Rounding probabilities: Maximum probability and minimum complexity multipliers

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Abstract

The choice of multipliers is studied, for multiplier methods of rounding that are based on rounding functions. Four multipliers are introduced and shown to be asymptotically equivalent, an easy-to-calculate multiplier, the exactly unbiased multiplier, the maximum probability multiplier, and the minimum complexity multiplier. The results are useful in assessing the rounding error when rounding probabilities to fractional proportions.

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1. Introduction

When rounding a finite set of probabilities to integral multiples of $1/n$, for a given denominator or accuracy n, standard rounding may well leave a nonvanishing discrepancy. That is, the rounded weights often fail to sum to one. For examples and details of the problem, see Mosteller, Youtz and Zahn (1967), Diaconis and Freedman (1979), Balinski and Young (1982), Happacher (1996), or Happacher and Pukelsheim (1996, 1998).

Table 1 shows the result of the 1996 Russian presidential vote region-by-region. The 11 categories comprise the valid votes for each of the ten candidates, and the vote against all candidates on the ballot. Using standard rounding, the counts are rounded to the tenth of a percent. In our notation, this is of the form n_i/n, with $n = 1000$. The last column gives the discrepancy, $D = \left(\sum_{i\leq 11} n_i\right) - 1000$.

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Table 1: Top portion [landscaped]

Table 1: Bottom portion [landscaped]

In this paper we discuss the problem of bringing the discrepancy close to zero, by making a good choice for a variable called *multiplier* to be introduced below. As in our previous work (Happacher 1996, Happacher and Pukelsheim 1996, 1998) we concentrate on a rounding function r_q, for some $q \in [0,1]$: For any integer $k \geq 0$, a number x in the interval $[k, k+1]$ is rounded to $r_q(x) = k$ if $x < k+q$, and to $r_q(x) = k+1$ if $x > k+q$. A tie occurs when $x = k+q$, but these form a nullset under the distributional assumptions that we adopt in the following.

For a fixed number of categories, c, we assume the probability vector $(W_1, \dots, W_c)$ to be uniformly distributed on the probability simplex of $\mathbb{R}^c$. This distributional assumption is fundamental to the sequel, and appears to be a natural starting point. The *total*

$$T_{c,q,\nu} = \sum\nolimits_{i \leq c} r_q(\nu W_i) \tag{1}$$

then is an integer-valued random variable, and crucially depends on the (continuous) *multiplier* $\nu > 0$. For given *accuracy* n, we seek to determine a multiplier ν_n so that the *discrepancy*

$$D_{c,q,n} = T_{c,q,\nu_n} - n \tag{2}$$

concentrates around zero, in some sense or other.

Table 1 presents an example for $c = 11$ categories, using standard rounding $q = 1/2$, accuracy $n = 1000$, and multiplier $\nu_n = n$. The 89 Constitutional Subjects of the Russian Federation, together with the votes cast abroad and the candidates' totals, yield the 91 realizations of the discrepancy $D = D_{11,1/2,1000}$ given in the last column of the table. The observed frequencies of the values of D are listed in Table 2.

For an individual set of weights $(w_1, \dots, w_c)$ one can always find a multiplier ν satisfying $\sum_{i \leq c} r_q(\nu w_i) = n$. This is what Balinski and Young (1982) call a *rounding method*. The method that comes with standard rounding, $q = 1/2$, is called the *Webster method*. Table 1 indicates the corrective action, following standard rounding, that is needed to obtain a solution according to the Webster method. A trailing sign + or − means to add or to subtract 0.1 percent, in order to make the discrepancy vanish.

Section 2 reviews our earlier results on the easy-to-calculate multipliers

$$\mu_{c,q,n} = n + c\left(q - \frac{1}{2}\right). \tag{3}$$

They achieve unbiasedness in an asymptotic sense, $\mathrm{E}[T_{c,q,\mu_{c,q,n}}] = n + O(1/n)$. Standard rounding has $\mu_{c,1/2,n} = n$. If the accuracy n is fixed then there is an *exactly unbiased multiplier*

$$\eta_{c,q,n}, \tag{4}$$

Table 2: Discrepancy Distribution for 11 Categories

Discrepancy $D_{11,1/2,1000}$	-4	-3	-2	-1	0	1	2	3	4
Observed frequency	0	0	9	18	37	20	6	1	0
Theoretical frequency	0	0	4	23	38	22	4	0	0
Probability	0.00002	0.00249	0.04845	0.24532	0.41096	0.24281	0.04751	0.00242	0.00002

The observed frequencies are from Table 1. The probabilities are calculated from the formula in Happacher (1996, page 66). They are rounded (Webster method, $n = 91$) to obtain the theoretical frequencies.

fulfilling $\mathrm{E}[T_{c,q,\eta_{c,q,n}}] = n$. This existence statement is of little merit for practical applications, as no closed form expression for $\eta_{c,q,n}$ is available.

