Slope rotatability over all directions designs for $k \geq 4$

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SUMMARY *Slope rotatability over all directions (SROAD) is a useful concept when the slope of a second-order response is to be studied. SROAD designs ensure that knowledge of the slope is acquired symmetrically, whatever direction later becomes of more interest as the data are analyzed. In a prior paper, we explored designs for $k = 2$ and 3 dimensions, which do not have the full symmetries of second-order designs but which still possess the SROAD property. Here, we discuss designs in higher dimensions. The introductory sections 1 and 2 are essentially identical to those of the prior paper.*

1 Introduction

In some response surface applications, attention focuses on the estimation of differences in response, or slopes rather than the absolute value of the response variable. It is then natural to consider the variance measure for the slope of the fitted surface at any given point. On the assumption that equal information in all directions about the design origin was important, Hader and Park (1978) introduced the idea of slope rotatability and discussed it in the context of central composite designs. Later, Park (1987) introduced the concept of second-order slope rotatability over all directions (SROAD), and gave necessary and sufficient conditions for a design to have this property, based on the precision matrix. However, only a few simple types of SROAD design have been discussed in the literature. The purpose of this paper is to investigate the moment structures of SROAD designs in detail for dimensions higher than three, and so to find alternative SROAD designs.

2 SROAD

Let us suppose that the response variable y satisfies a functional relationship in k predictor variables, $x_1, x_2, \ldots, x_k$, of the general form

$$y_i = \eta(\mathbf{x}_i) + \varepsilon_i$$

where y_i is the observed response taken at a selected combination, $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ki})'$ of the predictor variables $\mathbf{x} = (x_1, x_2, \ldots, x_k)'$ and $i = 1, 2, \ldots, N$. The ε_i terms are assumed to be uncorrelated random errors with zero means and constant variance σ^2. We assume that η can be represented adequately in a restricted region of interest by a second-order polynomial, i.e.

$$\eta(\mathbf{x}) = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \sum_{j \geq i}^{k} \beta_{ij} x_i x_j = \mathbf{z}_{\mathbf{x}}'\boldsymbol{\beta} \tag{1}$$

where $\mathbf{z}_{\mathbf{x}}' = (1, x_1, x_2, \ldots, x_k, x_1^2, x_2^2, \ldots, x_k^2, x_1x_2, \ldots, x_{k-1}x_k)$ and $\boldsymbol{\beta}$ is the vector of correspondingly subscripted coefficients. The least-squares estimator of $\boldsymbol{\beta}$ is $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, where $\mathbf{X}$ is the matrix of values of $\mathbf{z}_{\mathbf{x}}'$ taken at the N design points and $\mathbf{y}$ is the $N \times 1$ vector of y observations. The variance–covariance matrix of $\mathbf{b}$ is $\text{var}(\mathbf{b}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$. It is traditional to call $N^{-1}\mathbf{X}'\mathbf{X}$ the 'moment matrix' and its inverse $N(\mathbf{X}'\mathbf{X})^{-1}$ the 'precision matrix'.

The first derivatives (slopes) of $\eta(\mathbf{x})$ at a general point $\mathbf{u} = (u_1, \ldots, u_k)'$ in the x space are

$$\left.\frac{\partial \hat{y}}{\partial x_i}\right|_{\mathbf{u}} = b_i + 2b_{ii}u_i + \sum_{j \neq i} b_{ij}u_j \tag{2}$$

Let us denote the estimated slope vector by

$$\mathbf{g}(\mathbf{u}) = \left(\frac{\partial \hat{y}}{\partial x_1}, \frac{\partial \hat{y}}{\partial x_2}, \ldots, \frac{\partial \hat{y}}{\partial x_k}\right)_{\mathbf{u}}' = \mathbf{A}\mathbf{b} \tag{3}$$

say. Thus, the estimated derivative at any point $\mathbf{u}$ in the direction specified by a $k \times 1$ vector of direction cosines, i.e. $\mathbf{c}' = (c_1, c_2, \ldots, c_k)$, is $\mathbf{c}'\mathbf{g}(\mathbf{u})$, where $\mathbf{c}'\mathbf{c} = 1$. The variance of this is

$$V_c(\mathbf{u}) = \text{var}[\mathbf{c}'\mathbf{g}(\mathbf{u})] = \sigma^2\mathbf{c}'\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'\mathbf{c} \tag{4}$$

