Positive Scalar Curvature and a Smooth Variation of Baas-Sullivan Theory

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Contents

Abstract

We introduce a smooth variation of Baas-Sullivan theory, yielding an interpretation of singular homology and connective real K-theory by smooth manifolds with additional structures on their boundaries, called singular manifolds. This enables us to give a proof of the so-called Homology Theorem (with 2 inverted) which in many cases reduces the existence question of positive scalar curvature metrics on closed manifolds to the study of the singular homology resp. connective real K-theory of their fundamental groups. Subsequently, we consider the question of positive scalar curvature on simply connected singular manifolds and show existence theorems corresponding to statements in the closed case. Within the scope of our treatment of non-simply connected singular manifolds, we finally introduce the notion of a positive homology class, and we show that the atoral classes of the singular homology of an elementary Abelian p-group, p an odd prime, are positive.

Chapter 1

Introduction

1.1 Survey on Positive Scalar Curvature

An important area of differential geometry is concerned with the curvature of Riemannian manifolds. Curvature provides information about the local geometry of manifolds. Fundamental curvature invariants are sectional curvature, or equivalently, the Riemannian curvature tensor, Ricci curvature and scalar curvature.

Let (M, g) denote a smooth Riemannian manifold. Among the three curvatures mentioned above, sectional curvature contains the most information. Let v and w denote two linearly independent tangent vectors located at a point $p \in M$. The sectional curvature associates to v and w the classical Gauß curvature of the small surface S in M which is swept by shortest curves, geodesics, emanating from p tangent to the plane spanned by v and w. The Gauß curvature itself has in turn many characterizations. Up to a factor it can be identified with the scalar curvature of S at p which we will describe below. Sectional curvature makes strict demands on the geometry of the manifold. In the case of constant sectional curvature, it even determines the metric g locally.

If one fixes a single tangent vector v at p, one can consider the average of all sectional curvatures involving v. This yields the Ricci curvature in the direction v. Geometrically, it measures the change of the volume element in the direction v. Positive (negative) Ricci curvature in the direction v means that the volume of a small conical region around v in M is smaller (larger) than the volume of the corresponding Euclidean standard cone.

Finally, averaging over the Ricci curvatures of all tangent vectors at p, we obtain the scalar curvature at p. It is the weakest, and from the algebraic point of view simplest, curvature invariant. It is just a real valued function

$$scal_g : M \to \mathbb{R}$$

which is defined, rephrasing the above said, as the twofold contraction of the Riemannian curvature tensor. An intuitive geometric interpretation can be given by comparing the volume of a small geodesic ball of radius r in the manifold, $V_g(B_r(p))$, with the corresponding standard volume of an Euclidean ball, $V_{std}(B_r(0))$. More precisely, if the dimension of M is n, then there is an expansion

$$\frac{V_g(B_r(p))}{V_{std}(B_r(0))} = 1 - \frac{scal_g(p)}{6(n+2)} \cdot r^2 + \cdots$$

(see e.g. [Gra73, Theorem 3.1]). In particular, positive scalar curvature at p means that the volume of a sufficiently small ball in M centered at p is smaller than its Euclidean counterpart. This holds vice versa for negative scalar curvature.

For the remainder of this section M shall denote a smooth closed manifold. One poses the following question. Under which algebraic- resp. differential-topological conditions does M admit a metric g such that $scal_g$ is of a particular form. Problems concerning $scal_g$ are analytic by nature; namely, $scal_g$ involves the derivatives of g up to the second order. An answer of the foregoing question must therefore lead to connections between analytic, continuous considerations and, typically discrete, topological invariants.

For manifolds of dimension two a prototypal answer in this direction is given by the classical Gauß-Bonnet Theorem. If $\chi(S)$ denotes the Euler characteristic of an oriented surface S equipped with a metric g, then the Gauß-Bonnet formula reads

$$\int_S scal_g = 4\pi\chi(S).$$

For a given surface it is easy to compute its Euler characteristic. For the sphere S^2, the torus T^2 and the connected sum Γ_k of k tori, one has $\chi(S^2) = 2$, $\chi(T^2) = 0$ and $\chi(\Gamma_k) = 2 - 2k$. We conclude, for example, that S^2 does not admit a metric of non-positive scalar curvature. Further information could be obtained together with the Uniformization Theorem telling us that any surface admits a metric of *constant* scalar curvature. Let us note that in dimension two sectional, Ricci and scalar curvature coincide, up to a factor.

We now consider manifolds of arbitrary dimensions. The two dimensional case vaguely suggests that one divides manifolds into the three following classes:

1. Manifolds admitting metrics whose scalar curvatures are non-negative and not identically 0.
2. Manifolds admitting metrics of vanishing scalar curvature which do not lie in Class 1.
3. Manifolds which do not lie in Classes 1 or 2.

This separation is explained by the following theorem. As usual, let $\mathcal{C}^\infty(M)$ denote the set of all smooth functions on M. We set $\mathcal{C}^\infty_-(M)$ for the subset of $\mathcal{C}^\infty(M)$ containing functions which are negative at least at one point. In addition, the subset of $\mathcal{C}^\infty(M)$ of all functions occurring as scalar curvature functions at all is denoted by $\mathcal{SC}(M)$. Kazdan and Warner (see [KW75a, KW75b, KW75c]) proved to following remarkable

Trichotomy Theorem. *Let M be a connected manifold of dimension greater or equal than three.*

1. *If M lies in Class 1, then $\mathcal{SC}(M) = \mathcal{C}^\infty(M)$.*
2. *If M lies in Class 2, then $\mathcal{SC}(M) = \mathcal{C}^\infty_-(M) \cup \{0\}$.*
3. *If M lies in Class 3, then $\mathcal{SC}(M) = \mathcal{C}^\infty_-$.*

We note in particular that all manifolds of dimension greater or equal than three admit metrics of *negative* scalar curvature. This was already proved by Aubin [Aub70].

According to the Trichotomy Theorem, the question of whether a given manifold M lies in Class 1 or not is equivalent to the concise

Question 1.1.1. Does M admit a positive scalar curvature metric or not?

This problem is addressed in the sequel. We will not go into the remaining problem of whether M lies in Class 2 or not. Results of Futaki [Fut93] and Dessai [Des01] indicate that manifolds in Class 2, so-called strongly scalar-flat manifolds, seem to be quite special.

