Kshirsager & Cheng's rotatability measure

Norman R. Draper, Friedrich Pukelsheim

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Professor (em.) Dr. Friedrich Pukelsheim
Lehrstuhl für Stochastik und ihre Anwendungen

Institut für Mathematik der Universität Augsburg

Telefon: 0821 598-2205
Fax: 0821 598-2280
Universitätsstraße 14
Postadresse:
D–86135 Augsburg
Germany

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Dear Editor: The recent paper by Kshirsagar & Cheng (1996) surprised us in two ways. (1) It failed to mention Draper & Pukelsheim (1990), published before Cheng's Ph.D. thesis was completed in 1991. Nor did it mention other recent related work, such as Draper, Gaffke & Pukelsheim (1991, 1993), and Draper & Pukelsheim (1993). (2) The rotatability measure proposed by Kshirsagar & Cheng has a particular type of inadequacy which we discussed in 1990 with respect to criteria offered by Draper & Guttman (1988), and Khuri (1998). Among misprints in Kshirsagar & Cheng (1996) is a misspelling of Guttman; also, moments are defined without their usual divisor n.

The papers by Khuri (1988), Draper & Pukelsheim (1990), and Kshirsagar & Cheng (1996) have one definite point of agreement. That is, their rotatability measures are all R^2 statistics for regressions of the moment matrix of a given design on the moment matrix of a rotatable design. The essential differences between the three results are in the weights allocated to the moments in the regressions, and in the way the designs are scaled before the regression is applied.

In an ordinary $n^{-1}\mathbf{X}'\mathbf{X}$ matrix format, where quadratic and higher order terms are counted just once, Khuri's weights are chosen by ignoring all off-diagonal terms below the main diagonal, and weighting by the number of terms of various types that remain. Kshirsagar & Cheng (1996) use weights which are the squares of coefficients obtained by Box & Hunter (1957) in a generating function expansion of the moments of a rotatable design and having value (before squaring) of

$$\frac{(2d)!}{(2d-\delta)!\delta_1!\cdots\delta_k!} \qquad (1)$$

where d is the order of rotatability of the design (e.g. $d = 2$ for second order rotatability), and where $\delta = \delta_1 + \delta_2 + \cdots + \delta_k$. No rationale for using these latter weights is apparent to us.

Draper & Pukelsheim (1990) tackle the problem through a Kronecker algebra which makes rotatability simple to work with, and weight by the number of terms in their (expanded and singular) $\mathbf{X}'\mathbf{X}$ matrix. These weights are, in fact, those given by (1), and not the square of (1).

For scaling, Khuri (1988) chooses

$$\sum_{u=1}^{n} x_{iu}^2 = n, \quad i = 1, 2, \ldots, k$$

which has the problem that addition of a center point requires a rescaling. Kshirsagar & Cheng (1996) say they are doing what Khuri does but, confusingly, write

$$\sum_{u=1}^{n} x_{iu}^2 = 1, \quad i = 1, 2, \ldots, k$$

instead of the more usual $\sum x_{iu}^2 = n$.

Draper & Pukelsheim (1990) scale so that all design points lie on or within the unit sphere; such a scaling is not affected by center point additions, and their criterion is unaffected by rotating the design in the x-space. This is not true of the other measures. An illustrative comparison follows.

The 3^2 factorial design is used as an example in all three papers. Since a comparison of Khuri's (1988) rotatability measure and Draper & Pukelsheim's (1990) measure for this design appears in the latter paper, we discuss here only the inadequacy in Kshirsagar & Cheng's treatment. Suppose we follow Kshirsagar & Cheng (1996) and define

$$M(\delta_1, \delta_2, \ldots, \delta_k) = \sum_{u=1}^{n} x_{1u}^{\delta_1} x_{2u}^{\delta_2} \cdots x_{ku}^{\delta_k}$$

(although we do not refer to this as a "moment" as they do). We need only $k = 2$ here, for the example. Consider the nine points $(\pm 1, \pm 1), (\pm 1, 0), (0, \pm 1), (0, 0)$ of a standard 3^2 design. Rotate them about $(0, 0)$ through an angle θ and write $s = \sin\theta, c = \cos\theta$ to obtain points $(s-c, -s-c), (s+c, s-c), (-s-c, -s+c), (-s+c, s+c), (-c, -s), (c, s), (s, -c), (-s, c)$, and $(0, 0)$. There are now seven non-zero $M(\delta_1, \delta_2)$ for $\delta \leq 4$. After a rescaling to make $M(2, 0) = M(0, 2) = 1$, necessary to apply the Kshirsagar & Cheng (1996) criterion, we have

$$M(3,1) = -M(1,3) = sc(s^2 - c^2)/16,$$
$$M(4,0) = M(0,4) = (1 + 2s^2c^2)/6, \qquad \text{and} \qquad M(2,2) = (1 - 3s^2c^2)/9.$$

Kshirsagar & Chengs's rotatability measure (5.2) now becomes $0.9259z(\theta)$ where

$$z(\theta) = 9(1 - 2s^2c^2)/(9 - 28s^2c^2 + 12s^4c^4).$$

The value 0.9260 quoted by Kshirsagar & Cheng arises only when $s = 0$ or $c = 0$, i.e., when the design is not rotated other than through $90°$, or multiples of $90°$. It is evident that such an assessment of rotatability is not constant for the design, but depends on how the points are oriented. A similar criticism applies to the criteria of Draper & Guttman (1988), and Khuri (1988).

In summary, we believe that Kshirsagar & Cheng's rotatability measure is flawed and is not "useful in algorithms" as they claim.

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Yours sincerely,

Norman R. Draper
Department of Statistics
University of Wisconsin
Madison, WI 53706, USA

Friedrich Pukelsheim
Institut für Mathematik
Universität Augsburg
D-86135 Augsburg, Germany

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