The (integrated) average value of $V_c(\mathbf{u})$ over all possible directions, i.e. the averaged slope variance, is

$$\bar{V}(\mathbf{u}) = \frac{\sigma^2}{k}\,\text{tr}[\mathbf{A}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{A}'] \tag{5}$$

(see Park, 1987, pp. 450–451). $\bar{V}(\mathbf{u})$ is a function of the point $\mathbf{u}$, through $\mathbf{A}$, and also is a function of the design, through $\mathbf{X}$. A k-dimensional design is said to be SROAD if $\bar{V}(\mathbf{u})$ at the point $\mathbf{u}$ depends only on the distance of the point $\mathbf{u}$ from the design origin (Park, 1987). The necessary and sufficient conditions for a second-order design to be SROAD are as follows (Park, 1987):

(1) $2\,\mathrm{cov}(b_i, b_{ii}) + \sum_{j=1, j\neq i}^{k} \mathrm{cov}(b_j, b_{ij}) = 0$, for all i
(2) $2[\mathrm{cov}(b_{ii}, b_{ij}) + \mathrm{cov}(b_{jj}, b_{ij}) + \sum_{t=1, t\neq i,j}^{k} \mathrm{cov}(b_{it}, b_{jt}) = 0$,
for any (i, j), when $i \neq j$
(3) $4\,\mathrm{var}(b_{ii}) + \sum_{j=1, j\neq k}^{k} \mathrm{var}(b_{ij})$, equal for all i

Some designs given by Park (1987, p. 452) followed from his corollary.

Corollary 1. If the following moment conditions are satisfied, then the design is slope rotatable over all directions:

(1) all odd-order moments are 0 (i.e. $[i] = [ij] = [iij] = [iii] = [iiij] = \ldots = 0$);
(2) $[ii]$ are equal for all i;
(3) $[iiii]$ are equal for all i;
(4) $[iijj]$ are equal for all $i \neq j$.

The quantities in square brackets denote moments of the design. For example, we have $N^{-1}\sum_{u=1}^{N} x_{iu} = [i]$, $N^{-1}\sum_{u=1}^{N} x_{iu}^2 x_{ju} = [iij]$, and so on. If any subscript $i, j, \ldots$, appears an odd number of times, then the moment is odd; otherwise, it is even.

This corollary is framed using moment conditions for specific types of design that are known to be rotatable (i.e. $\mathrm{var}(\hat{y}(\mathbf{u}))$ is a function only of $\mathbf{u}'\mathbf{u}$; see Box & Hunter, 1957, p. 205) or slope rotatable over axial directions (i.e. $\mathrm{var}((\partial\hat{y}/\partial x_i)|_{\mathbf{u}})$ is a function only of $\mathbf{u}'\mathbf{u}$ and $\mathrm{var}((\partial\hat{y}/\partial x_i)|_{\mathbf{u}})$ are the same for all $i = 1, \ldots, k$; see Hader & Park, 1978, p. 414). The corollary thus implies that we have the following relationship among designs:

$$\text{rotatable} \subset \text{slope rotatable over all directions}$$

This means that the class of SROAD designs contains the class of rotatable designs. Thus, there must be a much wider choice of SROAD designs. In particular, we have found some with unbalanced moment structures and some with unbalanced point arrangements. Odd-order moments of an SROAD design could be non-zero. The types of SROAD designs vary from dimension to dimension. We discuss $k \geq 4$ dimensions here, giving examples of designs that are SROAD but which do not belong to the classes of rotatable designs and designs that are slope rotatable over axial directions. None of these designs has been given by previous authors.

3 Four-dimensional SROAD designs

In four dimensions, we have found some new types of moment structures for SROAD designs that are different from those in two and three dimensions. The results are summarized in Lemma 1 below. This lemma is obtained by finding a class of designs that satisfies the conditions on the precision matrix and inverting that matrix symbolically via the MAPLE matrix program to obtain conditions on the moment matrix.

Lemma 1. Suppose that the moments of a design satisfy the following

(1) all odd-order moments, except [123] and [1234], are 0;
(2) [33] = [22] = [11] = λ_{11};
(3) [3333] = [2222] = [1111] = λ_{1111};
(4) [3344] = [1122] = λ_{1122};
(5) [2244] = [1133] = λ_{1133};
(6) [2233] = [1144] = λ_{1144}.