A lot of work has been done towards Question 1.1.1. It turns out that *bordism theory* is an important approach to this problem. Bordism theory is concerned with the question of whether for two given closed manifolds M and N there exists a compact manifold W such that the boundary of W coincides with the disjoint union of M and N.

On the one hand, the relationship between the existence of positive scalar curvatures metrics and bordism theory is established by the *Surgery Lemma*, which was proved by Gromov-Lawson [GL80] and, independently, by Schoen-Yau [SY79]. It says that under mild conditions the existence of a positive scalar curvature metric is invariant under bordism. On the other hand, a certain characteristic number $\alpha(M)$, which is also invariant under (spin) bordism, yields an obstruction against positive scalar curvature metrics (see [Lic63, Hit74]). In this regard, the key observation is that according to the Atiyah-Singer index theorem, $\alpha(M)$ can be interpreted as the index of the Dirac operator which in turn is related to the scalar curvature function via the Weizenböck formula. This relationship is explained in more detail in Subsection 4.3.1.

The connection to bordism theory is very useful since via the Thom-Pontrjagin construction (see [Tho54]), bordism theory paves the way to stable homotopy theory, a branch where considerable tools of computation are available. Taken to an extreme, instead of having to solve complicated problems about partial differential equations or relations on manifolds, one can perform *easy*, or at least easier, algebraic computations in stable homotopy theory.

With respect to Question 1.1.1 the fundamental group of M plays an essential role. Manifolds with 'small' fundamental groups rather admit positive scalar curvature metrics than manifolds with 'big' fundamental groups. In order to give an overview of the answers of Question 1.1.1, let us first assume that the manifold in question is simply connected.

Due to the classification of surfaces, a simply connected manifold of dimension two is necessarily diffeomorphic to S^2, it therefore admits a positive scalar curvature metric. In fact, 'round' spheres of dimensions greater or equal than two are just the basic example spaces of positive (scalar) curvature. Much less trivial than in the two dimensional case, the Poincaré conjecture, proved by Perelman, implies that a simply connected manifold of dimension three is also diffeomorphic to the sphere.

A comprehensive answer of the question which, even simply connected, four dimensional manifolds admit positive scalar curvature metrics is not known. Perhaps the main problem is that the Surgery Lemma rather rarely applies. In addition, there are obstructions against positive scalar curvature metrics, *Seiberg-Witten invariants*, which only occur in dimension four.

In dimension greater or equal than five, the crucial observation is that we now are in a situation where surgery techniques from the so-called s-cobordism theorem are available. That often makes the Surgery Lemma applicable. We have the following

Theorem 1.1.2. *Let M be a closed simply connected manifold of dimension greater or equal than five, then the following hold:*

1. *If M admits a spin structure, then M admits a positive scalar curvature metric if and only if $\alpha(M)$ vanishes.*
2. *If M does not admit a spin structure, then M admits a positive scalar curvature metric.*

Theorem 1.1.2 completely answers Question 1.1.1 for simply connected manifolds of dimension greater or equal than five. Its proof excellently demonstrates the role of bordism theory in this context. Gromov and Lawson showed that the topological demands placed on M make the Surgery Lemma applicable (see [GL80]). This means that in case M is spin resp. oriented bordant to a manifold which admits a positive scalar curvature metric, then the same holds for M.

In the non-spin case, according to classical works on bordism theory, one knows explicit generators of the oriented bordism groups (see [Tho54], [Tho59, Mil60] for the torsion free part, see [Wal60, And66] for the torsion part). It is not difficult to construct positive scalar curvature metrics on these manifolds (see again [GL80]).

For the spin case, we noted above that the α-invariant yields an obstruction against positive scalar curvature metrics. It therefore remains to prove that a spin manifold M with vanishing α-invariant is spin bordant to a manifold admitting a positive scalar curvature metric. At odd primes this can be deduced from results in [Miy85]. Although explicit generators of the spin bordism groups at the prime 2 are not known, the task at hand is solved in the notable work [Sto92]. Stolz translated the question into a stable homotopy theoretic problem which is then solved by using the Adams spectral sequence.

For the sake of completeness, we note that both points of Theorem 1.1.2 are true in dimension two and three (see above) but wrong in dimension four (see [Weh03, Proposition 5.13]).

Now let us consider manifolds which are not necessarily simply connected. In contrast to the simply connected case, we do not have a complete answer of Question 1.1.1, in the fashion of Theorem 1.1.2. A comprehensive result can be found in [RS94, Theorem 3.3]. Loco cit. it is discussed how one can combine the Surgery Lemma with methods from the proof of the s-cobordism theorem to obtain the *Bordism Theorem.* To formulate the latter, let us consider the bordism groups $\Omega_*^G(X)$ of a space X. An element $[N, g] \in \Omega_n^G(X)$ is a bordism class of continuous maps $g\colon N \to X$ where N is a smooth closed manifold of dimension n equipped with a tangential structure G. We are interested in spin structures ($G = Spin$) and ordinary orientations ($G = SO$). Let us set

$$^+\Omega_n^G(X) = \left\{[N, g] \in \Omega_n^G(X) \,|\, N \text{ admits a positive scalar curvature metric}\right\}.$$

In $\Omega_n^G(X)$ addition is given by the disjoint union of manifolds and taking inverses by reversing the spin structure resp. orientation. Hence $^+\Omega_n^G(X)$ in fact becomes a subgroup of $\Omega_n^G(X)$.

Bordism Theorem. *Let M be a closed connected manifold of dimension n greater or equal than five with fundamental group π. Furthermore, let $B\pi$ be the classifying space of π, and let $f\colon M \to B\pi$ be the classifying map of the universal cover of M, then the following hold:*

1. *If M admits a spin structure, then M admits a positive scalar curvature metric if and only if $[M, f] \in {}^+\Omega_n^{Spin}(B\pi)$.*

2. *If M is orientable and totally non-spin, i.e. its universal cover does not admit a spin structure, then M admits a positive scalar curvature metric if and only if $[M, f] \in {}^+\Omega_n^{SO}(B\pi)$.*

There is a corresponding statement for non-spin manifolds whose universal covers *do* admit spin structures as well as for manifolds not being orientable (see [Sto, Definition 2.6], [RS94, p. 249]).

