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Deriving Focused Lattice Calculi

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Abstract We derive rewrite-based ordered resolution calculi for semilattices, distributive lattices and boolean lattices. Using ordered resolution as a metaprocedure, theory axioms are first transformed into independent bases. Focused inference rules are then extracted from inference patterns in refutations. The derivation is guided by mathematical and procedural background knowledge, in particular by ordered chaining calculi for quasiorderings (forgetting the lattice structure), by ordered resolution (forgetting the clause structure) and by Knuth-Bendix completion for non-symmetric transitive relations (forgetting both structures). Conversely, all three calculi are derived and proven complete in a transparent and generic way as special cases of the lattice calculi.
Keywords: automated deduction, lattice theory, term rewriting, ordered resolution, ordered chaining.

1 Introduction

We propose focused ordered resolution calculi for semilattices, distributive lattices and boolean lattices as theories of order. These calculi are relevant to many interesting applications, including set-theoretic and fixed-point reasoning, program analysis and construction, substructural logics and type systems. Focusing means integrating mathematical and procedural knowledge. Here, it is achieved via domain-specific inference rules, rewriting techniques and syntactic orderings on terms, atoms and clauses. The inference rules are specific superposition rules for lattice theory. They are constrained to manipulations with maximal terms in maximal atoms. Since lattices are quite complex for automated reasoning, focusing may not only drastically improve the proof-search in comparison with an axiomatic approach, it seems even indispensable. But it is also difficult. Only very few focused ordered resolution calculi are known so far. One reason is that existing methods require guessing the inference rules and justifying them a posteriori in rather involved completeness proofs (c.f. for instance [32,29]).

We therefore use an alternative method to *derive* the inference rules and prove refutational completeness by faithfulness of the derivation [27]. This derivation method uses ordered resolution as a metaprocedure. It has two steps. First, a specification is closed under ordered resolution with on the fly redundancy elimination. The resulting resolution basis has an independence property similar to set of support: No inference among its members is needed in a refutation. Second, the patterns arising in refutational inferences between non-theory clauses and the resolution basis are transformed into inference rules that entirely replace the resolution basis. Up to refutational completeness of the metaprocedure, the derivation method is purely constructive. It allows a fine-grained analysis of proofs. It supports variations, approximations and modular extensions.

The concision and efficiency of focused calculi depends on the quality of the mathematical and procedural background knowledge that is integrated. Here, it is achieved mainly as follows. First, via specific axioms: for instance, via a lattice-theoretic variant of a cut rule as a surprising operational characterization of distributivity [18]. Second, extending related procedures: ordered chaining for quasiorderings [4,27], ordered resolution calculi as solutions to lattice-theoretic uniform word problems [26] and Knuth-Bendix completion for quasiorderings [28]. On the one hand, ordered resolution is (via the cut rule) a critical pair computation for quasiorderings extended to distributive lattices. On the other hand, ordered chaining rules are critical pair computations extended to clauses. Extending both to lattices and clauses motivates an ordered resolution calculus with ordered chaining rules for lattices that include ordered resolution at the lattice level. Third, via syntactic orderings: these constrain inferences, for instance to the respective critical pair computations. We extend and combine those of the background procedures. We use in particular multisets as the natural data structure for lattice-theoretic resolution to build term orderings.

Briefly, our main results are the following.

- We propose refutationally complete ordered chaining calculi for finite semilattices, distributive lattices and boolean lattices. These calculi yield decision procedures. We also argue how to lift the calculi to semi-decision procedures for infinite structures.
- The lattice calculi comprise an ordered chaining calculus for quasiorderings, a Knuth-Bendix completion procedure for quasiorderings and ordered resolution calculi as special cases. Their formal derivation and proof of (refutational) completeness is therefore uniform and generic.
- As a peculiarity, we derive propositional ordered resolution (with redundancy elimination) formally as a rewrite-based solution to the uniform word problem for distributive lattices. This yields a constructive refutational completeness proof that uses ordered resolution reflexively as a metaprocedure.
- We present theory-specific simplification techniques that increase the efficiency of the calculi.
- Besides this, our results further demonstrate the power and applicability of the derivation method with a non-trivial example. In particular we show how easily focused calculi can be extended in a modular way.

We also include many examples that show the lattice calculi at work.

The remainder is organized as follows. Section 2 recalls some ordered resolution basics. Section 3 introduces some lattice theory and motivates our choice of the resolution basis with mathematical arguments. Section 4 introduces the syntactic orderings for our lattice calculi. Section 5 introduces the lattice calculi.

Here we restrict our attention to the ground case, that is the case of finite structures. Section 6 contains the first step of the derivation: the construction of the resolution basis. Section 7 contains the second step of the derivation: the extraction of the inference rules and therewith the proof of refutational completeness of the calculus for finite distributive lattices. Section 8 discusses refutational completeness of special cases and decidability issues. Section 9 shows that our calculi yield short and simple proofs of some typical textbook exercises from lattice theory. In section 10, we lift our calculi to the non-ground case. Section 11 discusses some simplification techniques for the lattice calculi. These are indispensable for implementations. Section 12 discusses some extensions of the lattice calculi. In particular to boolean lattices and to set-theoretic reasoning, which is an interesting application. Section 13 contains a conclusion and an outlook to further work.

2 Ordered Resolution and Redundancy

We first recall some well-known results about ordered resolution and redundancy elimination. Consider [3,5] for further reference.

Let $T_\Sigma(X)$ be a set of terms with signature Σ and variables in X, let P be a set of predicates. The set A of *atoms* consists of all expressions $p(t_1, \ldots, t_n)$, where p is an n-ary predicate and $t_1, \ldots, t_n$ are terms. A *clause* is an expression

$$\{\!|\phi_1, \ldots, \phi_m|\!\} \longrightarrow \{\!|\psi_1, \ldots, \psi_n|\!\}.$$

Its *antecedent* $\{\!|\phi_1, \ldots, \phi_m|\!\}$ and *succedent* $\{\!|\psi_1, \ldots, \psi_n|\!\}$ are finite multisets of atoms. Antecedents are schematically denoted by Γ, succedents by Δ. Brackets will usually be omitted. The above clause represents the closed universal formula

$$(\forall x_1 \ldots x_k)(\neg\phi_1 \vee \cdots \vee \neg\phi_m \vee \psi_1 \vee \cdots \vee \psi_n).$$

A *Horn clause* contains at most one atom in its succedent. We deliberately write Δ for $\longrightarrow \Delta$ and $\neg\Gamma$ for $\Gamma \longrightarrow$.

We consider calculi with inference rules constrained by syntactic orderings. A *term ordering* is a well-founded total ordering on ground terms. An *atom ordering* is a well-founded total ordering on ground atoms. For non-ground expressions e_1 and e_2 and a term or atom ordering $\prec$ we define $e_1 \prec e_2$ iff $e_1\sigma \prec e_2\sigma$ for all ground substitutions σ. Consequently, $s \not\succeq t$ if $t\sigma \succ s\sigma$ for some ground terms $s\sigma$ and $t\sigma$. An atom ϕ is *maximal* with respect to a multiset Γ of atoms, if $\phi \not\prec \psi$ for all $\psi \in \Gamma$. It is *strictly maximal* with respect to Γ, if $\phi \not\preceq \psi$ for all $\psi \in \Gamma$. The non-ground orderings are still well-founded, but need no longer be total.

While atom orderings suffice to constrain the inferences of the ordered resolution calculus, clause orderings are needed for redundancy elimination. To extend atom orderings to clauses, we measure clauses as multisets of their atoms and use the multiset extension of the atom ordering. To disambiguate occurrences of atoms in antecedents and succedents, we assign to those in the antecedent a greater weight than those in the succedent. See section 4 for more details.

The clause ordering then inherits totality and well-foundedness from the atom ordering. Again, the non-ground extension need not be total. In unambiguous situations we denote all orderings by $\prec$.

Definition 1 (Ordered Resolution Calculus). *Let $\prec$ be an atom ordering. The* ordered resolution calculus OR *consists of the following deduction inference rules. The* ordered resolution rule

$$\frac{\Gamma \longrightarrow \Delta, \phi \qquad \Gamma', \psi \longrightarrow \Delta'}{\Gamma\sigma, \Gamma'\sigma \longrightarrow \Delta\sigma, \Delta'\sigma}, \tag{Res}$$

where σ is a most general unifier of ϕ and ψ, $\phi\sigma$ is strictly maximal with respect to $\Gamma\sigma, \Delta\sigma$ and maximal with respect to $\Gamma'\sigma, \Delta'\sigma$. The ordered factoring rule

$$\frac{\Gamma \longrightarrow \Delta, \phi, \psi}{\Gamma\sigma \longrightarrow \Delta\sigma, \phi\sigma}, \tag{Fact}$$

where σ is a most general unifier of ϕ and ψ and $\phi\sigma$ is strictly maximal with respect to $\Gamma\sigma$ and maximal with respect to $\Delta\sigma$.

In all inference rules, *side formulas* are the parts of clauses denoted by capital Greek letters. Atoms occurring explicitly in the premises are called *minor formulas*, those in the conclusion *principal formulas.*

Let S be a clause set and $\prec$ a clause ordering. A clause C is *$\prec$-redundant* or simply redundant in S, if C is a semantic consequence of instances from S which are all smaller than C with respect to $\prec$. Closing S under OR-inferences and eliminating redundant clauses on the fly transforms S into a *resolution basis $rb(S)$*. The transformation need not terminate, but we have *refutational completeness*: All fair OR-strategies derive the empty clause within finitely many steps, if S is inconsistent. We call a proof to the empty clause an OR-*refutation.*

Proposition 1. *S is inconsistent iff $rb(S)$ contains the empty clause.*

A resolution basis B is a special basis. By definition it satisfies the independence property that all conclusions of *primary B-inferences*, that is OR-inferences with both premises from B, are redundant. However, it need not be unique.

Proposition 2. *Let B be a consistent resolution basis and T a clause set such that $B \cup T$ is inconsistent. There is an* OR*-refutation without primary B-inferences.*

In [3], variants of proposition 1 and proposition 2 have been shown for a stronger notion of redundancy. For deriving the lattice calculi, the weak notion suffices. But the strong notion can of course always be used.

By proposition 2, resolution bases allow ordered resolution strategies similar to set of support. The construction of a resolution basis will constitute the first step of our derivation of focused lattice calculi. OR will be used as a metaprocedure in the derivation. Its properties, like refutational completeness and avoidance of primary theory inferences, are essential ingredients.

3 Lattices

Here we are not concerned with arbitrary signatures and predicates. Let $\Sigma = \{\sqcup, \sqcap\}$ and $P = \{\leq\}$. $\sqcup$ and $\sqcap$ are varyadic operation symbols for the lattice join and meet operations. $\leq$ is a binary predicate symbol denoting a *quasiordering*—a reflexive transitive relation. A *join semilattice* is a quasiordered set closed under least upper bounds or *meets* for all pairs of elements. A*meet semilattice* is a quasiordered set closed under greatest lower bounds or *meets* for all pairs of elements. Join and meet semilattices are duals. *Lattice duality* means exchange of joins and meets and inversion of the ordering. A *lattice* is both a join and a meet semilattice. It is *distributive*, if (dist) holds (see below) or its dual and therewith both. A quasiordering is axiomatized by the set $Q = \{(\text{ref}), (\text{trans})\}$, the join and meet semilattice by $J = Q \cup \{(\text{lub}), (\text{ub})\}$ and $M = Q \cup \{(\text{glb}), (\text{lb})\}$, a lattice by $L = J \cup M$, a distributive lattice by $D = L \cup \{(\text{dist})\}$. Thereby

$$x \leq x \tag{ref}$$

$$x \leq y, y \leq z \longrightarrow x \leq z \tag{trans}$$

$$x \sqcap y \leq x \qquad\qquad x \sqcap y \leq y \tag{lb}$$

$$x \leq x \sqcup y \qquad\qquad y \leq x \sqcup y \tag{ub}$$

$$x \leq y, x \leq z \longrightarrow x \leq y \sqcap z \tag{glb}$$

$$x \leq z, y \leq z \longrightarrow x \sqcup y \leq z \tag{lub}$$

$$x \sqcap (y \sqcup z) \leq (x \sqcap y) \sqcup (x \sqcap z) \tag{dist}$$

For a quasiordering, joins and meets are unique up to the congruence $\sim = (\leq \cap \geq)$. Semantically, $\leq/\sim$ is a partial ordering, hence an antisymmetric quasiordering ($x \leq y, y \leq x \longrightarrow x = y$). Operationally, the only role of antisymmetry is splitting equalities into inequalities. We can therefore disregard it. Joins and meets are associative, commutative, idempotent ($x \sqcap x = x = x \sqcup x$) and monotonic in the associated partial ordering. We will henceforth consider all inequalities modulo AC (associativity and commutativity) and normalize with respect to I (idempotence). See [7,20] for further information on lattices.

In [25] we directly use J, M and D for deriving resolution bases. Here we choose a technically simpler way that uses more domain specific knowledge. First, the lattice axioms allow us to transform every inequality between lattice terms into a set of simpler expressions. Consider the following Horn clauses.

$$x \leq z \longrightarrow x \sqcap y \leq z \tag{ml}$$

$$x \leq y \longrightarrow x \leq y \sqcup z \tag{jr}$$

$$x \leq y \sqcap z \longrightarrow x \leq y \tag{imr}$$

$$x \sqcup y \leq z \longrightarrow x \leq z \tag{ijl}$$

Lemma 1. *Let A be a quasi-ordered set.*

(i) A is a meet-semilattice iff A satisfies (glb), (imr) iff A satisfies (glb), (ml).

(ii) A is a join-semilattice iff A satisfies (lub), (ijl) iff A satisfies (lub), (jr).

(glb) and (imr) and their duals (lub) and (ijl) together with (dist) allow us to restrict our attention to inequalities of a particular format.

Lemma 2. *Let S be a meet semilattice, L a distributive lattice and G a set of generators (that is a set of constants disjoint from Σ).*

(i) An inequality $s \leq t_1 \sqcap \ldots \sqcap t_n$ holds in S, iff $s \leq t_i$ holds for all $1 \leq i \leq n$.
(ii) An inequality $s < t$ holds in L, iff there is a set of inequalities of the form $s_1 \sqcap \ldots \sqcap s_m \leq t_1 \sqcup \ldots \sqcup t_n$ that hold in L and the s_i and t_i are generators occurring in s (t).