Then, the design is SROAD if and only if its moments satisfy the following equations:

$$\lambda_{123}\lambda_{1234}\sum_{s=2}^{4}(\lambda_{11ss}^{2}\lambda_{11}-\lambda_{11ss}\lambda_{123}^{2}-\lambda_{1234}^{2}\lambda_{11})^{-1}=0 \tag{6}$$

$$\begin{aligned}&4(\lambda_{1144}-\lambda_{11jj})[(\lambda_{11jj}+\lambda_{1144})(\lambda_{11}^{2}-\lambda_{44}^{2}-\lambda_{1111}+\lambda_{4444})\\&\quad+2\lambda_{11}\lambda_{44}(\lambda_{1111}+\lambda_{11ii})-2\lambda_{11}^{2}(\lambda_{4444}+\lambda_{11ii})]/\varphi\\&\quad+\lambda_{123}^{2}[(\lambda_{11jj}^{2}\lambda_{11}-\lambda_{11jj}\lambda_{123}^{2}-\lambda_{1234}^{2}\lambda_{11})^{-1}\\&\quad-(\lambda_{1144}^{2}\lambda_{11}-\lambda_{1144}\lambda_{123}^{2}-\lambda_{1234}^{2}\lambda_{11})^{-1}]=0\end{aligned} \tag{7}$$

for $i, j=2, 3$ and $i \neq j$. (There are two parts to equation (7), i.e. $(i, j)=(2, 3)$ and $(i, j)=(3, 2)$.) Also

$$\begin{aligned}&4(\lambda_{1144}-\lambda_{1111})[(\lambda_{1111}+\lambda_{1144})(\lambda_{11}^{2}-\lambda_{44}^{2}-\lambda_{1111}+\lambda_{4444})\\&\quad+2\lambda_{11}\lambda_{44}(\lambda_{1122}+\lambda_{1133})+2\lambda_{11}^{2}(\lambda_{1111}-\lambda_{4444}-\lambda_{1122}-\lambda_{1133})]/\varphi\\&\quad+\lambda_{123}^{2}\sum_{s=2}^{3}(\lambda_{11ss}^{2}\lambda_{11}-\lambda_{11ss}\lambda_{123}^{2}-\lambda_{1234}^{2}\lambda_{11})^{-1}=0,\end{aligned} \tag{8}$$

should hold, where

$$\begin{aligned}\varphi=&\lambda_{1111}(\lambda_{1111}+\lambda_{4444}-\lambda_{44}^{2})\sum_{s=2}^{4}\lambda_{11ss}^{2}+\lambda_{1111}^{3}(\lambda_{44}^{2}-\lambda_{4444})\\&-(\lambda_{1122}^{2}-\lambda_{1133}^{2}-\lambda_{1144}^{2})^{2}-\lambda_{11}^{2}\lambda_{4444}(\lambda_{1122}-\lambda_{1133}-\lambda_{1144})^{2}\\&+\lambda_{1122}\lambda_{1133}\lambda_{1144}[2(\lambda_{44}^{2}-\lambda_{4444})+6(\lambda_{11}^{2}-\lambda_{1111})]\\&+4\lambda_{1133}^{2}\lambda_{1144}^{2}-2\lambda_{11}\lambda_{44}\sum_{s=2}^{4}\lambda_{11ss}\sum_{t=1}^{4}(-1)^{I_{t+s}}\lambda_{11tt}\\&-2\lambda_{1111}\sum_{s\neq t\neq 1}\lambda_{11ss}\lambda_{11tt}-2\lambda_{11}^{2}\Bigg\{\lambda_{1111}\sum_{s=2}^{4}\lambda_{11ss}^{2}\\&+\sum_{s=2}^{4}\lambda_{11ss}\sum_{t=2}^{4}(-1)^{I_{t+s}}\lambda_{11tt}-\lambda_{4444}(3\lambda_{1111}^{2}/2+2\lambda_{1133}\lambda_{1144})\\&+\lambda_{1111}\left[\lambda_{4444}\sum_{s=2}^{4}\lambda_{11ss}-\lambda_{1122}(\lambda_{1133}+\lambda_{1144})-\lambda_{1133}\lambda_{1144}\right]\Bigg\}\end{aligned}$$

and $I_{(t=s)}$ is an indicator function such that

$$I_{(t=s)} = \begin{cases} 1, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases}$$

This lemma shows that the even moments $[iijj]$ of a four-dimensional SROAD design would be separated into several groups and other same-order even moments would not all be equal. Some special cases of this lemma are shown in the examples below.

