CONTROL THEORY AND DYNAMICAL SYSTEMS

FRITZ COLONIUS
Institut für Mathematik
Universität Augsburg
86159 Augsburg, Germany

WOLFGANG KLIEMANN
Department of Mathematics
Iowa State University
Ames, Iowa 50011 USA

ABSTRACT. Control systems can be considered as dynamical systems on the space $\mathcal{U} \times M$, where $\mathcal{U}$ is the set of admissible control functions, and the manifold M is the state space of the control system. This point of view allows us to study in this survey paper single trajectories depending on the control and the initial value, and it results in a number of connections between control theory and the theory of dynamical systems. On the one hand, one obtains new results for control theoretic questions, on the other hand one finds new (explicit) examples for many concepts in dynamical systems. Of particular importance are concepts like topological transitivity, chain recurrence, limit sets, Morse decompositions, linearized flows, subbundle decompositions and spectral theory (Lyapunov exponents), and their control theoretic analogues. This approach also yields, among others, new perturbation theorems for ordinary differential equations. We will study in detail applications to the (feedback) control about hyperbolic fixed points and the controllability around the attractor in the Lorenz equations ('control of chaos').

1. Introduction

The analysis of complex behavior in dynamical systems has made rapid progress in the last years, and it has attracted considerable interest with mathematics (and also general scientific) audiences. In the center of this analysis is the study of nontrivial limit behavior (attractors, fractals, ergodic and entropy theory) and its parameter dependence (bifurcation theory, perburbation theory). The recent text books of Mañé [62], Ruelle [75], and Schuster [77] illustrate these points.

If a dynamical system (in continuous time) is given as an ordinary differential equation $\dot{x} = X_0(x)$, one can consider a family of differential equations

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x), \tag{1}$$

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where the X_i, $i = 1 \dots m$ are vector fields, and the u_i, $i = 1 \dots m$ are functions with values in $\mathbb{R}$, i.e. $(u_i)_{i=1\dots m} = u \in \mathcal{U} := \{u : \mathbb{R} \to U \subset \mathbb{R}^m\}$. The additional term on the right hand side of (1) can be interpreted in different ways, leading to different but closely connected mathematical problems:

(i) as global, time dependent perturbations of an ordinary differential equation by the vector fields $X_1 \dots X_m$. Here the perturbation is given, and one wants to study the limit behavior of the trajectories of (1) for $t \to \infty$ relative to the limit behavior of $\dot{x} = X_0(x)$. For $u \in \mathcal{U}$ time invariant and small one obtains the typical setup of perturbation theory.

(ii) as control, where the vectorfields $X_1 \dots X_m$ determine the input structure and $U \subset \mathbb{R}^m$ is the given control range. Here the problem is to choose among the admissible control functions $u \in \mathcal{U}$ one function in such a way that the system (1) has a particular behavior, e.g. reachability of a point y starting from x, stabilization at a fixed point or a periodic trajectory, sometimes with additional properties.

(iii) as parameter uncertainty, where the question is e.g.: Given $X_1 \dots X_m$, how large can the uncertainty range U be, such that a stable fixed point of the "nominal system" $\dot{x} = X_0(x)$ remains stable? This problem can be combined with (ii): Consider the equation $\dot{x} = X_0(x) + \Sigma u_i(t)X_i(x) + \Sigma v_j(t)Y_j(x)$, where the second term on the r.h.s. describes the control, the third term is the parameter uncertainty. Now we can study the question, for which uncertainty range $V \subset \mathbb{R}^n$, i.e. $(v_j)_{j=1,\dots,n} \in \mathcal{V} = \{v : \mathbb{R} \to V \subset \mathbb{R}^n\}$ is there a $u \in \mathcal{U}$, such that the system with this control u is stable for all $v \in \mathcal{V}$ (stabilization of uncertain systems).

(iv) as stochastic process, e.g. as "white noise", leading to an Itō stochastic differential equation. Here one can discuss e.g. questions about the existence of a stationary (ergodic) solution, about the law of large numbers, large deviations, etc.

In this paper we will discuss in particular the aspects (i) and (ii), and its relations to the theory of dynamical systems. The interpretation of (1) as a dynamical system (control flow) is the starting point of the analysis, which will lead, among others, to two quite close connections between control and chaos. With respect to (iii) and (iv) a few short remarks must suffice here.

The ergodic theory of Markov stochastic processes, e.g. as solutions of stochastic differential equations, has always been developed in close vicinity to the ergodic theory of dynamical systems (cp. e.g. Ventcel [93] and Mañé [62]) by analyzing the limit behavior of Markov transition semigroups on the corresponding function spaces. The so-called support theorem for stochastic differential equations (cp. Stroock and Varadhan [90], Kunita [56],[57]) allows furthermore, to associate the trajectories of a diffusion process with those of a control system, and to characterize in a control theoretic way the supports of transition probabilities or of invariant measures, see Kunita [57] or Kliemann [54] and the domains of multistability, Colonius et al. [33]. For certain problems, like the exponential convergence of trajectories, the analysis of transition probabilities or invariant measures on the state space is too rough (see Baxendale [7]). Here the dynamics in form of the induced stochastic flow (random dynamical system) plays a crucial role: To study exponential convergence, one linearizes the stochastic system over a stationary solution, and characterizes it locally using the theorem of Oseledeč [67] and associated invariant manifolds (see e.g. Ruelle [73], Carverhill [18], Boxler [16], Dahlke [40]). In the same way, by associating stochastic flows and control flows, one can clarify the fine structure of invariant measures etc. (Colonius and Kliemann [24]), also for non-Markov stochastic systems. In this approach, Markov processes are characterized by certain measurability and hyperbolicity properties (Crauel [36]; [37]). All in all, these few remarks concerning (iv) show the close proximity of approaches for the analysis of the long term behavior of stochastic, dynamical , and control systems. Many more detailed results can be found in the conference proceedings [3] and [71]. Meanwhile, a consistent stochastic bifurcation theory (Arnold and Boxler [2], Boxler [17]), a theory of stochastic normal forms (Arnold and Kedai [4]) and entropy theory (Bogenschütz [10]) has been developed on the basis of stochastic flows, see also Arnold [1].

Concerning (iii), just a few remarks shall suffice, since a theory of nonlinear systems with uncertain parameters is still in its infancy. In uncertainty theory, the state or output of a system are affected in a noticable way (which is different from the problem of disturbance decoupling), but for stability or stabilization the entire, uncertain system is to remain stable. This leads to a worst case analysis: If one knows in a system the growth behavior with respect to all possible uncertainties about, say, a fixed point, then the maximal (exponential) growth rate determines, whether the uncertain system is stable at this point. For linear systems, such considerations are made e.g. in H^∞ theory, (see e.g. Francis [43], Doyle et al. [42]), or by Hinrichsen and Pritchard [49], Rotea and Khargonekar [72], and Townley and Ryan [92]. See also Knobloch et al. [55] for an exposition of linear H^∞ theory and an approach to the nonlinear counterpart. Precise results are obtained by using the Lyapunov exponents of the associated control flow, see [22],[28],[34]. In the case of stochastic uncertainties one can analyze again a control system, and the connections are described using the theory of large deviations, see Arnold and Kliemann [5] and Colonius and Kliemann [22].

In this paper we will treat the system (1) in particular with respect to (i) and (ii), i.e. from a control theoretic and perturbation point of view. It will become clear that control theoretic aspects are also crucial for (i), together with an analysis of (1) as a dynamical system (control flow). In various ways connections between control systems and complex dynamics will show up:

On the one hand, controllable systems themselves are "chaotic", i.e. they have a dense set of periodic points, are topologically mixing and transitive, and show sensitive dependence on initial conditions. (The fact that an embedded chaotic behavior helps in controlling a system, is emphasized also in some applications, see e.g. Ott et al. [66] or Shinbrot et al. [85].) We will introduce in Section 2., after the discussion of some basic control theoretic questions, the concept of control flows, and relate control theoretic concepts to those from the theory of dynamical systems. This will allow for a finer analysis of control systems than the usual reachability and controllability sets.

Using those concepts and a weak form of local accessibility, we will analyze the limit sets of control systems in Section 3. As applications we obtain.

- Generically the limit sets of globally perturbed ordinary differential equations are contained in certain control sets, in which the periodic points are again dense. "Generic" is understood here in the strong, topological sense of "valid on an open and dense set". Genericity results of this kind are rather rare in the theory of dynamical systems (they hold e.g. for the hyperbolicity of linear systems, or for Peixoto-systems on 2-manifolds, see [69]), where one often has to settle for statements on residual sets (see e.g. Kupka-Smale systems, Pugh's closing lemma, Lyapunov regularity in the sense of Oseledeč). For small perturbations this result can be strengthened to include the relation with the finest Morse decomposition of $\dot{x} = X_0(x)$. One obtains a topological analogue to the measure theoretic result of Ruelle [74], which says that dynamical systems under small stochastic perturbations live basically on attractors.

- Chain recurrent components of ordinary differential equations $\dot{x} = X_0(x)$ can be steered to periodic solutions, which can be chosen arbitrarily in the control sets that belong to these components. This result is closely related to the 'control of chaos', as described e.g. in Ott et al. [66] or Shinbrot et al. [85], see also experiments in Ditto et al. [41]. Here we present as an example the controllability of the (chaotic) attractor in the Lorenz equation.

The last Sections 6. and 7. are devoted to the discussion of linearized systems on the tangent and the projective bundle. They show, on the one hand, how the control structures on these bundles are related to Morse decompositions (and we obtain a new class of examples, for which finest Morse decompositions exist); on the other hand, we develop a spectral theory for control systems and discuss connections with other spectral concepts. Applications of this theory to stabilization of bilinear systems, and to linear feedback systems with bounded feedback again show how one can utilize the ideas and techniques developed here to answer classical control theoretic questions — results in this direction can be found e.g. in [22], [31], [46]. Furthermore, robust feedback design for nonlinear systems can be based on the detailed analysis of the linearized system and on the global structure of (1) using invariant manifolds.

2. Control Systems as Dynamical Systems

In the following, we consider a family of differential equations

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \tag{2.1}$$

on a Riemannian C^∞ manifold M, with $\dim M = d < \infty$. Here the X_i, $i = 0 \dots m$ are C^∞ vector fields and as functions u_i we take $(u_i)_{i=1\dots m} = u \in \mathcal{U} := \{u\colon \mathbb{R} \to U \subset \mathbb{R}^m$, locally integrable$\}$. We assume that (2.1) has a unique solution $\varphi(t,x,u)$ for all $x \in M$, $u \in \mathcal{U}$ with $\varphi(0,x,u) = x$ and being defined for all $t \in \mathbb{R}$.

If one interprets (2.1) as a control system, the first important question is that of reachability, i.e. denote by

$$\mathcal{O}^+(x) = \{y \in M;\ \text{there is } u \in \mathcal{U} \text{ and } t \geq 0 \text{ with } \varphi(t,x,y) = y\}$$

the positive orbit (reachable set) from $x \in M$, and analogously by

$$\begin{aligned} \mathcal{O}^-(x) &= \{y \in M;\ \text{there is } u \in \mathcal{U} \text{ and } t \leq 0 \text{ with } \varphi(t,x,u) = y\} \\ &= \{y \in M;\ \text{there is } u \in \mathcal{U} \text{ and } t \geq 0 \text{ with } \varphi(t,y,u) = x\} \end{aligned}$$

the negative orbit, then we are looking for conditions such that, given $x, y \in M$, we have $y \in \mathcal{O}^+(x)$ (or $x \in \mathcal{O}^-(y)$). Problems concerning reachability under additional assumptions, like stability, optimality, etc., can be treated once some basic results in this direction are obtained.

In general, the answer to this question is very difficult. For linear systems of the form

$$\dot{x} = Ax + Bu \quad \text{in } \mathbb{R}^d \text{ with } U = \mathbb{R}^m$$

a satisfactory criterion can be given (Kalman's controllability rank condition): (2.2) is completely controllable (for all $x, y \in \mathbb{R}^d$) iff rank $(B, AB, \dots, A^{d-1}B) = d$. The proof simply uses the explicit solution of (2.2) (variation of constants formula) and the Cayley-Hamilton theorem. If the rank of the reachability matrix $(B, AB, \dots, A^{d-1}B) < d$, one can steer x into y, if both points are in the smallest A-invariant subspace of $\mathbb{R}^d$ which contains B, i.e. in the linear space generated by the columns of the reachability matrix.

Complete controllability of linear systems (with $U = \mathbb{R}^m$) is equivalent to a seemingly weaker condition, namely *int* $\mathcal{O}^+(x) \neq \phi$ for all $x \in \mathbb{R}^d$ (accessibility), where '*int*' denotes the interior of a set. The equivalence is proved via the observation that accessibility holds iff the Lie-algebra

$$\mathcal{L} = \mathcal{LA}\{Ax + Bu;\ u \in \mathbb{R}^m\} = \mathcal{LA}\{Ax, b_1, \dots, b_m\}$$

has rank d for all $x \in \mathbb{R}^d$. (Here the $b_1, \ldots, b_m$ are the columns of B, and for a set $\mathcal{H}$ of vector fields $\mathcal{LA}\{\mathcal{H}\}$ denotes the Lie algebra generated by the elements of $\mathcal{H}$.) For $x = 0$ this is exactly Kalman's criterion. In this sense accessibility and controllability are the same for linear systems (2.2).

In the case of nonlinear systems (2.1) there is a gap between these two concepts: accessibility does not imply controllability. The accessibility problem is very well understood these days with the help of geometric control theory: A somewhat sharper version (local accessibility, i.e. for all neighborhoods $N \subset M$ of x and all $t > 0$ it holds that $int\, \mathcal{O}^+_{\le t}(x) \cap N \neq \phi$, where $\mathcal{O}^+_{\le t}(x)$ consists of the points reachable up to time $t > 0$) can be described again by a Lie algebraic criterion, namely

$$\mathcal{L} = \mathcal{LA}\{X_0 + \Sigma u_i X_i(x);\ (u_i) = u \in U\} \text{ has rank } d \text{ for all } x \in M.$$

(These are the theorems of Frobenius [44] and Chow [20].) If $\mathcal{L}$ has the maximal integral manifolds property then the state space of (2.1) can be reduced to such a maximal integral manifold, since these are invariant for (2.1), and there the above Lie-algebraic condition holds, see e.g. Sussmann [88]. If the Lie-algebra $\mathcal{L}$ is not integrable, the orbits of (2.1) in M are still immersed submanifolds; however, one has to give up the connection between accessibility and the rank criterion.

For complete controllability there is — under the assumption of accessibility — a series of sufficient conditions, all of which use specific properties of the system's vector fields, see e.g. the textbooks of Isidori [51] or Nijmeijer and van der Schaft [65]. However, in general this property is not even satisfied for relatively simple classes of systems, such as systems on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, which are obtained as projections of linear systems in $\mathbb{R}^d$ onto $\mathbb{S}^{d-1}$, cp. Section 6. Therefore it is important to clarify the control structure of accessible systems, i.e. to identify those regions in M, where any two points are reachable from each other. At least on compact manifolds, or in compact, invariant sets of (2.1), such regions always exist.