A *reduced join (meet) semilattice inequality* is a join (meet) semilattice inequality whose left-hand (right-hand) side is a generator. A *reduced lattice inequality* is a lattice inequality whose left-hand side is a meet and whose right-hand side is a join of generators. Reduced semilattice and lattice inequalities are lattice-theoretic analogs of Horn clauses and clauses. In particular, the clausal arrow is a quasiordering. We usually write more compactly

$$s_1 \ldots s_m \leq t_1 \ldots t_n$$

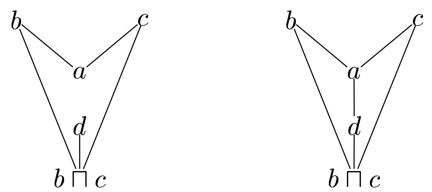
instead of $s_1 \sqcap \ldots \sqcap s_m \leq t_1 \sqcup \ldots \sqcup t_n$. By a lattice-theoretic variant of the Tseitin transformation [30], there is a linear reduction of distributive lattice terms. We speak of *reduced clausal theories*, when all inequalities in clauses are reduced.

(ml) and (jr) are procedurally preferable to (lb) and (ub), since they allow splitting at left-hand and right-hand sides of inequalities. This splitting is also important for Whitman's algorithm for solving the word problem for the free lattice [33,34,11]. It is also similar to the situation in the sequent calculus [13,22].

For the remainder of this text we assume that all lattice and semilattice inequalities are reduced. We will also restrict our inference rules to reduced terms. But then, some axioms of the join and meet semilattice as well as the distributivity axiom can no longer be used, since they do not operate on reduced clauses. However, one cannot completely dispense with their effect in the initial specification.

Lemma 3. *J without (lub) and M without (glb) are incomplete for the reduced clausal theory of the join and meet semilattice.*

Proof. By duality we only consider the meet semilattice. Let $M^- = M - \{(\text{glb})\}$. The quasi-orderings A_1 and A_2



are models of M^- but not of M, since by (glb), $a \leq b$, $a \leq c$ and $b \sqcap c \leq d$ imply $a \leq d$. a, b, c and d can be taken as generators. Thus the statement holds in particular in the reduced clausal theory. □

Lemma 4. *In the reduced clausal theory, a quasiordering is a model of M^-, but not of M iff it contains a sub-quasiordering isomorphic with A_1 or A_2.*

Proof. It remains to show the only if direction. Assume that A is a quasiordering that does not contain a substructure isomorphic with A_1 and A_2. Assume further that A is a model of M^-. We show that O is also a model of M. Whenever A contains an element a such that $a \leq b$ and $a \leq c$, then it contain also $b \sqcap c$ by (lb). Since A does not contain a substructure isomorphic with A_1, a and $b \sqcap c$ must be comparable. Since A does not contain a substructure isomorphic with A_2, $b \sqcap c \not\leq a$. Consequently, $a \leq b \sqcap c$. In reduced clausal theory, we cannot write $a \leq b \wedge c$. But whenever there is a d such that $b \sqcap c \leq d$, then our argument must hold. This follows from the same considerations. In particular it holds for $d = b \sqcap c$. □

Again a dual statement holds for the join semilattice.

At this point, background knowledge comes into play to find the appropriate initial specification for reduced clausal theories. Following [18] we use the cut rule

$$x_1 \leq y_1 z, x_2 z \leq y_2 \longrightarrow x_1 x_2 \leq y_1 y_2. \qquad \text{(cut)}$$

as an alternative characterization of distributivity.

Lemma 5. *A lattice is distributive iff it satisfies (cut).*

Written as an inference rule, this Horn clause is a lattice-theoretic variant of resolution. It combines the effect of transitivity, distributivity and monotonicity of join and meet. See [26] for a discussion and a derivation. In the context of Knuth-Bendix completion for quasiorderings, it can be restricted by ordering constraints and used as a critical pair computation to solve the uniform word problem for distributive lattices [26]. There are similar cut rules (not involving distributivity)

$$x_1 \leq y_1 z, z \leq y_2 \longrightarrow x_1 \leq y_1 y_2, \qquad \text{(jcut)}$$

$$x_1 \leq z, x_2 z \leq y_2 \longrightarrow x_1 x_2 \leq y_2, \qquad \text{(mcut)}$$

for the join and meet semilattice. These are used for solving the respective semilattice word problems. (cut), (jcut) and (mcut) operate on reduced inequalities. The expression z which is cut out is necessarily a generator.

Lemma 6. *The following sets axiomatize their reduced clausal theories up to normalization with I and modulo AC.*

(i) $J' = \{(ref), (trans), (jr), (jcut)\}$ *for join semilattices,*
(ii) $M' = \{(ref), (trans), (ml), (mcut)\}$ *for meet semilattices,*
(iii) $D' = \{(ref), (trans), (jr), (ml), (jcut), (mcut), (cut)\}$ *for distributive lattices.*

Proof. (ad i) The proof is dual to that of (ii).

(ad ii) First, we show that (mcut) holds in every semilattice. Let $x_1 \leq z$. Since meet is commutative and monotonic in every lattice, we also have $x_1 \sqcap x_2 \leq x_2 \sqcap z$. Then (trans) together with $x_2 \sqcap z \leq y_2$ yields $x_1 \sqcap x_2 \leq y_2$.

Let now $a \leq b$, $a \leq c$ and $bc \leq d$. By (mcut) $a \leq b$ and $bc \leq d$ imply $ac \leq d$; $a \leq c$ and $ac \leq d$ imply $aa \leq d$. Thus $a \leq d$ by normalization with idempotence. Hence (mcut) rules out sublattices isomorphic with A_1 and A_2.

(ad iii) This follows from combination of (i) with (ii) and from lemma 5. □

One can even further restrict the axiom sets.

Lemma 7. *J' and M' without (trans) and D' without (trans), (jcut) and (mcut) axiomatize their reduced clausal theories up to normalization with I and modulo AC.*

Proof. For M', we show that (mcut) makes (trans) superfluous. We show that we can derive $a \leq c$ from $a \leq b$ and $b \leq c$ without (trans). (ml) allows us to derive $ab \leq c$ from $b \leq c$. This yields $a \leq c$ via (mcut) and normalization with respect to idempotence. The proof for J' is dual.

For D', we only consider (mcut). We show that we can derive $a_1a_2 \leq b_2$ from $a_1 \leq x$ and $a_2x \leq b_2$ without (mcut). (jr) allows us to derive $a_1 \leq xb_2$ from $a_1 \leq x$. This yields $a_1a_2 \leq b_2$ via (cut) and normalization with respect to idempotence. □

In implementations, normalization with respect to I and also with respect of 0 and 1, when these constants are present, must be explicit. In finite semilattices and lattices, on can always define 1 and 0 as the join and meet of all generators. We use the canonical equational system modulo AC

$$\begin{aligned} T = \{x \sqcap x \to x, &\quad x \sqcup x \to x, \\ x \sqcup 1 \to 1, &\quad x \sqcap 1 \to x, \\ x \sqcup 0 \to x, &\quad x \sqcap 0 \to 0\}. \end{aligned}$$

The orientation of the rewrite rules is compatible with the term orderings from section 2. T induces reduced lattice inequalities modulo ACI01. AC and I01 are however treated completely differently. AC is handled by a compatible ordering (c.f. section 4), not by rewrite rules (the laws are not orientable). I01 is handled by the normalizing rewrite system T, not by a compatible ordering (which does not exist [19]). In the general non-ground case, some inference rules of our calculi use ACI- or ACI01-unification [1]. This is in the spirit of normalized equational rewriting [19]. ACI-unification is however very prolific. In presence of many variables, it leads to enormous numbers of most general unifiers. However, the unifiability test is very simple. Using a calculus with constraints [15,31,21], only unifiability must be tested in most steps of a refutation. This is important for the applicability of the calculus. But we do not treat this issue here. In the ground case, much cheaper operations, like string matching, suffice for comparing terms in the inference rules. Moreover, in presence of more structure, the most prolific inference rules completely disappear (c.f. section 12.

We denote the normal form of an expression e (a term, an atom, a clause) with respect to T by $(e)\downarrow_T$. In the ground case we may sort joins and meets of generators after each normalization. All consequences of a reduced clause set and J, M and D are again reduced. However, consequences of a reduced clause set in T-normal form may not be in T-normal form.

At the end of this section we again consider background knowledge to motivate the lattice calculi. As already mentioned, (cut) is used to solve the uniform word problem for distributive lattices. In this context, all non-theory clauses are atoms. We however want to admit arbitrary non-theory clauses. This is analogous to ordered chaining calculi for quasiordering, where a Knuth-Bendix completion procedure for quasiorderings, operating entirely on positive atoms, is extended to clauses via a (ground) positive chaining rule.

$$\frac{\Gamma \longrightarrow \Delta, r < s \qquad \Gamma' \longrightarrow \Delta', s < t}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta', r < t}$$

Thereby s is the maximal term in the minor formulas and the minor formulas are strictly maximal with respect to Γ, Γ', Δ and Δ'. The rule is a clausal extension of a critical pair computation. A chaining rule for distributive lattices can then analogously be motivated as an extension of (cut) to the lattice level.

$$\frac{\Gamma \longrightarrow \Delta, s_1 < t_1 x \qquad \Gamma' \longrightarrow \Delta', s_2 x < t_2}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta', s_1 s_2 < t_1 t_2}$$

But which other inference rules are needed and what are the ordering constraints? These questions require further consideration. They will be answered with the help of the derivation method.

Our third way of including background information is the construction of the specific syntactic orderings. In the following section, we will base them again on those of ordered resolution and ordered chaining. In section 6 we will show that J', M' and D' are resolution bases for these orderings and therefore independent sets. Thus the Horn clauses from these sets are independent, no inference between them is needed in refutations and their effect in refutations can be completely internalized into focused inference rules. This is the subject of section 7.

4 The Syntactic Orderings

The computation of a resolution basis, its termination and the procedural behavior of focused calculi crucially depend on the syntactic term, atom and clause orderings. In section 3 we have developed our initial axiomatizations with regard to the ordered chaining calculi for quasiorderings and ordered resolution in lattice theory. Here we build syntactic orderings for lattices that refine those of the background procedures and turn (jcut), and (cut) into lattice-theoretic variants of ordered Horn resolution and resolution.

Like in section 2, we first define a term ordering and then extend it to atoms and clauses. For the term ordering, let $\prec$ be the multiset extension of some total ordering on the set of generators. We assign minimal weight to 0 and 1, if present. $\prec$ is trivially well-founded, if the set is finite or denumerably infinite. We measure both joins and meets of generators by their multisets. This clearly suffices for the terms occurring in reduced lattice inequalities. By construction, $\prec$ is well-founded and compatible with AC: terms which are equal modulo AC are assigned the same measure. $\prec$ is natural for resolution, since multisets are a natural data structure for clauses. This choice of $\prec$ will force the calculus under construction to become resolution-like[1]. $\prec$ also appears, when resolution is modeled as a critical pair computation for distributive lattices.

We now consider the atom ordering. Let $\mathbb{B}$ be the two-element boolean algebra with ordering $<_{\mathbb{B}}$. Let

$$m = G \times \mathbb{B} \times \mathbb{B} \times G,$$

where G denotes a multiset of generators. Let A be a set of atoms occurring in some clause $C = \Gamma \longrightarrow \Delta$. The ordering $\prec_1 \subseteq m \times m$ is the lexicographic combination of $\prec$ for the first and last component of m and $<_{\mathbb{B}}$ for the others. A ground *atom measure* (for clause C) is the mapping $\mu_C : A \to m$ defined by

$$\mu_C : \phi \mapsto (t_\nu(\phi), p(\phi), s(\phi), t_\mu(\phi))$$

for each (ground) atom $\phi \in A$ occurring in C. Hereby $t_\nu(\phi)$ denotes the maximal term with respect to $\prec$ in ϕ and $t_\mu(\phi)$ denotes the minimal term with respect to $\prec$ in ϕ. $p(\phi) = 1$ if ϕ occurs in Γ and $p(\phi) = 0$, if ϕ occurs in Δ. $s(\phi) = 1$ if $\phi = s < t$ and $s \succeq t$and $s(\phi) = 0$, if $\phi = s < t$ and $s \prec t$. The (ground) *atom ordering* $\prec_2 \subseteq A \times A$ is defined by $\phi \prec_2 \psi$ iff $\mu_C(\phi) \prec_1 \mu_C(\psi)$ for $\phi, \psi \in A$. Hence $\prec_2$ is embedded in $\prec_1$ via the atom measure. The ordering $\prec_1$ is total and well-founded by construction. Via the embedding, $\prec_2$ inherits these properties. The definition of the atom measure follows those of the ordered chaining calculi for quasiorderings and of Knuth-Bendix completion for quasiorderings. This forces the calculus to become a chaining calculus at the clausal level and a completion procedure at the lattice level. The polarity p is not needed for comparing atoms, but it is crucial for the extension to clauses, to integrate redundancy elimination.

All these orderings are extended to the non-ground level and the clause level according to section 2. In particular the polarity p assigns greater weight to an occurrence of a term in the antecedent than to an occurrence in the succedent. In unambiguous situations we denote all orderings by $\prec$.

5 The Lattice Chaining Calculi

In this section we restrict our attention to finite lattices. Then all non-theory clauses are ground, since existential quantification can be replaced by a finite disjunction and universal quantification by a finite conjunction. The non-ground

[1] In [25], we alternatively force tableau calculi for distributive lattices with a term ordering emphasizing the subterm ordering.

clauses in J', M' and D' are completely internalized into the focused inference rules. Therefore, our entire calculi are then ground. The extension to the non-ground case is discussed in section 10.