Note that the Bordism Theorem is not as conclusive as Theorem 1.1.2. Although it reduces the question of positive scalar curvature to the study of certain bordism groups, we cannot yet restrict ourselves to the computation of characteristic numbers. To show that a given manifold M with classifying map f admits a positive scalar curvature metric, one is faced with the task of constructing manifolds of positive scalar curvature in the bordism class of $[M, f]$. In addition, the Bordism Theorem does not yield obstructions against positive scalar curvature metrics.

It is desirable to pass from the bordism groups of $B\pi$ to simpler groups which are easier to compute. Let $H_*(X)$ denote the singular homology of X with integer coefficients. There is a map

$$U\colon \Omega_n^{SO}(X) \to H_n(X)$$

sending an element $[M, f]$ to the image of the fundamental class $[M] \in H_n(M)$ under the induced map of f in homology. One has a corresponding map in the spin case, the Atiyah-Bott-Shapiro orientation (see [ABS64])

$$A\colon \Omega_n^{Spin}(X) \to ko_n(X)$$

where $ko_n(X)$ denotes the connective real K-theory of X. We note that the Atiyah-Bott-Shapiro orientation evaluated at a point coincides with the α-invariant. One sets $ko_n^+(X) = \operatorname{im} A|_{+\Omega_n^{Spin}(X)}$ and $H_n^+(X) = \operatorname{im} U|_{+\Omega_n^{SO}(X)}$. Theorem 4.11 in [RS01] states:

Homology Theorem. *Under the same assumptions as in the Bordism Theorem, the following hold:*

1. *If M admits a spin structure, then M admits a positive scalar curvature metric if and only if $A[M, f] \in ko_n^+(B\pi)$.*
2. *If M is orientable and totally non-spin, then M admits a positive scalar curvature metric if and only if $U[M, f] \in H_n^+(B\pi)$.*

The advantage of the Homology Theorem is that $ko_*(B\pi)$ and $H_*(B\pi)$ are much smaller than the corresponding bordism groups. For example, it is not difficult to see that lens spaces, discrete quotients of spheres, generate the homology of $B\mathbb{Z}_k$. Since curvature is a local property, lens spaces inherit positive scalar curvature metrics from the covering spheres, in dimensions greater than one. We may conclude that all orientable, totally non-spin manifolds of dimension greater or equal than five with fundamental group $\mathbb{Z}_k$ admit positive scalar curvature metrics.

As explained more precisely in the next section, a complete proof of the Homology Theorem has not appeared in the literature so far. The present thesis will provide missing details.

There is, once again, a refinement of the Homology Theorem. Rosenberg introduced a more general Dirac operator which takes account the fundamental group of the manifold. The corresponding index now takes values in $KO_*(C^*(\pi))$, the real K-theory of the C^*-algebra of π. It can be interpreted as the composition

$$\alpha_{\mathbb{R}} \colon \Omega_*^{Spin}(B\pi) \xrightarrow{A} ko_*(B\pi) \xrightarrow{per} KO_*(B\pi) \xrightarrow{\mathcal{A}} KO_*(C^*(\pi))$$

(see [Ros86]) where *per* denotes the periodization map from connective to periodic real K-theory and $\mathcal{A}$ the *assembly map* (see again [Ros86]). The *Gromov-Lawson-Rosenberg Conjecture* claims that M admits a positive scalar curvature metric if and only if $\alpha_{\mathbb{R}}[M, f] = 0$. Rosenberg proved that $\alpha_{\mathbb{R}}$ defines an obstruction against positive scalar curvature metrics. The vanishing of $\alpha_{\mathbb{R}}[M, f]$ is in general, however, not sufficient for the existence of a positive scalar curvature metric on M. In [Sch98] Schick gave a corresponding counterexample by using the *minimal hypersurface method* (see [SY79]), the only known tool (in dimensions greater or equal than five) to disprove the existence of positive scalar curvature metrics, when index-theoretic obstructions vanish. Nevertheless, the GLR Conjecture could be verified for several fundamental groups (for an overview, see e.g. [Sto95]). We note that the minimal hypersurface method applies only in cases where the fundamental groups are infinite. Therefore no counterexamples to the GLR Conjecture with finite fundamental groups are known.

The most far reaching general result about the existence of positive scalar curvature metrics on spin manifolds of dimension greater or equal than five is presented by Stolz in [Sto02, Theorem 3.10]. Under the assumption of the injectivity of the *Baum-Connes map* (see [BCH94]), Stolz sketched the proof of the *stable* Gromov-Lawson-Rosenberg Conjecture, which takes into account the periodic nature of the indices involved. Let B denote an arbitrary closed spin manifold of dimension eight such that $\alpha(B)$ is a generator of $ko_8 \cong \mathbb{Z}$ (such a B with trivial fundamental group is typically called *Bott-manifold*). One can show that the Bott periodicity isomorphism $KO_n(B\pi) \xrightarrow{\cong} KO_{n+8}(B\pi)$ is given by multiplication with B.

Stable GLR Conjecture. *Assume that the Baum-Connes map for π is injective. If $\alpha_{\mathbb{R}}[M,f]$ vanishes, then the product of M with sufficiently many copies of B admits a positive scalar curvature metric.*

Let $KO_n^+(B\pi)$ denote the image of $\oplus_{k\geq 0} ko_{n+8k}^+(B\pi)$ under the periodization map. By periodicity, we may consider $KO_n^+(B\pi)$ as a subgroup of $KO_n(B\pi)$. Stolz confirmed the stable conjecture by showing that the kernel of $\mathcal{A}$ lies in $KO_n^+(B\pi)$, under the assumption placed on π. We note that there exist elements in the kernel of the periodization map which are *not* representable by manifolds of positive scalar curvature (see [DSS03]).

The Gromov-Lawson(-Rosenberg) Conjecture for orientable, totally non-spin manifolds claims that all those manifolds of dimension greater or equal than five admit positive scalar curvature metrics. Again, this is wrong in general. By using the minimal hypersurface method, one can show, for example, that the orientable, totally non-spin manifolds $T^6 \sharp (S^2 \times \mathbb{C}\mathrm{P}^2)$ and $T^8 \sharp \mathbb{C}\mathrm{P}^4$ do not admit positive scalar curvature metrics. Let us note that a corresponding *stable* conjecture is trivially true because B is oriented bordant to a manifold of positive scalar curvature.