For the rest of this paper we always assume (local) accessibility, i.e.

$$\dim \mathcal{LA}\{X_0 + \Sigma u_i X_i;\ (u_i) = u \in U\}(x) = d \text{ for all } x \in M \tag{H}$$

holds, and also $U \subset \mathbb{R}^m$ is compact and convex.

2.1. Definition. A set $D \subset M$ is a *control set* of (2.1) if

(i) $D \subset cl\mathcal{O}^+(x)$ for all $x \in D$,
(ii) for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \in \mathbb{R}$,
(iii) D is maximal (with respect to set inclusion) with the properties (i) and (ii).

(Here 'cl' denotes the closure of a set.)

In the context of control theory, in particular for the construction of suitable feedback laws, only those control sets D play a role, for which $int\, D \neq \phi$ holds. For these control sets (ii) is automatically satisfied, and we have furthermore: Control sets are pairwise disjoint, connected, $c\ell(int\ D) = c\ell\ D$, and $int\ D \subset \mathcal{O}^+(x)$ for all $x \in D$, i.e. each point in $int\ D$ is precisely reachable from any point in D.

Besides controllability the concept of limit sets is crucial for the analysis of the long term behavior as $t \to \pm\infty$, such as stability, global perturbations, control of complex behavior. For this aspect, a study of the control sets is, in general, not sufficient: Reachability is a finite time concept, i.e. y is reachable from x if there is $u \in \mathcal{U}$ and $t \geq 0$ with $\varphi(t, x, u) = y$. If a system is completely controllable, then M is the control set, and all limit points are again contained in this control set. If $D \subsetneqq M$, this fact need not hold. Therefore, one defines in analogy to the concept of chain recurrence in the theory of dynamical systems, so-called chain control sets:

2.2 Definition. A set $E \subset M$ is a *chain control set* of (2.1) if

(i) for all $x, y \in E$ and all $\varepsilon, T > 0$ there are $n \in \mathbb{N}$, $x = x_0, \ldots, x_n = y$ in M, $t_0 \ldots t_{n-1} > T$ and $u_0 \ldots u_{n-1} \in \mathcal{U}$ with $d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon$ for all $i = 0 \ldots n-1$, where $d(\cdot, \cdot)$ is a Riemannian metric on M,
(ii) for all $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$,
(iii) E is maximal with the properties (i) and (ii).

Chain control sets are pairwise disjoint, connected, and closed. The following example illustrates the difference between control sets and chain control sets.

2.3. Example. Consider the control system on $M = \mathbb{S}^1$, given by

$$\dot{x} = -\sin^2 x + a\cos^2 x - u(t)\cos^2 x,$$
$$x \in \mathbb{R} \bmod 2\pi, \quad a > 0, \quad U = [A, a] \subset \mathbb{R}.$$

This system has four control sets with nonvoid interior (see Figure 1.)

$$D_1 = \left[0, \arctan(a-A)^{1/2}\right], \qquad D_2 = \left(\pi - \arctan(a-A)^{1/2}, \pi\right),$$
$$D_3 = D_1 + \pi, \qquad D_4 = D_2 + \pi.$$

However, there is only one chain control set $E = \mathbb{S}^1$, cp. [24], where a general procedure is described to determine the control sets and chain control sets on one-dimensional manifolds.

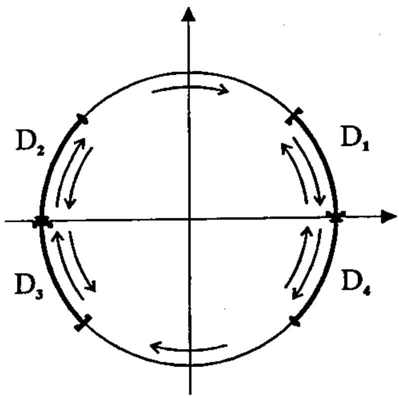

Figure 1. Control sets and dynamics for Example 2.3.

This example shows in particular that chain control sets do not necessarily reflect the reachability structure of a system. Therefore, the relation between the control and the limit structure plays an important role in the sequel, in particular we look for conditions, under which the limit sets are contained in the interior of control sets. This will enable us to analyze various aspects of the long term behavior also for systems that are not completely controllable.

So far, we have argued with the orbits and chain orbits of the system (2.1), where the contribution of each single trajectory $\{\varphi(t,x,u);\ t \in \mathbb{R}\}$ is lost. A precise analysis of the global fine structure of control systems is not possible with these concepts alone, just as for stochastic systems, where one has to use the associated stochastic flow to be able to answer certain questions. In complete analogy we will define the control flow on $\mathcal{U} \times M$, which describes the precise behavior of (2.1) and which yields via the theory of dynamical systems, results for control systems. In this way another mathematical area, which has enjoyed substantial progress over the last years, can be utilized for control theory, in addition to the theory of ordinary differential equations, differential geometry, Lie groups and Lie algebras, linearization techniques and linear algebra.

A continuous dynamical system (or flow) on a metric space S is given by a continuous map $\psi\colon \mathbb{R} \times S \to S$ with (i) $\psi_0 = id$ and (ii) $\psi_{t+s} = \psi_t \circ \psi_s$ for all $t, s \in \mathbb{R}$. (Here we have used $\psi_t = \psi(t,\cdot)$.) For a given, nonconstant function $u \in \mathcal{U}$ the map $(t,x) \mapsto \varphi(t,x,u)$ does not define a flow, as u is time varying. If one considers, however, the entire family of differential equations (2.1), indexed by $u \in \mathcal{U}$, then one can define a corresponding control flow on $S = \mathcal{U} \times M$, in the following manner:

$$\Phi\colon \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi(t,u,x) = (u(t+\cdot), \varphi(t,x,u)).$$

The shift $\theta_t u(\cdot) = u(t+\cdot)$ on $\mathcal{U}$, which is itself a dynamical system, guarantees the desired group property (ii), and turns $\Phi = (\theta, \varphi)$ into a skew product flow. For compact and convex $U \subset \mathbb{R}^m$ the set $\mathcal{U}$ is a compact, complete, metric space, if equipped with the weak* topology of $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m) = \left(L^1(\mathbb{R}, \mathbb{R}^m)\right)^*$. This also yields continuity of Φ. The use of the weak*-topology on $\mathcal{U}$ is appropriate for control theoretic considerations, because convergence of $u_n \to u$ in $\mathcal{U}$ implies uniform convergence of $\varphi(\cdot, x, u_n) \to \varphi(\cdot, x, u)$ on compact time intervals.

2.4. Remark. For a fixed feedback $u = F(x) \in U$ with F Lipschitz continuous, the equation $\dot{x} = X_0(x) + \Sigma F_i(x) X_i(x)$ defines, of course, a dynamical system on M. But for this one has to choose a certain feedback, which restricts the possible control functions *a priori*. Furthermore, for nonlinear systems it is not appropriate to consider only continuous feedback laws, see e.g. Sontag [84]. Sussmann [89] and [86] describe a construction to obtain piecewise smooth feedbacks from open-loop controllability of points for controllable systems.

2.5. Remark. The control flow Φ consists in its first component $(\mathcal{U}, \theta)$ of the time shift, for which only the concepts of topological dynamics are appropriate. In the second component, which by itself is not a flow, we find the smooth dynamics of the vector fields — and hence linearization techniques etc. can be used there, cp. Sections 6. and 7. Similar constructions for the definition of flows corresponding to non-autonomous differential equations were described in the 60's by Sell [79] and Miller [63], in particular for the almost periodic case. In the theory of stochastic differential equations with random perturbations ξ_t instead of $u(t)$ in (2.1), it is common place to consider the system over the trajectory space of ξ_t. The classical construction of Kolmogorov leads, for stationary ξ_t, to a shift invariant probability measure P on this space, which is considered given in the stochastic theory. In contrast, for the ergodic theory of control flows one first has to construct shift-invariant measures, e.g. via the Krylov-Bogolgubov device, see e.g. [CK4].

Let us recall the following concepts from the topological theory of dynamical systems (see e.g. Mañé [63]). In the following (S, ψ) denotes a flow on a metric space.

2.6. Definition. The *limit set* $\omega(s)$ of $s \in S$ is given by

$$\omega(s) = \{y \in S;\ \text{there is } t_k \to \infty \text{ with } \psi(t_k, s) \to y\}.$$

(S, ψ) is *topologically transitive*, if there exists $s \in S$ with $\omega(s) = S$.
(S, ψ) is *topologically mixing*, if for all open sets $V_1, V_2 \subset S$ there exist times $T_0 \in \mathbb{R}$, $T_1 > 0$ such that for all $n \in \mathbb{N}$ we have $\psi(-nT_1 + T_0, V_1) \cap V_2 \neq \phi$.

A closed ψ-invariant subset $W \subset S$ is *maximal topologically transitive*, if $(W, \psi|_W)$ is topologically transitive, and if for each closed $W' \supset W$ with $(W', \psi|_{W'})$ topologically transitive, we have $W' = W$. Similarly maximal topologically mixing sets are defined.

We obtain a relation between these concepts for the control flow $(\mathcal{U} \times M, \Phi)$ and the control sets of (2.1) in the following manner: Define the lift of a control set $D \subset M$ with $int\, D \neq \phi$ to a Φ-invariant set $\mathcal{D}$ in $\mathcal{U} \times M$ by

$$\mathcal{D} = c\ell\{(u,x) \in \mathcal{U} \times M;\ \varphi(t,x,u) \in intD \text{ for all } t \in \mathbb{R}\}. \tag{2.3}$$

2.7. Theorem. *Let $U \subset \mathbb{R}^m$ be compact and convex, and assume* (H).

(i) *If $D \subset M$ is a control set with $int\, D \neq \phi$, then $\mathcal{D}$, defined by* (2.3), *is a* maximal, topologically transitive (and mixing) set with

$$int\, D = int\, \pi_M \mathcal{D} \qquad \text{and} \qquad c\ell\, D = c\ell\, \pi_M\, \mathcal{D}. \tag{2.4}$$

Here π_M denotes the projection from $\mathcal{U} \times M$ onto the second component.

(ii) If $\mathcal{D} \subset \mathcal{U} \times M$ is a maximal, topologically transitive (or mixing) set of $(\mathcal{U} \times M, \Phi)$ with $int\, \pi_M\, \mathcal{D} \neq \phi$, then there is a control set $D \subset M$ satisfying (2.4).

2.8. Remark. In fact, under the assumptions of Theorem 2.7.(i) one obtains more: The periodic points are dense in $(\mathcal{D}, \Phi|_{\mathcal{D}})$ and Φ has sensitive dependence on initial conditions in $\mathcal{D}$. Furthermore, one obtains immediately from this theorem: The system (2.1) is completely controllable on M iff the flow $(\mathcal{U} \times M, \Phi)$ is topologically transitive (or mixing).

For a proof see [25], Theorem 3.9.

Next we describe the limit set structure of a control system and define for a flow (S, ψ) (compare, in particular, Conley [35]):

2.9. Definition. Let $\varepsilon, T > 0$. An (ε, T)-*chain* from $x \in S$ to $y \in S$ is given by $n \in \mathbb{N}$, $x = x_0 \ldots x_n = y$ in S, $t_0, \ldots, t_{n-1} > T$ such that $d(\psi(t_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0 \ldots n-1$. Here $d(\cdot,\cdot)$ is the metric on S.
The *chain limit set* of $x \in S$ is

$$\Omega(x) = \{y \in S;\ \text{for all } \varepsilon, T > 0 \text{ there is an } (\varepsilon, T)\text{-chain from } x \text{ to } y\},$$

and the *chain recurrent set* is defined as $CR = \{x \in S;\ x \in \Omega(x)\}$.
The flow (S, ψ) is *chain recurrent*, if $S = CR$, and *chain transitive*, if $y \in \Omega(x)$ for all $x, y \in S$.

For the control flow $(\mathcal{U} \times M, \Phi)$ we lift again the chain control sets $E \subset M$ to $\mathcal{U} \times M$ via

$$\mathcal{E} = c\ell\{(u,x) \in \mathcal{U} \times M;\ \varphi(t,x,u) \in E \text{ for all } t \in \mathbb{R}\}. \tag{2.5}$$

The analogue of Theorem 2.7 is the following result.

2.10. Theorem. *Let $U \subset \mathbf{R}^m$ be compact and convex, and assume* (H). *For a set $\mathcal{E} \subset \mathcal{U} \times M$, the set $E = \pi_M\, \mathcal{E}$ is a chain control set in M iff $\mathcal{E}$ is a maximal, chain transitive set of $(\mathcal{U} \times M, \Phi)$. In particular it holds:*

(i) *For all $(u,x) \in \mathcal{U} \times M$ with $\{\varphi(t,x,u),\ t \geq 0\}$ bounded, there exists a chain control set $E \subset M$ with $\omega(u,x) \in \mathcal{E}$, the corresponding lift, and hence $\pi_M \omega(u,x) \subset E$. (Here the limit set is with respect to the flow $(\mathcal{U} \times M, \Phi)$.)*

(ii) *For all minimal, Φ-invariant sets $W \subset \mathcal{U} \times M$ there is a control set $D \subset M$ such that $\pi_M\, W \subset c\ell\, D$.*

For the proof of Theorem 2.10 see [25], Theorem 4.1 and Lemma 5.3.

Theorems 2.7 and 2.10 clarify some basic relations between the control and limit structure of (2.1) on the one hand, and the control flow on the other hand. In the next section we analyze the connections between these two structures, aiming at the control of (complicated) limit behavior.

3. Limit Behavior of Controlled Trajectories

In this section we discuss the behavior of controlled trajectories as $t \to \pm\infty$. According to Theorem 2.10, the limit sets are contained in chain control sets and their lifts. For control theoretical purposes it is important to know, under which conditions trajectories enter areas of complete controllability. A characterization of this property is the aim of this section, various applications will be discussed in the remaining parts.