We introduce indexed brackets to abbreviate the presentation of the calculi. A pair of brackets [.] in a clause denotes alternatively the clause without the brackets and the clause, where the brackets together with their content have been deleted. The clause

$$\Gamma \longrightarrow \Delta, [r]s \leq t,$$

for instance, denotes alternatively the clauses

$$\Gamma \longrightarrow \Delta, rs \leq t \qquad\qquad \Gamma \longrightarrow \Delta, s \leq t.$$

In inference rules, brackets with the same index are synchronized. The inference rule

$$\frac{\Gamma \longrightarrow \Delta, [r]_i s \leq t}{\Gamma', [u]_i v \leq w \longrightarrow \Delta'},$$

for instance, denotes alternatively the inference rules

$$\frac{\Gamma \longrightarrow \Delta, rs \leq t}{\Gamma', uv \leq w \longrightarrow \Delta'} \qquad\qquad \frac{\Gamma \longrightarrow \Delta, s \leq t}{\Gamma', v \leq w \longrightarrow \Delta'}$$

Definition 2 (Distributive Lattice Chaining). *Let $\succ$ be the atom and clause ordering of section 4. Let all clauses be reduced. The* ordered chaining calculus for finite distributive lattices DC *consists of the deductive inference rules and the redundancy elimination rules of* OR[2] *and the following inference rules.*

$$\frac{\Gamma, t \leq t \longrightarrow \Delta}{\Gamma \longrightarrow \Delta} \qquad \text{(Ref)}$$

$$\frac{\Gamma, rs \leq t \longrightarrow \Delta}{\Gamma, r \leq t \longrightarrow \Delta} \quad \text{(ML)} \qquad\qquad \frac{\Gamma, r \leq st \longrightarrow \Delta}{\Gamma, r \leq s \longrightarrow \Delta} \quad \text{(JR)}$$

Here the minor formula is maximal with respect to Γ and strictly maximal with respect to Δ.

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq [t_1]_j x \qquad \Gamma' \longrightarrow \Delta', [s_2]_m x \leq t_2}{(\Gamma, \Gamma' \longrightarrow \Delta, \Delta', s_1[s_2]_m \leq [t_1]_j t_2) \downarrow_T} \qquad \text{(Cut+)}$$

Here the terms containing x are strictly maximal in the minor formulas. The minor formulas are strictly maximal with respect to the side formulas in their respective premises.

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq [t_1]_j x \qquad \Gamma', s_1[s_2]_m \leq [t_1]_j t_2 \longrightarrow \Delta'}{(\Gamma, \Gamma', [u]_m x \leq t_2 \longrightarrow \Delta, \Delta') \downarrow_T} \qquad \text{(Cut-)}$$

Here the terms containing s_1 are strictly maximal in the minor formulas. In the first premise, the minor formula is strictly maximal with respect to the side

[2] Section 2 only defines a semantic *notion* of redundancy. Every set of inference rules implementing this notion is admitted.

formulas. In the second premise, the minor formula is maximal with respect to the side formulas. Moreover, $[u]_m x \neq t_2 \mod AC$, $u = s_2$ *or else* $u = s_1$, *if* s_2 *is absent in the minor formula.*

$$\frac{\Gamma, [s_2]_m x \leq t_2 \longrightarrow \Delta \qquad \Gamma' \longrightarrow \Delta', s_1[s_2]_m \leq [t_1]_j t_2}{(\Gamma, \Gamma', s_1 \leq [u]_j x \longrightarrow \Delta, \Delta') \downarrow_T} \quad \text{(Cut-)}$$

Here the terms containing t_1 *are strictly maximal in the minor formulas. In the first premise, the minor formula is maximal with respect to the side formulas. In the second premise, the minor formula is strictly maximal with respect to the side formulas. Moreover,* $s_1 \neq [u]_j x \mod AC$, $u = t_1$ *or else* $u = t_2$, *if* t_1 *is absent in the minor formula.*

$$\frac{\Gamma \longrightarrow \Delta, s \leq [t_1]_m x, s \leq [t_1]_m t_2}{(\Gamma, [s]_j x \leq t_2 \longrightarrow \Delta, s \leq [t_1]_m t_2) \downarrow_T} \quad \text{(DF)}$$

Here, x *is a generator, either* t_1 *is strictly maximal in the minor formulas or* s *is strictly maximal in the minor formulas and* s *can be set to* 1 *in the antecedent of the conclusion. The leftmost minor formula is strictly maximal with respect to the side formulas and the rightmost minor formula.*

$$\frac{\Gamma \longrightarrow \Delta, [s_1]_j x \leq t, [s_1]_j s_2 \leq t}{(\Gamma, s_2 \leq [t]_m x \longrightarrow \Delta, [s_1]_j s_2 \leq t) \downarrow_T} \quad \text{(DF)}$$

Here, x *is a generator, either* s_1 *is strictly maximal in the minor formulas or* t *is strictly maximal in the minor formulas and* t *can be set to* 1 *in the conclusion. The leftmost minor formula is strictly maximal with respect to the side formulas and the rightmost minor formula.*

The calculus is meant modulo AC *at the lattice level.*

(Ref) stands for *reflexivity*, (JR) and (ML) for *join right* and *meet left*, in analogy to the sequent calculus. (Cut+) and (Cut-) stand for *positive* and *negative cut*, (DF) for *distributivity factoring*. The two (Cut-) rules and the two (DF) rules are dual, if also the indices of brackets are exchanged. We immediately obtain the following specializations of DC.

Definition 3. *Under the conditions of definition 2, the calculus* DC *specializes to the following calculi.*

(i) An ordered chaining calculus for join semilattices JC, *removing the inference rule (ML) and discarding in the inference rules of* DC *the contents of all brackets indexed with* m.
(ii) An ordered chaining calculus for meet semilattices MC, *removing the inference rule (JR) and discarding in the inference rules of* DC *the contents of all brackets indexed with* j.
(iii) An ordered chaining calculus for quasiorderings QC, *removing the inference rules (JR) and (ML) and discarding in the inference rules of* DC *the contents of all brackets indexed with* j *and* m.

(iv) An ordered chaining calculus for transitive relations TC*, removing the inference rule* (*Ref*) *from* QC.

Definition 4 (Lattice Ordered Resolution [26]). *Under the conditions of definition 2, the deduction inference rules of the* ordered Horn resolution calculus HOR *and the* ordered resolution calculus OR *of [26]*[3] *as rewrite-based solutions to the uniform word problem of semilattices and distributive lattices are restrictions of* DC *and* MC *to non-theory clauses consisting of positive atoms.*

In this case, the only applicable rule is (Cut+). It computes a lattice-theoretic variant of a Gröbner basis from the given presentation (the set of positive atoms). The query inequalities, which are in question as consequences of the presentation and the axioms, can then be shown to be redundant by a search method (c.f. [26]). An alternative solution to the word problem transforms the query into a set of negative atoms and then uses DC. We will see in section 8 that the resolution calculi are indeed a decision procedure.

Definition 5 (Knuth-Bendix Completion [28]). *Under the conditions of definition 2, the deduction inference rules of the* Knuth-Bendix completion procedures for quasiorderings *and* non-symmetric transitive relations *(without term structure), are restrictions either of* QC *and* TC *to non-theory clauses consisting of positive atoms or* OR*, all lattice terms being generators.*

Soundness and completeness of DC are the subject of section 6 and section 7, the other calculi are dealt with in section 12. The unordered variant of DC is an instance of theory resolution. We will derive the DC-rules as an ordered variant thereof. DC is more focused than mere reasoning with resolution bases.

6 Construction of the Resolution Bases

We now perform the first step of the derivation of the chaining calculi. We start with the axiom sets J', M' and D'. With the orderings of section 4, we compute the respective resolution bases; closing the rules with respect to OR and eliminating redundant clauses on the fly. Use of duality prevents us from repetitions. It turns out that J', M' and D' are almost the resolution bases. However, in presence of syntactic orderings, we must also consider restricted variants of (jcut), (mcut) and (cut). We define the following sets of Horn clauses.

$$J'' = J' \cup \{(\text{trans})\}, \qquad M'' = M' \cup \{(\text{trans})\}, \qquad D'' = J'' \cup M'' \cup \{\text{cut}\}.$$

The fact that also the restricted variants are needed is evident from the construction in the proof of lemma 8. These restrictions are of course unnecessary, when the structures under consideration contain a zero and a unity. Then the instance

$$s_1 \leq 0x, s_2x \leq t_2 \longrightarrow s_1s_2 \leq 0t_2$$

[3] The ordering constraints of these calculi are weaker as those of ordered resolution given in this text in section 2.

of (cut) normalizes with respect to T to the instance

$$s_1 \le x, s_2x \le t_2 \longrightarrow s_1s_2 \le t_2$$

of (mcut) and the instance

$$s_1 \le x, 1x \le t_2 \longrightarrow s_1 1 \le t_2$$

of (mcut) normalizes to the instance

$$s_1 \le x, x \le t_2 \longrightarrow s_1 \le t_2$$

of (trans).

Lemma 8. *Let $\prec$ be the atom ordering defined in section 4. Let* OR *be the ordered resolution calculus defined in section 2.*

(i) M'' is a resolution basis for the reduced clausal theory of meet semilattices. $M'' = rb(M'')$.
(ii) J'' is a resolution basis for the reduced clausal theory of join semilattices. $J'' = rb(J'')$.
(iii) D'' is a resolution basis for the reduced clausal theory of distributive lattices. $D'' = rb(D'')$.

We always implicitly normalize with respect to I.

Proof. By lemma 1 and lemma 6, J', M' and D' are sound and complete for the respective reduced clausal theories in the unordered case. Soundness and completeness of J'', M'' and D'' is an evident consequence.

(ad i) We first determine how M'' is ordered with respect to $\prec$. We assign an index i (increasing) to the clausal arrow, if the right-hand side of the clause is greater than the left-hand side, d (decreasing), if the converse holds and ?, if the clause must be ordered instance-wise. We identify clauses that are equivalent up to associativity and commutativity.

$$\longrightarrow_i x \le x \qquad \text{(ref)}$$
$$x \le z \longrightarrow_i xy \le z \qquad \text{(ml)}$$
$$x \le y, y \le z \longrightarrow_d x \le z \qquad \text{(trans)}$$
$$x_1 \le z, x_2z \le y_2 \longrightarrow_? x_1x_2 \le y_2 \qquad \text{(horncut)}$$

The simple computation leading to this orientation are left implicit. In case of (ml), we implicitly assume that z occurs twice at the right-hand side of the inequality of the antecedent and is then normalized in refutations.

We now show that all conclusions of OR-inferences between clauses in M'' are $\prec$-redundant. For the sake of simplicity we assume a zero and a unity. This allows us, without loss of generality, to treat (trans) as a special case of (mcut). In the computations below this means that all the cases of (trans) are covered by cases of (mcut).

(a) The only OR-inference between (ref) and another rule from M' is

$$\frac{\longrightarrow_i s \le s \qquad s \le s, s \le t \longrightarrow_d s \le t}{s \le t \longrightarrow_d s \le t}$$

with (trans) as a special case of (mcut). It yields a tautology. In particular, an inference between $\longrightarrow a \le a$ and an instance

$$r \le r, rs \le t \longrightarrow rs \le t$$

of (mcut) is ruled out by the ordering constraints. An inference between

$$\begin{aligned} &\longrightarrow t \sqcap s \le t \sqcap s, \\ r \le s, s \sqcap t \le s \sqcap t &\longrightarrow r \sqcap t \le s \sqcap t \end{aligned}$$

is ruled out, since only reduced clauses are considered.

(b) There are three main OR-inference between (ml) and (mcut). The first one is

$$\frac{s_1 \le x \longrightarrow_i s_1 s_2 \le x \qquad s_1 s_2 \le x, x s_3 \le t \longrightarrow_d s_1 s_2 s_3 \le t}{s_1 \le x, x s_2 \le t \longrightarrow s_1 s_2 s_3 \le t}$$

The conclusion is redundant. It follows semantically from the smaller instances

$$\begin{aligned} s_1 \le x, x s_3 \le t &\longrightarrow_d s_1 s_3 \le t, \\ s_1 s_3 \le t &\longrightarrow_i s_1 s_2 s_3 \le t \end{aligned}$$

of (mcut) and (ml). There is a similar inference with the clauses

$$\begin{aligned} s_1 \le x &\longrightarrow_i s_1 s_2 s_3 \le x \\ s_1 s_2 s_3 \le x, x s_3 \le t &\longrightarrow_d s_1 s_2 s_3 \le t \end{aligned}$$

which also leads to a redundant conclusion essentially in the same way. The second one is

$$\frac{x \le t \longrightarrow_i x s_2 \le t \qquad s_1 \le x, x s_2 \le t \longrightarrow_d s_1 s_2 \le t}{s_1 \le x, x \le t \longrightarrow s_1 s_2 \le t}$$

The conclusion semantically follows from the smaller instances

$$\begin{aligned} s_1 \le x, x \le t &\longrightarrow_d s_1 \le t, \\ s_1 \le t &\longrightarrow_i s_1 s_2 \le t \end{aligned}$$

of (mcut), that is (trans), and (ml). The third one is

$$\frac{s_1 \le t \longrightarrow_i s_2 x \le t \qquad s_1 \le x, x s_2 \le t \longrightarrow_d s_1 s_2 \le t}{s_1 \le x, s_2 \le t \longrightarrow s_1 s_2 \le t}$$

The conclusion semantically follows from the smaller instance of (ml) alone.

(c) There are three main OR-inference between (mcut) and (mcut). The first one is

$$\frac{s_1 \le x, xs_2 \le y \longrightarrow_i s_1s_2 \le y \qquad s_1s_2 \le y, ys_3 \le t \longrightarrow_d s_1s_2s_3 \le t}{s_1 \le x, xs_2 \le y, ys_3 \le t \longrightarrow s_1s_2s_3 \le t}$$

The conclusion semantically follows from the smaller instances

$$xs_2 \le y, ys_2 \le t \longrightarrow_d xs_2s_3 \le t,$$
$$s_1 \le x, xs_2s_3 \le t \longrightarrow_i s_1s_2s_3 \le t$$

of (mcut). There are similar inference with the clauses

$$s_1 \le x, xs_2s_3 \le y \longrightarrow_i s_1s_2s_3 \le y,$$
$$s_1s_2s_3 \le y, ys_3 \le t \longrightarrow_d s_1s_2s_3 \le t,$$

for instance, which also lead to a redundant conclusion essentially in the same way. The second one is

$$\frac{s_2 \le x, xy \le t \longrightarrow_i s_2y \le t \qquad s_1 \le y, ys_2 \le t \longrightarrow_d s_1 \le t}{s_1 \le y, s_2 \le x, xy \le t \longrightarrow s_1s_2 \le t}$$

The conclusion semantically follows from the smaller instances

$$s_1 \le y, xy \le t \longrightarrow_d s_1x \le t,$$
$$s_2 \le x, s_1x \le t \longrightarrow_i s_1s_2 \le t$$

of (mcut). The third one is

$$\frac{y \le x, s_1x \le t \longrightarrow_i s_1y \le t \qquad s_2 \le y, s_1y \le t \longrightarrow_d s_1 \le t}{s_2 \le y, y \le x, s_1x \le t \longrightarrow s_1s_2 \le t}$$

The conclusion semantically follows from the smaller instances

$$s_2 \le y, y \le x \longrightarrow_d s_2 \le x,$$
$$s_2 \le x, s_1x \le t \longrightarrow_i s_1s_2 \le t$$

of (mcut).