In Sections 3 and 4 we introduce two new optimality concepts. In Section 3 we prove that, for a given accuracy n, there is a multiplier

$$\pi_{c,q,n} \tag{5}$$

maximizing the probability of a vanishing discrepancy. This *maximum probability multiplier* $\pi_{c,q,n}$ is again hard to calculate. The same is true of the *minimum complexity multiplier*

$$\alpha_{c,q,n} \tag{6}$$

in Section 4, minimizing the expectation of the absolute value of the discrepancy. Table 3 illustrates the small numerical differences between the four multipliers (3)–(6). Figure 1 suggests that the differences between (4)–(6) and (3) are bounded of the order $1/n$.

Section 5 is devoted to the asymptotic discrepancy distribution, as the accuracy n tends to infinity. Theorem 6 shows that, under mild assumptions on the multiplier sequence $(\nu_n)_{n\geq 1}$, the discrepancies $D_{c,q,n}$ from (2) have a limiting distribution that does not depend on q and that is given by the density of the convolution of c uniform distributions on the interval $(-1/2, 1/2)$. The convolution of uniform distributions is a frequently used model for the sum of rounding errors. See, for example, Mosteller, Youtz and Zahn (1967), Diaconis and Freedman (1979), or Johnson, Kotz and Balakrishnan (1995, Chapter 26.9). Table 4 lists the asymptotic probabilities for $c = 3, 5, 7, 9, 11$ categories.

Section 6 comes to the conclusion that, asymptotically as $n \to \infty$, the multiplier sequence from (3) is of maximum probability and minimum complexity, besides being unbiased. In summary, we recommend the multipliers $\mu_{c,q,n}$ from (3).

2. Unbiased Multipliers

Unbiasedness relates to the moments of the total (1). For $n \geq c$, the existence of a unique exactly unbiased multiplier (4) is established by Happacher (1996, page 29), or Happacher and Pukelsheim (1996).

For the asymptotic statements we rely on Happacher (1996, pages 33, 36), or Happacher and Pukelsheim (1996). As ν tends to infinity, the first two moments of the total satisfy

$$\mathrm{E}[T_{c,q,\nu}] = \nu - c\left(q - \frac{1}{2}\right) + \binom{c}{2}\frac{1/6 + q(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right), \tag{7}$$

$$\mathrm{Var}[T_{c,q,\nu}] = \frac{c}{12} + \frac{2}{3}\binom{c}{2}\frac{q(q-1/2)(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right). \tag{8}$$

Hence the multiplier $\nu = \mu_{c,q,n}$ from (3) leads to the expectation $n + O(1/n)$ in (7). This is the property of asymptotic unbiasedness.

The moments in (7) and (8) depend on the onedimensional and twodimensional marginal distributions of the random vector $(W_1, \ldots, W_c)$. In general, the marginal distributions have a simple structure.

Lemma 1 (Marginals). *Fix $\ell \in \{1, \ldots, c\}$. The ℓ-dimensional marginal distributions of $(W_1, \ldots, W_c)$ are all identical,*

$$\mathrm{P}\left(W_{i_1} > y_1, \ldots, W_{i_\ell} > y_\ell\right) = \left(1 - \sum\nolimits_{i \leq \ell} y_i\right)^{c-1},$$

with $y_1, \ldots, y_\ell \in (0,1)$ such that $\sum_{i \leq \ell} y_i < 1$.

Proof. Exchangeability leads to identical marginal distributions. The formula itself is not hard to derive by a geometric argument, see Happacher (1996, page 26). □

3. Maximum probability multipliers

For a given accuracy n, a maximum probability multiplier $\pi_{c,q,n}$ must fulfill

$$\mathrm{P}(T_{c,q,\pi_{c,q,n}} = n) = \max\nolimits_{\nu>0} \mathrm{P}(T_{c,q,\nu} = n). \tag{9}$$

The following theorem shows that such a multiplier exists.

Theorem 2 (Maximum probability). *For every accuracy $n \geq c$, there exists a maximum probability multiplier $\pi_{c,q,n}$. All maximum probability multipliers lie in the interval* $\Big(n - c(1-q), n + cq\Big)$.

Proof. The function $g_n(\nu) = \mathrm{P}(T_{c,q,\nu} = n)$ is continuous on $(0, \infty)$. Indeed, the positive quadrant $(0,\infty)^c$ is tiled by cubes of the form $(k_1 - 1 + q, k_1 + q) \times \cdots \times (k_c - 1 + q, k_c + q)$, consisting of the vectors $(x_1, \ldots, x_c)$ that are rounded to $(k_1, \ldots, k_c)$. Let $C(n)$ be the union of the cubes with $\sum_{i \leq c} k_i = n$. We have

$$T_{c,q,\nu} = n \qquad \Longleftrightarrow \qquad \nu(W_1, \ldots, W_c) \in C(n).$$

Let $S(c)$ be the probability simplex in $\mathbb{R}^c$. The representation

$$g_n(\nu) = \frac{\mathrm{vol}_{c-1}\Big(C(n) \cap \nu S(c)\Big)}{\mathrm{vol}_{c-1}\Big(\nu S(c)\Big)} \tag{10}$$

shows that the function g_n is continuous on $(0, \infty)$.

