Timelike incompleteness of spacetimes

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Abstract. We discuss a conjecture of R. Bartnik saying that a spacetime containing a compact spacelike acausal hypersurface S and satisfying the strong energy condition is static or timelike geodesically incomplete. We show that this is the Lorentzian analogue of a theorem of Cheeger and Gromoll which states that a Riemannian manifold with nonnegative Ricci curvature and two or more ends is geodesically incomplete or isometric to some cylinder. Transferring the Riemannian arguments to the Lorentzian case, we show that Bartnik's conjecture can be proved if S lies in the past of points with sufficiently large Lorentzian distance from S.

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Riemannian and Lorentzian geometry display many similarities. However, at a first glance , the singularity or incompleteness theorems which have been discovered by Hawking and Penrose (cf. [10]) seem to describe a phenomenon which only occurs in spacetime geometry. But this is not really true. As an example, consider the well known Splitting Theorem in Riemannian geometry (cf. [2], [7]):

Theorem. (J. Cheeger, D. Gromoll 1972) *Let (M,g) be a noncompact Riemannian manifold with* (a) *two or more ends and* (b) *nonnegative Ricci curvature. Then either (M,g) is geodesically incomplete or (M,g) is a cylinder:*

$$(M,g) = (\mathbb{R} \times M', dt^2 + g')$$

where (M', g') is a compact Riemannian manifold.

Recall that an *end* of a noncompact manifold M is roughly a connected component of the complement of a sufficiently large compact subset. More precisely, an end is a mapping E which assigns to each compact subset $K \subset M$ a connected component $E(K)$ of $M \setminus K$ such that $E(K') \subset E(K)$ whenever $K \subset K'$. *Nonnegative Ricci curvature* means that the Ricci tensor $R_{ij} = R^k_{ijk}$ is positive semi-definite. M is called *geodesically incomplete* if there are inextendible geodesics with an affine parameter defined on a proper subset of the real line.

For the proof, one assumes that (M,g) is geodesically complete. Then there are two steps.

Step 1. Choose diverging sequences (p_i), (q_i) which run into two different ends. "Divergence" means that no subsequences are contained in a compact subset. Let γ_i be a minimal (i.e. length-minimizing) geodesic connecting p_i and q_i; this exists by the completeness assumption (theorem of Hopf and Rinow). Since all γ_i are passing through the same compact region K separating the ends, they converge to a minimal geodesic γ which is inextendible. Hence by geodesic completeness, γ is a so called *line*, i.e. a complete geodesic minimizing length on each of its segments.

Step 2. Recall that on a Riemannian manifold, an interior metric (distance) d is defined as follows: The distance $d(p,q)$ between two points p,q is the infimum of the lengths of all curves connecting p and q. Let γ be a line. We consider two open metric balls B_+, B_- in M with large radius r and centered at points $p_+, p_- \in \gamma$ with distance $d(p_+, p_-) = 2r$. They are seperated by a point $p \in \gamma$ which has distance r from p_+ and p_-. Since γ is a line, the triangle inequality says that $B_+ \cap B_- = \emptyset$. On the other hand, the curvature condition ("nonnegative Ricci curvature") says that metric balls B of large radius r become mean concave in the limit $r \to \infty$. Recall that a domain is called *mean concave* if its complement is mean convex, and a domain is *mean convex* if at each boundary point, there is a smooth supporting hypersurface whose mean curvature vector η satisfies $g(N, \eta) \leq \epsilon$ where N is the outer unit normal vector (cf. [5]). E.g. in the euclidean unit sphere, the metric balls are concave for $r \geq \pi/2$. In euclidean space, the balls of any finite radius are strictly convex, but a "ball of infinite radius", i.e. the union of all balls with a common tangent hyperplane (a half space) is mean concave. In M we consider balls B_+ and B_- with larger and larger radii while fixing the separating point p; in fact we take the union over all such balls. These limit domains D_+ and D_- are still separated by p and mean concave. But no two mean concave domains are separated by a point unless their boundaries agree (maximum principle, cf. [5]). Hence we get an equidistant family of minimal hypersurfaces perpendicular to γ. This together with the curvature hypothesis shows that M is isometric to $\mathbb{R} \times M'$ with $M' = \partial D_+ = \partial D_-$ (see [7] for details). M' is compact since M has two ends.

There is an analogue in Lorentzian geometry where the two ends are replaced with the past and the future of a "closed universe":

Conjecture. (R. Bartnik, 1988, [1]) *Let (M, g) be a spacetime with* (a) *a compact acausal spacelike Cauchy hypersurface $S \subset M$ and* (b) *nonnegative timelike Ricci curvature. Then either (M, g) is timelike geodesically incomplete or (M, g) describes a static closed universe:*

$$(M, g) = (\mathbb{R} \times M', -dt^2 + g')$$

where (M', g') is a compact Riemannian manifold.

By a *spacetime* we mean a time oriented Lorentzian manifold. Recall that a subset $S \subset M$ is called *acausal* if there is no causal curve which starts and ends on S. It is called a *Cauchy set* if any inextendible causal (i.e. everywhere time-like or lightlike) curve meets S. *Nonnegative timelike Ricci curvature* means that

$R_{ij}v^iv^j \geq 0$ for any timelike tangent vector $v = (v^i)$ on M. This is sometimes called *strong energy condition.* So the conjecture can be rephrased as follows: *Any nonstatic compact universe with strong energy condition is timelike geodesically incomplete.* This would generalize the Singularity Theorem 2, Ch. 8.2 of [10]; it omits the "generic condition".

Which of the previous Riemannian arguments do work in the new Lorentzian context?