Thus all OR-inference between members of M'' are redundant and M'' is a resolution basis for the reduced clausal theory of the meet semilattice. In a real construction, which starts with M', one will notice in (a) and the second case of (b), that there are irredundant conclusions (at least in absence of a unity). Adding (trans) makes these conclusions redundant.

(ad ii) This follows immediately from (i). The inferences are dual. In the redundancy check, the asymmetry in the third component s of the atom measure does not change the relative size of the clauses.

(ad iii) D' is the union of J' and M', with (mcut) replaced by (cut). (cut) can again be ordered only instance-wise. Obviously for reduced clauses, the clauses in J' and M' are mutually independent. Moreover the case analysis in the proof of (i) can be performed with (cut) exactly as with (mcut). In particular there are no new cases from resolving (cut) with (cut). Thus our results for J' and M' are modular and D' is a resolution basis for the reduced clausal theory of the distributive lattice. □

In principle, the construction of the resolution basis and in particular the verification, that a given set of axioms is a resolution basis, can be automated with a saturation-based theorem prover. The construction is in general quite involved. Probably, different resolution strategies might lead to different resolution bases of different procedural quality. A guidance by hand seems more realistic in many cases. Even the verification might be problematic for state of the art automated theorem provers, since we work modulo ACI or even ACI01. Even a formal proof with a higher-order proof checker seems involved. Complex ordering constraints must be specified and analyzed and again at least ACI unification must be explicitly considered.

Proposition 2 and lemma 8 immediately imply the following fact, which is essential for our further derivation.

Corollary 1. *For every inconsistent reduced clause set containing J'', M'' or D'' there exists a refutation without primary theory inferences in* OR.

By corollary 1, working with the resolution bases is already a strong improvement over plain axiomatic reasoning. In particular, the use of theory-specific knowledge leads to a restricted representation of the initial axioms in terms of reduced inequalities and to the use of rather sophisticated axioms in D' or D''. This should result in a much better performance that reasoning with the usual lattice axioms, even in the case of unordered resolution. We will see in the following section that the ordered lattice chaining calculi yield even more restrictive proofs, when axioms are completely avoided in favor of focused inference rules.

7 Deriving the Chaining Rules

We now turn to the second step in the derivation, namely the extraction of the inference rules of DC from OR-derivations with D''. Our main assumptions are refutational completeness of OC (theorem 1) and the fact that our ordering constraints rule out primary theory inferences (corollary 1). The main idea is to consider all possible interactions between non-theory clauses and the elements of D'' in a refutation. Disregarding the ordering constraints, the inference rule (Cut+) of DC can be derived, for instance, as

$$\dfrac{\dfrac{\Gamma \longrightarrow \Delta, s_1 \leq xt_1 \qquad \boxed{s_1 \leq xt_1}, \boxed{s_2x \leq t_2} \longrightarrow s_1s_2 \leq t_1t_2}{\Gamma, s_2x \leq t_2 \longrightarrow s_1s_2 \leq t_1t_2} \qquad \Gamma' \longrightarrow \Delta', s_2x \leq t_2}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta', s_1s_2 \leq t_1t_2}$$

from two non-theory clauses and an instance of (cut). Here, the atoms of (cut) that are cut out are in boxes. In the rule (Cut+), the effect of (cut) is then completely internalized. The derivations of (Cut-) are similar

$$\dfrac{\dfrac{\Gamma, s_1s_2 \leq t_1t_2 \longrightarrow \Delta \qquad \boxed{s_1 \leq t_1x}, s_2x \leq t_2 \longrightarrow \boxed{s_1s_2 \leq t_1t_2}}{\Gamma, s_1 \leq t_1x, s_2x \leq t_2 \longrightarrow \Delta} \qquad \Gamma' \longrightarrow \Delta', s_1 \leq t_1x}{\Gamma, \Gamma', s_2x \leq t_2 \longrightarrow \Delta, \Delta'}$$

$$\dfrac{\dfrac{\Gamma, s_1s_2 \leq t_1t_2 \longrightarrow \Delta \qquad s_1 \leq t_1x, \boxed{s_2x \leq t_2} \longrightarrow \boxed{s_1s_2 \leq t_1t_2}}{\Gamma, s_1 \leq t_1x, s_2x \leq t_2 \longrightarrow \Delta} \qquad \Gamma' \longrightarrow \Delta', s_2x \leq t_2}{\Gamma, \Gamma', s_1 \leq t_1x \longrightarrow \Delta, \Delta'}$$

The (DF) rules arise from a resolution step followed by a factoring step. (JR) and (ML) arise from resolution steps with (jr) and (ml). The restrictions in DC modeled by brackets arise from derivation with (jcut), (mcut) or (trans). If it is possible to partition all refutations into these macro inferences, then the respective rules from D can be completely discarded in favor of the focused inference rules. Looking more precisely, this partition in patterns is however not straightforward. First, it must respect the ordering constraints of ordered resolution. Second, there are certain legal proof steps in refutations that may violate the pattern construction. We first argue that these unwanted proof steps can be avoided. We then show how the partition of arbitrary OR-refutations into patterns corresponding to the DC-rules is obtained.

To understand the obstacles to the macro inferences, consider again the above derivation of (Cut-). First, $\Gamma' \longrightarrow \Delta', s_2x \leq t_2$ may be another instance of (cut). We call this situation a *secondary theory inference*. Second, Δ may contain an atom bigger than $s_1 \leq t_1x$. Then the derivation of (Cut-) does not continue as above, but with a “bad” resolution step cutting out this bigger atom. We call this situation a *blocking inference*. In a blocking inference, only the first step uses the theory axiom. In DC, secondary theory inferences can occur in connection with (cut), (jcut), (mcut) and (trans). An OR-derivation is *regular*, when it contains neither primary and secondary theory inferences nor blocking inferences. See [27] and the references given there for a more formal definition and discussion.

We first prove a technical lemma that allows us to restrict our attention to instances of (cut), (jcut), (mcut) and (trans) that are decreasing from left to right in a refutation and thus can be indexed with d.

Lemma 9. *For every two-step proof in a* OR*-refutation that cuts out two atoms of (cut), (jcut), (mcut) or (trans) there is another* OR *proof from the same non-theory premises to the same conclusion, in which (another instance of) (cut) (jcut), (mcut) or (trans) decreases from left to right.*

Proof. It suffices to inspect cases, where (cut) is not already in d. For an instance

$$s_1 \leq t_1x, s_2x \leq t_2 \longrightarrow s_1s_2 \leq t_1t_2$$

this is the case, when either s_1 or t_2 are maximal, or when s_2 is maximal and $s_1 \succ x$ or when t_1 is maximal and $t_2 \succ x$.

(case i) Let s_1 be maximal. We put the occurance of s_1 in the atom which is cut out into a box. Consider the two-step proof

$$\dfrac{\dfrac{\Gamma, \boxed{s_1} s_2 \le t_1 t_2 \longrightarrow \Delta \quad s_1 \le t_1 x, s_2 x \le t_2 \longrightarrow \boxed{s_1} s_2 \le t_1 t_2}{\Gamma, \boxed{s_1} \le t_1 x, s_2 x \le t_2 \longrightarrow \Delta} \quad \Gamma' \longrightarrow \Delta', \boxed{s_1} \le t_1 x}{\Gamma, \Gamma', s_2 x \le t_2 \longrightarrow \Delta, \Delta'}$$

We replace it by the three-step proof

$$\dfrac{\dfrac{\dfrac{\Gamma' \longrightarrow \Delta', \boxed{s_1} \le t_1 x}{\Gamma' \longrightarrow \Delta', \boxed{s_1} s_2 \le t_1 x} \quad \boxed{s_1} s_2 \le t_1 x, s_2 x \le t_2 \longrightarrow s_1 s_2 \le t_1 t_2}{\Gamma', s_2 x \le t_2 \longrightarrow \Delta', \boxed{s_1} s_2 \le t_1 t_2} \quad \Gamma, \boxed{s_1} s_2 \le t_1 t_2 \longrightarrow \Delta}{\Gamma, \Gamma', s_2 x \le t_2 \longrightarrow \Delta, \Delta'}$$

The leftmost uppermost inference uses an unoriented variant of meet left. Idempotence has been implicitly applied in (cut).

(case ii) Let t_2 be maximal. This case is completely dual to (case i).

(case iii) Let t_1 be maximal and $t_2 \succ x$. Consider the two-step proof

$$\dfrac{\dfrac{\Gamma, s_1 s_2 \le \boxed{t_1} t_2 \longrightarrow \Delta \quad s_1 \le t_1 x, s_2 x \le t_2 \longrightarrow s_1 s_2 \le t_1 t_2}{\Gamma, s_1 \le \boxed{t_1} x, s_2 x \le t_2 \longrightarrow \Delta} \quad \Gamma' \longrightarrow \Delta', s_1 \le \boxed{t_1} x}{\Gamma, \Gamma', s_2 x \le t_2 \longrightarrow \Delta, \Delta'}$$

We replace it by the three-step proof

$$\dfrac{\dfrac{\dfrac{\Gamma' \longrightarrow \Delta', s_1 \le \boxed{t_1} x}{\Gamma' \longrightarrow \Delta', s_1 \le \boxed{t_1} t_2 x} \quad s_1 \le \boxed{t_1} t_2 x, s_2 x \le t_2 \longrightarrow s_1 s_2 \le t_1 t_2}{\Gamma', s_2 x \le t_2 \longrightarrow \Delta', s_1 s_2 \le \boxed{t_1} t_2} \quad \Gamma, s_1 s_2 \le \boxed{t_1} t_2 \longrightarrow \Delta}{\Gamma, \Gamma' s_2 x \le t_2 \longrightarrow \Delta, \Delta'}$$

The leftmost uppermost inference uses an unoriented variant of meet left. Idempotence has been implicitly applied in (cut).

(case iv) Let s_2 be maximal This case is completely dual to (case iii). □

This proof transformation is completely local with respect to two-step and three-step proofs. Therefore, when internalizing the rules of D'', the transformation is completely inside of the macro inference, hence completely hidden to the outside. We use lemma 9 to transform a given refutation into one, where (jcut), (mcut) and (cut) are always decreasing. This can be done in a local way, just by inserting join left or meet right inferences at the appropriate places and rearranging some subtrees in the refutation. This transformation increases the number of proof-steps in a refutation (measured as a tree) only by the number of non-decreasing instances of (jcut), (mcut) and (cut). Thus the number of proof steps can at most be doubled. For the remainder of this section we assume that all proofs are of this particular shape.

Lemma 10. *For every inconsistent clause set there exists a regular* OR*-refutation (possibly violating the ordering constraints).*

Proof. We show that (i) there is a refutation without blocking inferences, (ii) there is a refutation without secondary theory inferences and (iii) there is a refutation without both these inferences. By lemma 9 we assume a refutation in which all instances of (cut) are in d.

(ad i) By induction on the size of clauses in a refutation. The argument is independent from the particular structure of D''. It has already been given in [26]. The main idea is that the needed inference can be permuted up in the refutation tree, whereas the second step of the blocking inference can be permuted down. One thereby obtains a new refutation with the desired macro inference at the appropriate place. Due to factoring it may be the case that some subtrees of the proof tree must be copied. This construction is then iterated on the proof tree.

(ad ii) By induction on the size of clauses in a refutation. Here the argument depends on the rules in D'' and in particular on the given precedence on generators.

(case a) Let s_1 be strictly maximal. Consider the ordered resolution inference

$$\frac{\Gamma \longrightarrow \Delta, \boxed{s_1} s_2 \leq t_1 x \qquad \boxed{s_1} s_2 \leq t_1 x, s_2 x \leq t_2 \longrightarrow s_1 s_2 \leq t_1 t_2}{\Gamma, s_2 x \leq t_2 \longrightarrow \Delta, \boxed{s_1} s_2 \leq t_1 t_2} \tag{1}$$

using (cut). There are three possible further resolution inferences that lead to a secondary theory inference.

First, consider the inference

$$\frac{\Gamma, s_2 x \leq t_2 \longrightarrow \Delta, \boxed{s_1} s_2 s_3 \leq t_1 t_2 \qquad \boxed{s_1} s_2 s_3 \leq t_1 t_2, s_3 t_2 \leq t_3 \longrightarrow s_1 s_2 s_3 \leq t_1 t_3}{\Gamma, s_2 x \leq t_2, s_3 t_2 \leq t_3 \longrightarrow \Delta, s_1 s_2 s_3 \leq t_1 t_3} \tag{2}$$

We replace the two step inference by the one-step inference

$$\frac{\Gamma \longrightarrow \Delta, \boxed{s_1} s_2 \leq t_1 x \qquad \boxed{s_1} s_2 \leq t_1 x, s_2 s_3 x \leq t_3 \longrightarrow s_1 s_2 s_3 \leq t_1 t_3}{\Gamma, s_2 s_3 x \leq t_3 \longrightarrow \Delta, s_1 s_2 s_3 \leq t_1 t_3} \tag{3}$$

It differs from the conclusion of (7) only in the antecedent. In a refutation, there are necessarily two clauses

$$\Gamma' \longrightarrow \Delta', s_2 x \leq t_2, \qquad \Gamma'' \longrightarrow \Delta'', s_3 t_2 \leq t_2$$

which cut out the atoms $s_2 x \leq t_2$ and $s_3 t_2 \leq t_3$ later in the refutation. But using the smaller instance

$$s_2 x \leq t_2, s_3 t_2 \leq t_3 \longrightarrow s_2 s_3 x \leq t_3$$

of (cut) we can then also cut out the atom $s_2 s_3 x \leq t_3$ in the antecedent of (3). By the induction hypothesis, these new instances do not introduce secondary theory inferences.