If one tries to verify the conjecture for a given group π, one must study how much of $H_*(B\pi)$ is exhausted by $H_*^+(B\pi)$. For example, if π is an elementary Abelian p-group for some prime p, one can show that $U[M,f]$ lies in $H_*^+(B\pi)$ provided that $U[M,f]$ is *atoral* (see [BR02, Theorem 5.8]). In Chapter 5 we will prove a more general version of this statement for p odd.

Further information about the positive scalar curvature question can be found in the (survey) papers [RS94, Sto95, RS01, Sto02, Ros07].

1.2 Overview and Main Results

This thesis is concerned with the existence question of positive scalar curvature metrics on closed and, more generally, *singular* manifolds. Our treatment of this question involves a *smooth variation* of Baas-Sullivan theory, which is potentially of independent interest and is presented in Chapter 2. Among other things, classical Baas-Sullivan theory provides a geometric description of singular homology by manifolds with singularities. A brief survey on this subject is given in Section 2.1.

Let $\mathscr{P}$ be a family of smooth closed manifolds. In Section 2.2 we introduce a homology theory $\mathcal{P}_*(\_)$ which we call the *bordism spanned by* $\mathscr{P}$. Let X be an arbitrary space. Elements in $\mathcal{P}_*(X)$ are represented by *smooth* closed manifolds, which are built up using the elements of $\mathscr{P}$, together with specified maps to X. These representatives are called *$\mathscr{P}$-manifolds in X*.

In Section 2.3 we then define a related theory $\Omega_*^{\mathscr{P}}(\_)$, called *singular $\mathscr{P}$-bordism*. An element in $\Omega_*^{\mathscr{P}}(X)$ is represented by a map from a *smooth* compact manifold to X whose restriction to the boundary is a $\mathscr{P}$-manifold in X. The label *singular $\mathscr{P}$-bordism* is suitable since $\Omega_*^{\mathscr{P}}(\_)$ is in fact a smooth version of the homology theory introduced by Baas in [Baa73], which goes by the name of bordism with *singularities*. We call the representatives of elements in $\Omega_n^{\mathscr{P}}(X)$ *singular $\mathscr{P}$-manifolds in X*.

Let $\Omega_*(\_)$ denote ordinary bordism theory. Elements of $\Omega_*(X)$ are represented by continuous maps from smooth closed manifolds to X. The relationship of these homology theories is best expressed by the so-called generalized Bockstein exact sequence

$$\cdots \to \mathcal{P}_n(X) \xrightarrow{\iota} \Omega_n(X) \xrightarrow{\pi} \Omega_n^{\mathscr{P}}(X) \xrightarrow{\partial} \mathcal{P}_{n-1}(X) \to \cdots \tag{1.2.1}$$

where ι associates to a $\mathscr{P}$-manifold its underlying smooth closed manifold, π interprets a closed as a singular $\mathscr{P}$-manifold (the defining condition is empty), and ∂ denotes the obvious boundary map.

In Section 2.4 we show that the map ι is injective on coefficients, provided that $\mathscr{P}$ is a *regular sequence*. As a consequence, we can determine the coefficients $\mathcal{P}_*$ and $\Omega_*^{\mathscr{P}}$ in our cases of interest.

We finally show in Section 2.5 that there exist families $\mathscr{P}^u$ resp. $\mathscr{P}^\alpha$ of oriented resp. spin manifolds such that, after inverting 2, the singular $\mathscr{P}^u$-resp. $\mathscr{P}^\alpha$-bordism can be identified with singular homology resp. connective real K-theory.

We note that our treatment of $\mathcal{P}_*(\_)$ and $\Omega_*^{\mathscr{P}}(\_)$ is self-contained and can be considered as an alternative approach to Baas-Sullivan theory.

Chapter 3 is concerned with general existence questions of positive scalar curvature metrics on closed manifolds. In Section 3.1 we recall consequences of the Surgery Lemma, these include the Bordism Theorem mentioned above. We give a proof of the Bordism Theorem, correcting a minor mistake in [RS95, Proof of Theorem 1.5].

It turns out that in order to deduce the Homology Theorem from the Bordism Theorem, one has to show that the elements in the kernels of the orientation maps

$$A\colon \Omega_n^{Spin}(B\pi) \to ko_n(B\pi) \quad \text{and} \quad U\colon \Omega_n^{SO}(B\pi) \to H_n(B\pi)$$

could be represented by manifolds of positive scalar curvature. This can be shown by proving it *localized* at 2, i.e. after tensoring the groups involved with $\mathbb{Z}_{(2)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ prime to } 2 \right\}$ and after *inverting* 2, i.e. after tensoring the groups involved with $\mathbb{Z}\left[\frac{1}{2}\right]$.

1. $\ker A \otimes \mathbb{Z}_{(2)} \subset {}^+\Omega_*^{Spin}(B\pi) \otimes \mathbb{Z}_{(2)}$ is proved by Stolz by means of splitting results of $MSpin$-module spectra (see [Sto94]) relying on computations in [Sto92]).
2. $\ker U \otimes \mathbb{Z}_{(2)} \subset {}^+\Omega_*^{SO}(B\pi) \otimes \mathbb{Z}_{(2)}$ can be deduced from the Atiyah-Hirzebruch spectral sequence (sketched in [RS01]).

After inverting 2 it is mentioned in [RS01] that there is a proof by Rainer Jung [Jun], for both the spin and the oriented case, based on Baas-Sullivan theory. To the best of our knowledge, experts in this field agree that Jung's proof is probably correct. However, this proof is not available to the public (and in fact unknown to us). Hence one cannot verify its details, it is unclear how much technical effort is needed and generalizations or modifications cannot be carried out. One matter of our thesis is to fill this gap in the literature.

We show that with 2 inverted the transformations A and U coincide with the map π in the Bockstein exact sequence (1.2.1) - under the identifications of singular $\mathscr{P}^\alpha$- resp. $\mathscr{P}^u$-bordism with connective real K-theory resp. singular homology, obtained at the end of Chapter 2. This implies that the elements in the kernel of A resp. U are representable by $\mathscr{P}^\alpha$- resp. $\mathscr{P}^u$-manifolds.