Case 1. A simple subcase of Lemma 1 is that a design is SROAD if, additionally, $[123]=0$, $[ii]$ are all equal and $[iiii]$ are all equal. Equations (6)–(8) are then all satisfied.

Example 1. A design with the following points is SROAD, for any a, b, α and n_0:

(1) $\frac{1}{2}S(a, a, a, a)$ with $\mathbf{I} = -1234$ (eight points);
(2) $\frac{1}{2}S(b, b, b, b)$ with $\mathbf{I} = 1234$ (eight points);
(3) $S(\alpha, 0, 0, 0)$ (eight points);
(4) n_0 center points.

The notation $S(a, a, a, a)$ denotes the 16 factorial points $(\pm a, \pm a, \pm a, \pm a)$. Thus, set (1) is a 2_{III}^{4-1} design. $S(\alpha, 0, 0, 0)$ denotes the $2k$ axial points $(\pm\alpha, 0, 0, 0)$, …, $(0, 0, 0, \pm\alpha)$. For this example, $N = 24 + n_0$, $[ii]N = 8(a^2+b^2)+2\alpha^2$ and $[iiii]N = 8(a^4+b^4)+2\alpha^4$, for all i; $[iijj]N = 8(a^4+b^4)$ for all $i \neq j$; and $[1234]N = 8(a^4-b^4)$. When $a=b=1$, this design becomes a 2^4 central composite design.

Example 2. Draper and McGregor (1967) gave an infinite class of second-order rotatable designs which combined point sets with non-equal fourth moments $[iijj]$ for four dimensions. We can use part of such a design and add a set of axial points. Our selected design contains the following points.

(1) 32 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u})$ with coordinates chosen from

$$(\pm x, \pm y, \pm z, \pm w)$$
$$(\pm y, \pm z, \pm w, \pm x)$$
$$(\pm z, \pm w, \pm x, \pm y)$$
$$(\pm w, \pm x, \pm y, \pm z)$$

where only combinations of signs are used which make $x_{1u}x_{2u}x_{3u}x_{4u} = xyzw$; an alternative description is that it is generated by the cyclic permutation $\{(x, y, z, w) \Rightarrow (y, z, w, x) \Rightarrow (z, w, x, y) \Rightarrow (w, x, y, z)\}$ of the matrix $\mathbf{D}_1$, i.e.

$$\mathbf{D}_1 = \begin{pmatrix} -x & -y & -z & -w \\ x & y & -z & -w \\ -x & y & z & -w \\ x & -y & z & -w \\ -x & -y & z & w \\ x & -y & -z & w \\ -x & y & -z & w \\ x & y & z & w \end{pmatrix}$$

(2) eight axial points with distance α;
(3) n_0 center points.

For this design, $N = 40 + n_0$, $[ii]N = 8(x^2 + y^2 + z^2 + w^2) + 2\alpha^2$, $[iiii]N = 8(x^4 + y^4 + z^4 + w^4) + 2\alpha^4$ and $[iijj]$ are separated into two groups, i.e. $[iijj]N = 16(x^2z^2 + y^2w^2)$ for $(i, j) = (1, 3), (2, 4)$ and $[iijj]N = 8(x^2 + z^2)(y^2 + w^2)$ for $(i, j) = (1, 2), (2, 3), (3, 4), (1, 4)$; $[1234]N = 32xyzw$. The design is SROAD for any values of x, y, z, w and α, and any n_0 (as is the full set of Draper and McGregor's design).

Example 3. Let us suppose a design contains the following points.