Second, consider the inference

$$\frac{\Gamma, s_2x \le t_2 \longrightarrow \Delta, \boxed{s_1}s_2s_3 \le t_1t_2 \qquad s_3 \le t_3s_2, \boxed{s_1}s_2s_3 \le t_1t_2 \longrightarrow s_1s_3 \le t_1t_2t_3}{\Gamma, s_2x \le t_2, s_3 \le t_3s_2 \longrightarrow \Delta, s_1s_3 \le t_1t_2t_3} \tag{4}$$

which continues inference (1). We transform the proof like in the first case, using an instance

$$s_1s_2s_3 \le t_1x, s_3x \le t_2t_3 \longrightarrow s_1s_3 \le t_1t_2t_3$$

of (cut). We can again use the smaller instance

$$s_3 \le t_3s_2, s_2x \le t_2 \longrightarrow s_3x \le t_2t_3$$

of (cut) to cut out the atom $s_3x \le t_2t_3$, using the clauses cutting out $s_2x \le t_2$ and $s_3 \le t_3s_2$ in (4).

Third, consider the inference

$$\frac{\Gamma, s_2x \le t_2 \longrightarrow \Delta, \boxed{s_1}s_2 \le t_1t_2 \qquad s_3 \le t_3s_1, \boxed{s_1}s_2 \le t_1t_2 \longrightarrow s_2s_3 \le t_2t_3}{\Gamma, s_2x \le t_2, s_3 \le t_3s_1 \longrightarrow \Delta, s_2s_3 \le t_1t_2t_3} \tag{5}$$

We now transform the proof using an instance

$$s_3 \le t_2s_1, s_1 \le t_1x \longrightarrow s_3 \le t_1t_3x$$

of (cut). We can again use the smaller instance

$$s_3 \le t_1t_3x, s_2x \le t_2 \longrightarrow s_2s_3 \le t_2t_3$$

of (cut) to cut out the atom $s_3 \le t_1t_3x$, using the clauses cutting out $s_2x \le t_2$ and $s_2s_3 \le t_2t_3$ in (5) together with (ml).

(case b) Let t_1 be maximal and let $x \succ t_2$. Then the first inference is

$$\frac{\Gamma \longrightarrow \Delta, s_1 \le \boxed{t_1}x \qquad s_1 \le \boxed{t_1}x, s_2x \le t_2 \longrightarrow s_1s_2 \le t_1t_2}{\Gamma, s_2x \le t_2 \longrightarrow \Delta, s_1s_2 \le t_1t_2}. \tag{6}$$

Thee are four ways to produce a secondary theory inference.

First, consider the inference

$$\frac{\Gamma, s_2x \le t_2 \longrightarrow \Delta, s_1s_2 \le \boxed{t_1}t_2t_3 \qquad s_1s_2 \le \boxed{t_1}t_2t_3, s_3t_2 \le t_3 \longrightarrow s_1s_2s_3 \le t_1t_3}{\Gamma, s_2x \le t_2, s_3t_2 \le t_3 \longrightarrow \Delta, s_1s_2s_3 \le t_1t_3} \tag{7}$$

We now transform the proof using an instance

$$s_1 \le t_1x, s_2s_3x \le t_3 \longrightarrow s_1s_2s_3 \le t_1t_3$$

of (cut). We can again use the smaller instance

$$s_2x \le t_2, s_3t_2 \le t_3 \longrightarrow s_2s_3x \le t_3$$

of (cut) to obtain another refutation.

Second, consider the inference

$$\frac{\Gamma, s_2x \leq t_2 \longrightarrow \Delta, s_1s_2 \leq \boxed{t_1}t_2 \qquad s_1s_2 \leq \boxed{t_1}t_2, s_3t_1 \leq t_3 \longrightarrow s_1s_2s_3 \leq t_2t_3}{\Gamma, s_2x \leq t_2, s_3t_1 \leq t_3 \longrightarrow \Delta, s_1s_2s_3 \leq t_2t_3} \tag{8}$$

We now transform the proof using an instance

$$s_1 \leq t_1x, s_3t_1 \leq t_3 \longrightarrow s_1s_3 \leq t_3x$$

of (cut). We can again use the smaller instance

$$s_1s_3 \leq t_3x, s_2x \leq t_2 \longrightarrow s_1s_2s_3 \leq t_2t_3$$

of (cut) to obtain another refutation.

Third, consider the inference

$$\frac{\Gamma, s_2x \leq t_2 \longrightarrow \Delta, s_1s_2 \leq \boxed{t_1}t_2 \qquad s_3 \leq s_1t_3, s_1s_2 \leq \boxed{t_1}t_2 \longrightarrow s_2s_3 \leq t_1t_2t_3}{\Gamma, s_2x \leq t_2, s_3 \leq s_1t_3 \longrightarrow s_2s_3 \leq t_1t_2t_3} \tag{9}$$

We now transform the proof using an instance

$$s_3 \leq s_1t_3, s_1s_3 \leq t_1t_3x \longrightarrow s_3 \leq t_1t_3x$$

of (cut). We can again use the smaller instance

$$s_3 \leq t_1t_3x, s_2x \leq t_2 \longrightarrow s_2s_3 \leq t_1t_2t_3$$

of (cut) to obtain another refutation.

Fourth, consider the inference

$$\frac{\Gamma, s_2x \leq t_2 \longrightarrow \Delta, s_1s_2 \leq \boxed{t_1}t_2 \qquad s_3 \leq s_2t_3, s_1s_2 \leq \boxed{t_1}t_2 \longrightarrow s_1s_3 \leq t_1t_2t_3}{\Gamma, s_2x \leq t_2, s_3 \leq s_2t_3 \longrightarrow \Delta, s_1s_3 \leq t_1t_2t_3} \tag{10}$$

We now transform the proof using an instance

$$s_1 \leq t_1x, s_3x \leq t_2t_3 \longrightarrow s_1s_2 \leq t_1t_2t_3$$

of (cut). We can again use the smaller instance

$$s_2x \leq t_2, s_3 \leq s_2t_3 \longrightarrow s_3x \leq t_2t_3$$

of (cut) to obtain another refutation.

(case c) t_1 maximal and $x \succ t_2$. This case is the same as (case b), with some more uses of (ml).

(case d) Let x be maximal. There are two main cases. Consider the inference

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq t_1\boxed{x} \qquad s_1 \leq t_1\boxed{x}, s_2x \leq t_2 \longrightarrow s_1s_2 \leq t_1t_2}{\Gamma, s_2x \leq t_1 \longrightarrow \Delta, s_1s_2 \leq t_1t_2} \tag{11}$$

Since now the antecedent is maximal, there can at most be inferences using (ml) or (jr). There are three subcases, namely

$$\frac{\Gamma, s_2\boxed{x} \leq t_1 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2}{\Gamma, s_2 \leq t_1 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2} \qquad (12)$$

$$\frac{\Gamma, s_2\boxed{x} \leq t_1 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2}{\Gamma, x \leq t_1 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2} \qquad (13)$$

$$\frac{\Gamma, s_2\boxed{x} \leq t_1' t_1'' \longrightarrow \Delta, s_1 s_2 \leq t_1' t_1'' t_2}{\Gamma, s_2\boxed{x} \leq t_1' \longrightarrow \Delta, s_1 s_2 \leq t_1' t_1'' t_2} \qquad (14)$$

In (14), $t_1 = t_1' t_1''$. The conclusion of (12) is a lattice theoretic tautology, from which the empty clause can never be derived. This inference is not needed in any refutation and can therefore be disregarded.

We transform the proof with (13) using the instance

$$s_1 \leq t_1 x, x \leq t_1 \longrightarrow s_1 \leq t_1$$

of (cut) and the fact that the atom $x \leq t_1$ can be eventually eliminated. The atom $s_1 \leq t_1$ in the conclusion can also be eliminated, since $s_1 s_2 \leq t_1 t_2$ can be eliminated and (ml) and (jr) can be used. The argument for (14) is similar.

The second main case, where $\Gamma \longrightarrow \Delta, s_1 \leq t_1\boxed{x}$ cuts out the second atom in the antecedent of (cut) is completely dual to the first one, using in particular (jr).

(case d) Let s_2 be maximal. This case is completely dual to (case b).

(case e) Let t_2 be maximal. This case is completely dual to (case a).

Moreover, the cases for (jcut), (mcut) and (trans) are straightforward restrictions of the cases for (cut).

(ad iii) The argument is again identical to that in [26]. By induction on the size of clauses. Primary theory inferences are ruled out *ab initio* by the ordering constraints. We inspect the proof bottom up. We first transform secondary theory inferences (if they exist) up to the first blocking inference. This does not introduce new blocking inferences, by the induction hypothesis. We then transform the first blocking inference. This permutation introduces at most one secondary theory inference at the top level. We then transform all secondary theory inferences up to the second blocking inference. Whenever we copy a proof tree in the transformation, we simultaneously transform all copies. Therefore the procedure terminates after finitely many steps for each proof and yields a regular derivation. □

We are now prepared for our main theorem.

Theorem 1. . *Let all clauses be reduced. The ground ordered chaining calculus* DC *is refutationally complete for finite distributive lattices: For every reduced ground clause set that is inconsistent in the first-order theory of distributive lattices there exists a refutational derivation in* DC.

Proof. Consider a regular refutation. Such a derivation exists by lemma 10. Hence in all inferences either both premises are non-theory clauses (with respect to D'') or one premise is a non-theory clause and the other an instance of a clause in D''. The former inferences are handled by specializations of (Cut+) and (DF) to (Res) and (Fact). The latter yield the inference rules of the ordered chaining calculus DC, as the following argument shows. Consider the (Res) and (Fact) steps of non-theory clauses with D''. By duality we can restrict our attention to steps involving (ref), (ml) and some cases for (cut). We sometimes put terms, at which the chaining takes place into boxes.

(case i) The resolution inference of a non-theory clause $\Gamma, s \leq s \longrightarrow \Delta$ with a ground instance of (ref) is

$$\frac{\longrightarrow_i s \leq s \qquad \Gamma, s \leq s \longrightarrow_d \Delta}{\Gamma \longrightarrow \Delta},$$

where $s \leq s$ is maximal in the premise. Internalization of (ref) yields (Ref).

(case ii) The resolution inference of a non-theory clause $\Gamma, s_1 s_2 \leq t \longrightarrow_d \Delta$ with a ground instance of (ml) is

$$\frac{s_1 \leq t \longrightarrow_i s_1 s_2 \leq t \qquad \Gamma, s_1 s_2 \leq t \longrightarrow_d \Delta}{\Gamma, s_1 \leq t \longrightarrow \Delta},$$

where the minor formula is maximal with respect to the side formulas in the second premise. Internalization of (ml) yields (ML).

(case iii) The derivation of (JR) from (jr) is dual to (case ii).

(case iv) Consider the resolution steps with a ground instance

$$s_1 \leq t_1 x, s_2 x \leq t_2 \longrightarrow s_1 s_2 \leq t_1 t_2 \tag{15}$$

of (cut), where s_1, s_2 are meets of generators, t_1, t_2 are joins of generators and x is a generator. By the proof of lemma 10, we can put (cut) in d by adding appropriate contexts to left- and right-hand sides of instances of non-theory expressions and instances of (cut). We use this fact only implicitly in proofs.

(case iv a) Let s_1 be maximal in (15) and assume that there is a clause $\Gamma \longrightarrow \Delta, s_1 \leq t_1 x$ in which the atom $s_1 \leq t_1 x$ is strictly maximal. Then, a possible resolution inference is

$$\frac{\Gamma \longrightarrow \Delta, \boxed{s_1} \leq t_1 x \qquad \boxed{s_1} \leq t_1 x, s_2 x \leq t_2 \longrightarrow s_1 s_2 \leq t_1 t_2}{\Gamma, s_2 x \leq t_2 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2}. \tag{16}$$

In the conclusion, the maximal element s_1 may occur more than once, but only in Δ. The situation is completely analogous to that of (case i) in the refutational completeness proof for the ordered chaining calculus for transitive relations in [26]. The difference between (cut) and (trans) is not visible at this stage. Again there a several subcases. By assumption, there are no secondary theory or blocking inferences. So in the first case, $s_1 s_2 \leq t_1 t_2$ may be strictly maximal in the conclusion and does not occur in Δ. Then there may be a non-theory clause

$\Gamma', \boxed{s_1} s_2 \leq t_1 t_2 \longrightarrow \Delta'$ and a resolution inference

$$\frac{\Gamma', \boxed{s_1} s_2 \leq t_1 t_2 \longrightarrow \Delta' \quad \Gamma, s_2 x \leq t_2 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2}{\Gamma, \Gamma', s_2 x \leq t_2 \longrightarrow \Delta, \Delta'},$$

thus deriving a ground instance of the first (Cut-) rule. Moreover it may be the case that $s_1 = s_2$. Then s_2 does not appear in the premise of this last inference, however it must appear in the conclusion. This yields the last proviso of the first (Cut-) rule.

In the second case, Δ may contain the clause $s_1 s_2 \leq t_1 t_2$. In order to be smaller than $s_1 \leq t_1 x$, it is necessary that s_2 is identical to s_1 or to 1. Moreover $t_2 \prec x$ must hold. Hence $\Delta = \overline{\Delta}, s_1 \leq t_1 t_2$ and the conclusion of (16) specializes to

$$\Gamma, s_1 x \leq t_2 \longrightarrow s_1 \leq t_1 t_2, s_1 \leq t_1 t_2.$$

Still, a (Fact) inference is not possible, since $s_1 x \leq t_2$ is the maximal atom. However, the s_1 in the antecedent stems from the instance of (cut) and not from the non-theory clause. But in the instance of (cut) we are free to set this occurrence of s_1 to 1. Then (16) specializes to

$$\frac{\Gamma \longrightarrow \overline{\Delta}, \boxed{s_1} \leq t_1 x, s_1 \leq t_1 t_2 \qquad \boxed{s_1} \leq t_1 x, x \leq t_2 \longrightarrow s_1 \leq t_1 t_2}{\Gamma, x \leq t_2 \longrightarrow \Delta, s_1 \leq t_1 t_2, s_1 \leq t_1 t_2}.$$

It can be continued by (Fact) as

$$\frac{\Gamma, x \leq t_2 \longrightarrow \overline{\Delta}, s_1 \leq t_1 t_2, s_1 \leq t_1 t_2}{\Gamma, x \leq t_2 \longrightarrow \overline{\Delta}, s_1 \leq t_1 t_2}.$$

This yields a first of the (DF) rules.