A rounding function r_q comes with the basic relation $r_q(\nu W_i) - 1 + q \leq \nu W_i \leq r_q(\nu W_i) + q$, for all $i \leq c$. Summation yields

$$T_{c,q,\nu} - c(1-q) \leq \nu \leq T_{c,q,\nu} + cq. \tag{11}$$

On the set $\{T_{c,q,\nu} = n\}$, the multiplier ν then lies in the interval $K = [n - c(1-q), n + cq] \subset (0, \infty)$. For ν outside K we have $\mathrm{P}(T_{c,q,\nu} = n) = 0$. This extends to the endpoints $\nu = n - c(1-q)$ and $\nu = n + cq$, by continuity. Thus $\pi_{c,q,n}$ exists, and any such multiplier must lie in the interior of K. □

The function g_n in the proof fails to be everywhere differentiable. Cubes that stick out through one of the bounding faces of the positive quadrant are cut off. On the boundary it is therefore not cubes, but rectangular subsets that are relevant. At such values of ν where the scaled simplex $\nu S(c)$ just touches some cube or some boundary rectangle, the function g_n is not differentiable.

The first part of the proof makes no use of the special rounding functions r_q of the present paper. Hence the existence result carries over to general rounding functions r that are determined by a signpost sequence $s(k)$, as discussed in Happacher and Pukelsheim (1996).

4. Minimum complexity multipliers

The rounding algorithm in Dorfleitner and Klein (1999) relies on an initial multiplier ν to calculate the total $t = T_{c,q,\nu}$. The first step, called the multiplier start, may leave a nonzero discrepancy $d = t - n$. The second step, the discrepancy finish, needs $|d|$ iterations to work the discrepancy up or down to zero. The expected absolute discrepancy $\mathrm{E}[|D_{c,q,n}|]$ thus measures the complexity of the algorithm. For this reason a multiplier $\alpha_{c,q,n}$ with

$$\mathrm{E}[|T_{c,q,\alpha_{c,q,n}} - n|] = \min_{\nu>0} \mathrm{E}[|T_{c,q,\nu} - n|] \tag{12}$$

is called a minimum complexity multiplier. The following statement parallels Theorem 2.

Theorem 3 (Minimum complexity). *For every accuracy $n \geq c$, there exists a minimum complexity multiplier $\alpha_{c,q,n}$. All minimum complexity multipliers lie in the interval $\Big(n - c(1-q), n + cq\Big)$.*

Proof. We need to minimize the function $h(\nu) = \mathrm{E}[|T_{c,q,\nu} - n|]$. From (11) we obtain a lower bound and an upper bound for the support of the total,

$$\nu - cq \leq T_{c,q,\nu} \leq \nu + c(1-q). \tag{13}$$

For $\nu \in (0, n - c(1-q)]$ this entails $T_{c,q,\nu} \leq n$; here $h(\nu) = n - \mathrm{E}[T_{c,q,\nu}]$ is nonincreasing. For $\nu \in [n + cq, \infty)$ we get $T_{c,q,\nu} \geq n$; here $h(\nu) = \mathrm{E}[T_{c,q,\nu}] - n$ is nondecreasing. Hence h is minimized in-between.

For $\nu \leq n + cq$ we have $T_{c,q,\nu} \leq n + c$ and

$$h(\nu) = \sum_{t=0}^{n-1} (n-t)\mathrm{P}(T_{c,q,\nu} = t) + \sum_{t=n+1}^{n+c} (t-n)\mathrm{P}(T_{c,q,\nu} = t).$$

The functions $g_t(\nu) = \mathrm{P}(T_{c,q,\nu} = t)$ are continuous, admitting representations similar to (10). Hence h is also continuous, and attains a minimum. □

The objective function h has value $c/2 + O(1/n)$ at $\nu = n - c(1-q)$ and at $\nu = n + cq$, as follows from (7). At $\nu = \eta_{c,q,n}$, the trivial estimate $|T_{c,q,\nu} - n| \leq (T_{c,q,\nu} - n)^2$ and (8) yield the upper bound

$$\frac{c}{12} + O\left(\frac{1}{n}\right). \tag{14}$$

Table 3: Numerical Examples of Various Multipliers

q	0	1/4	1/2	3/4	1
$\mu_{11,q,100}$	94.5	97.25	100	102.75	105.5
$\eta_{11,q,100}$	94.40291	97.26260	100.04580	102.76042	105.41305
$\pi_{11,q,100}$	94.39741	97.26310	100.05039	102.76046	105.40812
$\alpha_{11,q,100}$	94.40068	97.26286	100.04764	102.76039	105.41106

The numerical differences between the unbiased multipliers (3)–(4) and the optimal multipliers (5)–(6) are small, which is true beyond the special cases for $c = 11$ and $n = 100$ that are shown in the table.

The Jensen inequality provides the alternative bound

$$\sqrt{\frac{c}{12}} + O\left(\frac{1}{\sqrt{n}}\right). \tag{15}$$

Therefore, up to terms of higher order, the minimum complexity lies below (14) for $c \leq 12$, and below (15) for $c \geq 12$.