We start by defining (partial) orders on the sets of control sets and of chain control sets in the following way:

Let $D_1, D_2 \subset M$ be control sets of (2.1), define

$$D_1 \prec D_2 \text{ if there is } x_1 \in D_1 \text{ with } \mathcal{O}^+(x_1) \cap D_2 \neq \phi. \tag{3.1}$$

"$\prec$" is a partial order on the set $\{D \subset M;\ D \text{ is a control set of } (2.1)\}$, and under the hypothesis (H) we have:

3.1. Lemma.

(i) *Open control sets are minimal, and closed control sets are maximal elements w.r.t. $\prec$.*

(ii) *Invariant control sets $C \subset M$, i.e. $c\ell\, \mathcal{O}^+(x) = c\ell\, C$ for all $x \in C$, are always closed, and hence maximal w.r.t. $\prec$.*

(iii) *If M is compact, or if $K \subset M$ is a compact, invariant set of (2.1), then there exist (at least) one open and one closed control set in M (or in $K \subset M$, respectively), and these are exactly the minimal and the maximal elements w.r.t. $\prec$.*

In a similar way we define for chain control sets $E, E' \subset M$

$$E \prec E' \quad \begin{array}{l}\text{if there are chain control sets } E = E_1 \ldots E_k = E' \text{ and} \\ (u_1, x_1) \ldots (u_k, x_k) \text{ in } \mathcal{U} \times M \text{ with } \pi_M \omega^*(u_i, x_i) \subset E_i \\ \text{and } \pi_M \omega(u_i, x_i) \subset E_{i+1} \text{ for } i = 1 \ldots k-1. \\ \text{(Here } \omega^* \text{ denotes the limit sets for } t \to -\infty.\text{)}\end{array} \tag{3.2}$$

'$\prec$' is a partial order on the set $\{E \subset M;\ E \text{ is chain control set of (2.1)}\}$, and it holds: For each control set $D \subset M$ there exists exactly on chain control set $E(D)$ with $D \subset E(D)$. If $D_1 \prec D_2$, then $E(D_1) \preccurlyeq E(D_2)$, and as Example 2.3 shows, equality may hold here. The connection with Morse decompositions is given by

3.2 Proposition. *The number k of chain control sets is finite iff the control flow $(\mathcal{U} \times M, \Phi)$ has a finest Morse decomposition, i.e. iff the number k' of the connected components of the chain recurrent set is finite. In this case we also have $k = k'$.*

The proof follows from Theorem 2.10. This theorem and Proposition 3.2. show that chain control sets and their order describe the limit behavior of the trajectories. For control sets we know first of all that each minimal set of the flow Φ, projected onto M, is contained in the closure of a control set (Theorem 2.10(ii)). From this fact we see immediately that for all $(u, x) \in \mathcal{U} \times M$ there is a control set D with $\pi_M \omega(u, x) \cap cl D \neq \phi$. But this does not guarantee that $int\, D \neq \phi$ or that $\pi_M \omega(u, x) \cap int\, D \neq \phi$ hold. This last statement would imply that $\pi_M \omega(u, x) \subset cl\, D$ for some control set with $int\, D \neq \phi$, and then one would be able to control the limit behavior of this trajectory. The following example shows the difficulties that can arise, if these conditions are not fulfilled.

3.3 Example. Consider the linear system in $\mathbb{R}^2$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & u_1(t) \\ u_1(t) & u_2(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{with } U = \left[0, \frac{1}{2}\right] \times [1, 2],$$

and its projection onto the unit sphere $\mathbb{S}^1 \subset \mathbb{R}^2$. We parametrize $\mathbb{S}^1$ by the angle $\alpha \in \mathbb{R} \bmod 2\pi$ with $\alpha = 0$ for $x = 1$, $y = 0$, and obtain for the projected system four control sets with nonvoid interior

$$D_1 = \left(\frac{3\pi}{4}, \pi\right), \quad D_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \quad D_3 = D_1 + \pi, \quad D_4 = D_2 + \pi,$$

and a continuum of one-point control sets

$$D_\alpha = \{\alpha\}, \quad \alpha \in \mathbb{S}^1 \setminus \left(\bigcup_{i=1}^{4} D_i\right) =: A.$$

The points in A are fixed points on $\mathbf{S}^1$ corresponding to $u_1(t) \equiv 0$ and $u_2(t) \equiv 1$, see [26], Example 5.3 and [24].

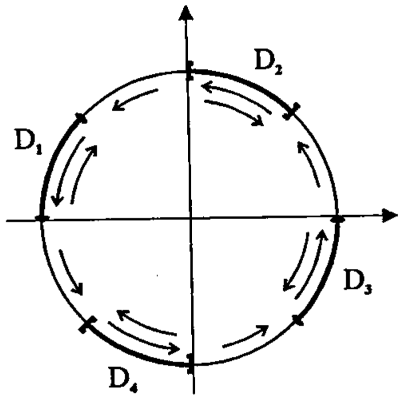

Figure 2. Control sets and dynamics for Example 3.3.

For $\alpha \in A$ and $u(t) \equiv \binom{0}{1}$ we have $\pi_{\mathbf{S}^1}\omega(u,\alpha) = D_\alpha$, but the projected system cannot be steered back to α after a small perturbation to $\beta > \alpha$, and it cannot be stabilized at the point α.

Therefore, we try to find conditions, under which $\pi_M\omega(u,x) \subset \mathit{int}\ D$ for some control set D holds.

3.4. Definition. The pair $(u,x) \in \mathcal{U} \times M$ is called an *inner pair*, if there are $T > 0$ and $S \geq 0$ with $\varphi(T,x,u) \in \mathit{int}\ \mathcal{O}^+_{\leq T+S}(x)$.

It follows from hypothesis (H) that $\mathcal{O}^+_{\leq t}(x)$ has nonvoid interior for all $t > 0$. Now the property of inner pairs means that the trajectory $\varphi(\cdot,x,u)$ is eventually in the interior of the reachable set from x. This excludes basically those trajectories that appear as optimal ones in time optimal control problems. Using the concept above we obtain the following characterization:

3.5. Theorem. *For $(u,x) \in \mathcal{U} \times M$ let the positive trajectory $\varphi^+(x,u) := \{\varphi(t,x,u);\ t \geq 0\}$ be bounded. Then the following statements are equivalent:*

(i) *$\omega(u,x)$ consists of inner pairs,*
(ii) *$\pi_M\omega(u,x) \subset \mathit{int}\ D$ for some control set $D \subset M$ of (2.1).*

The proof of this theorem can be found in [28], Theorem 4.4. The remaining problem is to find simple criteria for (u,x) such that $\omega(u,x)$ consists of inner pairs. In Section 5. we will demonstrate strategies for two examples, the model of a chemical reaction and the Lorenz equation. Here we point at two rather general criteria.

3.6. Remark. The maps $\varphi(t,\cdot,u): M \to M$ are diffeomorphisms on M, forming for $t \geq 0$ a semigroup $\mathcal{S}$. If $\mathcal{S}$ is a subsemigroup of a Lie group acting on a homogeneous space M, then (H) implies that $\mathcal{S}$ has nonvoid interior in $\mathcal{G}$, the group generated by $\mathcal{S}$, see e.g. Brockett [12]. In this case it is easy to see that we have for $g \in int\mathcal{S}$: (u_g, x) is an inner pair for all $x \in M$, where $u_g \in \mathcal{U}$ is a control associated with the diffeomorphism g. This includes in particular the action of matrix groups on spheres $\mathbb{S}^{d-1}$ or on flag manifolds.

3.7. Remark. Consider the control system (2.1) with a constant control $u^0 \in \mathcal{U}$. We can analyze the system 'translated by u^0': $\dot{x} = X_0(x) + \Sigma u_i^0 X_i(x) + \Sigma v_i(t) X_i(x)$, where $v(t) \in V := U - u^0$. According to Theorem 3.5. the limit set $\pi_M \omega(u^0, x)$ is contained in the interior of a control set, if all points $y \in \pi_M \omega(u^0, x)$ are locally controllable for the translated system. A survey of results on local controllability can be found in Bacciotti [6]. In particular, let us mention the following criterium, which is particularly simple to verify: Let $Y(x) = X_0(x) + \Sigma u_i^0 X(x)$ and denote by $ad_Y^k X_i$ the k-th Lie derivative of the vectorfield X_i along Y. Then we have for $u^0 \in int\, U$: $y \in \pi_M \omega(u^0, x)$ is locally controllable for the translated system, if linear span $\{ad_Y^k X_i,\ i = 1 \ldots m,\ k = 0, 1, 2, \ldots\}(y) = T_y M$.

Summing up, we obtain for the results above for control theoretic applications: If
$\pi_M \omega(u, x) \subset int\ D$ for some control set D, then the system can be steered from x to any point $y \in int\ D$; in particular, trajectories with a simpler limit behavior can be realized in $int\ D$, e.g. periodic orbits with sufficiently long period. Recall that through any point $y \in int\ D$ there passes a variety of periodic trajectories of the control system, and vice versa all periodic trajectories of (2.1) are contained in some control set. In Section 5. we will discuss two examples, but first of all we present some consequences for the perturbation theory of ordinary differential equations.

4. Two Perturbation Theorems for Ordinary Differential Equations

In this section we consider time varying perturbations of ordinary differential equations, and we study their limit behavior using the concepts from the previous sections. In the literature the perturbation problem is usually formulated like this: Given a small, possibly periodic perturbation of a differential equation, which qualitative properties (e.g. the existence of periodic solutions) remain valid under the perturbation (persistence, see e.g. Murdock [64]), and which ones are changed (structural stability, bifurcation, see e.g. Ruelle [75]). Here we ask the question in the following form: What is the behavior under all possible, time varying perturbations with values in a given set? It turns out that under certain richness assumptions on the perturbation, the limit sets of the perturbed equations and their relation to the limit sets of the original, unperturbed equation can be described in detail.

As a first step we consider perturbations with values in a bounded set. We formulate the results for compact manifolds, but all hold as well for compact, invariant sets.

Consider the differential equation

$$\dot{x} = X_0(x) \tag{4.1}$$

on a compact manifold M, together with time varying perturbed right hand sides of the form

$$X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x), \tag{4.2}$$

where, as in Section 2., $u \in \mathcal{U}$ with $U \subset \mathbb{R}^m$ compact and convex. We analyze the limit behavior of the trajectories $\varphi(t,x,u)$ for $t \to \pm\infty$. According to Lemma 3.1.(iii), there are on M (at least) one open and one closed control set of (4.2), and these are exactly the minimal (and maximal, respectively) elements with respect to the order $\prec$, defined in (3.1). Furthermore, on a compact manifold there are at most finitely many open or closed control sets with nonvoid interior.

4.1. Theorem. *Under* Hypothesis (H) *for (4.2) we have: The set*

$$\{(u,x) \in \mathcal{U} \times M;\ \textit{there is } T > 0 \textit{ with: for all } t \leq -T \textit{ we have}$$
$$\varphi(t,x,u) \in \textit{int } C^*,\ \textit{some open control set, and for all } t \geq T \textit{ we have}$$
$$\varphi(t,x,u) \in \textit{int } C,\ \textit{some closed (i.e. invariant) control set}\}$$

is open and dense in $\mathcal{U} \times M$. *In particular we have for the limit sets* $\pi_M\omega^*(u,x)$ *and* $\pi_M\omega(u,x)$ *in an open and dense subset of* $\mathcal{U} \times M$: $\pi_M\omega^*(u,x) \subset cl\ C^*$ *and* $\pi_M\omega(u,x) \subset C$.

This result, the proof of which can be found in [28], has various stochastic and measure theoretic analogues, see e.g. Ruelle [74] or Kliemann [54]. It says that generically all limit sets are concentrated on the finitely many open or closed control sets; it therefore describes a relation between limit sets and the minimal and maximal topologically transitive components of the associated control flow. Outside of these control sets there are therefore only limit sets of a "thin" subset of perturbations $u \in \mathcal{U}$ and initial values $x \in M$ (and they are contained in the lifts of chain control sets).

4.2. Remark. If $D \subset M$ is a control set (with $int\ D \neq \phi$), then we have, of course, for all $(u,x) \in \mathcal{D}$, (the lift of D to $\mathcal{U} \times M$) that $\pi_M\omega(u,x) \subset cl\ D$ holds. Now we have with respect to the behavior of $\omega(u,x)$ in $\mathcal{D}$: The set $\{(u,x) \in \mathcal{D};\ \pi_M\omega(u,x) = cl\ D\}$ is residual in $\mathcal{D}$, i.e. this set contains a countable intersection of open and dense subsets, in particular it is dense in $\mathcal{D}$.

4.3. Remark. If $D \subset M$ is a control set (with $int\ D \neq \phi$), then for each $x \in D$ there is $u \in \mathcal{U}$ with $\pi_M\omega(u,x) \subset int\ D$, and $\bar{u} \in \mathcal{U}$ with $\pi_M\omega(\bar{u},x) = clD$. Indeed, any point $y \in \bigcup_{D \preccurlyeq D'} D'$ can be in the projection of a limit set with initial value $x \in X$. The converse problem, i.e. is there for a control set D and a perturbation $u \in \mathcal{U}$ a point $x \in D$ with $\pi_M \omega(u, x) \subset cl\ D$, is a central question in the ergodic theory of control and stochastic systems, see [24]. Simple examples show that this need not be the case. This property is satisfied, however, for the action of linear semigroups on spheres or projective spaces, which will be considered in more detail in Section 6.

We now turn to the problem of small perturbations of the differential equation (4.1). We embed (4.1) into the following family of differential equations

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \tag{4.3$^\rho$}$$

with $u = (u_i)_{i=1\dots m} \in \mathcal{U}^\rho := \{u\colon \mathbb{R} \to \mathbb{R}^m;\ |u(t)| \leq \rho\}$, where $\rho \geq 0$, and $|\cdot|$ is any norm on $\mathbb{R}^m$. We are interested in the connection between the Morse sets of (4.1) and the control structure of (4.3$^\rho$).

Let us assume that (4.1) posses a (finite) finest Morse decomposition $\mathcal{M} = \{M_1, \dots, M_n\}$, i.e. the M_i, $i = 1 \dots n$, are the connected components of the chain recurrent set associated with the flow of (4.1) on M. In particular, the attractors of (4.1) are the maximal elements of $\mathcal{M}$, see e.g. Conley [35] or Ruelle [75]. We obtain the following result:

4.4. Theorem. *Assume that the chain recurrent set of (4.1) consists of finitely many connected components $M_1 \dots M_n$. For all $\rho > 0$ and all $x \in \bigcup_{i=1}^{n} M_i$ let $(0, x) \in \mathcal{U}^\rho \times M$ be an inner pair of (4.3$^\rho$). Then we have:*

(i) *For all $i = 1 \dots n$ there is an increasing family D_i^ρ of control sets of (4.3$^\rho$) with $M_i \subset int\ D_i^\rho$, such that $M_i = \bigcap_{\rho > 0} D_i^\rho$.*

(ii) *Vice versa, if for a sequence $\rho_k \to 0$ and for control sets D^{ρ_k} of (4.3$^{\rho_k}$) the set of limit points $L := \{y \in M;\ \text{there is } x^k \in D^{\rho_k} \text{ with } x_k \to y\}$ is not empty, then $L = M_i$ for some $i = 1 \dots n$.*

For a proof see [28]. This result says that the chain recurrent components of the differential equation (4.1) are 'blown up' to the topologically transitive components of the control flow associated with (4.3$^\rho$), if the embedding of (4.1) into the family (4.3$^\rho$) is sufficiently rich, i.e. if (H) holds and if the points in the Morse sets are inner pairs of the flow. We would like to point out a connection of this result with two ideas in the theory of stochastic dynamical systems: Ruelle [74] embeds a differential equation into a stochastic system with small noise, which satisfies a nondegeneracy condition, and he shows that for $t \to \infty$ this system lives basically on the attractors. If we combine Theorems 4.4 and 4.1, then we have a topological analogue of Ruelle's result.