The crucial statement for our proof of the Homology Theorem, which is shown in Section 3.2, is

Theorem 1.2.1. *Let $\mathscr{P}$ denote a locally finite family of closed manifolds admitting positive scalar curvature metrics. Then a $\mathscr{P}$-manifold, considered as a smooth manifold with additional structure, also admits a positive scalar curvature metric.*

It turns out that the families $\mathscr{P}^u$ and $\mathscr{P}^\alpha$ can be chosen to consist of manifolds admitting positive scalar curvature metrics. This completes the proof of the Homology Theorem.

In Chapter 4 we address the question of positive scalar curvature on simply connected singular $\mathscr{P}^u$- and $\mathscr{P}^\alpha$-manifolds. Our aim is to show under which conditions the *particular* positive scalar curvature metrics on the boundaries of these manifolds, constructed in the proof of Theorem 1.2.1, extend to positive scalar curvature metrics on the whole manifolds. *In this case* we speak of positive scalar curvature metrics on singular $\mathscr{P}^u$- and $\mathscr{P}^\alpha$-manifolds.

In Section 4.1 we show that after having fixed positive scalar curvature metrics on the elements of some family of closed manifolds $\mathscr{P}$, the aforementioned particular positive scalar curvature metrics on $\mathscr{P}$-manifolds are canonical with respect to the bordism relation in $\mathcal{P}_*(\_)$.

Afterwards, in Section 4.2, we are able to prove an analogue of Gromov and Lawson' Theorem on simply connected non-spin manifolds (see Theorem 1.1.2 above) for singular $\mathscr{P}^u$-manifolds.

Theorem 1.2.2. *Let M be a simply connected singular $\mathscr{P}^u$-manifold of dimension greater or equal than five which does not admit a spin structure. Then M admits a positive scalar curvature metric.*

In Section 4.3 we turn to the spin case. There is an obvious index theoretic obstruction
$$\alpha^{\mathscr{P}^\alpha} : \Omega_*^{\mathscr{P}^\alpha} \to ko_*$$
against positive scalar curvature metrics on singular $\mathscr{P}^\alpha$-manifolds. Although we could not show (or refute) that the kernel of $\alpha^{\mathscr{P}^\alpha}$ is generated by singular $\mathscr{P}^\alpha$-manifolds of positive scalar curvature (the analogue for closed manifolds is true, see Theorem 1.1.2 above), we do have the following, more restrictive statement which treats the case of a single singularity.

Theorem 1.2.3. *Let $R \in \mathscr{P}^\alpha$ and let M denote a simply connected singular R-manifold of dimension greater or equal than five. Then M admits a positive scalar curvature metric if and only if $\alpha^R(M)$ vanishes.*

The question of positive scalar curvature on singular spin manifolds is also treated in [Bot01], from a slightly different point of view. We show that under certain assumptions a sufficient obstruction to the existence of positive scalar curvature metrics, defined loc. cit., always vanishes.

Chapter 5 is based on joint work with Bernhard Hanke. We consider the question of positive scalar curvature on non-simply connected singular manifolds. In Section 5.1 we introduce our notion of a positive homology class. Singular $\mathscr{P}^u$-manifolds represent elements in $H_*(X)$. A homology class will be called *positive* if it is representable by a singular $\mathscr{P}^u$-manifold of positive scalar curvature, and the subgroup of positive homology classes is denoted by $H_*^{\oplus}(X)$. That is, we enlarge the groups $H_*^+(X)$ by adding classes which are not representable by closed manifolds but by singular $\mathscr{P}^u$-manifolds of positive scalar curvature. The benefit of our broader definition of positivity becomes apparent by our success in proving the following theorem.

Let p denote an odd prime. For an elementary Abelian p-group of rank $r \geq 1$, i.e. $\mathbb{Z}_p^r = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, r factors, we show in Section 5.2

Theorem 1.2.4. *The* atoral classes *of $H_*(B\mathbb{Z}_p^r)$ are positive, i.e.*

$$H_*^{atoral}(B\mathbb{Z}_p^r) \subset H_*^{\oplus}(B\mathbb{Z}_p^r).$$

Here atoral classes are defined as a complement to the elements of $H_*(B\mathbb{Z}_p^r)$ which are represented by sums of tori (see Definition 5.1.4).

To indicate the advantage of our notion of positivity, we note that the proof of Theorem 1.2.4 uses the Künneth formula for singular homology. If one tries to prove a similar statement involving $H_*^+(B\mathbb{Z}_p^r)$, one will be faced with the more complicated Künneth formula for oriented bordism (see [BR02, Theorem 5.8]).

Our notion of positive homology enables us to draw conclusions for the question of positive scalar curvature on closed manifolds. It turns out that in the Homology Theorem, $H_*^+(B\pi)$ can be replaced by $H_*^{\oplus}(B\pi)$. We therefore obtain the following

Corollary 1.2.5. *Let M be a closed connected orientable totally non-spin manifold of dimension greater or equal than five with fundamental group $\mathbb{Z}_p^r$, and let $f\colon M \to B\mathbb{Z}_p^r$ be the classifying map of the universal cover of M.*

Then M admits a positive scalar curvature metric if $f_*[M] \in H_*^{atoral}(B\mathbb{Z}_p^r)$.

This was proved before by Botvinnik-Rosenberg in [BR02, Theorem 5.8].

We believe that our notion of positive homology, and its analogue for the spin case, positive connective real K-theory, are suitable to prove further existence theorems concerning positive scalar curvature metrics.

The proof of the Homology Theorem (with 2 inverted), which is carried out in Chapter 2 and Section 3.2, has been accepted for publication in *Mathematische Zeitschrift* (see [Füh]).

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Chapter 2

A Smooth Variation of Baas-Sullivan Theory

2.1 Introduction

Baas-Sullivan theory is located at the intersection of geometric and algebraic topology. In the latter one associates algebraic objects to topological spaces, and these objects should be invariant under deformations of the spaces involved. Perhaps the most basic invariant is the singular homology $H_*(X)$ of a space X. In case X is a CW-complex, it is quite easy to determine. Namely, using cellular homology, it turns out that one merely has to count the cells of X according to their dimensions and compute the mapping degrees of the attaching maps. In spite of their simplicity, singular homology is a powerful tool in, most notably, homotopy theoretic problems, for example if one wants to determine whether two spaces are homotopy equivalent or not.