(1) 32 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u})$ with coordinates

$$(\pm x, \pm y, \pm z, \pm w)$$
$$(\pm y, \pm z, \pm w, \pm x)$$
$$(\pm z, \pm w, \pm x, \pm y)$$
$$(\pm w, \pm x, \pm y, \pm z)$$

where only combinations of signs are used which make $x_{1u}x_{2u}x_{3u}x_{4u} = xyzw$;
(2) 32 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u})$, similar to point (1), with coordinate values a, b, c, d replacing (x, y, z, w);
(3) eight axial points with distance α;
(4) n_0 center points.

In this case, $N = 72 + n_0$, $[ii]N = 8(x^2 + y^2 + z^2 + w^2 + a^2 + b^2 + c^2 + d^2) + 2\alpha^2$, $[iiii]N = 8(x^4 + y^4 + z^4 + w^4 + a^4 + b^4 + c^4 + d^4) + 2\alpha^4$ and $[iijj]$ are separated into three groups, i.e. $[iijj]N = 16(x^2z^2 + y^2w^2) + 8(a^2 + d^2)(b^2 + c^2)$ for $(i, j) = (1, 3), (2, 4)$; $[iijj]N = 16(a^2d^2 + b^2c^2) + 8(x^2 + z^2)(y^2 + w^2)$ for $(i, j) = (1, 4), (2, 3)$; and $[iijj]N = 8[(x^2 + z^2)(y^2 + w^2) + (a^2 + d^2)(b^2 + c^2)]$ for $(i, j) = (1, 2), (3, 4)$; $[1234]N = 32(xyzw + abcd)$. The design is SROAD for any values of x, y, z, w, a, b, c, d and α, and any n_0.

Case 2. Another subcase of Lemma 1 occurs when, additionally, $[1234] = 0$ and the $[iijj]$ are all equal. In this case, equations (6) and (7) are satisfied, and equation (8) reduces to

$$4[(\lambda_{1122} + \lambda_{1111})(\lambda_{1111} - \lambda_{4444} + \lambda_{44}^2) + 2\lambda_{11}(\lambda_{11}\lambda_{4444} - 2\lambda_{44}\lambda_{1122}) + 3\lambda_{11}^2(\lambda_{1122} - \lambda_{1111})]/\varphi + 2\lambda_{123}^2/(\lambda_{1122}^2\lambda_{11} - \lambda_{1122}\lambda_{123}^2) = 0 \qquad (9)$$

where λ_{1122} denotes $[iijj]$ and

$$\varphi=(\lambda_{1111}-\lambda_{1122})\{\lambda_{1122}[3(\lambda_{1122}-2\lambda_{11}\lambda_{44})+2(\lambda_{44}^2-\lambda_{4444})] + 3\lambda_{11}^2\lambda_{4444}+\lambda_{1111}(\lambda_{44}^2-\lambda_{4444})\}$$

Example 4. Let us consider the design containing {a 2^{4-1} design with $\mathbf{I}=123$} plus {star points with distance α for factors 1, 2 and 3 and γ for factor 4} plus {n_0 center points}:

$$\mathbf{D}=\begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ -\alpha & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The design is essentially an extension into $k=4$ dimensions of Hartley's $k=3$ design (see Ying *et al.*, 1995). For this design, $[11]N=[22]N=[33]N=2(4+\alpha^2)$, $[44]N=2(4+\gamma^2)$; $[1111]N=[2222]N=[3333]N=2(4+\alpha^4)$, $[4444]N=2(4+\gamma^4)$; and $[iijj]N=[123]N=8$, for all $i\neq j$. The design is SROAD if and only if

$$[\alpha^6(n_0+8)+2\alpha^4(n_0-14)+4\alpha^2(3n_0+2)+16(n_0+6)]\gamma^4 - 16\alpha^2(\alpha^4-\alpha^2-4)\gamma^2-2\alpha^4[(n_0+10)(\alpha^4+8)-2\alpha^2(n_0+26)]=0 \qquad (10)$$

Values of γ which make the design SROAD versus α for selected n_0 are shown in Table 1 and Fig. 1. As was the case for the Hartley designs, rotation of the x axes can make all the third- and fourth-order odd moments of the above SROAD designs non-zero (after rotation).

4 k-Dimensional SROAD designs ($k\geq 5$)

Some results in three and four dimensions can be extended to higher dimensions.