(case iv b) Let t_1 be maximal in (15) and assume that there is a non-theory clause $\Gamma \longrightarrow \Delta, s_1 \leq t_1 x$ in which the atom $s_1 \leq t_1 x$ is strictly maximal. Then there is a possible resolution inference

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq \boxed{t_1} x \quad s_1 \leq \boxed{t_1} x, s_2 x \leq t_2 \longrightarrow s_1 s_2 \leq t_1 t_2}{\Gamma, s_2 x \leq t_2 \longrightarrow \Delta, s_1 s_2 \leq t_1 t_2}.$$

As in (case iv a), it can either be continued as

$$\frac{\Gamma, s_2 x \leq t_2 \longrightarrow \Delta, s_1 s_2 \leq \boxed{t_1} t_2 \quad \Gamma' s_1 s_2 \leq \boxed{t_1} t_2 \longrightarrow \Delta'}{\Gamma, \Gamma', s_2 x \leq t_2 \longrightarrow \Delta, \Delta'},$$

which yields the first of the (Cut-) rules, or we obtain a first instance of (DF):

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq t_1 x, s_1 \leq t_1 t_2}{\Gamma, s_1 x \leq t_2 \longrightarrow \Delta, s_1 \leq t_1 t_2}$$

(case iv c) Let x be maximal in (15) and assume that there is a clause $\Gamma \longrightarrow \Delta, s_2 x \leq t_2$ in which the atom $s_2 x \leq t_2$ is strictly maximal. The situation is completely analogous to the completeness proof of the Positive Chaining rule of OC in in [26]. There must be a clause $\Gamma' \longrightarrow \Delta', s_1 \leq t_1 x$ that leads to an instance of (Cut+) in a two-step proof with (cut).

The cases, where s_2 or t_2 are maximal in (15), are dual to (case iv a) and (case iv b). They yield instances of the second (Cut-) and (DF) rules. □

Soundness of DC is trivial from the completeness proof, since all rules have been derived from OR with rules from D. Again, it should be possible, but involved, to verify this proof at least with a higher-order proof checker.

8 Decidability and Special Cases

In this section we discuss various special cases of theorem 1, including chaining calculi for semilattices, quasiorderings and transitive relations. We also address decidability of these calculi, which follow almost immediately from the finiteness of the structures under consideration. In section 12, we argue how DC can be extended to the non-ground case. Our first corollary addresses specializations.

Corollary 2. *(i)* DC *without (DF) is refutationally complete for the reduced Horn theory of finite distributive lattices.*

(ii) JC *and* MC *are refutationally complete for the reduced clausal theories of the finite join and meet semilattice.*

(iii) QC *and* TC *are refutationally complete for the theory of quasiorderings and transitive relations.*

This follows immediately from refutational completeness of DC, discarding the respective brackets in the inference rules.

Our second corollary addresses decidability.

Corollary 3. DC *decides the elementary theory of finite distributive lattices.*

Proof. Finitely presented distributive lattices are finite. Modulo ACI the inference rules of DC do not introduce any new variables or constants. Thus there are only finitely many inferences that lead to irredundant conclusions. Together with refutational completeness this implies that the procedure terminates after finitely many steps. The resulting resolution basis contains the empty clause if and only if the initial clause set was inconsistent. □

This result also immediately specializes to decidability of JC, MC, QC and TC. Remember that we only consider ground calculi. Termination of DC also has the following consequence.

Corollary 4. HOR *and* OR *of [26] are solutions to the uniform word problem for semilattices and distributive lattices.*

In particular, a variant of propositional ordered resolution has been formally derived as a rewrite-based lattice-theoretic decision problem using ordered resolution resolution as a metaprocedure. This implies a reflexive refutational completeness proof. However, the ordering constraints at the lattice-level are weaker than those of the logical level. An alternative (less formal and generic) derivation of HOR and OR as Knuth-Bendix completion procedures for quasiorderings that are specialized to semilattices and distributive lattices can be found in [26]. These calculi also use the weak ordering constraints in the context of the uniform word problem. The strong constraints are only appropriate for refutations, which is the context of uniform word problems means a restriction to the query $1 \leq 0$.

Corollary 5. *The variants of Knuth-Bendix completion procedures for quasiorderings and transitive relations of definition 5 terminate.*

Our variants are only special cases of more general completion procedures for quasiorderings and transitive relations (c.f. [28]). Here, we neither deal with non-ground terms nor with monotonic functions. In presence of monotonic functions, the procedures need not terminate even in the ground case. An example can be found in [28]. If the system is moreover non-ground, the situation becomes even more problematic. Then also variable critical pairs occur [17,24]. In presence of the rules

$$a \longrightarrow_i b, \qquad x \sqcap x \longrightarrow_d x,$$

for instance, there is the two-step rewrite proof

$$a \sqcap b \longrightarrow_i b \sqcap b \longrightarrow_d b,$$

which is a critical pair in the sense of non-symmetric rewriting, and which cannot be replaced by a rewrite proof (a possibly empty sequence of decreasing steps followed by a possibly empty sequence of increasing steps). So far, there is no way to finitely represent these variable critical pairs within first-order logic. Since critical pair computations appear in Knuth-Bendix completion and also in chaining rules, this fact is very important for the extension of DC to the non-ground case in section 10.

9 Examples

We now consider some examples that show DC at work. They are as difficult as typical exercises from introductory textbooks on lattice theory or typical lemmata that arise everywhere in lattice theory. The example show that DC can easily handle such proofs. Sometimes the resolution proofs are even more direct and concise than standard textbook proofs.

Example 1. We show that DC can appropriately handle the axioms of D and D'.

Lemma 11. DC *refutes the axioms (ref), (trans), (lb), (ub), (jr) and (ml) of lattice theory.*

Proof. (dist) (as an equation) and (cut).

(ad ref). The clause to be refuted is $x \not\leq x$. The empty clause is obtained in one step by (Ref).

(ad trans). The clauses to be refuted are $x \leq z$, $z \leq y$ and $x \not\leq z$. Depending on the ordering constraints, we obtain a two-step refutation with either (Cut+) or (Cut-) and ordered resolution.

(ad lb). The clause to be refuted is $xy \not\leq x$. We obtain a two-step refutation using (ML) and (Ref).

(ad ub) This case is dual to that of (lb).

(ad jr). The clauses to be refuted are $x \leq z$ and $xy \not\leq z$. We obtain a two-step refutation using (JR) and ordered resolution.

(ad ml) This case is dual to that of (jr). □

We now show that DC can appropriately handle the distributivity axiom.

Lemma 12. DC *refutes the distributivity law*

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z). \tag{17}$$

Proof. We first split (17) into a conjunction of two inequalities

$$x \sqcup (y \sqcap z) \leq (x \sqcup y) \sqcap (x \sqcup z), \tag{18}$$

$$(x \sqcup y) \sqcap (x \sqcup z) \leq x \sqcup (y \sqcap z). \tag{19}$$

In order to refute it, we negate it and then simply multiply it out to obtain reduced lattice inequalities. This yields the clause

$$\Gamma, \Delta \longrightarrow$$

where

$$\Gamma = x \leq xy, x \leq xz, yz \leq xy, yz \leq xz,$$
$$\Delta = x \leq xy, x \leq xz, xz \leq xy, xz \leq xz, xy \leq xy, xy \leq xz, yz \leq xy, yz \leq xz.$$

Γ stems from (18 and Δ stems from (19). We consider only Γ. We obtain the refutation

$$
\begin{array}{ll}
& x \leq xy, x \leq xz, yz \leq xy, yz \leq xz, \Delta \longrightarrow \\
\vdash & \quad \{\text{(ML) with third and fourth atom}\} \\
& x \leq xy, x \leq xz, y \leq xy, z \leq xz, \Delta \longrightarrow \\
\vdash & \quad \{\text{(JR) with all atoms}\} \\
& x \leq x, x \leq x, y \leq y, z \leq z, \Delta \longrightarrow \\
\vdash & \quad \{\text{(Ref) with all atoms}\} \\
& \Delta \longrightarrow
\end{array}
$$

The refutation of $\Delta \rightarrow$ is similar. □

No cut rule has been used in the proof. For (18), this is only understandable, since this inequality holds in all lattices. (19) also does not use any of the cut rules, although it only holds in distributive lattices. This shows that in some cases the effect of distributivity is sufficiently handled by the transformation to reduced inequalities.

Lemma 13. DC *refutes (cut).*

Proof. The clauses to be refuted are $x_1 \leq y_1 z$, $x_2 z \leq y_2$ and $x_1 x_2 \not\leq y_1 y_2$. Like in the case of (trans) we obtain two-step refutation using either (Cut+) or (Cut-) and ordered resolution. □

We now show a further important property of the join and meet operation.

Lemma 14. *Join and meet are monotonic operation in a lattice.*

$$x \leq y \longrightarrow x \sqcup z \leq y \sqcup z, \tag{20}$$

$$x \leq y \longrightarrow x \sqcap z \leq y \sqcap z \tag{21}$$

hold for all elements x, y and z of the lattice.

Proof. (ad 20) For the refutation, we obtain the reduced clauses $x \leq y$, $x \not\leq yz$ and $z \leq yz$. The elimination of the first two clauses is identical to the refutation of (jr). The elimination of the third clause is identical to the refutation of (ub).

(ad 21) The refutation is dual to that of (20).

Example 2. We now show that DC can also handle the equational axioms of lattices.

Lemma 15. DC *refutes the following equations.*

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z, \tag{22}$$

$$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z, \tag{23}$$

$$x \sqcup y = y \sqcup x, \tag{24}$$

$$x \sqcap y = y \sqcap x, \tag{25}$$

$$x \sqcup x = x, \tag{26}$$

$$x \sqcap x = x, \tag{27}$$

$$x \sqcup (x \sqcap y) = x, \tag{28}$$

$$x \sqcap (x \sqcup y) = x. \tag{29}$$

Proof. (ad 22) We consider only the inequality $x \sqcup (x \sqcup z) \leq (x \sqcup y) \sqcup z$. For the refutation, we obtain the reduced clause

$$x \leq xyz, y \leq xyz, z \leq xyz \longrightarrow .$$

The empty clause follows immediately, using (JR) and (Ref). The proof for the other inequality is similar.

(ad 23) The proof is dual to that of (ad 22).

(ad 24) We consider only the inequality $x \sqcup y \leq y \sqcup x$. For the refutation we obtain the reduced clause

$$x \leq xy, y \leq xy \longrightarrow .$$

The empty clause follows immediately, using (JR) and (Ref). The proof for the other inequality is similar.

(ad 25) The proof is dual to that of (ad 24).

(ad 26) For the refutation, we obtain the reduced clause

$$x \leq x, x \leq xx \longrightarrow .$$

The first atom can be eliminated by (Ref), the second one by (JR) and (Ref). This yields the empty clause.

(ad 27) The proof is dual to that of (ad 26).

(ad 28) For the refutation, we obtain the reduced clause

$$x \leq x, xy \leq x, x \leq xy, x \leq xx \longrightarrow .$$

The first atom can be eliminated by (Ref), the second one by (ML) and (Ref), the third and fourth one by (JR) and (Ref). This yields the empty clause.

(ad 29) The proof is dual to that of (ad 28). □

We have therewith shown the following

Lemma 16. *Every distributive lattice as a theory of order is an equational distributive lattice.*

Proof. We have refuted the equational axioms of lattices in lemma 15 and the distributivity axiom in lemma 12. By refutational completeness of DC (theorem 1 and lemma 6, these axioms are therefore consequences of the axioms for the reduced clausal theory of distributive lattices. By lemma 2, the case of reduced clauses can be lifted to the first-order theory of lattices. □

Example 3. We now present one more example that shows that in many cases, the use of distributivity in the transformation to reduced lattice inequalities already suffices for the refutation. This makes proofs very simple.

Lemma 17. *Every distributive lattice A is* modular. *That is,*

$$z \leq x \Longrightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup z, \tag{30}$$

holds for every $x, y, z \in A$.

Proof. We first show that $x \sqcap (y \sqcup z) \leq (x \sqcap y) \sqcup z$ holds without the assumption $z \leq x$. Transforming to reduced lattice inequalities yields

$$xy \leq xy, xy \leq yz, xz \leq xz, xz \leq yz \longrightarrow . \tag{31}$$

This clause can be reduced by a simple refutation to the empty clause using mainly (JR) and (ML), which hold in every lattice, but without any of the cut rules.

We now show that the converse inequality $(x \sqcap y) \sqcup z \leq x \sqcap (y \sqcup z)$ follows from the assumption. Transformation to reduced lattice inequalities (without using distributivity) yields

$$z \leq x, \tag{32}$$

$$xy \leq x, xy \leq yz, z \leq x, z \leq yz \longrightarrow . \tag{33}$$

Using (JR), (ML) and (Ref), (33) reduces to $z \leq x \longrightarrow$. The empty clause then follows immediately from (32) by ordered resolution. □

Example 4. We now show that in some cases, proofs in DC can be surprisingly simple.

Lemma 18. *Complements in distributive lattices with* 0 *and* 1 *are uniquely defined (provided they exist).*

Proof. We first present a typical textbook proof. Let b and c be two complements of an element a of the distributive lattice. By definition of complements, therefore we have two elements 0 and 1 such that

$$a \sqcup b = 1, \tag{34}$$

$$a \sqcap b = 0, \tag{35}$$

$$a \sqcup c = 1, \tag{36}$$

$$a \sqcap c = 0. \tag{37}$$

We show that under these conditions $b = c$. Now

$$\begin{array}{ll} & b \\ = & \{\text{Definition of } 1\} \\ & b \sqcap 1 \\ = & \{\text{by } (36)\} \\ & b \sqcap (a \sqcup c) \\ = & \{\text{Distributivity}\} \\ & (b \sqcap a) \sqcup (b \sqcap c) \\ = & \{\text{by } (35)\} \\ & 0 \sqcup (b \sqcap c) \\ = & \{\text{Definition of } 0\} \\ & b \sqcap c \\ \leq & \{\text{Definition of lower bound}\} \\ & c. \end{array}$$

A dual argument, of the same size, using (34) and (37) shows that $c \leq b$ and therewith $b = c$. The proof obviously requires some creative steps.