An algebraic invariant with a more geometric flavor is the oriented bordism $\Omega_*^{SO}(X)$. An element of $\Omega_*^{SO}(X)$ consists of an equivalence class of pairs $[M, f]$ where M is a smooth closed oriented manifold and $f\colon M \to X$ is a continuous map. The pair (M, f) is called an *oriented manifold in* X[1]. Let (M, f) and (N, g) denote two oriented manifolds in X. They are called equivalent or *bordant* if there exists a compact oriented manifold W, an ori-

[1]We note that in the present paper the term *singular manifold in* X is reserved for objects introduced in Section 2.3.

entation preserving diffeomorphism $\phi\colon M \dot{\cup} (-N) \to \partial W$ and a continuous map $h\colon W \to X$ such that $h \circ \phi = f \cup g$. Connor (see [Con79, p. 11-13]) showed that oriented bordism shares formal properties of singular homology, i.e. he verified the so-called Eilenberg-Steenrod axioms. In contrast to singular homology, bordism does not satisfy the *dimension axiom*, which makes bordism groups more difficult to compute than singular homology groups. As mentioned in the introduction, bordism groups apply in Riemannian geometry. They play an important role in the theory of positive scalar curvature metrics.

How can one relate $\Omega_n^{SO}(X)$ to $H_n(X)$? The first step is to construct a *fundamental class* $[M] \in H_n(M)$. For this take an arbitrary triangulation of M. Then the sum of the coherent oriented simplices defines the desired class. In this way one obtains the orientation map

$$U\colon \Omega_n^{SO}(X) \to H_n(X), \quad [M, f] \mapsto f_*[M].$$

Much work has been devoted to the study of U. An overly naive desire would be to think that U is an isomorphism. Namely, the singular homology of a point is concentrated in the zeroth degree, but there exist in general nontrivial elements in Ω_n^{SO} for $n > 0$; there are manifolds that are not the boundary of another manifold, for example the complex projective space $\mathbb{C}\mathrm{P}^2$. More surprisingly, it turns out that U is not even surjective. According to [Tho54] (see e.g. also [Rud98, Theorem 7.35]), there exists an element in $H_7(K(\mathbb{Z}_3, 7))$ which does not lie in the image of U, here $K(\mathbb{Z}_3, 7)$ is an Eilenberg-MacLane space.

In his work on the Hauptvermutung (see [Sul67], [Sul96]), Sullivan introduced objects, called polyhedra, in order to represent homology classes by such polyhedra. Let P denote a closed oriented manifold. A *polyhedron of singularity type* P is a space of the form

$$M \cup_{\partial M} (CP \times B)$$

where B is a closed oriented manifold, M is a compact oriented manifold with boundary $\partial M = -P \times B$, and CP is the cone over P. Sullivan suggested to consider a homology theory where polyhedra should play the role of closed manifolds.

Baas successfully accomplished this idea in [Baa73]. Let $S = (P_1, P_2, \ldots)$ denote a sequence of closed manifolds. Baas introduced a bordism theory of manifolds with *singularity type* S, denoted by $MG(S)_*(\_)$, where G refers to some kind of tangential structure. Baas proved that $MG(S)_*(\_)$ defines a homology theory, i.e. he verified that $MG(S)_*(\_)$ satisfies the Eilenberg-Steenrod axioms. In the case of unitary bordism, and S a sequence of generators of Ω_*^U, he computed $MU(S)_0 = \mathbb{Z}$ and $MU(S)_n = 0$ for $n \neq 0$. One can show (see e.g. [Hat02, Theorem 4.59]) that if the coefficients of a homology theory are concentrated in the zeroth degree, then the theory is uniquely determined by them. This implies that $MU(S)_*(\_)$ coincides with singular homology, with integer coefficients.

Apart from a geometric description of singular homology, Baas-Sullivan theory, and hence its smooth variation carried out below, allows interpretations and constructions of various more theories, like Johnson-Wilson theories (see [JW73]), Morava K-theories (see [JW75]) or elliptic homology (see [Tho99]). Baas-Sullivan theory also provides a geometric description of connective real K-theory (after inverting two), which we will obtain as a byproduct in Section 2.5.

Another characterization of the homology theory $MG(S)_*(\_)$ can be obtained by removing neighborhoods of the singularities. This description is explained by Botvinnik (see [Bot92, Chapter 1]). Elements in the resulting homology theory are now represented by manifolds with corners, which carry specified structures on their boundaries. Let us call this theory $MG_*^S(\_)$. In the next section but one we will give a smooth variation of $MG_*^S(\_)$. Elements in our theory will be represented by smooth manifolds right away.

Also in [Bot92, Chapter 1] the homology theory which is related to $MG_*^S(\_)$ via the generalized Bockstein exact sequence (1.2.1) is introduced and discussed. Our smooth description of this related theory is denoted by $\mathcal{P}_*(\_)$. We find it convenient to begin our constructions with $\mathcal{P}_*(\_)$.

2.2 The Bordism Spanned by $\mathscr{P}$

We start with some preliminary remarks. Recall that in bordism theory one typically considers manifolds equipped with some kind of tangential structure G. By a tangential structure G one understands a sequence of compatible fibrations over the classifying spaces $BO(n)$. Such fibrations are usually induced by a sequence of compatible group homomorphisms $G(n) \to O(n)$. For example, in the next chapters we are interested in ordinary orientation, $G(n) = SO(n)$ and spin structures, $G(n) = Spin(n)$. Until the end of Section 2.4 we fix a tangential structure G, and we assume that every manifold is equipped with G which means that a sequence of classifying maps of normal bundles, being induced by an embedding of the manifold in Euclidean space, can be lifted compatibly to the $BG(n)$'s (see e.g. [Swi02, Chapter 12]). In addition, we assume that all diffeomorphisms between manifolds preserve these G-structures.

Let $\mathbb{H}_i^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$ denote the i-th upper half space. As usual, a smooth n-dimensional manifold M with boundary is modeled on $\mathbb{H}_n^n$.