Table 3 conveys some impression of how the multipliers (3)–(6) compare numerically, for $c = 11$ categories, accuracy $n = 100$, and five values of q. The numbers were calculated using the exact distribution of Happacher (1996, page 66). Figure 1 provides additional insight for growing accuracy $n = 11, \ldots, 300$, in the special case $c = 11$ and $q = 1/2$, by exhibiting the scaled remainder sequences

$$\begin{aligned} UB(n) &= n(\eta_{c,q,n} - \mu_{c,q,n}), \\ MP(n) &= n(\pi_{c,q,n} - \mu_{c,q,n}), \\ MC(n) &= n(\alpha_{c,q,n} - \mu_{c,q,n}). \end{aligned} \tag{17}$$

The graphs seem to indicate that the differences between (4)–(6) and (3) stay bounded of order $1/n$. We were unable to confirm this result theoretically.

5. Asymptotic discrepancy distribution

The natural domain of definition of a rounding function is the positive half line $(0, \infty)$. Standard rounding, however, permits an unambiguous extension to the full real line by setting $r_{1/2}(y) = z$ if $y \in (z - 1/2, z + 1/2)$, for all integers z and for all $y \in \mathbb{R}$. This extension is "stationary", in that we have $r_{1/2}(z + y) = z + r_{1/2}(y)$.

Lemma 5 parallels a result of Diaconis and Freedman (1979, Lemma 2). It reduces the rounding function r_q to standard rounding of appropriately shifted roundoff errors $V_{q,n,i}$.

Figure 1: Scaled Remainder Sequences

5.0 $MP(n) = n(\pi_{c,q,n} - \mu_{c,q,n})$

4.8

$MC(n) = n(\alpha_{c,q,n} - \mu_{c,q,n})$

4.6

$UB(n) = n(\eta_{c,q,n} - \mu_{c,q,n})$

4.4

n

11 100 200 300

For increasing accuracy $n = 11, \ldots, 300$, the remainder sequences (17) that are scaled by n appear to be bounded. The graphs are for the special case of $c = 11$ categories and standard rounding, $q = 1/2$.

Lemma 5 (Convolutionlike representation). *Let $\nu_n > 0$ be an arbitrary multiplier. Then the random variables $V_{q,n,i} = r_q(\nu_n W_i) - \nu_n W_i + q - 1/2$ take values in $(-1/2, 1/2)$, for $i = 1, \ldots, c-1$, and satisfy*

$$D_{c,q,n} = r_{1/2}\left(\nu_n - \mu_{c,q,n} + \sum\nolimits_{i<c} V_{q,n,i}\right). \tag{18}$$

Proof. From $\nu_n W_i = r_q(\nu_n W_i) - V_{q,n,i} + q - 1/2$ and $W_c = 1 - \sum_{i<c} W_i$, we get

$$\nu_n W_c = \nu_n - \sum\nolimits_{i<c} r_q(\nu_n W_i) + \sum\nolimits_{i<c} V_{q,n,i} - c\left(q - \frac{1}{2}\right) + q - \frac{1}{2}.$$

Using $r_q(x) = r_{1/2}(x - q + 1/2)$ and the stationarity of $r_{1/2}$ on $\mathbb{R}$, this rounds to

$$r_q(\nu_n W_c) = -\sum\nolimits_{i<c} r_q(\nu_n W_i) + r_{1/2}\left(\nu_n - c\left(q - \frac{1}{2}\right) + \sum\nolimits_{i<c} V_{q,n,i}\right).$$

Collecting terms and again exploiting the stationarity of $r_{1/2}$ on $\mathbb{R}$ establishes (18). □

It is tempting to conjecture that the cumulated roundoff errors $\sum_{i<c} V_{q,n,i}$ behave asymptotically like $\sum_{i<c} U_i$, where $U_1, \ldots, U_{c-1}$ are independent random variables with a uniform distribution on $(-1/2, 1/2)$. For the discrepancy $D_{c,q,n}$, however, one more degree of freeedom is caused by the standard rounding operation in (18). To be precise, let f_c denote the density of the c-fold convolution of the uniform distribution on $(-1/2, 1/2)$, see Johnson, Kotz and Balakrishnan (1995, Chapter 26.9).

Table 4: Distribution of the Asymptotic Discrepancy D_c

c	0	±1	±2	±3	±4
3	0.75	0.125			
5	0.59896	0.19792	0.00260		
7	0.51102	0.22880	0.01567	0.00002	
9	0.45292	0.24078	0.03213	0.00063	0.0
11	0.41096	0.24407	0.04798	0.00245	0.00002

The probabilities are calculated from (21). For $c = 11$ categories, symmetrization of the exact probabilities in Table 2 yields almost precisely the present numbers; the support points ±5 have probability $0.27 \cdot 10^{-9}$.