Case 1. This is for dimension $k=3p$, where p is an integer. If the following moment conditions are satisfied, then the design is SROAD:

(1) all odd-order moments with order less than 5 except $[(3j-2)(3j-1)(3j)]$, $j=1,\ldots,p$, are 0;

TABLE 1. Values of γ for selected α and n_0 for example 4

α	γ				
	$n_0=1$	$n_0=2$	$n_0=3$	$n_0=4$	$n_0=5$
1.00	0.806	0.808	0.810	0.811	0.812
1.25	0.922	0.929	0.933	0.936	0.938
1.50	1.081	1.180	1.079	1.079	1.078
1.75	1.391	1.328	1.296	1.278	1.266
2.00	1.714	1.601	1.539	1.501	1.475
2.25	1.932	1.816	1.747	1.701	1.668
2.50	2.080	1.978	1.912	1.866	1.831

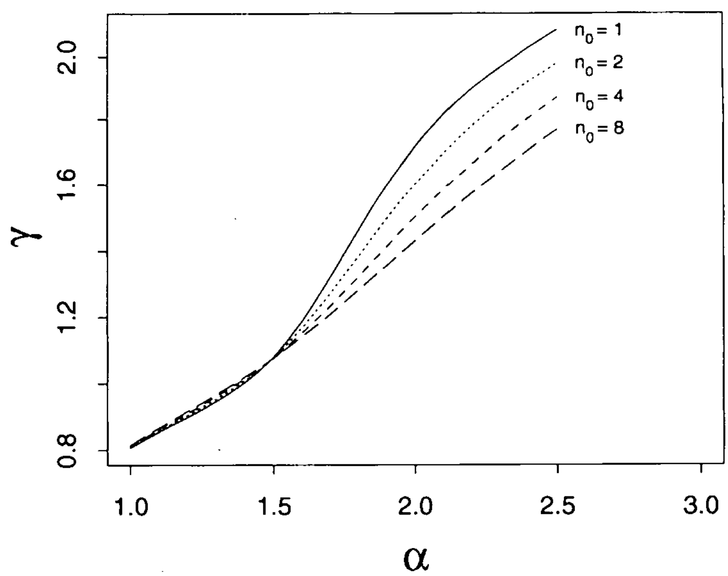

FIG. 1. Plot of the values of γ versus α for selected n_0 for example 4.

(2) $[(3j-2)(3j-1)(3j)]$ are equal for all $j=1, \ldots, p$;
(3) $[ii]$ are equal for all i;
(4) $[iiii]$ are equal for all i;
(5) $[iijj]$ are equal for all $i \neq j$.

(The proof is omitted.) The $k=3p$ Hartley designs which consist of a 2^{k-p} design with $\mathbf{I}=123=456=\ldots=(3p-2)(3p-1)3p$, plus $2k$ star points at distance α, plus n_0 center points, satisfy the above moment conditions. The Hartley designs can also be regarded as members of a larger specific general class involving 2^{k-p} fractional factorial designs.

Case 2. Let us consider designs that contain the following:

(1) a 2^{k-p} design with $p=[k/3]$ (the integer part of $k/3$) and $\mathbf{I}=123=456=\ldots=(3p-2)(3p-1)3p$, where $k \geq 3$;

(2) star points with distance α for factors $1, \ldots, 3p$ and γ for factors $3p+1, \ldots, k$;
(3) n_0 center points.

Then, if $k=3p$, the design is a Hartley design and is SROAD for any α. However, for $k=3p+1$, the design is SROAD if α, γ and n_0 satisfy

$$\begin{aligned}&\{\alpha^6(n_0+2^{k-p})+2\alpha^4[n_0-(k-2)2^{k-p}+2]\\&\quad+2^{k-p-1}\alpha^2[(k-1)n_0+18p^2-24p+8]+2^{k-p}(k-2)(n_0+6p)\}\gamma^4\\&\quad-2^{k-p+1}\alpha^2\{\alpha^4-(k-3)\alpha^2-2(k-2)\}\gamma^2\\&\quad-2\alpha^4\{\alpha^4(n_0+2^{k-p}+2)-2^{k-p-2}\alpha^2(n_0+24p+2)\\&\quad+2^{k-p-1}(k-2)[n_0+2(k+1)]\}=0\end{aligned} \tag{11}$$

Example 5 $(k=7)$. Let us consider the 2_{III}^{7-2} design with $\mathbf{I}=123=456$. Values of γ for selected α and n_0 are listed in Table 2. The plot of these values would be similar to Fig. 1, except that the curves diverge later at about $\alpha=2.15$ (rather than at 1.5).