We call subsets $N_1, \dots, N_k$ of M a *transverse family of submanifolds* if for all $1 \leq i_1 < \cdots < i_l \leq k$ around every point in $N_{i_1} \cap \cdots \cap N_{i_l}$ there exists a chart $\psi\colon U \to \mathbb{H}_n^n$ of M and an injective map $s\colon \{1, \dots, l\} \to \{1, \dots, n-1\}$ such that

$$\psi(U \cap N_{i_j}) = \psi(U) \cap \mathbb{H}_{s(j)}^n$$

simultaneously for all $1 \leq j \leq l$.

Let M and N denote smooth manifolds and let $A \subset M$ be a subset. A map $f\colon A \to N$ is called *smooth* if f is the restriction of a smooth map $M \to N$.

Now let $\mathscr{P} = (P_1, P_2, \dots, P_k)$ be a finite family of smooth closed manifolds. For $I \subset \{1, \dots, k\}$ we set $P_I = \prod_{i \in I} P_i$.

Definition 2.2.1. An n-dimensional *$\mathscr{P}$-manifold* consists of a tuple $(M, (A_i)_{1 \leq i \leq k}, (B_I, \phi_I)_{I \subset \{1, \dots, k\}})$ such that

- M is a smooth n-dimensional manifold,
- $A_1, \dots, A_k$ is a transverse family of smooth n-dimensional submanifolds, closed as subsets, such that the interiors of A_i cover M,
- for all $I \subset \{1, \dots, k\}$, B_I is a subset of some smooth manifold C_I and ϕ_I is a map $A_I = \cap_{i \in I} A_i \to P_I \times C_I$ which is a diffeomorphism onto $P_I \times B_I$,
- for all $J \subset I$ the map

  $$\phi_J \circ \phi_I^{-1}\colon P_J \times P_{I-J} \times B_I = P_I \times B_I \to P_J \times B_J$$

  is of the form $(x, y) \mapsto (x, \phi_J^I(y))$ where $x \in P_J$, $y \in P_{I-J} \times B_I$ and $\phi_J^I\colon P_{I-J} \times B_I \hookrightarrow B_J$ is some map.

We assume that $\mathbb{H}^n_n$ always denotes the model space of the ambient manifold M. Let us call $A_i \subset M$ the *P_i-part* of a $\mathscr{P}$-manifold M. If all B_j are empty except B_i, we call M a *P_i-manifold*.

Definition 2.2.2. Let X be a space and $A \subset X$. An n-dimensional *$\mathscr{P}$-manifold in* (X, A) is a tuple $(M, f, (A_i)_{1\le i\le k}, (B_I, \phi_I)_{I\subset\{1,\dots,k\}}, (f_i)_{1\le i\le k})$ such that

- the tupel $(M, (A_i)_{1\le i\le k}, (B_I, \phi_I)_{I\subset\{1,\dots,k\}})$ is a compact n-dimensional $\mathscr{P}$-manifold,
- $f\colon (M, \partial M) \to (X, A)$ and $f_i\colon B_i \to X$ are continuous maps such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "f"] \arrow[d, "\phi_i"] & X \\ P_i \times B_i \arrow[r, "pr_{B_i}"] & B_i \arrow[u, "f_i"'] \end{tikzcd}$$

  commutes for all i.

In the sequel we fix a family $\mathscr{P} = (P_1, P_2, \dots, P_k)$ of smooth closed manifolds and write $(M, f, A_i, B_I, \phi_I, f_i)$, (M, f, A_i) or (M, f) short for a $\mathscr{P}$-manifold in (X, A).

Definition 2.2.3. An n-dimensional $\mathscr{P}$-manifold $(M, f, A_i, B_I, \phi_I, f_i)$ in (X, A) is said to *$\mathscr{P}$-bord* if there exists a tuple $(\hat{M}, \hat{f}, \hat{A}_i, \hat{B}_I, \hat{\phi}_I, \hat{f}_i)$ such that

- $(\hat{M}, \hat{A}_i, \hat{B}_I, \hat{\phi}_I)$ is a smooth compact $(n+1)$-dimensional $\mathscr{P}$-manifold,
- $M \subset \partial\hat{M}$ and $\hat{A}_i \cap M = A_i$ for all i; moreover, each

$$\hat{\phi}_i \circ \phi_i^{-1}\colon P_i \times B_i \to P_i \times \hat{B}_i$$

  is of the form $(x, y) \mapsto (x, \omega_i(y))$ for some map ω_i,
- $\hat{f}\colon \hat{M} \to X$ and $\hat{f}_i\colon \hat{B}_i \to X$ are continuous maps such that $\hat{f}(\partial\hat{M} - M) \subset A$, $\hat{f}|_M = f$ and the diagram

$$\begin{tikzcd} \hat{A}_i \arrow[r, "\hat{f}"] \arrow[d, "\hat{\phi}_i"] & X \\ P_i \times \hat{B}_i \arrow[r, "pr_{\hat{B}_i}"] & \hat{B}_i \arrow[u, "\hat{f}_i"'] \end{tikzcd}$$

  commutes for all i.

We call $(\hat{M}, \hat{f}, \hat{A}_i, \hat{B}_I, \hat{\phi}_I, \hat{f}_i)$, $(\hat{M}, \hat{f}, \hat{A}_i)$ or $(\hat{M}, \hat{f})$ a *zero $\mathscr{P}$-bordism* for (M, f).

One then proceeds as in ordinary bordism homology. The disjoint union of two $\mathscr{P}$-manifolds in (X, A) is again a $\mathscr{P}$-manifold in (X, A). We say that two $\mathscr{P}$-manifolds (M, f) and (N, g) in (X, A) are *$\mathscr{P}$-bordant* if $(M, f) \dot{\cup} (-N, g)$ $\mathscr{P}$-bords.

Lemma 2.2.4. *$\mathscr{P}$-bordism defines an equivalence relation.*

Proof. Let $(M, f, A_i, B_I, \phi_I, f_i)$ be a $\mathscr{P}$-manifold in (X, A). For the proof of reflexivity we consider

$$(M \times [0,1], f \circ pr_M, A_i \times [0,1], B_I \times [0,1], \phi_I \times \mathrm{id}, f_i \circ pr_{B_i}). \qquad (2.2.1)$$

By 'straightening the angle' [Con79, p. 8], $M \times [0,1]$ can be given a differentiable structure. By doing so, $A_1 \times [0,1], \ldots, A_k \times [0,1]$ becomes a transverse family of submanifolds. There is an induced straightening of B_I for all I, and thus (2.2.1) becomes a $\mathscr{P}$-bordism between (M, f) and itself.