Theorem 6 (Asymptotic discrepancy distribution). *Let $q \in [0,1]$ be arbitrary and let $(\nu_n)_{n\geq 1}$ be a multiplier sequence satisfying*

$$\lim_{n\to\infty} (\nu_n - \mu_{c,q,n}) = \lambda \in \mathbb{R}. \tag{19}$$

Then we have, for every integer d,

$$\lim_{n\to\infty} \mathrm{P}(D_{c,q,n} = d) = \int_{d-1/2-\lambda}^{d+1/2-\lambda} f_{c-1}(y)\, dy. \tag{20}$$

Proof. It is a consequence of Lemma 3 of Diaconis and Freedman (1979) that $\sum_{i<c} V_{q,n,i}$ converges in distribution to $\sum_{i<c} U_i$. Thus representation (18) and assumption (19) yield (20),

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}(D_{c,q,n} = d) &= \mathrm{P}\left(r_{1/2}\left(\lambda + \sum\nolimits_{i<c} U_i\right) = d \right)\\ &= \mathrm{P}\left(\sum\nolimits_{i<c} U_i \in \left(d - \frac{1}{2} - \lambda, d + \frac{1}{2} - \lambda\right)\right).\end{aligned}$$

Happacher (1996, page 81) provides an alternative proof based on the exact finite distribution of $D_{c,q,n}$. □

Let D_c be an integer-valued random variable with distribution

$$\mathrm{P}(D_c = d) = \int_{d-1/2}^{d+1/2} f_{c-1}(y)\, dy = f_c(d), \tag{21}$$

on the support points $d = -\lfloor (c-1)/2 \rfloor, \ldots, \lfloor (c-1)/2 \rfloor$. According to (20) with $\lambda = 0$, the discrepancies $D_{c,q,n}$ converge in distribution to D_c as the accuracy n tends to infinity. Table 4 gives the distribution of D_c for $c = 3, 5, 7, 9, 11$ categories.

6. Asymptotically optimal multiplier sequences

For asymptotic comparisons we may restrict attention to multiplier sequences $(\nu_n)_{n\geq 1}$ that satisfy the convergence condition (19).

Lemma 7 (Limiting unimodality). *For every multiplier sequence* $(\nu_n)_{n\geq 1}$ *that satisfies* (19) *and for every* $k \geq 0$, *we have*

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}(|T_{c,q,\nu_n} - n| \leq k) &= \int_{-k-1/2-\lambda}^{k+1/2-\lambda} f_{c-1}(y)\,dy \\ &\leq \int_{-k-1/2}^{k+1/2} f_{c-1}(y)\,dy = \lim_{n\to\infty} \mathrm{P}(|T_{c,q,\mu_{c,q,n}} - n| \leq k).\end{aligned} \tag{22}$$

Proof. The two equalities result from Theorem 6. The densities f_{c-1} are symmetric and unimodal about 0. Therefore the integral is maximized when the interval of integration is centered at 0. This is the inequality in (22). □

The special case $k = 0$ shows that the multipliers from (3) are asymptotically of maximum probability among the sequences (19),

$$\lim_{n\to\infty} \mathrm{P}\left(T_{c,q,\nu_n} = n\right) \leq \lim_{n\to\infty} \mathrm{P}\left(T_{c,q,\mu_{c,q,n}} = n\right). \tag{23}$$

The multipliers in (4)–(6) are asymptotically maximum probability sequences as well.

From $\mathrm{E}[|T_{c,q,\nu_n} - n|] = \sum_{k\geq 1} \mathrm{P}\left(|T_{c,q,\nu_n} - n| \geq k\right)$ we infer that the multipliers (3) asymptotically also minimize the complexity,

$$\lim_{n\to\infty} \mathrm{E}[|T_{c,q,\nu_n} - n|] \geq \lim_{n\to\infty} \mathrm{E}[|T_{c,q,\mu_{c,q,n}} - n|]. \tag{24}$$

Again the same is true of the multipliers in (4)–(6).

Our results comprise the type of inverse problem considered by Athanasopoulos (1994, Theorem 1.2). She fixes c and k, chooses the multiplier $\nu_n = n$, and then determines the parameter $q \in [0, 1]$ that maximizes $\lim_{n\to\infty} \mathrm{P}\left(|T_{c,q,n} - n| \leq k\right)$. Our Theorem 6 states that the limiting shift is $\lambda = c(q - 1/2)$. This probability is maximized when the shift vanishes, forcing $q = 1/2$.

In summary our results strongly advocate the multiplier $\mu_{c,q,n}$ from (3). It is easy to calculate and, asymptotically, it achieves unbiasedness, maximizes the probability of a vanishing discrepancy, and minimizes the complexity of our generic algorithm.

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Note added in proof. In the theory of apportionment, the rounding method with $q = 1$ is known as the method of d'Hondt, or Jefferson (Balinski and Young 1982, page 18). For this method, Gfeller (1890, page 130) proposes to use as multiplier "le nombre des candidats *plus la moité du nombre des listes*", that is, $\mu_{c,1,n} = n + c/2$ as in (3). For the same method Hagenbach-Bischoff (1905, page 15), who advocates the multiplier $\nu_n = n + 1$ and thus generates a negative shift $\lambda = 1 - c/2$ in (19), calculates the asymptotic discrepancy distributions of Theorem 6 for $c = 3, 4, 5$.