Also for case 2, if $k=3p+2$, then the design is SROAD when α, γ and n_0 satisfy

$$\begin{aligned}&\{\alpha^6(n_0+2^{k-p})+2\alpha^4[n_0-(k-3)2^{k-p}+2]\\&\quad+2^{k-p-1}\alpha^2[(k-2)n_0+18p^2-24p+8]+2^{k-p}(k-3)(n_0+6p)\}\gamma^8\\&\quad-2^{k-p+2}\alpha^2[\alpha^4-(k-4)\alpha^2-2(k-3)]\gamma^6\\&\quad-2\alpha^4\{\alpha^2(n_0+2^{k-p}+2)-2^{k-p-1}\alpha^2(n_0+12p+4)\\&\quad+2^{k-p-1}[(k-4)n_0+6p(k-1)-12)]\}\gamma^4\\&\quad+2^{k-p+2}\alpha^6(\alpha^2-3p)\gamma^2-2^{k-p}\alpha^8(n_0+4)=0\end{aligned} \tag{12}$$

Example 6 $(k=5)$. Let us consider the 2_{III}^{5-1} design with $\mathbf{I}=123$. We see that we again have an extension of the $k=3$ Hartley design—this time into five dimensions rather than four as previously. Equation (12) becomes

$$\begin{aligned}&[\alpha^6(n_0+16)+2\alpha^4(n_0-30)+8\alpha^2(3n_0+2)+32(n_0+6)]\gamma^8\\&\quad-64\alpha^2(\alpha^4-\alpha^2-4)\gamma^6-2\alpha^4[\alpha^4(n_0+18)-8\alpha^2(n_0+16)\\&\quad+8(n_0+12)]\gamma^2+64\alpha^6(\alpha^2-3)\gamma^2-16\alpha^8(n_0+4)=0\end{aligned} \tag{13}$$

TABLE 2. Values of γ for selected α and n_0 ($k=7$, $p=2$) for example 5

	α			
α	$n_0=1$	$n_0=2$	$n_0=4$	$n_0=8$
1.00	0.865	0.866	0.866	0.866
1.25	1.014	1.014	1.015	1.015
1.50	1.134	1.135	1.136	1.138
1.75	1.228	1.231	1.235	1.239
2.00	1.315	1.320	1.325	1.330
2.25	1.474	1.456	1.441	1.431
2.50	1.845	1.719	1.625	1.566

Some values of γ which make the design SROAD for selected α and n_0 are listed in Table 3. The plot of these values resembles Fig. 1 but the curves diverge now at $\alpha = 1.65$.

Proofs of the results for $k = 3p$, $k = 3p + 1$ and $k = 3p + 2$ are omitted.

Case 3. This is for SROAD designs with $[iijj]$ not all equal. In k dimensions ($k \geq 4$), a design is SROAD if the following moment conditions are satisfied.

(1) All odd-order moments up to and including four are 0.
(2) $[ii]$ are equal for all i and $[iiii]$ are equal for all i.
(3) $[iijj]$, $i \neq j$, are separated to n groups ($n < k$). Each group contains kp or $kp/2$ of $[iijj]$ with equal value τ_{4t}, $t = 1, \ldots, n$, where p is an integer, and each number of $1, \ldots, k$ appears exactly the same number of times in the $[iijj]$ of each group.
(4) Let $\mathbf{V}_t$ denote a $(k+1) \times (k+1)$ matrix with a 1 in each position $[(i+1), (j+1)]$, where $[iijj]$ belongs to the tth group, and 0 elsewhere, $t = 1, 2, \ldots, n$. Let $\mathbf{V}_{03}$ denote a $(k+1) \times (k+1)$ matrix with a 1 in each position (i, i), $i = 2, 3, \ldots, k+1$. Let $\mathbf{M}_1 = \sum_{t=1}^{n} \alpha_t \mathbf{V}_t$. For any real numbers α_t, $t = 1, \ldots, n$, there exist β_t, $t = 1, \ldots, n$, and β_{03} such that

$$\mathbf{M}_1^2 = \sum_{t=1}^{n} \beta_t \mathbf{V}_t + \beta_{03} \mathbf{V}_3$$

The moment matrix $(\mathbf{X}'\mathbf{X})/N$ of this type of design, and its inverse, have exactly the same structure, i.e., var (b_i) are equal for all i, var (b_{ii}) are equal for all i, var (b_{ij}) are equal for all i, j in the same group, cov (b_0, b_{ii}) are equal for all i, cov (b_{ii}, b_{jj}) are equal for all i, j in the same group, and the other covariances are 0.

