ is even, and is 0 when m is odd. Since $W_0^- = (S^2\mathfrak{p}_-)^H$ is one-dimensional, the indeterminacy in this case has dimension $2 - 1 = 1$ and it occurs with x_{2-}. The choice of initial metric is unique up to homothety since G/H is isotropy irreducible. Since V is not a summand in $S^2(\mathfrak{p}_-)$, the only choice for the initial shape operator is 0, i.e., the singular orbit must be totally geodesic. There is, though, a one-dimensional freedom in choosing x_{2+}.

Next we give an example to show that the dimension of the space of indeterminacy can be quite high.

Example 5.2. $G = E_7$, $H = SO(3) \times SU(2)$, and $K = SO(2) \times SU(2)$.

In this case, the Lie algebra $\mathfrak{h}$ is one of the non-regular, non-simple maximal subalgebras of the exceptional Lie algebra E_7. For more information, see [10], p. 226. We will denote the unique irreducible representation of $SU(2)$ of dimension $k+1$ by k below. Then k factors through $SO(3)$ iff k is even. With this shorthand notation, $\mathfrak{p}_-$ is the H−representation

$$(8 \otimes 2) \oplus (6 \otimes 4) \oplus (4 \otimes 6) \oplus (4 \otimes 2) \oplus (2 \otimes 4) .$$

The representation V is $2 \otimes 0$. So our assumption (A) (cf. Section 1) is satisfied. It follows from the decomposition of S^mV in Example 5.1 that

$$S^{2m}V = (4m \otimes 0) \oplus ((4m-4) \otimes 0) \oplus \cdots \oplus (0 \otimes 0) ,$$

and

$$S^{2m+1}V = ((4m+2)\otimes 0) \oplus ((4m-2)\otimes 0) \oplus \cdots \oplus (2\otimes 0)\ .$$

We can then use the Clebsch–Gordon formula (exercise 11.11, p. 151 of [13]) to decompose the representation $S^2(\mathfrak{p}_-)$. It turns out that the irreducible summands on which the $SU(2)$ factor acts trivially are

$$(16\otimes 0) \oplus 3(12\otimes 0) \oplus (10\otimes 0) \oplus 6(8\otimes 0) \oplus 2(6\otimes 0) \oplus 7(4\otimes 0) \oplus 4(0\otimes 0)\ .$$

Again by Schur's Lemma we conclude that W_m^- stablizes for even m starting at $m = 8$ and for odd m starting at $m = 5$. Also, we have

$$\dim\left(W_{2m_0}^- / W_0^-\right) = 1+3+6+7 = 17$$

and

$$\dim\left(W_{2m_1+1}^- / W_1^-\right) = 1+2 = 3\ .$$

Therefore, there is a 20-parameter family of formal solutions for any choice of initial metric, trace-free part of x_{2+}, and initial shape operator. Of course, there is, up to homothety, a 4-parameter family of metrics and a 1-parameter family of the trace-free part of x_{2+} from which to choose the initial data. Notice that again since $S^2(\mathfrak{p}_-)$ does not contain a copy of V, the singular orbit must be totally geodesic in E in all cases.

Finally, we give an example to show that stablization of W_m^- can occur at arbitarily high order m. This means that one has to prescribe initial derivatives of arbitrarily high order to determine a unique solution for the $(-)$-part of the metric.

Example 5.3. $G = Sp(\frac{m+1}{2}) \times Sp(\frac{m+1}{2})$ with m odd, $H = SU(2)\times SU(2)$ embedded by the representation m of $SU(2)$ in each factor, a symplectic representation, and $K = SU(2)$ embedded diagonally into H. The manifold is then the associated 4-plane bundle of a 3-sphere bundle over the product manifold $N \times N$, where $N = Sp(\frac{m+1}{2})/SU(2)$.