Table 1. Russian Presidential Vote of 16 June 1996

Constitutional Subject	*Yeltsin*	*Zyuganov*	*Lebed*	*Yavlinsky*	*Zirinovsky*	*Fedorov*	*Go*
Republics [Respubliky]							
Adygeya	45 374:20.3	116 701:52.1	31 710:14.2	11 977: 5.4	11 494: 5.1	2 245:1.0	5
Altay	27 562:29.1+	42 204:44.6+	12 614:13.3	3 347: 3.5	4 671: 4.9	836:0.9	9
Bashkortostan	769 089:34.9	941 539:42.7+	200 859: 9.1	152 557: 6.9	64 541: 2.9	12 256:0.6	17 4
Buryatia	134 856:31.3	177 293:41.2	46 609:10.8	33 451: 7.8	21 329: 5.0	5 464:1.3	2 5
Checheniya	239 905:68.1	60 119:17.1	9 371: 2.7	15 666: 4.4+	5 172: 1.5	3 804:1.1	6 5
Chuvashia	132 422:21.3+	347 524:56.0	49 296: 7.9+	29 446: 4.7	27 381: 4.4	20 906:3.4	2 3
Dagestan	230 614:29.3	511 202:64.9	10 799: 1.4	13 753: 1.7	9 041: 1.1	2 208:0.3	2 7
Ingushetiya	37 129:47.2	19 653:25.0	1 796: 2.3	12 195:15.5	1 398: 1.8	616:0.8	3 5
Kabardino–Balkaria	163 872:44.8	139 521:38.2	36 685:10.0	12 590: 3.4	5 358: 1.5	1 809:0.5	1 2
Karachay–Cherkessia	54 823:26.4−	117 677:56.6	18 624: 9.0	6 527: 3.1	5 286: 2.5	1 014:0.5	1 0
Kareliya	165 584:43.0	66 428:17.3	47 053:12.2	55 768:14.5	33 134: 8.6	3 817:1.0	1 9
Khakassia	75 801:29.7	91 956:36.0	32 491:12.7	18 784: 7.4	25 108: 9.8	3 098:1.2	1 6
Khal'mg Tangc	88 615:59.9−	38 964:26.3	8 215: 5.5	3 791: 2.6	5 407: 3.7	633:0.4	5
Komi	202 373:41.2−	81 572:16.6	90 830:18.5	47 240: 9.6	49 103:10.0	4 262:0.9	2 9
Mari-El	93 124:24.8	166 131:44.2	41 948:11.2−	28 179: 7.5	28 418: 7.6	5 047:1.3	1 7
Mordvinia	116 693:25.0	240 263:51.4+	51 434:11.0	14 493: 3.1	33 138: 7.1	3 323:0.7	1 4
North Ossetia	57 849:19.5	187 007:63.1	28 795: 9.7	5 390: 1.8	9 703: 3.3	1 705:0.6	8
Sakha (Yakutia)	228 398:53.2	90 529:21.1	55 551:12.9	20 620: 4.8	16 099: 3.8−	4 647:1.1	3 4
Tatarstan	745 181:39.4	740 451:39.2	143 429: 7.6	134 161: 7.1	50 119: 2.7	17 895:0.9	15 7
Tyva	69 971:62.5	24 716:22.1	5 297: 4.7	4 926: 4.4	3 529: 3.2	532:0.5	1 1
Udmurtia	271 865:37.4	225 074:30.9+	85 125:11.7	68 215: 9.4	44 243: 6.1	6 802:0.9	5 0
Territories [Kraya]							
Altay	300 499:22.1	578 478:42.5	267 216:19.6+	69 619: 5.1	101 669: 7.5	9 439:0.7	6 3
Khabarovsk	288 585:39.4	169 586:23.2	90 550:12.4	77 077:10.5	64 007: 8.7	15 991:2.2	5 0