We now consider the proof in DC. The clauses that are necessary for representing the problem are

$$ab \leq 0, \tag{38}$$
$$ac \leq 0, \tag{39}$$
$$1 \leq ab, \tag{40}$$
$$1 \leq ac, \tag{41}$$
$$b \leq c, c \leq b \longrightarrow . \tag{42}$$

Let $a \succ b \succ c \succ 1 \succ 0$. The refutation is

$$\vdash \quad \{\text{by (Cut+) of (41) and (38)}\}$$
$$b \leq c \tag{43}$$
$$\vdash \quad \{\text{by (Cut+) of (40) and (39)}\}$$
$$c \leq b \tag{44}$$
$$\vdash \quad \{\text{resolving (43) and (42)}\}$$
$$c \leq b \longrightarrow . \tag{45}$$
$$\vdash \quad \{\text{resolving (44) and (45)}\}$$
$$\square$$

Other choices of $\prec$ lead to similar proofs. □

Here, the DC-proof is obviously simpler and shorter than the textbook proof. In particular, there are no creative steps. The search possibilities at each stage of the proof are strongly restricted. This shows very well that focusing pays.

Example 5. Here is another example that is surprisingly simple with DC. In particular it shows the necessity of the special proviso in the (Cut-) rules.

Lemma 19. *Let A be a distributive lattice. Then*

$$a \sqcup b = c \sqcup b \wedge a \sqcap b = c \sqcap b \Longrightarrow a = c \tag{46}$$

holds for all $a, b, c \in A$.

Proof. Transforming (46) to reduced lattice inequalities yields the clauses

$$a \leq bc, \tag{47}$$
$$c \leq ab, \tag{48}$$
$$ab \leq c, \tag{49}$$
$$cb \leq a, \tag{50}$$
$$a \leq c, c \leq a \longrightarrow . \tag{51}$$

Let $a \succ b \succ c$. The refutation is

$$
\begin{aligned}
&\vdash \quad \{\text{by (Cut-) of (47) and (51)}\} \\
&\quad ab \leq c, c \leq a \longrightarrow && (52) \\
&\vdash \quad \{\text{by (Cut-) of (50) and (52)}\} \\
&\quad ab \leq c, c \leq ab \longrightarrow && (53) \\
&\vdash \quad \{\text{resolving (49) and (53)}\} \\
&\quad c \leq ab \longrightarrow && (54) \\
&\vdash \quad \{\text{resolving (48) and (54)}\} \\
&\quad \Box
\end{aligned}
$$

$\Box$

Beyond such simple examples, the number of clauses that are needed for the problem specification increase so quickly that a calculation by hand becomes hopeless. A real verification of the claim that focused calculi are strongly superior to axiomatic reasoning therefore depends on extensive experimentation with an efficient implementation.

The experience with these examples and also the fact that the (Cut-) rules can be quite prolific shows that cut rules should be applied in a lazy way in implementations.

10 The Non-Ground Case

We now consider the extension of the calculi to infinite structures. Then quantifiers can no longer be eliminated and first-order variables appear. Our calculi become semi-decision procedures, since the corresponding first-order theories are undecidable. We have already discussed the problems that may arise with monotonic functions and non-linear variables. Here we do not build in chaining at subterms. Monotonicity laws, like for instance

$$x \leq y \longrightarrow f(x) \leq f(y)$$

must therefore be treated in an axiomatic way. We will always use reduction orderings that contain the subterm ordering. Then monotonicity axioms always increase from left to right. Monotonicity axioms may introduce unshielded variables (c.f section 11) that make chaining very prolific. A more sophisticated alternative would be to restrict subterm chaining to linear variables and use monotonicity axioms for the non-linear ones. This may still be simpler than resorting to higher-order techniques like context unification [9], which has been proposed for handling variable critical pairs [17].

Fortunately, Skolem functions, which appear in the infinite case from existential quantification, are non-monotonic.

For the remainder of this text, we assume that monotonicity is not built in for any free function. Then, our calculi can be lifted in a straightforward way.

However some care has to be taken with idempotence, which can no longer be used implicitly as a simplification. This yields factoring rules also at the lattice level.

$$\frac{C, [\neg] s_1 \dots s_i \dots s_k \dots s_m \leq t}{C\sigma, [\neg] s_1\sigma \dots s_i\sigma \dots s_m\sigma \leq t\sigma},$$

where σ is a most general unifier of s_i and s_k and the σ-instance of the minor formula is strictly maximal in the left-hand rule and maximal in the right-hand rule.

$$\frac{C, [\neg] s \leq t_1 \dots t_i \dots t_k \dots t_m}{C\sigma, [\neg] s\sigma \leq t_1\sigma \dots t_i\sigma \dots t_m\sigma},$$

where σ is a most general unifier of t_i and t_k and the σ-instance of the minor formula is strictly maximal in the left-hand rule and maximal in the right-hand rule. Also the remaining rules of DC now have unification constraints. The (Cut+) rule now is of the form

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq [t_1]_j x \qquad \Gamma' \longrightarrow \Delta', [s_2]_m x' \leq t_2}{\Gamma\sigma, \Gamma'\sigma \longrightarrow \Delta\sigma, \Delta'\sigma, s_1\sigma[s_2\sigma]_m \leq [t_1\sigma]_j t_2\sigma}$$

Thereby σ is a most general unifier of x and x', $[t_1\sigma]_j x\sigma \not\preceq s_1\sigma$, $[s_2\sigma]_m x'\sigma \not\preceq t_2\sigma$ and the σ-instances of the minor formulas are strictly maximal with respect to the σ-instances of the side formulas in their respective instances.

The first of the (Cut-) rules is now of the form

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq [t_1]_j x \qquad \Gamma', s_1'[s_2]_m \leq [t_1']_j t_2 \longrightarrow \Delta'}{\Gamma\sigma, \Gamma'\sigma, [u]_m x\sigma \leq t_2\sigma \longrightarrow \Delta\sigma, \Delta\sigma'}$$

Thereby σ is a most general unifier of s_1 and s_1' and of t_1 and t_1', $s_1\sigma \not\preceq [t_1\sigma]_j x\sigma$, $s_1\sigma[s_2\sigma]_m \not\preceq [t_1'\sigma]_j t_2\sigma$. In the first premise, the σ-instance of the minor formula is strictly maximal with respect to σ-instance of the side formulas. In the second premise, the σ-instance of the minor formula is maximal with respect to the σ-instance of the side formulas. Moreover, $[u]_m x\sigma \neq t_2\sigma \mod AC$, $u = s_2\sigma$ or else $u = s_1\sigma$, if s_2 is absent in the minor formula.

The specification of the second (Cut-)-rule is dual.

The first of the (DF) rules is now of the form

$$\frac{\Gamma \longrightarrow \Delta, s \leq [t_1]_m x, s' \leq [t_1']_m t_2}{\Gamma\sigma, [s\sigma]_j x\sigma \leq t_2\sigma \longrightarrow \Delta, s\sigma \leq [t_1\sigma]_m t_2\sigma)}$$

Here, x is not a join, σ is a most general unifier of s and s' and of t_1 and t_1', either $t_1\sigma \not\preceq s\sigma x\sigma$ and $t_1'\sigma \not\preceq s'\sigma t_2\sigma$ or $s\sigma \not\preceq [t_1\sigma] x\sigma$ and $s\sigma \not\preceq [t_1'\sigma]_m t_2\sigma$ and s can be set to 1 in the conclusion. The σ-instance of the leftmost minor formula is strictly maximal with respect to the σ-instances of the side formulas and the σ-instance of the rightmost minor formula.

The specification of the second (DF) rule is dual.

Substitutions may instantiate variables at lefthand sides of reduced lattice inequalities with joins and those at righthand sides with meets. Thus the rules of

(DC) must be intertwined with reduction rules that transform these expressions again into reduced inequalities.

$$\frac{C, [\neg] s_1 \dots (s_{i1} \sqcup \dots \sqcup s_{ik}) \dots s_m \leq t}{\bigwedge_{1 \leq j \leq k} C.s_1 \dots s_{ij} \dots s_m \leq t}$$

$$\frac{C, [\neg] s \leq t_1 \dots (t_{i1} \sqcap \dots \sqcap t_{ik}) \dots t_m}{\bigwedge_{1 \leq j \leq k} C.s \leq t_1 \dots t_{ij} \dots t_m}.$$

As already discussed in section 3, in particular the ACI-unifications of the (Cut-) rules can be extremely prolific. This can however be avoided using unification constraints, where unification can often be replaced by a simple test for unifiability. See [31,21] for constraint superposition calculi. For the sake of simplicity, we do not add constraints to the calculi given in this text. But for implementations, they are definitely important. Fortunately, in presence of more structure, like for instance set theory, the most prolific rules can be completely discarded, as we will see in section 12.

11 Simplification

Simplification techniques are indispensable for efficient ordered resolution calculi. Some techniques, like for instance subsumption, concern only the clausal structure. They are common to all ordered resolution calculi. We do not discuss them in this text (c.f. [3]). We restrict our attention to theory-specific and therefore focused simplification techniques.

The main idea of simplification is that a clause C in a clause set S is simplified by a clause C', if adding C' to S makes C redundant and if C' is a consequence of S. Then of course C' can be added to S and C' can be discarded. Here we may assume that S contains both the non-theory clauses and the lattice axioms from D''. But if every inference of C with members of D'' and the new clause C' is redundant, then in particular every application of a DC-rule leads to a redundant conclusion, since it is nothing but a macro inference with rules from D''.

In this section, we not only discuss simplifications in this strong sense, but also avoidance of prolific inferences that arises through the ordering constraints and term-dependent redundancy mechanisms. Situations, where the calculi themselves can be simplified, are discussed in section 12.

We first consider simplification techniques. Consider, for instance, the clause

$$\Gamma, s_1 \dots u \dots s_m \leq t_1 \dots u \dots t_n \longrightarrow \Delta. \tag{55}$$

It becomes redundant, when the (smaller) clause

$$\Gamma \longrightarrow \Delta \tag{56}$$

is added (by subsumption). Moreover (56) semantically follows from (55), since $s_1 \dots u \dots s_m \leq t_1 \dots u \dots t_n$ holds in lattice theory. Thus (55) may be deleted as soon as (56) is added. Similarily, a clause

$$\Gamma \longrightarrow \Delta, s_1 \ldots u \ldots s_m \leq t_1 \ldots u \ldots t_n$$

can be deleted, since it is subsumed by $s_1 \ldots u \ldots s_m \leq t_1 \ldots u \ldots t_n$ and this clause semantically follows from the lattice axioms.

Moreover, all instances of axioms from D'' can be deleted, when they are generated by chance during the procedure. More generally, we may use reduced clauses that hold in lattice theory for simplification, provided they are compatible with the ordering constraints of redundancy. This is for instance the case with lemma 19. We have the clause

$$a \leq bc, ab \leq c \longrightarrow a \leq c.$$

which is a special case of (cut). So if there are clauses

$$\Gamma \longrightarrow \Delta, a \leq bc,$$
$$\Gamma' \longrightarrow \Delta', ab \leq c,$$

such that the explicit inequalities are strictly maximal, we may replace them by the clause

$$\Gamma, \Gamma' \longrightarrow \Delta, \Delta', a \leq c,$$

since the latter follows from the formers and the lattice axioms and the latter makes the formers redundant. There are also the obvious simplification rules for 0 and 1.

We now discuss possibilities to discard clauses that are redundant in presence of others. For example, we may discard the clause

$$\Gamma \longrightarrow \Delta, ab \leq cd$$

in presence of

$$\Gamma' \longrightarrow \Delta', a \leq c,$$

provided $\Gamma' \longrightarrow \Delta'$ subsumes $\Gamma \longrightarrow \Delta$.

We now show that the ordering constraints rule out some prolific inferences, namely chaining with variables. See [4] for a discussion in the context of ordered chaining calculi for transitive relations. The situation for lattices is very similar. We say that a variable that occurs in an atom or a clause is *shielded*, if it occurs at least once below a free function symbol. Otherwise it is called *unshielded*. Shielded variables cannot be maximal in an atom or a clause, since we assume an ordering that contains the subterm ordering. Therefore shielded variables cannot be cut out by (Cut+) inferences. However there can be (Cut+) inferences with unshielded variables.

$$\frac{\Gamma \longrightarrow \Delta, s_1 \leq t_1 x \qquad \Gamma' \longrightarrow \Delta', s_2 s_2' \leq t_2}{\Gamma\sigma, \Gamma' \longrightarrow \Delta\sigma, \Delta, s_1 \sigma s_2 \leq t_1 \sigma t_2}$$

At least, if x occurs neither in s_1 and t_1, nor in Γ and Δ, then the conclusion of (Cut+) reduces to

$$\Gamma, \Gamma' \longrightarrow \Delta, \Delta', s_1 s_2 \leq t_1 t_2.$$

It is already properly subsumed by the first premise according to our above considerations or it is identical to it up to renaming of variables. This yields an partial restriction of variable chaining. However, as in the case of ordered chaining calculi for transitive relations, chaining with variables cannot be completely avoided.

We have seen that also simplification techniques depend on the integration of theory-specific knowledge. Here, mathematical ingenuity may have drastic impact on the efficiency of the calculi. In opposition to the derivation of the focused inference rules, it is now however integrated via the notion of redundancy and therefore the syntactic ordering. Considerations are now not based on proof transformation, but on semantical consideration. In general, there may be further possibilities of simplification for distributive lattices. Their appropriateness and usefulness is best confirmed by experimenting with an implementation and practical examples.

12 Extensions

In this section we consider some extensions of DC. These include focused reasoning with boolean lattices and fragments of set theory and the integration of reasoning with filters and ideals.

A first extension concerns boolean lattices.