On the other hand, the Ventcel-Freidlin theory is concerned with small, random perturbations of dynamical systems and their transient and limit behavior, see [94]. The difference between our approach and the stochastic theories is that we consider perturbations of bounded size $U^\rho \subset \mathbb{R}^m$, while in the latter the strength of a nondegenerate noise goes to zero.

We will make a few remarks concerning the meaning of Theorem 4.4, and draw a control theoretic consequence.

4.5. Remark. One could try to draw from Theorem 4.4 the conclusion that for $\rho > 0$ sufficiently small the number of control sets of (4.3^ρ) coincides with the number of chain recurrent components of (4.1). In general, this is false, as the following example shows. The reason is that even for small ρ the control system can change the dynamics of the differential equation substantially.

4.6. Example. Consider the following system on the unit circle $M = \mathbb{S}^1$

$$\dot{x} = X_0(x) - 3u_1 + 6u_2 =: X(x, u_1, u_2), \quad x \in \mathbb{R} \bmod 2\pi,$$

with $U = [-1,1] \times [-1,1]$. We construct the vector field X_0 in the following way: Define for $n \in \mathbb{N}$

$$x_n := \pi + \frac{1}{n}$$
$$I_n := \left(x - \frac{1}{n^2}, x + \frac{1}{n^2}\right)$$
$$J_n := \left(\frac{1}{2}(x_n + x_{n+1}), \frac{1}{2}(x_n + x_{n-1})\right).$$

Then it holds for n_0 sufficiently large and for all $n \geq n_0$: $\quad x_n \in I_n \subset J_n \subset (0, 2\pi)$, and $J_n \cap J_{n+1} = \phi$. Choose a C^∞ vector field X_0 on $[0, 2\pi)$ such that

$$-\frac{2}{n} < X_0(x) < -\frac{1}{n} \qquad \text{for } x \in I_n$$
$$X_0(x) \leq -\frac{2}{n} \qquad \text{for } x \in J_n \setminus I_n,$$

and such that for some $y_n \in J_n \setminus I_n$, $y_n < x_n$ we have

$$X_0(y_n) = -\frac{10}{n}.$$

Then $X_0(\pi) = 0$ and we require furthermore that there is $z_0 \in (0, \pi)$ with

$$X_0(z_0) - 3 - 6 > 0$$

so that the system cannot be steered from $x = \pi$ to $y = 2\pi = 0$. With this construction we obtain the following behavior:

For $x \in (\pi, 2\pi)$ and $u_1 \geq 0$ one has

$$X(x, u_1, 0) = X_0(x) - 3u_1 < 0.$$

Take $\rho_N = \frac{1}{N}$, i.e. $U^{\rho_N} = \left[-\frac{1}{N}, \frac{1}{N}\right] \times \left[-\frac{1}{N}, \frac{1}{N}\right]$. Then we have for all $n \in \mathbb{N}$ with $\frac{N}{2} \leq n \leq N$, and for $u_1 = -\frac{1}{N}$, $u_2 \geq 0$, $x \in I_n$

$$X\left(x, -\frac{1}{N}, u_2\right) = X_0(x) - 3u_1 + 6u_2 > -\frac{2}{N} + \frac{3}{N} + 6u_2 \geq \frac{1}{N} + 6u_2 > 0.$$

Hence I_n is contained in the interior of a control set D_n^N. Furthermore, π is in the interior of some control set D_0^N:

$$X\left(\pi, \frac{1}{N}, 0\right) = X_0(\pi) - \frac{3}{N} = -\frac{3}{N} < 0 \quad \text{and}$$
$$X\left(\pi, 0, \frac{1}{N}\right) = X_0(\pi) + \frac{6}{N} = \frac{6}{N} > 0.$$

Let $N > 2n_0$ be given and take $j, n \in \mathbb{N}$ such that $\frac{N}{2} \leq j < n \leq N$, then one cannot steer the system from D_j^N to D_n^N: The point y_n is in between D_j^N and D_n^N and we have for all $(u_1, u_2) \in U^{\rho_N}$:

$$X(y_n, u_1, u_2) = X_0(y_n) - 3u_1 + 6u_2 \leq -\frac{10}{N} + \frac{3}{N} + \frac{6}{N} \leq -\frac{1}{N} < 0,$$

and $D_j^N \cap D_n^N = \phi$. Similarly one obtains for $n \in \mathbb{N}$ with $\frac{N}{2} < n < N$ that $D_0^N \cap D_n^N = \phi$ holds. Then we have for even $N \in \mathbb{N}$ at least $\frac{N}{2} + 2 - n_0$ control sets with nonvoid interior, namely $D_0^N, D_{N/2}^N \dots D_N^N$, while X_0 can be chosen on $[0, 2\pi)$ in such a way that the differential equation $\dot{x} = X_0(x)$ has only two connected components of its chain recurrent set (e.g. a stable and an unstable fixed point). Note that in this example the number of chain control sets goes to infinity with $\rho_N = \frac{1}{N} \to 0$. For any given control range U^ρ a control system has only finitely many control sets (with nonvoid interior) on a one dimensional compact state space, if (H) holds, see [24].

4.7. Remark. Between the order on the Morse decomposition $\mathcal{M} = \{M_1 \dots M_n\}$ (see Conley [35]) and the associated control sets D_i^ρ we have the following relation: $M_i \prec M_j$ implies $D_i^\rho \preccurlyeq D_j^\rho$ for all $\rho > 0$. If $M_i \neq M_j$, then there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$ it holds that $D_i^\rho \neq D_j^\rho$, and hence $D_i^\rho \cap D_j^\rho = \phi$. Vice versa, if $\bigcap_{\rho>0} D_i^\rho$ and $\bigcap_{\rho>0} D_j^\rho$ are nonvoid and if for ρ small enough $D_i^\rho \prec D_j^\rho$ holds, then $M_i \prec M_j$. In particular we obtain: The maximal (and the minimal) Morse sets of (4.1) correspond exactly to the maximal (and the minimal, respectively) control sets of (4.3^ρ) for ρ small enough. Hence attractors of (4.1) are in one-to-one correspondence with the invariant control sets. Additional control sets, like those in Example 4.6, can be neither open nor closed.

As a last remark in this section we draw a control theoretic consequence from Theorem 4.4, concerning complete controllability with arbitrarily small controls.

4.8. Corollary. *Under the assumptions of* Theorem 4.4 *we have: The system* (4.3^ρ) *is completely controllable for all* $\rho > 0$ *iff the flow of the differential equation* (4.1) *is chain recurrent on* M.

This corollary generalizes results of Lobry [59], Cheng et al. [39] and Jurdjevic and Quinn [53].

5. Two Examples

In this section we discuss examples that show how to use the theory developed so far for control of the dynamical behavior of systems. The first example is the model of well-stirred tank reactor, which is to be controlled around a hyperbolic fixed point, and the second example concerns the controlled Lorenz equation, for which we want to change the (numerically observed) strange attractor to a simpler limit set, namely a periodic trajectory. In both cases the system is not completely controllable, but the limit sets of the uncontrolled system are contained in the interior of control set, in which we can accomplish the desired changes. For the Lorenz equation we will use a generalization of Theorem 3.5.

The model of a well-stirred tank ractor is given by the equations (see e.g. Golubitski and Schaeffer [45] or Poore [70])

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1-x_2)e^{x_1} \\ -x_2 + \alpha(1-x_2)e^{x_1} \end{pmatrix} + u(t)\begin{pmatrix} x_c - x_1 \\ 0 \end{pmatrix} =: X_0(x) + u(t)X_1(x). \tag{5.1}$$

Here x_1 is the (dimensionless) temperature, x_2 is the product concentration, and a, α, B, x_c are positive constants. As control we use the heat transfer coefficient, and take $u(t) \in U = [-0.15, 0.15] \subset \mathbb{R}$. The state space for (5.1) is $M = (0,\infty) \times (0,1)$. For our analysis we have taken $a = 0.15$, $\alpha = 0.05$, $B = 7.0$, and coolant temperature $x_c = 1.0$, see Poore [70] for the system behavior with different parameter values.

The uncontrolled equation for $u \equiv 0$ has 3 fixed points in M, namely

$$x^0 = \begin{pmatrix} z^0 \\ \alpha e^{z^0}/(1+\alpha e^{z^0}) \end{pmatrix}, \quad \text{stable}$$

$$x^1 = \begin{pmatrix} z^1 \\ \alpha e^{z^1}/(1+\alpha e^{z^1}) \end{pmatrix}, \quad \text{hyperbolic, i.e. the linearization about } x^1 \text{ has one negative and one positive eigenvalue}$$

$$x^2 = \begin{pmatrix} z^2 \\ \alpha e^{z^2}/(1+\alpha e^{z^2}) \end{pmatrix}, \quad \text{stable.}$$

Here $z^0 < z^1 < z^2$ are the zeros of the transcendental equation

$$-x - a(x - x_c) + B\alpha\left(1 - \frac{\alpha e^x}{1 + \alpha e^x}\right)e^x = 0.$$

Figure 3. shows the phase portrait of (5.1) for the parameter values chosen above.

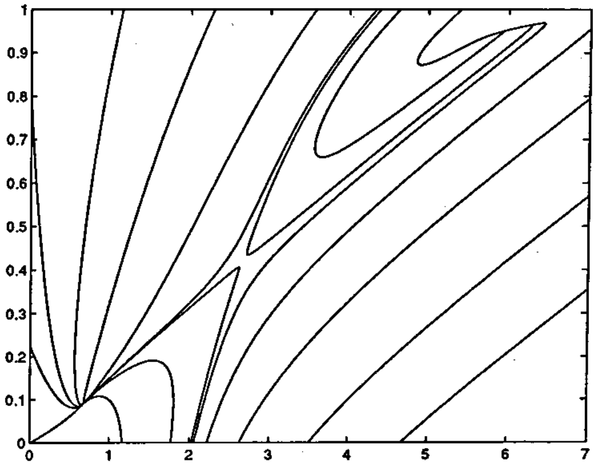

Figure 3. Phase portrait of (5.1) for $a = 0.15$, $\alpha = 0.05$, $B = 7.0$, $x_c = 1.0$, $u \equiv 0$.

The interesting feature of this system is that the stable fixed point x^2 with the highest product concentration cannot be realized because of technical reasons, see Bellman et al. [8], while the fixed point x^1 is unstable, and hence it cannot be used for a technical realization of the system without modification. However, if one can embed x^1 into the interior of a control set D, then we can steer the system to x^1 from all $x \in D$, and from the entire domain of attraction of D, and stabilize the system there.

In order to use Theorem 3.5 and Remark 3.7, we compute the corresponding Lie derivatives of X_0 and X_1, with these vector fields defined as in (5.1):

$$ad^0_{X_0}X_1 = (1 - x_1)\frac{\partial}{\partial x_1}$$

$$ad^1_{X_0}X_1 = [1 + B\alpha(1 - x_2)e^{x_1}(-2 + x)]\frac{\partial}{\partial x_1} + (\alpha - x_2)e^{x_1}(-1 + x_1)\frac{\partial}{\partial x_2}$$

$$
\begin{aligned}
ad^2_{X_0}X_1 = &[(-1.15x_1 + 0.15 + B\alpha(1-x_2)e^{x_1})\, B\alpha(1-x_2)(-1+x_1) \\
& - (1 + B\alpha(1-x_2)e^{x_1}(-2+x_1))(-1.15 + B\alpha(1-x_2)e^{x_1}) \\
& + B\alpha e^{x_1}(2-x_1)(-x_2 + \alpha(1-x_2)e^{x_1}) + B\alpha^2 e^{2x_1}(1-x_2)(-1+x_1)] \frac{\partial}{\partial x_1} \\
& + [\alpha x_1(1-x_2)e^{x_1}(-1.15x_1 + 0.15 + B\alpha(1-x_2)e^{x_1}) - \alpha(1-x_2)e^{x_1} \\
& \cdot (1 + B\alpha(1-x_2)e^{x_1}(-2+x_1)) + \alpha e^{x_1}(1-x_1)(-x_2 + \alpha(1-x_2)e^{x_1}) \\
& + \alpha(1-x_2)e^{x_1}(-1+x_1)(1+\alpha e^{x_1})] \frac{\partial}{\partial x_2}.
\end{aligned}
$$

One sees easily that for $x = (x_1, x_2) \in (0, \infty) \times (0, 1)$ with $x_1 \neq 1$ the vectorfields $X_1(x)$ and $ad^1_{X_0}X_1(x)$ span the tangent space $T_x\mathbb{R}^2$. For $x_1 = 1$ we have

$$(ad^1_{X_0}X_1)(x) \doteq (1 - B\alpha(1-x_2)e)\frac{\partial}{\partial x_1} + 0 \cdot \frac{\partial}{\partial x_2},$$

and the second component of $(ad^2_{X_0}X_1)(x)$ vanishes iff $1 - B\alpha(1-x_2)e = 0$. But for the choice of the parameters above, this equation has no zero in the interval $(0,1)$. Hence the three vector fields $ad^i_{X_0}X_1$, $i = 0, 1, 2$, span the entire tangent space for all $(x_1, x_2) \in M$.

Using the results from Sections 3. and 4. we see that there exist control sets D^0, D^1, D^2 around the fixed points x^0, x^1, x^2 with $x^i \in int\, D^i$ for $i = 1, 2, 3$. Here D^0 and D^2 are closed (invariant) control sets, and D^1 is variant, i.e. for all $x \in D^1$ there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \notin D^1$ for t sufficiently large, see Theorem 4.4 with respect to these remarks.

Figure 4. shows the numerically computed control sets D^0, D^1, D^2, and Figure 5. contains the domain of attraction $\mathcal{A}(D^1)$ of the variant control set D^1, i.e.

$$\mathcal{A}(D) = \{x \in M;\ \mathcal{O}^+(x) \cap D \neq \phi\}.$$

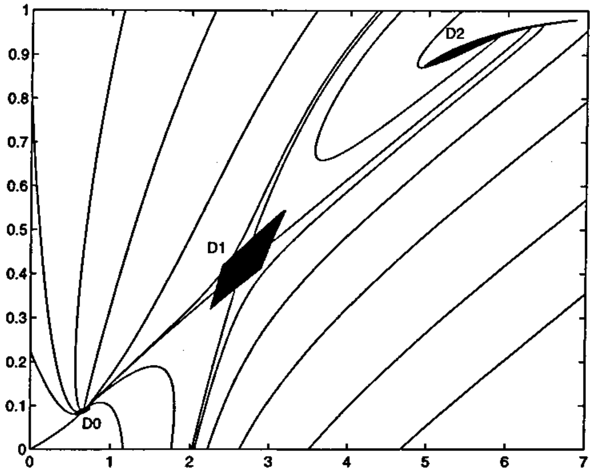

Figure 4. The control sets D^0, D^1, D^2 of the system (5.1).