Krasnodar	682 602:26.6	1 024 603:39.9	454 555:17.7	165 231: 6.4	165 721: 6.5−	23 266:0.9	8 0
Krasnoyarsk	523 135:35.3	428 781:28.9	208 494:14.0	150 527:10.1	113 953: 7.7	13 264:0.9	8 8
Primor'ye	308 747:29.9	256 574:24.9	203 384:19.7	74 840: 7.3−	133 029:12.9	13 094:1.3	5 7
Stavropol'	302 236:22.3−	603 570:44.5−	265 729:19.6	56 353: 4.2−	84 991: 6.3	10 654:0.8	8 2
Regions [Oblasti]							
Amur	127 233:26.9	200 186:42.4	56 610:12.0	28 985: 6.1	37 852: 8.0	5 651:1.2	2 3
Arkhangel'sk	288 225:41.3+	129 299:18.5	121 910:17.5	76 136:10.9	46 277: 6.6	11 037:1.6	3 9
Astrachan'	150 190:30.0	185 925:37.1	82 140:16.4	30 710: 6.1	36 407: 7.3	4 674:0.9	1 6
Belgorod	189 320:23.2	383 688:46.9+	140 322:17.2	47 592: 5.8	35 666: 4.4	4 336:0.5	2 7
Bryansk	210 257:26.6	397 454:50.3	92 948:11.8	27 904: 3.5	40 777: 5.2	4 746:0.6	2 6
Chelyabinsk	685 273:37.2	463 071:25.1+	371 120:20.1+	164 230: 8.9	97 937: 5.3	13 732:0.7	8 9
Chita	130 011:24.9	207 282:39.8−	61 981:11.9	29 071: 5.6	68 603:13.2	6 688:1.3	2 8
Irkutsk	363 648:32.7	311 353:28.0	183 962:16.5+	100 075: 9.0	95 810: 8.6	22 271:2.0	7 1
Ivanovo	204 084:30.0	160 105:23.5	203 997:30.0−	41 938: 6.2	48 275: 7.1	4 215:0.6	2 5
Kaliningrad	173 769:33.8	119 830:23.3	100 264:19.5	66 703:13.0	37 412: 7.3	3 189:0.6	2 2
Kaluga	190 706:31.9−	214 933:35.9	94 650:15.8	45 258: 7.6	31 018: 5.2	5 249:0.9	2 3
Kamchatka	57 435:34.7	31 307:18.9	23 549:14.2	28 935:17.5	16 689:10.1	1 731:1.0	8
Kemerovo	332 376:23.4	561 397:39.5−	220 789:15.5	77 099: 5.4	167 925:11.8	23 566:1.7	7 1
Kirov	272 471:31.6+	252 624:29.3+	119 504:13.9	105 934:12.3	75 155: 8.7	7 232:0.8	3 7
Kostroma	122 971:28.4	125 399:29.0−	102 078:23.6−	34 112: 7.9	33 426: 7.7	3 357:0.8	2 0
Kurgan	170 311:29.7	218 464:38.0	64 877:11.3	38 479: 6.7	58 143:10.1	4 582:0.8	3 1
Kursk	177 328:24.5	376 880:52.1−	81 555:11.3−	39 641: 5.5	28 666: 4.0	4 280:0.6	2 6
Leningrad	348 505:37.9	215 511:23.4+	168 540:18.3	107 896:11.7	39 882: 4.3	11 038:1.2	5 7
Lipetsk	168 077:25.5	310 671:47.1	88 165:13.4	37 251: 5.6	35 638: 5.4	4 616:0.7	1 8
Magadan	40 679:37.3−	17 666:16.2	26 288:24.1	6 770: 6.2	12 021:11.0	1 570:1.4	5