Corollary 6. DC *is refutationally complete for the reduced clausal theory of (finite) boolean lattices.*

Proof. Boolean lattices are complemented distributive lattices with 1 and 0. A lattice is complemented, if every element a has a complement a', that is $a \wedge a' = 0$ and $a \vee a' = 1$ hold. For lattice inequalities $a \wedge b' \leq c$ is equivalent to $a \leq b \vee c$ and dually $a \leq b' \vee c$ to $a \wedge b \leq c$. Thus complements can be eliminated while reducing clauses and DC suffices for the boolean case. □

An alternative would be equational reasoning with a boolean ring. Such a calculus can be obtained from the superposition calculi for rings in [29]. These calculi extend the Gröbner-base approach to the uniform word problems for certain commutative rings [6]. Performance of both methods should be investigated by experiment. One might expect, that in some examples, lattice-theoretic reasoning with orderings pays, whereas others are more efficiently handled by a ring-based approach. The transformation from boolean lattices to boolean rings uses essentially the complement. It is therefore not possible for non-boolean lattices. Equational reasoning with boolean lattices in the sense of superposition calculi is impossible, since there is no canonical term rewrite system for this class [23].

Another extension are linear orderings. A partial ordering is *linear*, if $a \leq b$ or $b \leq a$ holds for all elements a and b of the universe. This further axiom allows one to transform every negative inequality $a \leq b \longrightarrow$ into a positive inequality $\longrightarrow b \leq a$. This means that all clauses in a specification consist solely of positive atoms. In the focused calculi, therefore, all inference rules that use negative atoms are no longer applicable. This holds in particular for (Cut-). Thus a main source of prolificy can be avoided in presence of more structure. In this case, however, the use of focused rules offers no further advantage over plain ordered resolution with a resolution basis. The main effect of focusing hereby comes from the appropriate choice of the axioms and the syntactic ordering. In the context of lattices, linear orderings are trivial, but standing alone and in particular in combination with further axioms like density and end-point properties linear orderings have interesting applications [8,2]. We have added this remark, since ordered chaining calculi for transitive relations and quasiorderings arise as specialization of our calculi.

An interesting application of DC and its extension to boolean lattices is reasoning in set theory. Since sets form a boolean lattice under union, intersection, complement an inclusion, DC immediately applies to this case. It has already been noticed in [14] that an unordered variant of (Cut+) can be used for set-theoretic reasoning. In [14], a single further interpreted operation $s(.)$ for singleton sets is used for a calculus based on unordered resolution for reasoning with sets. Moreover, one assumes a lattice with a 0 (the empty set), but without a 1. Unfortunately, no precise axioms for the additional structure are explicitly given in [14]. At least, negative inequalities can again be transformed into positive ones:

$$s \not\subseteq t \Longleftrightarrow \exists x.s(x) \subseteq s \wedge s(x) \cap t \subseteq \emptyset.$$

This equivalence can be used as a rewrite rule in the transformation to reduced inequalities. Then in particular the prolific (Cut-) rules can again be completely avoided. A reconstruction of the calculus in [14], in particular of its simplification rules, in the context of ordered resolution or the development of similar calculi based on DC seems very promising.

Finally, we show that reasoning with filters and ideals can be easily integrated into the calculi. This example demonstrates in particular that the derivation method supports a simple and transparent modular approach to focusing. This is also a major advantage over previous approaches. An *ideal* I of a lattice A is a non-empty subset of A that is downward closed ($a \in I \wedge b \leq y \Longrightarrow b \in I$) and closed under finite joins ($a \in I \wedge b \in I \Longrightarrow a \sqcup b \in I$.) Dually, a *filter* F is a non-empty subset of A that is upward closed and closed under finite meets. This yields the clauses

$$I(a), b \leq a \longrightarrow I(b), \tag{57}$$

$$I(a), I(b) \longrightarrow I(a \sqcup b), \tag{58}$$

$$F(a), a \leq b \longrightarrow F(b), \tag{59}$$

$$F(a), F(b) \longrightarrow F(a \sqcap b). \tag{60}$$

We use an atom ordering in which I and F have greater weight than $\leq$. Then there are no ordered resolution inferences between these axioms and D''. It is also easy to see that all inferences between these axioms lead to redundant clauses.

Hence also D'' plus the filter and ideal axioms is a resolution basis. Then the axioms can immediately be turned into the inference rules

$$\frac{\Gamma, I(a \sqcup b) \longrightarrow \Delta}{\Gamma, I(a), I(b) \longrightarrow \Delta}, \tag{61}$$

$$\frac{\Gamma, I(b) \longrightarrow \Delta \qquad \Gamma' \longrightarrow \Delta', I(a)}{\Gamma, \Gamma', b \leq a \longrightarrow \Delta, \Delta'}, \tag{62}$$

$$\frac{\Gamma, F(a \sqcap b) \longrightarrow \Delta}{\Gamma, F(a), F(b) \longrightarrow \Delta}, \tag{63}$$

$$\frac{\Gamma, F(b) \longrightarrow \Delta \qquad \Gamma' \longrightarrow \Delta', F(a)}{\Gamma, \Gamma', a \leq b \longrightarrow \Delta, \Delta'}. \tag{64}$$

We must again consider secondary theory inferences. For ideals, the only possibility is

$$\frac{\Gamma, I(b) \longrightarrow \Delta \qquad I(a_1 \sqcup a_2), a_1 \leq b, a_2 \leq b \longrightarrow I(b)}{\Gamma, I(a_1 \sqcup a_2), a_1 \leq b, a_2 \leq b \longrightarrow \Delta}.$$

The inference can be continued with an instance of (58) as

$$\frac{\Gamma, I(a_1 \sqcup a_2), a_1 \leq b, a_2 \leq b \longrightarrow \Delta \qquad I(a_1), I(a_2) \longrightarrow I(a_1 \sqcup a_2)}{\Gamma, I(a_1), I(a_2), a_1 \leq b, a_2 \leq b \longrightarrow \Delta}.$$

But in a refutation, also the atoms $I(a_1)$ and $I(a_2)$ can be eventually be eliminated. We can then use the smaller instances

$$I(a_1), b \leq a \longrightarrow I(b),$$
$$I(a_2), b \leq a \longrightarrow I(b)$$

of (57) for a new refutation, like in the proof of lemma 10. The argument for filters is dual.

Our above axiomatization of filter and ideals is not complete. We disregard the fact that filters and ideals must be non-empty. If necessary, the clausal variant of the axiom $\exists x. I(x)$ (including Skolemization) can be added. In most applications, however, this axiom is probably unnecessary.

An alternative would be the inclusion of set-theoretic reasoning in a second calculus based on DC that uses another ordering $\subseteq$ for set inclusion and other joins and meets for union and intersection. Since these calculi work on different orderings, their combination is trivial. In particular one could add a comprehension axiom of the form

$$I(p) \Longleftrightarrow s(p) \subseteq \iota(I),$$

where $\iota(I)$ denotes the set corresponding to the predicate I to shift between logic and set-theory in a well-defined way.

We finish this section with a simple example for reasoning with ideals.

Example 6. We prove the simple fact that for every ideal I in a lattice A we have that $a \in I$ and $b \in A$ implies $a \sqcap b \in I$. The clauses for the refutation are $I(a)$ and $\neg I(a \sqcap b)$. Then (62) yields $ab \not\leq a$. This clause can be refuted in two steps using (ML) and (Ref).

13 Conclusion

We have derived focused ordered resolution calculi for semilattices, distributive lattices and boolean lattices with a novel method. Using ordered resolution as a metaprocedure, theory-specific mathematical and procedural knowledge has been integrated in a smooth and controlled way. Boolean lattices are probably the most complex structures that have been integrated into ordered resolution procedures so far. As the title says, the main emphasis of this text has been on the construction of the inference systems. But we have also presented theory-specific simplification techniques to enhance practical proof performance. Moreover, a series of examples at the niveau of textbooks on lattice theory show that our calculi yield simple and concise proofs in practice. In case of finite structures, our calculi specialize to decision procedures. In case of infinite structures, some inference rules may become quite prolific, since we unify modulo ACI. Working with unification constraints should yield a drastic improvement in implementations. But, as often in computer science, practical experiments should be the ultimate criterion of applicability.

Our calculi are also interesting in (at least) two more respects. First, we developed and thereby proved (refutational) completeness of several other calculi as special cases in a uniform and generic way. In particular, we formally developed and proved refutation completeness of (ground) resolution as a lattice-theoretic uniform word problem reflexively within ordered resolution at the metalevel. Second, the lattice calculi are the basis to many interesting practical applications. Our example of focused reasoning with filters and ideals shows that with the derivation method, the lattice calculi can be extended in a simple and modular way. Moreover, in presence of more structure the calculi can be considerably simplified. In particular, our most prolific inference rules completely disappear in lattice-theoretic reasoning with sets. Besides a more extensive integration of set-theoretic reasoning (exploiting, for instance, the approach based on unordered resolution in [14]), we plan to investigate certain embedding of semilattices and distributive lattices into semirings [16,10] or allegories [12] and to include fixed point reasoning. Such structures are important for the construction and analysis of hard- and software systems. Proof support for these endeavors is very challenging.

References

1. F. Baader and W. Büttner. Unification in commutative idempotent monoids. *J. Theoretical Computer Science*, 56:345–352, 1988.
2. L. Bachmair and H. Ganzinger. Ordered chaining for total orderings. Technical Report MPI-I-93-250, Max-Planck-Institut für Informatik, 1993.
3. L. Bachmair and H. Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *J. Logic and Comp utation*, 4(3):217–247, 1994.
4. L. Bachmair and H. Ganzinger. Rewrite techniques for transitive relations. In *Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 384–393. IEEE Computer Society Press, 1994.

5. D. Basin and H. Ganzinger. Automated complexity analysis based on ordered resolution. *Journal of the Association of Computing Machinery*, 48(1):70–109, January 2001.
6. T. Becker and V. Weispfenning. *Gröbner Bases : A Computational Approach to Commutative Algebra*, volume 141 of *Graduate Texts in Mathematics*. Springer-Verlag, 1993.
7. G. Birkhoff. *Lattice Theory*, volume 25 of *Colloquium Publications*. American Mathematical Society, 1984. Reprint.
8. W. Bledsoe, K. Kunen, and R. Shostak. Completeness results for inequality provers. *Artificial Intelligence*, 27:255–288, 1985.
9. H. Comon. Completion of rewrite systems with membership constraints. In *Int. Coll. on Automata, Languages and Programming, ICALP'92*, volume 623 of *LNCS*. Springer-Verlag, 1992.
10. J. H. Conway. *Regular Algebras and Finite State Machines*. Chapman and Hall, 1971.
11. R. Freese, J. Ježek, and J.B. Nation. Term rewrite systems for lattice theory. *J. Symbolic Computation*, 16:279–288, 1993.
12. P. J. Freyd and A. Scedrov. *Categories, Allegories*. North-Holland, 1990.
13. G. Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 1935.
14. L. Hines. Str+ve$\subseteq$: The Str+ve-based Subset Prover. In M. E. Stickel, editor, *10th International Conference on Automated Deduction*, volume 449 of *LNAI*, pages 193–206. Springer-Verlag, 1990.
15. C. Kirchner, H. Kirchner, and M. Rusinowitch. Deduction with symbolic constraints. *Revue Française d'Intelligence Artificielle*, 4(3):9–52, 1990.
16. D. Kozen. Kleene algebra with tests. *Transactions on Programming Languages and Systems*, 19(3):427–443, 1997.
17. J. Levy and J. Agustí. Bi-rewrite systems. *J. Symbolic Computation*, 22:279–314, 1996.
18. P. Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *The Journal of Symbolic Logic*, 16(2):81–106, 1951.
19. C. Marché. Normalized rewriting and normalized completion. In *Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 394–403. IEEE Computer Society Press, 1994.
20. R.N. McKenzie, G.F. McNulty, and W.F. Taylor. *Algebras, Varieties and Lattices*, volume I. Wadsworth & Brooks/Cole, 1987.
21. R. Nieuwenhuis and A. Rubio. AC-superposition with constraints: no AC-unifiers needed. In A. Bundy, editor, *Proc. 12th International Conference on Automated Deduction*, volume 814 of *LNAI*, pages 545–559. Springer-Verlag, 1994.
22. H. Schwichtenberg. Proof theory: Some applications of cut-elimination. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 867–895. North-Holland, 1977.
23. R. Socher-Ambrosius. Boolean algebra admits no convergent term rewriting system. In R.E. Book, editor, *4th International Conference on Rewriting Techniques and Applications*, volume 488 of *LNCS*, pages 264–274. Springer-Verlag, 1991.
24. G. Struth. Non-symmetric rewriting. Technical Report MPI-I-96-2-004, Max-Planck-Institut für Informatik, Saarbrücken, 1996.
25. G. Struth. *Canonical Transformations in Algebra, Universal Algebra and Logic*. PhD thesis, Institut für Informatik, Universität des Saarlandes, 1998.
26. G. Struth. An algebra of resolution. In L. Bachmair, editor, *Rewriting Techniques and Applications, 11th International Conference*, volume 1833 of *LNCS*, pages 214–228. Springer-Verlag, 2000.

27. G. Struth. Deriving focused calculi for transitive relations. In A. Middeldorp, editor, *Rewriting Techniques and Applications, 12th International Conference*, volume 2051 of *LNCS*, pages 291–305. Springer-Verlag, 2001.
28. G. Struth. Knuth-Bendix completion for non-symmetric transitive relations. In M. van den Brand and R. Verma, editors, *Second International Workshop on Rule-Based Programming (RULE2001)*, volume 59 of *Electronic Notes in Theoretical Computer Science*. Elsevier Science Publishers, 2001.
29. J. Stuber. *Superposition Theorem Proving for Commutative Algebraic Theories*. PhD thesis, Institut für Informatik, Universität des Saarlandes, 1999.
30. G. S. Tseitin. On the complexity of derivations in propositional calculus. In J. Siekmann and G. Wrightson, editors, *Automation of Reasoning: Classical Papers on Computational Logic*, pages 466–483. Springer-Verlag, 1983. reprint.
31. L. Vigneron. Associative-commutative deduction with constraints. In A. Bundy, editor, *Proc. 12th International Conference on Automated Deduction*, volume 814 of *LNAI*, pages 530–544. Springer-Verlag, 1994.
32. U. Waldmann. *Cancellative Abelian Moioids in Refutational Theorem Proving*. PhD thesis, Institut für Informatik, Universität des Saarlandes, 1997.
33. Ph.M. Whitman. Free lattices. *Ann. of Math.*, 42(2):325–330, 1941.
34. Ph.M. Whitman. Free lattices (II). *Ann. of Math.*, 43(2):104–115, 1942.