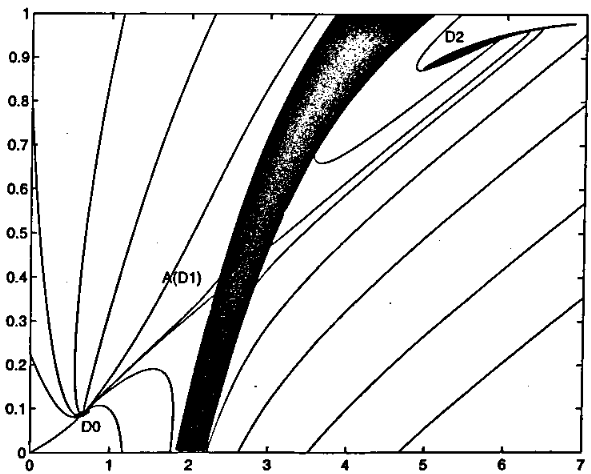

Figure 5. The domain of attraction $\mathcal{A}(D^1)$ of the variant control set D^1 of (5.1).

As a result of these considerations we obtain that the reactor (5.1) can be steered to the interesting point x^1 from $\mathcal{A}(D^1)$, and it can be kept at that point, as long as disturbances do not perturb the system out of $\mathcal{A}(D^1)$. Furthermore, the control set D^1 and the domain of attraction $\mathcal{A}(D^1)$ can be characterized precisely via the stable and unstable manifolds of the hyperbolic fixed points $x^1(u)$, $u \in U$. This allows the construction of a (discontinuous) feedback law $u = F(x)$, such that $x^1 = x^1\ (u = 0)$ becomes a globally asymptotically stable equilibrium on $\mathcal{A}(D^1)$ for the feedback system. This feedback has certain maximal robustness properties with respect to disturbances. The explicit construction can be seen in [31].

Our second example concerns the Lorenz equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} p(x_2 - x_1) \\ -x_1x_3 + rx_1 - x_2 \\ x_1x_2 - bx_3 \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix} =: X_0(x) + u(t)X_1(x) \tag{5.2}$$

with $U = [-\rho, \rho]$, $p = 10$, $b = \frac{8}{3}$, $r = 28$, where $\rho > 0$ is a constant. This system serves as a finite-dimensional model of the Rayleigh-Bénard convection, and we have chosen as control u the Rayleigh coefficient, i.e. the temperature difference applied at the boundary, see e.g. Lorenz [60] or Bergé et al. [11].

Equation (5.2) exhibits for the parameter values above a (numerically observed) strange attractor $A \subset \mathbb{R}^3$, and we want to answer the question, whether it is possible to reduce this complex limit behavior to a simpler one, e.g. to periodic trajectories, using small controls $u(t) \in U$ ('control of chaos'). We will need the following facts, which can be found e.g. in Hairer et al. [48] or in Sparrow [87].

For constant controls $U(t) \equiv u \in U$ the set

$$R = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 + (x_3 - p - (r+u))^2 \le c^2\}$$

is positively invariant for c sufficiently large. Hence the positive trajectories $\{\varphi(t, x, u);\ t \ge 0\}$ are bounded for $x \in R$. If $r + u > 1$, the equation has three (hyperbolic) fixed points, namely the origin $0 \in \mathbb{R}^3$ and

$$x^* = \left(\sqrt{b(r+u-1)}, \sqrt{b(r+u-1)}, r+u-1\right) \quad \text{and}$$
$$x^{**} = \left(-\sqrt{b(r+u-1)}, -\sqrt{b(r+u-1)}, r+u-1\right).$$

In order to show the existence of a control set D with nonvoid interior, which contains the strange attractor, it would be tempting to use Theorem 3.5 and the criterion from Remark 3.7, just as in the first example. However, this approach is not successful because of the invariance of the x_3-axis. Hence we will use the following generalization, which does not need the hypothesis (H).

5.1. Theorem. *Assume that there exists an open set $L \subset M$, such that we have for all $x \in L$ and all $T' > T > 0$: $\varphi(T, x, 0) \in \operatorname{int} \mathcal{O}^+_{\le T'}(x)$ and $x \in \operatorname{int}\mathcal{O}^-_{\le T'}(\varphi(T, x, 0))$. Let $K \subset L$ be a compact set. Then for all $x \in M$ with $\{\varphi(t, x, 0);\ t \ge 0\}$ bounded there exists a control set $D \subset M$ such that $K \cap \pi_M\omega(0, x) \subset \operatorname{int} D$.*

We will now check the assumptions of this theorem, whose proof can be found in [CK8], for equation (5.2) with $L = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3;\ x_1 = x_2 = 0\}$. Let X_0 and X_1 be defined as in (5.2). Then one computes

$$ad_{X_0}X_1 = p(x_2 - x_1)\frac{\partial}{\partial x_1} + (x_1 - px_1 - x_1^2)\frac{\partial}{\partial x_2}$$

$$ad^2_{X_0}X_1 = p(x_2 - x_1)(1 - p - 2x_1 + x_3 - r - x_2)\frac{\partial}{\partial x_1} + \left[(-x_1x_3 + rx_1 - x_2)p + (x_1 - px_1 - x_1^2)(1 - p - x_1)\right]\frac{\partial}{\partial x_2}$$

$$ad^3_{X_0}X_1 = f_1(x)\frac{\partial}{\partial x_1} + f_2(x)\frac{\partial}{\partial x_2} + p(x_1x_2 - bx_3)(x_2 - 2x_1)\frac{\partial}{\partial x_3},$$

where the precise form of the functions f_1 and f_2 is not important. We have for

(i) $x_1 \neq 0$, and

(ii) $x_1 \neq x_2$

that $X_1(x)$ and $(ad_{X_0}X_1)(x)$ are linearly independent. Furthermore, if

(iii) $x_1x_2 - bx_3 \neq 0$, and

(iv) $x_2 - 2x_1 \neq 0$,

then the linear span of $\{X_1(x), (ad_{X_0}X_1)(x), (ad^3_{X_0}X_1)(x)\}$ is all of the tangent space $T_x\mathbb{R}^3$ for those x that satisfy (i)–(iv). Hence the system is locally controllable at these points, and therefore the assumptions in Theorem 5.1 are satisfied on this set. For all points that do not satisfy (i)–(iv) and that are not on the x_3-axis, one verifies that the vectorfield X_0 is nowhere tangential to the manifold described by (i)–(iv). Hence for all points outside the x_3-axis one can steer the system with the control $u \equiv 0$ immediately into a region of local controllability. This verifies the assumptions of Theorem 5.1 in the current set up.

Therefore, we obtain the following result: Let $x \in \mathbb{R}^3$ be a point outside of the x_3-axis. Then there exists a control that steers the system from x into the interior of a control set D, which contains the strange attractor in its interior (actually the strange attractor intersected with any compact set that does not contain the x_3-axis). In *int* D one can follow any periodic trajectory, which is realized in (5.2) by appropriate steering. Let us recall that through any point in *int* D there exists a variety of periodic trajectories.

The simplification of the limit behavior follows this recipe: Embed the attractor of dimension between 1 and 3 into a control set of dimension 3, and realize there a limit set of dimension 1.

6. Analysis of Linearized Control Systems

Linearization techniques play an important role in the theory of dynamical systems for the analysis of local behavior, such as stability statements, Lyapunov exponents, invariant manifolds, entropy theory, etc. It seems possible to use all these methods also for the analysis of control systems, but here one faces the problem that a control system always contains trajectories that are not Lyapunov regular. Furthermore, there is no 'natural' shift-invariant measure on the space $\mathcal{U}$ of admissible control functions. But such a measure is needed in the theorem of Oseledeč

[67] for the characterization of the spectrum. In this section we will present briefly some ideas concerning the control theoretic analysis of linearized systems, and then state some results on the spectrum in the next part.

Let us start again with a nonlinear control system

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) =: f(x, u(t)) \tag{2.1}$$

as explained in Section 2. In control theory the linearization of (2.1) is usually done in the following way: Pick a fixed point $x^0 = x^0(u^0)$, $u^0 \in \mathbb{R}^m$ of (2.1) and linearize the system about this point with respect to x and u. One obtains a "linear control system" of the form

$$\dot{v} = Av + Bu \tag{6.1}$$

where $A = D_x f(x^0, u^0)$, $B = D_u f(x^0, u^0)$, compare the basic paper by Brockett [13] and the survey by Sontag [84] for stability results that can be obtained via this approach.

Instead of following this procedure, we will linearize the system (2.1) only with respect to x to obtain a system on the tangent bundle TM. This allows for global results on M, and will lead, as in the theory of dynamical systems, to a construction of invariant manifolds via the linearization, see e.g. [15] and [32]; compare also Ruelle [73] and Dahlke [40] in the context of stochastic flows.

On the tangent bundle TM we obtain the equation

$$(Tx)^{\cdot} = TX_0(Tx) + \sum_{i=1}^{m} u_i(t) TX_i(Tx) \tag{6.2}$$

with initial values $Tx_0 = (x_0, v_0) \in T_{x_0}M$, where Tx denotes points in TM, and for a smooth vector field X on M we write its linearization as $TX = (X, DX)$. In local coordinates on a chart in M this means: If $X_j = \sum_{k=1}^{d} \alpha_{kj}(x) \frac{\partial}{\partial x_k}$ for $j = 0 \ldots m$, denote by $A_j(x) = \left(\frac{\partial \alpha_{k,j}(x)}{\partial x_i}\right)_{i,k}$ its Jacobi matrix. With this notation we have $TX_j(x) = (\alpha_j(x), A_j(x)v)$, where α_j is the column $(\alpha_{kj})_{k=1\ldots d}$. Then (6.2) is locally a pair of coupled differential equations

$$\begin{aligned} \dot{x} &= \alpha_0(x) + \sum_{i=1}^{m} u_i(t) \alpha_i(x) \\ \dot{v} &= A_0(x)v + \sum_{i=1}^{m} u_i(t) A_i(x) v. \end{aligned} \tag{6.3}$$

If $x^0 \in M$ is a fixed point of the control system (2.1), linearization about x^0 yields a so- called bilinear control system

$$\dot{v} = A_0(x^0)v + \sum_{i=1}^{m} u_i(t)A_i(x^0)v \quad \text{in } \mathbb{R}^d. \tag{6.4}$$

Its analysis gives information about local stability and stabilization for (2.1) in a neighborhood of x^0. For this reason, bilinear systems play an important role for control theoretic problems, when one uses this linearization approach.

Before we are able to make statements about the structure of (6.2), we need some further concepts: (6.2) induces a control system on the projective bundle $\mathbb{P}M$, which is given by

$$(\mathbb{P}x)^{\cdot} = \mathbb{P}X_0(\mathbb{P}x) + \sum_{i=1}^{m} u_i(t)\mathbb{P}X_i(\mathbb{P}x) \tag{6.5}$$

with initial values $\mathbb{P}x_0 = (x_0, s_0) \in \mathbb{P}_{x_0}M$, the projective space in $T_{x_0}M$. Here $\mathbb{P}X$ is the projection of a vector field TX from TM onto $\mathbb{P}M$, i.e. for $j = 0 \dots m$ the $\mathbb{P}X_j$ are given locally by

$$\begin{aligned} \mathbb{P}X_j(x,s) &= (\alpha_j(x), h(A_j(x), s)) \\ h(A_j(x), s) &= \left(A_j(x) - s^T A_j(x)s \cdot Id\right) s, \end{aligned} \tag{6.6}$$

where ^{T} denotes transposition and Id is the $d \times d$ identity matrix. In the following we denote the solutions of (6.2) by $T\varphi(\cdot, Tx, u)$, and those of (6.6) by $\mathbb{P}\varphi(\cdot, \mathbb{P}x, u)$.

Just like the basic control system (2.1), the linearized system (6.2) and the projected system (6.6) induce associated control flows

$$\begin{aligned} T\Phi &: \mathbb{R} \times \mathcal{U} \times TM \to \mathcal{U} \times TM, \quad T\Phi_t(u, Tx) = (\theta_t u, T\varphi(t, Tx, u)) \\ \mathbb{P}\Phi &: \mathbb{R} \times \mathcal{U} \times \mathbb{P}M \to \mathcal{U} \times \mathbb{P}M, \quad \mathbb{P}\Phi_t(u, \mathbb{P}x) = (\theta_t u, \mathbb{P}\varphi(t, \mathbb{P}x, u)). \end{aligned} \tag{6.7}$$

First we describe the chain control sets of the projectivized system. We have to introduce some notation. For a chain control set $E \subset M$ and its lift $\mathcal{E} \subset \mathcal{U} \times M$ (cp. (2.5)) let

$$\begin{aligned} T\mathcal{E} &= \{(u, (x, v)) \in \mathcal{U} \times TM; (u, x) \in \mathcal{E}\} \\ \mathbb{P}\mathcal{E} &= \{u, (x, \mathbb{P}v)) \in \mathcal{U} \times \mathbb{P}M; (u, x) \in \mathcal{E}\}. \end{aligned}$$

Note that $T\mathcal{E}$ is $T\Phi$-invariant and the dimension of the fibers is equal to $\dim M = d$. In fact, $T\mathcal{E} \to \mathcal{E}$ is a vectorbundle. For a chain control set ${}_{\mathbb{P}}E \subset \mathbb{P}M$ denote the lift to $\mathcal{U} \times \mathbb{P}M$ by ${}_{\mathbb{P}}\mathcal{E}$ and define the lift to $\mathcal{U} \times TM$ by

$${}_{T}\mathcal{E} = \{(u, Tx) \in \mathcal{U} \times TM; Tx \notin Z \text{ implies } (u, \mathbb{P}(Tx)) \in {}_{\mathbb{P}}\mathcal{E}\},$$

where Z is the zero section in TM.

6.1. Theorem. *Let $E \subset M$ be a compact chain control set of system (2.1).*

(i) *There are $1 \leq l \leq d$ chain control sets ${}_{\mathbb{P}}E_i$ of the projective system (6.5) on the projective bundle $\mathbb{P}M$ with $\mathbb{P}\pi({}_{\mathbb{P}}E_i) \subset E$. The order on these sets (cp.(3.2)) is linear and we enumerate in such a way that ${}_{\mathbb{P}}E_1 \prec \cdots \prec_{\mathbb{P}} E_\ell$.*

(ii) *The lifts ${}_T\mathcal{E}_i, i = 1, \ldots, \ell$ are invariant subbundles of $(T\mathcal{E}, T\Phi|T\mathcal{E})$ with $T\mathcal{E} =_T \mathcal{E}_1 \oplus \cdots \oplus_T \mathcal{E}_\ell$ (Whitney sum).*

(iii) *$\{{}_{\mathbb{P}}\mathcal{E}_1, \ldots, {}_{\mathbb{P}}\mathcal{E}_\ell\}$ is the unique finest Morse decomposition of the flow $\mathbb{P}\Phi|\mathbb{P}\mathcal{E}$.*

The proof of this result follows from a theorem due to Selgrade [78] describing the chain recurrent components of projectivized linear flows over a chain recurrent base space. Recall that according to Theorem 2.10 the lifts ${}_{\mathbb{P}}\mathcal{E}_i$ of the chain control sets are the chain recurrent components of the projectivized linear control flow.