The representation V (as an H-module) is $1\otimes 1$ and as a K-module it is $2\oplus 0$. The isotropy representation of N is $S^2(m) - 2 = 2m \oplus (2m-4) \oplus \cdots \oplus 6$ (since m is odd). Note that the factor 2 is the adjoint representation of $SU(2)$ and must be removed. Hence as an H-module, $\mathfrak{p}_-$ is

$$(2m\otimes 0)\oplus\cdots\oplus(6\otimes 0)\oplus(0\otimes 2m)\oplus\cdots\oplus(0\otimes 6)\ .$$

Restricted to K, $\mathfrak{p}_-$ becomes

$$2(2m)\oplus\cdots\oplus 2(6)\ ,$$

and so condition A is clearly satisfied. Now $S^2(\mathfrak{p}_-)$ contains $2m\otimes 2m$ as an irreducible H-summand as does $S^{2m}V$. In fact, the summands in $S^2(\mathfrak{p}_-)$ are of two types. First, there are the ones from taking S^2 of the irreducible factors. These are representations either of $SU(2)\times\{1\}$ or of $\{1\}\times SU(2)$. The other summands are of the form $(2m-4k)\otimes(2m-4l)$. On the other hand, the summands in S^kV are of the form $p\otimes p$ for some p by exercise 19.21 in [13]. Hence, stablization starts with S^{2m} for W_{2m}^- with $\dim(W_{2m}^-/W_0^-) = m_0$, where $m = 2m_0+1$. Of course, $W_{2q+1}^- = 0$ for all q.

6. Real solution

In Section 5 we showed that the initial value problem given by Equations (5.1a), (5.1b), (5.1c), (5.1d) always has a formal power series solution, although not necessarily a unique one. Furthermore, by construction, the formal solution series has the property that if it is truncated at any order, the resulting "polynomial metric" corresponds to a smooth G-invariant Riemannian metric on $E = G \times_H V$. In this section, we continue to use the same notation as that in Section 5. By Theorem 7.1 in [17] (see pp. 164–165), one then obtains a real solution of the equations defined on a small interval $[0, T]$. In order to make this paper more self-contained, we outline below along the lines of [12] the Picard iteration argument that produces the real solution in our case, and then we give a short argument to show that the real solution corresponds to a *smooth* G-invariant Einstein metric in a small tube around the zero section of E.

Recall that we are considering the system

$$x(0) = a, \;\; x' \;=\; 2y\,, \tag{6.1a}$$

$$y(0) = b, \;\; y' \;=\; \frac{1}{t^2}A(x) + \frac{1}{t}B(t, x, y) \tag{6.1b}$$

for functions $x, y : [0, T] \to S^2(\mathfrak{p})$ for some fixed $T > 0$ where $a = I$ and $b = (0, L_1)$. We have absorbed the term $C(t, x, y)$ into B at the expense of making B depend explicitly on t as well. Note that A and B are bounded and uniformly Lipschitz in the variables x, y (i.e., with Lipschitz constant L independent of t) as long as $x - a$ and $y - b$ are bounded by some sufficiently small constant, say 2ϵ. As usual, we transform (6.1a) and (6.1b) into a system of integral equations

$$x(t) \;=\; a + \int_0^t 2y(s)ds\,, \tag{6.2a}$$

$$y(t) \;=\; b + \int_0^t \left(\frac{1}{s^2}A(x) + \frac{1}{s}B(s, x, y)\right) ds\,. \tag{6.2b}$$

For a fixed positive integer m, let (x_m, y_m) be a *solution of order* m of our initial value problem (6.1a) and (6.1b). By this we mean a pair of $S^2(\mathfrak{p})$-valued polynomials x_m, y_m of degree $m + 1$ and m, respectively, such that

$$x_m(0) = a, \;\; x_m' \;=\; 2y_m + O(m + 1)\,, \tag{6.3a}$$

$$y_m(0) = b, \;\; y_m' \;=\; \frac{1}{t^2}A\left(x_m\right) + \frac{1}{t}B\left(t, x_m, y_m\right) + O(m)\,. \tag{6.3b}$$

Here, $O(p)$ denotes an arbitrary $S^2(\mathfrak{p})$-valued function α on $[0, T]$ such that $\alpha(t)/t^p$ is bounded. Let us write

$$\xi = x - x_m, \;\; \eta = y - y_m\,.$$

Then our integral Equation (6.2a) and (6.2b) takes the form $(\xi, \eta) = \Theta(\xi, \eta)$ where $\Theta = (\Theta_1, \Theta_2)$ is defined as follows:

$$\Theta_1(\xi, \eta)(t) \;=\; a - x_m(t) + \int_0^t 2\left(y_m + \eta\right)(s)ds\,,$$

$$\Theta_2(\xi, \eta)(t) \;=\; b - y_m(t) + \int_0^t \left(\frac{1}{s^2}A\left(x_m + \xi\right) + \frac{1}{s}B\left(s, x_m + \xi, y_m + \eta\right)\right) ds\,,$$