Step 2: Yes! *If (M, g) is a timelike geodesically complete spacetime with strong energy condition containing a timelike line, i.e. a complete maximal (i.e. length maximizing) timelike geodesic, then*

$$(M, g) = (\mathbb{R} \times M', -dt^2 + g')$$

where (M', g') is a complete Riemannian manifold. This was proved in [4] under the assumption that M is also globally hyperbolic. Later, G. Galloway [9] and R.P. Newman [11] showed that one can omit either the timelike geodesic completeness assumption or the global hyperbolicity. The idea of the proof is the same as in the Riemannian case; however one has to replace the Riemannian distance $d = d_R$ by $-d_L$ where d_L is the Lorentzian distance: For two points $p, q \in M$ with q in the causal future of p, the distance $d_L(p, q)$ is the supremum of the lengths of all causal curves from p to q. This "distance" satisfies the reversed triangle inequality, so the ordinary triangle inequality holds for $-d_L$. The metric balls which are sublevel sets of d_R are replaced by sublevel sets of $-d_L$, and basically the same arguments work.

Step 1: No! P. Ehrlich and G. Galloway gave a 2-dimensional counterexample [3]. What goes wrong with Step 1? We start as before: We assume that (M, g) is timelike geodesically complete and we choose divergent sequences (p_i) in the past and (q_i) in the future of S. If q_i is in the future of p_i, there is a maximal timelike geodesics γ_i from p_i to q_i; recall that our spacetime is globally hyperbolic since it contains a Cauchy hypersurface. Since all γ_i pass through the compact set S, they accumulate to an inextendible maximal geodesic γ. But we are facing two problems:

(a) Is q_i really in the future of p_i?

(b) Is γ timelike?

In fact, in the counterexample of Ehrlich and Galloway, we cannot choose (p_i) and (q_i) such that (a) becomes true, so we do not get the geodesics γ_i. Therefore, a further hypothesis is necessary:

Assumption (A): *There is a constant R such that S is in the past of any point q with $d_L(S, q) \geq R$.*

This assumption certainly settles Problem (a). But it turns out that also Problem (b) is solved so that Step 1 can be executed: The limit geodesic γ of the geodesic segments γ_i from p_i to q_i is an inextendible maximal timelike or null geodesic. Choose a future directed diverging sequence (r_j) on γ. Since S is compact, there exists a geodesic segment λ_j from r_j to S of maximal length. In particular, λ_j meets S perpendicularly. Again by compactness, these geodesics accumulate to a future inextendible geodesic λ which is still perpendicular to S and hence timelike. By timelike geodesic completeness, λ has infinite length, and therefore $d_L(S, r_j) \to \infty$. Thus $d_L(S, r_j) > R$ for large j, and therefore S is in the timelike past of r_j by Assumption (A). In particular, the point r_0 where γ meets S is in the timelike past of r_j, so $d_L(r_0, r_j) > 0$. Since γ is a maximal geodesic between r_0 and r_j, it must be timelike. Thus we have shown:

Theorem. (cf. [6]) *Let (M,g) be a timelike geodesically complete spacetime containing a compact acausal spacelike Cauchy hypersurface S satisfying* (A). *Then M contains a timelike line, i.e. a complete maximal timelike geodesic.*

Corollary. *The conjecture is true if one further assumes* (A).

Remark 1. In [6] actually a stronger statement was proved. It is sufficient instead of (A) to assume that S lies in the past of a future complete timelike geodesic λ starting on S and maximizing the distance from S to each of its points (λ is a so called *S-ray*). Now we have to choose the sequence (q_i) on λ. Moreover, fthe Cauchy property of S is not needed. By a previous argument of Galloway [8], we may choose the sequence (p_i) inside the past Cauchy domain $D^-(S)$; this is the set of points $p \in M$ such that any future inextendible causal curve starting at p meets S. Since now M is no longer globally hyperbolic, we might have no maximal geodesics from p_i to q_i, but we may use almost maximizing timelike curves γ_i whose length is not less then, say, $d_L(p_i, q_i) - \frac{1}{i}$. These curves also accumulate to a line.

Remark 2. The counterexample of Ehrlich and Galloway does not disprove the original Bartnik conjecture since it does not satisfy the strong energy condition. So, this conjecture is still open.

References

[1] Bartnik, R., Remarks on cosmological spacetimes and constant mean curvature surfaces, Commun. Math. Phys. 117 (1988), 615–624

[2] Cheeger, J., D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differential Geometry 6 (1971), 119–128

[3] Ehrlich, P.E., G.J. Galloway, Timelike lines, Class. Quantum Grav. 7 (1990), 297–307

[4] Eschenburg, J.-H., The splitting theorem for space-times with strong energy condition, J. Differential Geometry 27 (1988), 477–491

[5] Eschenburg, J.-H., Maximum Principle for hypersurfaces. Manuscripta Math. 64 (1989), 55–75

[6] Eschenburg, J.-H., G.J. Galloway, Lines in space-times, Commun. Math. Phys. 148 (1992), 209–216

[7] Eschenburg, J.-H., E. Heintze, An elementary proof of the Cheeger-Gromoll Splitting Theorem. Ann. Glob. Anal. and Geom. 2 (1984), 141–151

[8] Galloway, G.J., Curvature, causality and completeness in space-times with causally complete spacelike slices, Math. Proc. Camb. Phil. Soc. 99 (1986), 367–375

[9] Galloway, G.J., The Lorentzian Splitting Theorem without the completeness assumption. J. Differential Geometry 29 (1989), 373–387

[10] Hawking, S.W., G.F.R. Ellis, The large scale structure of space-time. Cambridge University Press, Cambridge, 1973

[11] Newman, R.P.A.C., A proof of the splitting conjecture of S.T.Yau, J. Differential Geometry 31 (1990), 163–184