Next we will describe the control sets in $\mathbb{P}M$ and their relation to the chain control sets. We consider the linearized system (6.2) and extend the control sets from $\mathbb{P}M$ to TM

$$_TD_i = \{Tx \in TM;\ Tx \notin Z \text{ implies } \mathbb{P}(Tx) \in {}_{\mathbb{P}}D_i\}. \tag{6.8}$$

For a control set ${}_{\mathbb{P}}D$ and its extension ${}_TD$ we define the lift to $\mathcal{U} \times \mathbb{P}M$ (and $\mathcal{U} \times TM$) by

$${}_{\mathbb{P}}\mathcal{D} = c\ell\{u, \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}M;\ \mathbb{P}\varphi(t, \mathbb{P}x, u) \in int\, {}_{\mathbb{P}}D \text{ for all } t \in \mathbb{R}\}, \tag{6.9}$$

and analogously for ${}_T\mathcal{D}$.

Now we are faced with a question, which is particularly important for spectral theory and related problems: Do the ${}_T\mathcal{D}_i$ define a decomposition of $(\mathcal{U} \times TM, T\Phi)$ into invariant subbundles? The following example shows that this cannot be expected in general.

6.2. Example. Consider again the system from Example 3.3, i.e.

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v + u_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v + u_2(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} v$$

with $U = \left[0, \frac{1}{2}\right] \times [1, 2]$. For the projected system on the projective space $\mathbb{P}^1$ in $\mathbb{R}^2$ we obtain 2 main control sets D_1 and D_2, see Example 3.3., which are connected by a continuum of control sets with void interior. For constant controls $u_1(t) \equiv \alpha$, $u_2(t) \equiv \beta$ we get the eigenvalues of the system matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & \beta \end{pmatrix}$ as

$$\lambda_{1,2} = \frac{1}{2}(1 + \beta) \pm \sqrt{\alpha^2 - \beta + \frac{1}{4}(1 + \beta)^2}.$$

Hence for $\alpha \neq 0, \beta \neq 1$ the eigenspaces are one-dimensional, but for $\alpha = 0, \beta = 1$ the (generalized) eigenspace is $\mathbb{R}^2$. Subbundle decompositions necessarily have constant dimension, hence ${}_T\mathcal{D}_1$, ${}_T\mathcal{D}_2$ cannot yield such a decomposition.

Below we will show that this is an 'exceptional' situation. We embed – as in Section 4 – the control system with fixed control range U into a family of systems with varying control range $\rho U, \rho \geq 0$. Then we will show that under an inner pair assumption the control sets with nonvoid interior and the chain control sets coincide up to closure for all $\rho \geq 0$, with the possible exception of at most d ρ-values. For $U \subset \mathbb{R}^m$ compact and convex with $0 \in U$ denote

$$U^\rho = \rho U = \{\rho u; u \in U\} \text{ for } \rho \geq 0.$$

To complete the picture we also consider the case of unbounded perturbations. Let $\mathcal{U}^\infty = \bigcup_{\rho>0} \mathcal{U}^\rho = L_\infty(\mathbb{R}, \mathbb{R}^m)$ and denote the corresponding control system by $(2.1)^\infty$. Also for other quantities a superscript ρ will indicate their dependence on the control range U^ρ for $0 \leq \rho \leq \infty$. Note that for $\rho = 0$ we simply obtain the differential equation $\dot{x} = X_0(x)$.

Let E^0 be a chain recurrent component for $(2.1)^0$. For $\rho > 0$ there are unique chain control sets of $(2.1)^\rho$ with $E^0 \subset E^\rho$. Under an inner pair assumption (cp. Theorem 4.4) there are also unique control sets D^ρ with $E^0 \subset \text{int } D^\rho$. Similarly, we consider the chain recurrent components ${}_{\mathbb{P}}E_i^0$ in the projective bundle $\mathbb{P}M$ (with $\mathbb{P}\pi({}_{\mathbb{P}}E_i^0) \subset E^0$) of the uncontrolled projective system

$$(\mathbb{P}x)^\cdot = \mathbb{P}X_0(\mathbb{P}x), \; t \in \mathbb{R}.$$

Here $i \in I$, some index set according to Theorem 6.1. for $\rho = 0$. Let $\mathcal{C}(\mathbb{P}M)$ denote the set of compact subsets of $\mathbb{P}M$ with the Hausdorff metric, and consider for $i \in I$ the following maps associating to ρ a chain control set of $(6.5)^\rho$

$$\begin{aligned} {}_{\mathbb{P}}E_i : [0,\infty] \to \mathcal{C}(\mathbb{P}M), \quad \rho \mapsto {}_{\mathbb{P}}E_i^\rho \\ \text{where } {}_{\mathbb{P}}E_i^0 \subset {}_{\mathbb{P}}E_i^\rho \text{ for } \rho \geq 0. \end{aligned}$$

Similarly we define for $i \in I$ maps associating to ρ control sets of $(6.5)^\rho$ by

$${}_{\mathbb{P}}D_i : [0,\infty] \to \mathcal{C}(\mathbb{P}M), \quad \rho \mapsto c\ell_{\mathbb{P}}D_i^\rho$$

where ${}_{\mathbb{P}}D_i(0) = {}_{\mathbb{P}}E_i^0$ and ${}_{\mathbb{P}}D_i^\rho$ is a control set with ${}_{\mathbb{P}}E_i^0 \subset \text{int } {}_{\mathbb{P}}D_i^\rho$ for $\rho > 0$.

The following theorem shows in particular, that these maps are well defined. Note that some of the ${}_{\mathbb{P}}E_i$ (and ${}_{\mathbb{P}}D_i$, resp.) may coincide.

6.3. Theorem. *Fix $0 < \rho_1 \leq \infty$ and assume that for all $\rho \in (0, \rho_1]$ the systems $(6.5)^\rho$ are locally accessible. Let E^0 be a chain recurrent component for the uncontrolled system $(2.1)^0$ and for $\rho \in (0, \rho_1]$ let E^ρ denote the unique chain control set of $(2.1)^\rho$ with $E^0 \subset E^\rho$. Suppose that there is a compact set $L \subset M$ such that*

$E^{\rho_1} \subset L$ and the set L is positively invariant under all controls $u \in \mathcal{U}^{\rho_1}$. Assume that the following ρ-ρ'-inner pair condition holds:

$$\text{For all } \rho', \rho \in [0, \rho_1) \text{ with } \rho' > \rho \text{ and all chain control sets } {}_{\mathbb{P}}E_j^\rho$$
$$\text{every } (u, \mathbb{P}x) \in {}_{\mathbb{P}}\mathcal{E}_j^\rho \text{ is an inner pair for } (6.5)^{\rho'}.$$

(i) *Then for all $\rho \in (0, \rho_1]$ there are unique chain control sets E^ρ with $E^0 \subset E^\rho$ and $\ell(\rho)$ chain control sets ${}_{\mathbb{P}}E_i^\rho$ with $\mathbb{P}\pi({}_{\mathbb{P}}E_i^\rho) \cap E^0 \neq \emptyset$; their number $\ell(\rho)$ satisfies $1 \leq \ell(\rho) \leq d$ and one has $\mathbb{P}\pi({}_{\mathbb{P}}E_i^\rho) = E^\rho$.*

(ii) *For all $\rho \in (0, \rho_1]$ there are unique control sets D^ρ of $(2.1)^\rho$ with $E^0 \subset$ int D^ρ and control sets ${}_{\mathbb{P}}D_i^\rho$ of $(6.5)^\rho$ such that ${}_{\mathbb{P}}E_i^0 \subset$ int ${}_{\mathbb{P}}D_i^\rho$; for all but at most d ρ-values*

$$\text{int } D^\rho = \text{ int } E^\rho \text{ and int } {}_{\mathbb{P}}D_i^\rho = \text{ int } {}_{\mathbb{P}}E_i^\rho.$$

Furthermore, for all $\rho \in [0, \rho_1]$ the projections onto M satisfy $\mathbb{P}\pi(D_i^\rho) = D^\rho$.

The first part of this theorem is a consequence of Theorem 6.1. The proof of the second part which, additionally, uses control theoretic methods is given in [32] (a proof for the bilinear case is given in [30]). In the next section we will use these control sets and chain control sets in the projective bundle in order to describe the Lyapunov exponents of the linearized system.

6.4. Remark. Local accessibility of the projective linear system can be guaranteed if Hypothesis (H) holds for the system on $\mathbb{P}M$, i.e.

$$\dim \mathcal{LA}\{\mathbb{P}X_0 + \Sigma u_i \mathbb{P}X_i; (u_i) = u \in U\}(x, v) = 2d - 1 \text{ for all } (x, v) \in \mathbb{P}M,$$

compare San Martin and Arnold [82] and San Martin [81] for a discussion of this hypothesis. Furthermore, [82] shows uniqueness of the invariant control set in $\mathbb{P}M$ over an invariant control set in M. In [14], Barros and San Martin give more precise information about the number of control sets in bilinear systems.

6.5. Remark. In the bilinear case, the relation between control sets and chain control sets of the projected system (6.5) can be described precisely, without the inner pair condition, see [26]. Here we only mention that each chain control set ${}_{\mathbb{P}}E_j$ contains (at least) one control set ${}_{\mathbb{P}}D_i$ with nonvoid interior. Furthermore, control sets and chain control sets are described by eigenspaces that belong to the fundamental matrices of solutions which are periodic on $\mathbb{P}^{d-1}$ and associated with piecewise constant periodic controls.

7. Spectral Theory of Nonlinear Control Systems

For the analysis of dynamical and of stochastic systems via linearization spectral theory (Lyapunov exponents) play an important role. As far as we know, there is no approach to a systematicspectral theory for nonlinear control systems, although the well-known concepts of the Oseledeč spectrum [67], the dichotomy spectrum (see Sacker and Sell [76]) and the 'topological' spectrum (based on the condition that the zero section is isolated, see Selgrade [78], Salamon and Zehnder [91]) are applicable to this situation. We want to start a systematic investigation in this section and analyze the Lyapunov spectrum of nonlinear control systems, as well as its connections with other spectral concepts. There are several results for bilinear systems (see [21], [23], and the survey [34]), which, in particular, have applications to the stabilization of linear, uncertain systems via output feedback (see [22], [27]). Let us mention here, that spectral theory for control systems is first of all an open-loop theory, and one has to construct appropriate feedbacks from a detailed analysis afterwards.

Let us consider again the control system (2.1) and its linearization (6.2), where we can write, according to (6.3), the solutions of (6.2) as $T\varphi(t,Tx,u) = (\varphi(t,x,u), D\varphi(t,x,u)v)$ for $Tx = (x,v) \in T_xM$. The exponential growth behavior of the solutions of (6.2) is described by the Lyapunov exponents, which are defined on the linearized control flow (6.7)

$$\begin{aligned} &\lambda\colon \mathcal{U} \times TM \to \overline{\mathbb{R}} \\ &\lambda(u,x,v) = \limsup_{t\to\infty} \frac{1}{t} \log \|D\varphi(t,x,u)v\|, \quad v \neq 0 \end{aligned} \tag{7.1}$$

where $\overline{\mathbb{R}}$ is the extended real axis. (7.1) defines the forward Lyapunov exponent, i.e. for $t \to \infty$, and in a similar way the backward exponent $\lambda^-\colon \mathcal{U} \times TM \to \overline{\mathbb{R}}$ is defined for $t \to -\infty$.

In general it holds even for linear systems: There is, for (x,u) fixed, no decomposition $\mathbb{R}^d = \oplus V_i$ into linear subspaces with $\lambda(u,x,v) = \lambda^-(u,x,v)$ for all $v \in \cup V_i$; the lim sup need not be a lim; and (exponenetial) stability of a system does not necessarily imply stability under small perturbations of the vectorfield, since control systems have time dependent right hand sides, see e.g. Cesari [19] or Hahn [47] for examples in the context of ordinary differential equations. Lyapunov [61] introduced the concept of regularity, which guarantees the three properties above. The main result in this direction is the theorem of Oseledeč [67], for which one has to assume stationarity of the underlying flow Φ (i.e. the existence of a Φ-invariant probability measure P), together with an integrability condition on the cocycle $D\varphi$. Then there is a Φ-invariant set of full P-measure, such that all points in this set are Lyapunov regular, see e.g. Ruelle [73] for a presentation of this theory, and also Mañé [62] for the time discrete case. (Note that from a topological point of view the set of Lyapunov regular points, although of full measure, may be quite 'thin', namely residual, see e.g. Mañé [62].)

For control systems the situation is somewhat more complex, since one always has to consider non-regular points as well: Let the vectorfields $X_0 \dots X_m$ in (2.1) be linear, i.e. the system is bilinear $\dot{x} = A_0 x + \Sigma u_i(t) A_i x$, with non-zero matrices $A_0 \dots A_m$. Let the space $U \subset \mathbb{R}^m$ of admissible control values be a product of intervals. Then there is an admissible control function $u \in \mathcal{U}$, such that the matrix function $A_0 + \Sigma u_i(t) A_i x$ is not Lyapunov regular, (examples are easily constructed using the ideas in Cesari [19] or Hahn [47]). We are interested in the full Lyapunov spectrum

$$\Sigma_{Ly} = \{\lambda(u, x, v);\ (u, x, v) \in \mathcal{U} \times TM\} \tag{7.2}$$

and therefore we cannot use Oseledeč's Theorem. In the following we will analyze Σ using the projected linearized system (6.5) on the projective bundle $\mathbb{P}M$, and we will discuss relations with other spectral concepts. For applications to the stability theory of control systems we refer to the papers mentioned above.

In the following we construct for the Lyapunov spectrum an 'inner' approximation, the Floquet spectrum, and an 'outer' approximation, the Morse spectrum. Then the Lyapunov spectrum is sandwiched in between. Additional assumptions will imply that the (closure of) the Floquet spectrum and the Morse spectrum and hence the Lyapunov spectrum all coincide. In this situation, semicontinuity properties of the Floquet and the Morse spectrum yield continuity properties of the Lyapunov spectrum.

We start by defining the Floquet spectrum, which is based on periodic coefficient functions for the linear part. Clearly, this can only be guaranteed, if the corresponding control function together with the trajectory in the base space M is periodic. Then the Floquet exponents of the corresponding linear periodic differential equations will be special Lyapunov exponents. Essentially, the Floquet spectrum will consist of these Floquet exponents with some convenient additional properties, which we specify in the following formal definition of the Floquet spectrum of the system (2.1).