Let $\mathcal{O}(p)$ denote the Banach space of all $S^2(\mathfrak{p})$-valued continuous functions α on $[0, T]$ of Type

$O(p)$ with the norm

$$\|\alpha\|_p = \sup_{[0,T]} \frac{|\alpha(t)|}{t^p} ,$$

and let $\mathcal{B} = \mathcal{O}(m+1) \times \mathcal{O}(m)$ with norm

$$\|(\xi, \eta)\| = \|\xi\|_{m+1} + \|\eta\|_m .$$

We claim that the operator Θ defined above is a contraction on the closed ϵ-ball $\mathcal{B}_\epsilon \subset \mathcal{B}$ provided that T is small and m large enough. In fact, if $(\xi, \eta) \in \mathcal{B}$, then by (6.3a) and (6.3b) and the Lipschitz property of A and B we have

$$\begin{aligned} 2\,(y_m + \eta) &= x_m' + O(m) , \\ \frac{1}{t^2} A\,(x_m + \xi) + \frac{1}{t} B\,(t, x_m + \xi, y_m + \eta) &= y_m' + O(m-1) , \end{aligned}$$

as long as $|x_m + \xi - a|$, $|y_m + \eta - b| \le 2\epsilon$. Moreover, for $(\xi, \eta) = (0, 0)$, we may replace $O(m)$ by $O(m+1)$ and $O(m-1)$ by $O(m)$. Choose $T \in (0, 1]$ (depending on m) so small that $|x_m - a|$, $|y_m - b| \le \epsilon$ on $[0, T]$ and let $(\xi, \eta) \in \mathcal{B}_\epsilon$. Integration raises the order by one, hence $\Theta(\xi, \eta) \in \mathcal{B}$, and $\|\Theta(0, 0)\|$ is arbitrary small, say $\|\Theta(0, 0)\| < \epsilon/2$, if T is chosen small enough.

Further, if (ξ, η), $(\tilde{\xi}, \tilde{\eta}) \in \mathcal{B}_1$, we obtain from $|x_m - a|$, $|y_m - b| \le \epsilon$ and the Lipschitz property of A and B:

$$\begin{aligned} \left\| \Theta_1(\xi, \eta) - \Theta_1(\tilde{\xi}, \tilde{\eta}) \right\|_{m+1} &\le \frac{2C}{m+1} \|(\xi, \eta) - (\tilde{\xi}, \tilde{\eta})\| , \\ \left\| \Theta_2(\xi, \eta) - \Theta_2(\tilde{\xi}, \tilde{\eta}) \right\|_{m} &\le \frac{L}{m} \|(\xi, \eta) - (\tilde{\xi}, \tilde{\eta})\| , \end{aligned}$$

where C is some upper bound of $|a|$ and $|b|$. Recall that ϵ and L depend only on A and B, not on m. Now if we had chosen m so large that $\frac{2C}{m+1}$, $\frac{L}{m} \le 1/4$, then Θ becomes a contraction with factor $1/2$ on $\mathcal{B}_\epsilon$, and in particular, it maps $\mathcal{B}_\epsilon$ into itself. Therefore, we find a fixed point (ξ^*, η^*) of Θ which gives a solution $(x_m + \xi^*, y_m + \eta^*)$ of (6.2a) and (6.2b), and hence of (6.1a) and (6.1b).

Notice that $x_m(t)$ defines, by its construction in Section 5, a smooth G-invariant metric on the open normal tube of radius T about Q. On the other hand, ξ^* may be regarded as a smooth map

$$\xi^* : [0, T) \times S^k \longrightarrow S^2\left(V \oplus \mathfrak{p}_-\right) ,$$

where $\xi^*(t, v) = \xi^*(t)(v)$. Now ξ^* vanishes at $t = 0$ up to order $m+1$. So, using this fact and the chain rule, one can easily verify by counting orders of vanishing that the map

$$\hat{\xi}^* : V \setminus 0 \longrightarrow S^2\left(V \oplus \mathfrak{p}_-\right), \ \hat{\xi}^*(v) = \xi^*\left(|v|, \frac{v}{|v|}\right)$$

extends to V as a C^m map.

Since we could have chosen m to be at least 3, the solution x gives a C^3 G-invariant metric satisfying the appropriate equations. Therefore, by Corollary 2.6 in Section 2, x gives a smooth

G-invariant Einstein metric on the open normal tube of radius T about the singular orbit Q. This completes the proof of our main theorem.

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