Moskva	1 675 374:44.8−	912 684:24.4−	571 886:15.3	298 656: 8.0	113 883: 3.0	34 510:0.9	17 4
Murmansk	190 719:41.0	56 789:12.2	119 396:25.7	45 435: 9.8	32 775: 7.0	4 177:0.9	2 4
Nizhniy Novgorod	657 961:35.4	614 467:33.0	279 053:15.0	134 905: 7.3	102 621: 5.5	16 620:0.9	8 0
Novgorod	148 515:36.1	98 682:24.0	76 912:18.7	45 786:11.1	25 813: 6.3	3 398:0.8	2 4
Novosibirsk	371 210:26.0	506 791:35.5	144 918:10.1+	202 117:14.2	141 440: 9.9	14 609:1.0	16 1
Omsk	369 782:33.3	417 029:37.6	94 396: 8.5	101 027: 9.1	78 352: 7.1	8 693:0.8	5 0
Orenburg	288 865:26.4	468 689:42.8+	151 489:13.8+	65 027: 5.9	83 523: 7.6	10 316:0.9	7 0
Oryol	109 020:21.7	275 643:54.9	59 972:12.0	19 788: 3.9	22 402: 4.5	3 187:0.6	1 5
Penza	181 839:21.1	442 066:51.4	105 389:12.2	60 565: 7.0	46 188: 5.4	5 775:0.7	2 4
Perm'	742 968:56.1+	216 713:16.4	130 203: 9.8	96 926: 7.3	83 952: 6.3	12 410:0.9	8 3
Pskov	121 667:25.0	149 056:30.7	115 549:23.8	34 537: 7.1	49 999:10.3	3 319:0.7	2 0
Rostov-na-Donu	725 949:29.4	873 609:35.4	500 263:20.3	192 273: 7.8	115 162: 4.7	15 082:0.6	7 9
Ryazan'	186 477:25.0	302 484:40.5	149 544:20.0	42 242: 5.7	40 968: 5.5	4 981:0.7	2 6
Sakhalin	87 577:30.3	78 935:27.3	54 755:18.9	27 174: 9.4	26 581: 9.2	4 030:1.4	1 6
Samara	620 526:36.6−	604 110:35.6	200 054:11.8	105 776: 6.2	96 378: 5.7	16 932:1.0	8 1
Saratov	426 533:28.8−	624 996:42.1	191 822:12.9	79 404: 5.4	106 482: 7.2	14 135:1.0	5 4
Smolensk	141 854:22.2+	287 621:45.1	102 726:16.1	32 942: 5.2	53 764: 8.4	3 834:0.6	2 3
Sverdlovsk	1 302 951:60.1	255 514:11.8	310 841:14.3	117 496: 5.4	107 039: 4.9	23 103:1.1	9 3
Tambov	144 669:21.2	361 552:53.0	81 045:11.9	32 003: 4.7	42 183: 6.2	5 576:0.8	2 1
Tomsk	178 881:35.5+	113 281:22.5	100 788:20.0	55 780:11.1	36 419: 7.2	4 026:0.8	3 0
Tula	311 280:30.4+	314 098:30.7	249 663:24.4	68 439: 6.7	47 545: 4.6+	6 196:0.6	3 3
Tver'	299 435:32.5	313 168:33.9	159 813:17.3	64 843: 7.0	51 496: 5.6	6 799:0.7	3 5
Tyumen'	238 171:39.7−	166 491:27.7	80 961:13.5	34 750: 5.8	57 206: 9.5	4 988:0.8	3 2
Ul'yanovsk	184 218:24.1+	355 066:46.5+	95 559:12.5	45 748: 6.0	57 167: 7.5	7 158:0.9	2 5
Vladimir	270 736:31.4	261 808:30.3+	174 490:20.2	64 783: 7.5	58 774: 6.8	6 980:0.8	3 6
Volgograd	411 822:28.9	576 802:40.5	196 609:13.8	92 623: 6.5	94 418: 6.6	19 237:1.3+	6 0
Vologda	306 663:45.6	126 665:18.9	119 719:17.8	40 200: 6.0	48 338: 7.2	5 894:0.9	4 6
Voronezh	319 402:22.9	641 540:46.0	246 234:17.7	62 458: 4.5	82 429: 5.9	10 767:0.8	4 3
Yaroslavl'	260 919:33.3+	144 188:18.4+	245 613:31.4	65 886: 8.4	38 380: 4.9	4 896:0.6	3 3
Cities [Gorod]							
Moskva	2 861 058:61.7+	694 862:15.0	449 900: 9.7	372 524: 8.0	68 285: 1.5	37 790:0.8	23 5
Saint Peterburg	1 137 382:49.8−	342 466:15.0	321 244:14.1	347 488:15.2	49 273: 2.2	25 410:1.1	17 6
Autonomous Region [Avtonomnaja Oblast']							
Avt. Oblast' of Jews	28 859:30.8	31 220:33.3	14 544:15.5	6 134: 6.5	7 594: 8.1	1 725:1.8	6
Autonomous Districts [Avtonomny Okruga]							
Buryat of Aginskoye	13 647:45.7	10 903:36.5	1 630: 5.5	794: 2.7	1 732: 5.8	231:0.8	3
Buryat of Ust'-Ordynsk	21 827:37.8	23 604:40.9	5 041: 8.7	2 335: 4.0	2 691: 4.7	663:1.1	4
Chukchi	20 859:49.0	5 808:13.6+	7 337:17.2	2 741: 6.4	3 254: 7.6	844:2.0	2
Evenki	3 678:44.1−	1 694:20.3	1 390:16.7−	533: 6.4	597: 7.2	140:1.7	
Khanty and Mansy	271 345:53.2−−	66 241:13.0	78 175:15.3	34 138: 6.7	39 217: 7.7	7 178:1.4	2 9
Komi-Permyak	37 649:54.3	16 751:24.2−	3 850: 5.6−	2 116: 3.1	6 013: 8.7	360:0.5	6
Koryaki	7 270:46.8−	2 367:15.2	2 497:16.1	1 411: 9.1	1 028: 6.6	208:1.3	1
Nentsy	9 033:43.3+	3 891:18.7	2 537:12.2	1 619: 7.8	2 104:10.1	465:2.2	2
Taymyr' (Dolgany and Nentsy)	9 434:50.3−	2 304:12.3	2 843:15.1	1 234: 6.6	1 920:10.2	292:1.6	1
Yamal-Nentsy	104 486:56.0−	17 360: 9.3	29 789:16.0	11 824: 6.3	14 304: 7.7	2 975:1.6	1 2
Ballots cast in diplomatic representations outside the Russian Federation							
(abroad)	103 212:42.7	60 517:25.0+	40 589:16.8	14 830: 6.1	11 169: 4.6	2 862:1.2	1 9
Candidate's Total	26 665 495:35.8	24 211 686:32.5	10 974 736:14.7	5 550 752: 7.4	4 311 479: 5.8	699 158:0.9	386 0

Counts are turned into proportions to the tenth of a percent using standard rounding. E.g. in the Republic Adygeya, th
When row percentages do not total 100.0 *they leave a nonzero discrepancy D; trailing signs* +, − *indicate the corrective ac*
of Khanty and Mansy has total percentage 100.2 *and discrepancy 2; the Webster method assigns to Yeltsin* 53.0%. *I*