7.1. Definition. Let ${}_{\mathbb{P}}D$ be a control set with nonvoid interior of the system (6.5) on the projective bundle $\mathbb{P}M$ induced by the linearization (6.2) of system (2.1). The Floquet spectrum of the system (6.5) over ${}_{\mathbb{P}}D$ is defined as

$$\Sigma_{Fl}({}_{\mathbb{P}}D) = \left\{ \begin{array}{c} \lambda(u, x); (u, \mathbb{P}x) \in \mathcal{U} \times \text{ int } {}_{\mathbb{P}}D, u \text{ is piecewise constant,} \\ \text{periodic with period } \tau \text{ such that } \mathbb{P}\varphi(\tau, \mathbb{P}x) = \mathbb{P}x \end{array} \right\}.$$

The Floquet spectrum over a control set D in the base space M of the system (2.1) is

$$\Sigma_{Fl}(D) = \bigcup \left\{ \begin{array}{c} \Sigma_{Fl}({}_{\mathbb{P}}D);\ {}_{\mathbb{P}}D \text{ is a control set with nonvoid interior} \\ \text{of the system (6.5) with } \mathbb{P}\pi({}_{\mathbb{P}}D) \subset D \end{array} \right\}.$$

Obviously, one has for every control set $D \subset M$ the inclusion

$$\Sigma_{Fl}(D) \subset \Sigma_{Ly}.$$

Thus the Floquet spectrum furnishes an inner approximation to the Lyapunov spectrum.

Next we introduce an 'outer' approximation of the Lyapunov spectrum given by the Morse spectrum. This concept is based on topological considerations. Chain control sets in the projective bundle $\mathbb{P}M$ or the chain recurrent components of the corresponding control flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}M$ are an appropriate generalization of sums of generalized eigenspaces. (On the other hand, the Floquet spectrum is based on control sets corresponding to topologically transitive components of the control flow and to eigensapces of periodic matrix functions, see [26]). Observing that chain recurrent components are defined via chains, it may appear natural to base a corresponding concept of exponential growth rates on chains. This leads us to the following definition.

For $\varepsilon, T > 0$ an (ε, T)-chain ζ of $\mathbb{P}\Phi$ is given by $n \in \mathbb{N}, T_0, \ldots, T_n \geq T$, and $(u_0, \mathbb{P}x_0), \ldots, (u_n, \mathbb{P}x_n)$ in $\mathcal{U} \times \mathbb{P}M$ with $d(\mathbb{P}\Phi(T_i, (u_i, \mathbb{P}x_i)), (u_{i+1}, \mathbb{P}x_{i+1})) < \varepsilon$ for $i = 0, \ldots, n-1$. Define the finite time exponential growth rate of such a chain ζ (or 'chain exponent') by

$$\lambda(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} (\log |\varphi(T_i, x_i, u_i)| - \log |x_i|)$$

where $x_i \in \mathbb{P}^{-1}(\mathbb{P}x_i)$.

7.2. Definition. Let ${}_{\mathbb{P}}\mathcal{L} \subset \mathcal{U} \times \mathbb{P}M$ be a compact invariant set for the induced flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}M$ and assume that $\mathbb{P}\Phi|_{\mathbb{P}}\mathcal{L}$ is chain transitive. Then the Morse spectrum over ${}_{\mathbb{P}}\mathcal{L}$ is

$$\Sigma_{Mo}({}_{\mathbb{P}}\mathcal{L}) = \left\{ \begin{array}{c} \lambda \in \mathbb{R};\ \text{there are } \varepsilon^k \to 0,\ T^k \to \infty \text{ and } (\varepsilon^k, T^k)\text{ - chains } \zeta^k \\ \text{in } {}_{\mathbb{P}}\mathcal{L} \text{ with } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty \end{array} \right\}.$$

For a compact invariant set $\mathcal{L} \subset \mathcal{U} \times M$ define the Morse spectrum over $\mathcal{L}$ as

$$\Sigma_{Mo}(\mathcal{L}) = \bigcup \Sigma_{Mo}({}_{\mathbb{P}}\mathcal{E})$$

where the union is taken over all chain recurrent components ${}_{\mathbb{P}}\mathcal{E}$ of $\mathbb{P}\Phi|(\mathbb{P}\pi)^{-1}\mathcal{L}$. For $\mathcal{L} = \mathcal{U} \times M$ we call $\Sigma_{Mo}(\mathcal{U} \times M)$ the Morse spectrum of the system (2.1), denoted by Σ_{Mo}.

Of particular interest is the Morse spectrum over a control set $D \subset M$ with nonvoid interior and compact closure. Then ${}_{\mathbb{P}}\mathcal{L} = (\mathbb{P}\pi)^{-1}\mathcal{D} \subset \mathcal{U} \times \mathbb{P}_{clD}M \subset \mathcal{U} \times \mathbb{P}M$ is compact, where $\mathcal{D}$ is the lift of D. Define via this compact invariant set in $\mathcal{U} \times M$

$$\Sigma_{Mo}(D) := \Sigma_{Mo}(\mathcal{D})$$

Similarly, we define the Morse spectrum $\Sigma_{Mo}(E)$ over a chain control set $E \subset M$ via the lift to $\mathcal{U} \times M$. Theorem 3.7 in [29] implies that the Lyapunov spectrum is contained in the Morse spectrum, i.e.

$$\Sigma_{Fl} \subset \Sigma_{Ly} \subset \Sigma_{Mo}.$$

Hence the Lyapunov spectrum lies in between the Floquet and the Morse spectrum. We are mainly interested in the Lyapunov spectrum, hence we present conditions which imply that the closure of the Floquet spectrum and the Morse spectrum, and hence the Lyapunov spectrum all coincide.

7.3. Theorem. *Let the assumptions of Theorem 6.3 be satisfied. Then the following assertions hold:*

(i) *For all $\rho \in [0, \rho_1]$ and for each $i = 1, \dots, \ell$, the Morse spectrum has the form*

$$\Sigma_{Mo}(_{\mathbb{P}}E_i^\rho) = [\kappa^*(_{\mathbb{P}}E_i^\rho), \kappa(_{\mathbb{P}}E_i^\rho)]$$

with $\kappa^*(_{\mathbb{P}}E_i^\rho) < \kappa^*(_{\mathbb{P}}E_j^\rho)$ and $\kappa(_{\mathbb{P}}E_i^\rho) < \kappa(_{\mathbb{P}}E_j^\rho)$ if $_{\mathbb{P}}E_i^\rho \neq {_{\mathbb{P}}E_j^\rho}$ and $i < j$.

(ii) *For each $i = 1, \dots, \ell$ the sets of continuity points of the two maps $\rho \mapsto cl\Sigma_{Fl}(_{\mathbb{P}}D_i^\rho)$ and $\rho \mapsto \Sigma_{Mo}(_{\mathbb{P}}E_i^\rho)$ agree. At each continuity point we have $cl\Sigma_{Fl}(_{\mathbb{P}}D_i^\rho) = \Sigma_{Mo}(_{\mathbb{P}}E_i^\rho)$. In paraticular, if ρ is a continuity point for all $i = 1, \dots, \ell$, then*

$$cl\Sigma_{Fl}(D^\rho) = \bigcup_{i=1}^{\ell} cl\Sigma_{Fl}(_{\mathbb{P}}D_i^\rho) = \Sigma_{Ly}(D^\rho) = \bigcup_{i=1}^{\ell} \Sigma_{Mo}(_{\mathbb{P}}E_i^\rho) = \Sigma_{Mo}(E^\rho).$$

Note that there are at most countably many points of discontinuity.

This theorem gives - under the ρ-ρ'-inner pair condition - a complete characterization of the Lypunov spectrum and the corresponding 'eigenspaces' for bilinear control flows on vectorbundles. The proof follows from the separate analysis of the spectra together with the inner pair condition and the results in [28], see [32]. For the special case of bilinear systems, see [30]. We also note that the strict inequalities in assertion (i) follow from an important (but apparently little known) theorem which appears in Bronstein and Chernii [9], see also [15]. It sheds new light on Selgrade's Theorem [78] and states that a decomposition of a linear flow into subbundles corresponding to attractor-repeller pairs in the projective bundle is equivalent to a decomposition into exponentially separated subbundles.

7.4. Remark. Suppose, in addition to be assumptions of Theorem 6.3, that M is compact. Then the set

$$\{(u, x, v) \in \mathcal{U} \times TM; \lambda(u, x, v) \in \bigcup \{\Sigma_{Fl}(_{\mathbb{P}}C),\ _{\mathbb{P}}C \text{ is an invariant control set}\}\}$$

contains an open and dense subset of $\mathcal{U} \times TM$. Thus, generically, the Lyapunov exponents will belong to an invariant control set in $\mathbb{P}M$.

The spectra $\Sigma_{Ly}(_{\mathbb{P}}E)$ corresponding to different chain control sets in $\mathbb{P}M$ can overlap, as the following simple example shows:

7.5. Example. Consider the following 2-dimensional linearized (bilinear) system

$$\dot{v} = u_1(t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} v + u_2(t)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} v + u_3(t)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} v$$

with $U = [0,2] \times (\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. The control sets of the projected system on the projective space $\mathbb{P}$ in $\mathbb{R}^2$ are given by

$$_{\mathbb{P}}D_1 = \Pi_{\mathbb{P}}\{\begin{pmatrix}v_1\\ v_2\end{pmatrix} \in \mathbb{R}^2;\ v_2 = \alpha v_1,\ \alpha \in \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right)\}$$
$$_{\mathbb{P}}D_2 = \Pi_{\mathbb{P}}\{\begin{pmatrix}v_1\\ v_2\end{pmatrix} \in \mathbb{R}^2;\ v_2 = \alpha v_1,\ \alpha \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]\},$$

where $\Pi_{\mathbb{P}}$ is the projection of $\mathbb{R}^2$ onto $\mathbb{P}$. The chain control sets are

$$_{\mathbb{P}}E_1 = c\ell\ _{\mathbb{P}}D_1, \quad _{\mathbb{P}}E_2 =_{\mathbb{P}} D_2, \quad \text{see [CK4].}$$

For constant $u \in U$ denote the right hand side of the system equation by

$$A(u) = \begin{pmatrix}u_1 & u_2\\ u_3 & u_1\end{pmatrix}.$$

Furthermore, let $\lambda_1(u)$, $\lambda_2(u)$ be the real parts of the eigenvalues of $A(u)$ with $\lambda_1(u) \leq \lambda_2(u)$. Then we have $\{\lambda_1(u);\ u \in U\} = [-1, \frac{3}{2}]$ and $\{\lambda_2(y);\ u \in u\} = [\frac{1}{2}, 3]$. Since these intervals are contained in the spectral sets over $_{\mathbb{P}}E_1$, and $_{\mathbb{P}}E_2$ respectively, we obtain $[\frac{1}{2}, \frac{3}{2}] \subset \Sigma_{Ly}(_{\mathbb{P}}E_1) \cap \Sigma_{Ly}(_{\mathbb{P}}E_2)$.

Little can be said about the relation of the different spectra at the ρ-discontinuity points of Theorem 7.3. The next example shows that while the spectral intervals can be different at these points, the closure of the entire Floquet spectrum and the Morse spectrum may still agree.

7.6. Example. Consider the bilinear system

$$\dot{v} = \begin{pmatrix}0 & 1\\ a & 0\end{pmatrix} v + u(t)\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} v,$$

with $u \in U = [A, a] \subset \mathbb{R}$. The projection of this system onto the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ yields Example 2.3. If we identify the projective space with the angle $\varphi \in [0, \pi)$, then the projected system has two control sets, namely (with $0 = \pi$) $_{\mathbb{P}}D_1 = \left[0, \arctan(a-A)^{1/2}\right]$ and $_{\mathbb{P}}D_2 = \left(\pi - \arctan(a-A)^{1/2}, \pi\right)$, while the only chain control set is $_{\mathbb{P}}E = \mathbb{P}$. The associated spectral sets are

$$\Sigma(_{\mathbb{P}}D_1) = \left(0, \sqrt{a-A}\right), \quad \Sigma(_{\mathbb{P}}D_2) = \left(-\sqrt{a-A}, 0\right),$$
and
$$\Sigma(_{\mathbb{P}}E) = \left[-\sqrt{a-A}, \sqrt{a-A}\right] = I(_{\mathbb{P}}E) = \Sigma.$$

Hence the control spectrum decomposes the Morse spectrum $I(_{\mathbf{P}}E)$ into two intervals, corresponding to the controllability structure of the projected system. In particular, we have:

- $\{\lambda(u,v);\ u \in \mathcal{U}\} = c\ell\Sigma(_{\mathbf{P}}D_1)$ for $v \in \mathbf{P} \setminus \left(_{\mathbf{P}}D_2 \cup \{\pi - \arctan(a-A)^{1/2}\}\right)$,
- $\{\lambda(u,v);\ u \in \mathcal{U}\} = \Sigma(_{\mathbf{P}}E)$ for $v \in {}_{\mathbf{P}}D_2$,
- $\{\lambda(u,v);\ u \in \mathcal{U}\} = c\ell\Sigma(_{\mathbf{P}}D_1) \cup \{-\sqrt{a-A}\}$ for $v = \pi - \arctan(a-A)^{1/2}$.

Hence only for $v \in {}_{\mathbf{P}}D_2 \cup \{\pi - \arctan(a-A)^{1/2}\}$ is there an $u \in \mathcal{U}$, such that the Lyapunov exponent of the solution is negative, i.e. such that the system is open-loop stabilizable at the origin 0 from v.

Finally, we briefly discuss the relation to the dichotomy spectrum (or dynamical spectrum) introduced by Sacker and Sell [76], cp. also Selgrade [78], and the Oseledeč spectrum [67]. We note that a number of interesting connections between the dichotomy spectrum, the Oseledeč spectrum and the Lyapunov spectrum have been derived in [52]. The dichotomy spectrum is based on exponential dichotomies of the flow and, over a chain recurrent base space, this is equivalent to the topological spectrum. The Lyapunov spectrum is always contained in the dichotomy spectrum. The intervals of the dichotomy spectrum are obtained by taking the unions of overlapping intervals of the Morse spectrum, cp. [29]. Example 7.5 above shows that the intervals of the Lyapunov spectrum corresponding to different control sets (or chain control sets) may overlap. Hence the bundle decomposition of the Morse spectrum can be strictly finer than the one induced by the dichotomy (or the topological) spectrum. Furthermore, the bundle decomposition associated with the Morse spectrum is equivalent to a decomposition into exponentially separated subbundles. And exponential separation is equivalent to the existence of a scalar cocycle such that the linear flow multiplied by this cocycle admits an exponential dichotomy, cp. Palmer [68] and, in particular, Bronshtein [15].

For the case of a base space which is not chain transitive, we note that the topological spectrum is still associated with attractor-repeller pairs in $\mathbf{P}M$, (cf. [91], Theorem 2.7.), and the dichotomy spectrum corresponds to a subbundle decomposition via the projectors of exponential dichotomies, but the decompositions corresponding to the Morse spectrum are not defined globally.

The Oseledeč spectrum is defined for invariant measures μ on the base space. Hence, for a given μ, the associated (measurable) bundle decomposition can be finer than the one induced by the Morse spectrum. One of the main results of [52] shows that for all ergodic μ the Oseledeč spaces are contained in the subbundles induced by the dichotomy spectrum. Combining this with Bronshtein's result on the characterization of exponentially separated subbundles [15], one obtains that for an ergodic measure μ on the base space the Oseledeč bundles are contained in the Morse spectral bundles, with μ-probability one, see Latushkin [58].