Additionally, for $4p$ dimensions, we may have $[1234] = [5678] = \ldots = [(4p-3)(4p-2)(4p-1)4p]$ as non-zero; also, for $3p$ dimensions, we may have $[123] = \ldots = [(3p-2)(3p-1)3p]$ as non zero; where p is an integer.

Examples 1–3 satisfy these conditions for $k = 4$.

TABLE 3. Values of γ for selected α and n_0 ($k = 5$, $p = 1$) for example 6

	α			
α	$n_0 = 1$	$n_0 = 2$	$n_0 = 4$	$n_0 = 8$
1.00	0.851	0.852	0.853	0.854
1.25	1.001	1.002	1.004	1.006
1.50	1.137	1.139	1.141	1.143
1.75	1.280	1.278	1.277	1.275
2.00	1.472	1.445	1.423	1.409
2.25	1.807	1.673	1.594	1.550
2.50	2.085	1.921	1.781	1.698

Example 7. Let us consider a five-dimensional design with the following points:

(1) 80 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u}, x_{5u})$ with coordinates

$$(\pm x, \pm y, \pm z, \pm w, \pm s)$$
$$(\pm y, \pm z, \pm w, \pm s, \pm x)$$
$$(\pm z, \pm w, \pm s, \pm x, \pm y)$$
$$(\pm w, \pm s, \pm x, \pm y, \pm z)$$
$$(\pm s, \pm x, \pm y, \pm z, \pm w)$$

where only combinations of signs are used which make $x_{1u}x_{2u}x_{3u}x_{4u}x_{5u} = xyzws$;

(2) 10 axial points with distance α;

(3) n_0 center points.

In this case, $[ii]N = 16(x^2+y^2+z^2+w^2+s^2)+2\alpha^2$, $[iiii]N = 16(x^4+y^4+z^4+w^4+s^4)+2\alpha^4$, $[iijj]N = 16(z^2w^2+w^2s^2+x^2y^2+y^2z^2+s^2x^2)$ for $(i, j) = (1, 2)$, $(2, 3), (3, 4), (4, 5), (5, 1)$; and the other $[iijj]N = 16(x^2z^2+y^2w^2+z^2s^2+w^2x^2+s^2y^2)$. Other moments of orders up to and including four are 0. The design, with $(90+n_0)$ points, is SROAD for any values of x, y, z, w, s and α, and any number of center points.

Example 8. Let us consider this six-dimensional design:

(1) 48 points $(x_{1u}, x_{2u}, x_{3u}, x_{4u}, x_{5u}, x_{6u})$ with coordinates

$$(\pm x, \pm y, \pm z, \pm x, \pm y, \pm z)$$
$$(\pm y, \pm z, \pm x, \pm y, \pm z, \pm x)$$
$$(\pm z, \pm x, \pm y, \pm z, \pm x, \pm y)$$

where only combinations of signs are used which make $x_{1u}x_{2u}x_{3u} = x_{4u}x_{5u}x_{6u} = xyz$;

(2) 12 axial points with distance α;

(3) n_0 center points.

In this case, $[ii]N = 16(x^2+y^2+z^2)+2\alpha^2$, $[123]N = [456]N = 48xyz$, are non-zero odd moments; $[iiii]N = 16(x^4+y^4+z^4)+2\alpha^2$, $[iijj]N = 16(x^4+y^4+z^4)$ for $(i, j) = (1, 4)$, $(2, 5), (3, 6)$; and the other $[iijj]N = 16(x^2y^2+y^2z^2+z^2x^2)$. Other moments of orders up to and including four are zero. The design, which has $(60+n_0)$ points, is SROAD for any values of x, y, z and α, and any number of center points.

As we have seen, there exists a much wider range of SROAD designs than just the symmetrical designs given by Park.

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