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References.

1. Arnold, L. (1995), *Random Dynamical Systems*, in preparation.
2. Arnold, L., P. Boxler (1992), *Stochastic bifurction: Instructive examples in dimension one*, Stochastic Flows, Pinsky, M., V. Wihstutz (eds.), Birkhäuser, 241–255.
3. Arnold, L., H. Crauel, J.-P. Eckmann (eds.) (1991), *Lyapunov Exponents, Proceedings of the Oberwolfach Conference 1990*, Springer-Verlag (to appear).
4. Arnold, L., X. Kedai (1995), *Normal forms for random differential equations*, J. Diff. Equations (to appear).
5. Arnold, L., W. Kliemann (1987), *Large deviations of linear stochastic differential equations*, Stochastic Differential Systems, Engelbrecht, J. H., W. Schmidt (eds.), Springer-Verlag, 117–151.
6. Bacciotti, A. (1992), *Local Stabilizability of Nonlinear Control Systems*, World Scientific.
7. Baxendale, P. (1986), *Asymptotic behavior of stochastic flows of diffeomorphisms*, Proceedings of the 15th SPA, Nagoya 1985, LN Mathematics Vol. 1203, Springer-Verlag, 1–19.
8. Bellman, R., J. Bentsman, S. M. Meerkov (1986), *Vibrational control of nonlinear systems: Vibrational stabilizability*, IEEE Trans. Aut. Control **AC-31**, 710–716.
9. Bronshtein, I.U., V.F. Chernii (1978), *Linear extensions satisfying Perron's condition I*, Differential Equations 14, 1234–1243.
10. Bogenschuetz, T. (1990), *Entropy for random dynamical systems*, Rep. Nr. 235, Institut für Dynamische Systeme, Universität Bremen, Bremen, FRG.
11. Berge, P., Y. Pomeau, C. Vidal (1984), *Order within Chaos*, Wiley and Hermann.
12. Brockett, R. W. (1973), *Lie algebras and Lie groups in control theory*, Geometric Methods in Systems Theory, Brockett, R. W., D. Q. Mayne (eds.), Reidel, 43–82.
13. Brockett, R. W. (1983), *Asymptotic stability and feedback stabilization*, Differen-

tial Geometric Control Theory, Brockett, R. W., R. S. Millman, H. J. Sussmann (eds.), Birkhäuser, 181–191.

14. Barros, C.J.B. L.A.B. San Martin (1994), *On the number of control sets on projective spaces*, RP 67/94, Instituto de Matematica, Universidade Estadual de Campinas.
15. Bronshtein, I.U. (1984), *Nonautonomous Dynamical Systems*, Kishinev.
16. Boxler, P. (1989), *A stochastic version of center manifold theory*, Probab. Th. Related Fields **83**, 509–545.
17. Boxler, P. (1990), *A necessary condition for a stochastic bifurcation*, Report Nr. 237, Institut für Dynamische Systeme, Universität Bremen, Bremen, FRG.
18. Carverhill, A. (1985), *Flows of stochastic dynamical systems: ergodic theory*, Stochastics **14**, 273–317.
19. Cesari, L. (1971), *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 3rd ed., Springer-Verlag.
20. Chow, W. L. (1938), *Ueber Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117**, 98–105.
21. Colonius, F., W. Kliemann (1989), *Infinite time optimal control and periodicity*, Appl. Math. Optim. **20**, 113–130.
22. Colonius, F., W. Kliemann (1990), *Stability radii and Lyapunov exponents*, Control of Uncertain Systems, Hinrichsen, D., B. Martensson (eds.), Birkhäuser, pp. 19–55.
23. Colonius, F., W. Kliemann (1993), *Maximal and minimal Lyapunov exponents of bilinear control systems*, J. Diff. Equations **101**, 232–275.
24. Colonius, F., W. Kliemann (1992), *Remarks on ergodic theory of stochastic flows and control flows*, Stochastic Flows, Pinsky, M., V. Wihstutz (eds.), Birkhäuser, pp. 203–240.
25. Colonius, F., W. Kliemann (1993), *Some aspects of control systems as dynamical systems*, J. Dynamics and Differential Equations **5**, 469–494.
26. Colonius, F., W. Kliemann (1993), *Linear control semigroups acting on projective space*, J. Dynamics and Differential Equations **5**, 495–528.
27. Colonius, F., W. Kliemann (1991), *Stabilization of linear uncertain systems*, Modelling and Control of Uncertain Systems, G.B. DiMasi, A. Gombani, A. Kurzhanski (eds.), Birkhäuser, 76–90.
28. Colonius, F., W. Kliemann (1994), *Limit behavior and genericity for nonlinear control systems*, J. Diff. Equations **109**, 8–41.
29. Colonius, F., W. Kliemann (1995), *The Morse spectrum of linear flows on vector*

bundles, to appear in Trans. Amer. Math. Soc..

30. Colonius, F., W. Kliemann (1995), *The Lyapunov spectrum of families of time-varying matrices*, to appear in Trans. Amer. Math. Soc..
31. Colonius, F., W. Kliemann (1992), *On control sets and feedback for nonlinear systems*, Proceedings Nonlinear Control System Design Symposium, (Bordeaux, 24–26 June 1992) IFAC, 49–56.
32. Colonius, F., W. Kliemann (1995), *Control Flows*, (in preparation).
33. Colonius, F., F.J. DelaRubia, W. Kliemann (1995), *Stochastic models with multistability and extinction levels*, to appear in SIAM J. Appl. Math.
34. Colonius, F., W. Kliemann, S. Krull (1991), *Stability and stabilization of linear uncertain systems – a Lyapunov exponents approach*, Report No. 372 DFG Schwerpunkt 'Anwendungsbezogene Optimierung und Steuerung', Universität Augsburg.
35. Conley, C. (1978), *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference Series, no. 38, American Mathematical Society, Providence.
36. Crauel, H. (1991), *Markov measures for random dynamical systems*, Stochastics and Stochastics Reports **37**, 153–173.
37. Crauel, H. (1993), *Non-Markov invariant meausres are hyperbolic*, Stochastic Processes and Their Applications **45**, 13–28.
38. Crauel, H. (1986), *Lyapunov exponents and invariant measures of stochastic systems on manifolds*, Lyapunov Exponents, Arnold, L., V. Wihstutz (eds.), LN Mathematics Vol. 1186, Springer-Verlag, 271–291.
39. Cheng, G.-S.J., T. J. Tarn, D. L. Elliott (1975), *Controllability of bilinear systems*, Variable Structure Systems with Application to Economics and Biology, LN in Economics and Mathematical Systems Vol. 111, Springer-Verlag, 83–100.
40. Dahlke, S. (1989), *Invariante Mannigfaltigkeiten für Produkte zufälliger Diffeomorphismen*, Dissertation, Universität Bremen, Bremen, FRG.
41. Ditto, W. L., S. N. Rauseo, M. L. Spano (1990), *Experimental control of chaos*, Physical Review Letters **65**, 3211–3214.
42. Doyle, J.C., K. Glover, P.P. Khargonekar, B.A. Francis (1989), *State space solutions to standard H_2 and H_∞ control problems*, IEEE Trans. Autom. Control AC-**34**, 831–846.
43. Francis, B. A. (1987), *A Course in H-Infinity Control Theory*, LN Control and Information Sciences Vol. 88, Springer-Verlag.
44. Frobenius, G. (1911), *Über die unzerlegbaren diskreten Bewegungsgruppen*, Sitzungsberichte der Königl. Preuss. Akad. Wiss. **29**, 654–665.

45. Golubitsky, M., D. G. Schaeffer (1985), *Singularities and Groups in Bifurcation Theory*, Springer-Verlag.

46. Grüne, L. (1994), *Numerical stabilization of bilinear control systems*, Report No. 301, Institut für Mathematik, Universität Augsburg, Augsburg, submitted.

47. Hahn, W. (1967), *The Stability of Motion*, Springer-Verlag.

48. Hairer, E., S. P. Norsett, G. Wanner (1987), *Solving Ordinary Differential Equations I*, Springer-Verlag.

49. Hinrichsen, D., A. J. Pritchard (1990), *Real and complex stability radii: A survey*, Control of Uncertain Systems, Hinrichsen, D., B. Martensson (eds.), Birkhäuser, pp. 119–162.

50. Ichihara, K., H. Kunita (1974), *A classification of the second order degenerate elliptic operators and its probabilistic characterization*, Z. Wahrsch. verw. Gebiete **30**, 235–25; **39** (1977), 81–84.

51. Isidori, A. (1989), *Nonlinear Control Systems*, 2nd ed., Springer-Verlag.

52. Johnson, R. A., K. J. Palmer, G. R. Sell (1987), *Ergodic properties of linear dynamical systems*, SIAM J. Math. Anal. **18**, 1–33.

53. Jurdjevic, V., J. P. Quinn (1978), *Controllability and stability*, J. Diff. Equations **28**, 381–389.

54. Kliemann, W. (1987), *Recurrence and invariant measures for degenerate diffusions*, Ann. Prob. **15**, 690–707.

55. Knobloch, H.W., A. Isidori, D. Flockerzi (1993), *Topics in Control Theory*, DMV Seminar Band 22, Birkhäuser.

56. Kunita, H. (1974), *Diffusion Processes and Control Systems*, Lecture Notes, University of Paris.

57. Kunita, H. (1978), *Supports of diffusion processes*, Proceedings Int. Symp. Stochastic Diff. Equ., Ito, K. (ed.), Wiley, 163–185.

58. Latushkin, Y. (1992), *Exact Lyapunov exponents and exponentially separated subbundles*, Partial Differential Equations, Wiener, J., J. Hale (eds.), Pitman, 91–95.

59. Lobry, C. (1974), *Controllability of nonlinear systems on compact manifolds*, SIAM J. Control **12**, 1–4.

60. Lorenz, F. N. (1963), *Deterministic nonperiodic flow*, J. Atmospheric Sci. **20**, 130–141.

61. Lyapunov, M. A. (1893), *General Problems in the Stabiity of Motion*, (Russian), Comm. Soc. Math. Kharkow, (reprinted in Ann. Math. Studies **17**, Princeton (1949)).

62. Mañé, R. (1987), *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag.
63. Miller, R. K. (1965), *Almost periodic differential equations as dynamical systems with applications to the existence of a.p. solutions*, J. Diff. Equations **1**, 337–345.
64. Murdock, J. A. (1991), *Perturbations. Theory and Methods*, Wiley-Interscience.
65. Nijmeijer, H., A. J. van der Schaft (1990), *Nonlinear Dynamical Control Systems*, Springer-Verlag.
66. Ott, E., C. Grebogi, J. A. Yorke (1990), *Controlling Chaos*, Physical Review Letters **64**, 1196–1199.
67. Oseledeč, V. I. (1968), *A multiplicative ergodic theory. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19**, 197–231.
68. Palmer, K.J. (1982), *Exponential separation, exponential dichotomy, and spectral theory for linear systems of ordinary differential equations*, J. Diff. Equations **46**, 324–345.
69. Peixoto, M. (1962), *Structural stability on 2-dimensional manifolds*, Topology **1**, 101–120.
70. Poore, A. B. (1974), *A model equation arising from chemical reactor theory*, Arch. Rational Mech. Anal. **52**, 358–388.
71. Pinsky, M., V. Wihstutz (eds.) (1992), *Stochastic Flows*, Birkhäuser.
72. Rotea, M. A., P. P. Khargonekar (1989), *Stability of uncertain systems with norm bounded uncertainty - a control Lyapunov function approach*, SIAM J. Control Optim. **27**, 1462–1476.
73. Ruelle, D. (1979), *Ergodic theory of differentiable dynamical systems*, Publ. I.H.E.S. **50**, 275–320.
74. Ruelle, D. (1981), *Small random perturbations of dynamical systems and the definition of attractors*, Commun. Math. Phys. **20**, 167–192.
75. Ruelle, D. (1989), *Elements of Differentiable Dynamics and Bifurcation*, Academic Press.
76. Sacker, R. J., G. R. Sell (1978), *A spectral theory for linear differential systems*, J. Diff. Equations **37**, 320–358.
77. Schuster, H.G. (1988), *Deterministic Chaos, An Introduction*, 2nd ed., VCH-Verlags-gesellschaft, Weinheim.
78. Selgrade, J. (1975), *Isolated invariant sets for flows on vector bundles*, Trans. Amer. Math. Soc. **203**, 359–390.
79. Sell, G. R. (1967), *Nonautonomous differential equations and topological dynamics I. The basic theory*, J. Diff. Equations **3**, 241–262.

80. Sussmann, H. J., V. Jurdjevic (1972), *Controllability of nonlinear systems*, J. Diff. Equations **12**, 95–116.
81. San Martin, L. (1986), *Invariant Control Sets on Fibre Bundles*, Ph.D. dissertation, Warwick, England.
82. San Martin, L. A. B., L. Arnold (1986), *A control problem related to the Lyapunov spectrum of stochastic flows*, Matematica Aplicada e Computacional **5**, 31–64.
83. Sontag, E. D. (1990), *Mathematical Control Theory*, Springer-Verlag.
84. Sontag, E. D. (1990), *Feedback stabilization of nonlinear systems*, Robust Control of Linear Systems and Nonlinear Control, M. A. Kashoek, J. H. van Schuppen, A. C. Ran (eds.), Birkhäuser.
85. Shinbrot, T., E. Ott, C. Grebogi, J. A. Yorke (1990), *Using chaos to direct trajectories to targets*, Physical Review Letters **65**, 3215–3218.
86. Sontag, E. D., H. J. Sussmann (1980), *Remarks on continuous feedback*, Proceedings IEEE Conference on Decision and Control, Albuquerque 1980, 916–921.
87. Sparrow, C. (1982), *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer-Verlag.
88. Sussmann, H. J. (1973), *Orbits of families of vectorfields and integrability of distributions*, Trans. Amer. Math. Soc. **180**, 171–188.
89. Sussmann, H. J. (1979), *Subanalytic sets and feedback control*, J. Diff. Equations **31**, 31–52.
90. Stroock, D. W., S. R. S. Varadhan (1972), *On the support of diffusion processes with applications to the strong maximum principle*, Proc. 6th Berkeley Symp. Math. Stat. Probab. **3**, 333–359.
91. Salamon, D., E. Zehnder (1988), *Flows on vector bundles and hyperbolic sets*, Trans. Amer. Math. Soc. **306**, 623–649.
92. Townley, S., E. P. Ryan (1991), *On the supremally regulated complex stability radius as a real robustness measure*, Syst. Control Letters **16**, 249–259.
93. Ventcel, A. D. (1979), *Theorie zufälliger Prozesse*, Birkhäuser.
94. Ventcel, A. D., M. I. Freidlin (1984), *Random Perturbations of Dynamical Systems*, Springer-Verlag.