The symmetry relation is obvious.

To prove transitivity, let (M, f) and (N, g) resp. (N, g) and (O, h) be $\mathscr{P}$-bordant n-dimensional $\mathscr{P}$-manifolds in (X, A), say via $(V, F, A_i, B_I, \phi_I, f_i)$ resp. $(W, G, C_i, D_I, \psi_I, g_i)$. Because of transversality, one finds charts of V around $A_I \cap \partial V$ and of W around $C_I \cap \partial W$ in which the respective inner boundaries $\overline{\partial A_i - \partial V}$ and $\overline{\partial C_i - \partial W}$ of the P_i-parts lie on a common $\partial \mathbb{H}^{n+1}_j$, for some $j \leq n$ depending on i. Hence, for all i we can glue A_i and C_i along $(A_i \cap \partial V)|_N \cong (C_i \cap \partial W)|_N$ such that $A_1 \cup C_1, \ldots, A_k \cup C_k$ becomes a family of smooth manifolds. The definitions of the smooth structures involve choices of collar neighborhoods of $A_i \cap \partial V$ and $C_i \cap \partial W$. Since collar neighborhoods are unique up to isotopy, the smooth structures on the overlaps of $A_i \cup C_i$ and $A_j \cup C_j$ coincide, for all i, j. We therefore obtain an induced smooth structure on $V \cup W$, and $A_1 \cup C_1, \ldots, A_k \cup C_k$ is a transverse family of submanifolds. Let the P_i-part of N be diffeomorphic to $P_i \times E_i$. By means of point two of Definition 2.2.3, one recovers E_i as a submanifold of B_i and D_i. Thus, for all i we can also glue B_i and D_i along E_i. One obtains an induced gluing of B_I and D_I for all I. Now the desired $\mathscr{P}$-bordism between (M, f) and (O, h) is given by

$$(V \cup W, F \cup G, A_i \cup C_i, B_I \cup D_I, \phi_I \cup \psi_I, f_i \cup g_i).$$

□

Denote by $\mathcal{P}_n(X,A)$ the set of all $\mathscr{P}$-bordism classes of n-dimensional $\mathscr{P}$-manifolds in (X,A). Via the disjoint union it becomes an Abelian group with zero element the $\mathscr{P}$-bordism class which $\mathscr{P}$-bords. A map $g\colon (X,A) \to (Y,B)$ induces a group homomorphism $\mathcal{P}_n(X,A) \to \mathcal{P}_n(Y,B)$ by $[M,f] \mapsto [M, g\circ f]$. If (M,f) is a $\mathscr{P}$-manifold in (X,A), then the boundary of M becomes a $\mathscr{P}$-manifold in A by restriction. It is denoted by $\partial(M,f)$. We then have an induced map $\partial\colon \mathcal{P}_n(X,A) \to \mathcal{P}_{n-1}(A)$ defined by $[M,f] \mapsto [\partial(M,f)]$.

Proposition 2.2.5. *The* bordism spanned by $\mathscr{P}$

$$\mathcal{P}_*(X,A) = \bigoplus_{n\geq 0} \mathcal{P}_n(X,A)$$

is a homology theory.

Proof. We have to show that $\mathcal{P}_*(X,A)$ satisfies the Eilenberg-Steenrod axioms. One proceeds in the same way as in the case of ordinary bordism homology (see [Con79, p. 11-13]) and additionally takes the local product structures into account. As expected, the proof of the excision property requires the most work.

Obviously, $\mathcal{P}_*(\_)$ is a functor from the category of pairs of topological spaces (with continuous maps as morphisms) to the category of Abelian groups. It remains to show:

Homotopy axiom. Let g and $h\colon (X,A) \to (Y,B)$ be homotopic maps, and let $H\colon (X,A)\times[0,1] \to (Y,B)$ be a homotopy between g and h. Then $g_* = h_*\colon \mathcal{P}_n(X,A) \to \mathcal{P}_n(Y,B)$.

Let $(M, f, A_i, B_I, \phi_I, f_i)$ be a $\mathscr{P}$-manifold in (X,A). We define

$$G\colon M\times[0,1] \to Y, (x,t) \mapsto H(f(x),t).$$

By straightening the angle, $M\times[0,1]$ can be equipped with the structure of a $\mathscr{P}$-manifold. Then

$$(M\times[0,1], G, A_i\times[0,1], B_I\times[0,1], \phi_I\times \mathrm{id}, H\circ(f_i\times\mathrm{id}))$$

becomes a $\mathscr{P}$-bordism between $(M, g\circ f)$ and $(M, h\circ f)$.

Exactness axiom. Let $i\colon A \hookrightarrow X$ and $j\colon (X,\emptyset) \hookrightarrow (X,A)$ denote the inclusions, then the sequence

$$\cdots \xrightarrow{\partial} \mathcal{P}_n(A) \xrightarrow{i_*} \mathcal{P}_n(X) \xrightarrow{j_*} \mathcal{P}_n(X,A) \xrightarrow{\partial} \mathcal{P}_{n-1}(A) \xrightarrow{i_*} \cdots$$

is exact.

It is clear that $\partial \circ j_* = 0$ and $i_* \circ \partial = 0$. Let $[M, f] \in \mathcal{P}_n(A)$. A zero $\mathscr{P}$-bordism for $(M, j \circ i \circ f)$ is given by $(M \times [0,1], j \circ i \circ f \circ pr_M)$, hence $j_* \circ i_* = 0$.

Let $[M, f, A_i, B_I, \phi_I, f_i] \in \mathcal{P}_n(X, A)$ be in the kernel of ∂. Then $\partial(M, f)$ bords in A, i.e. there exists a zero $\mathscr{P}$-bordism $(W, g, C_i, D_I, \psi_I, g_i)$ for $\partial(M, F)$ in A. As in the proof of transitivity in Lemma 2.2.4, we can glue A_i and C_i along $A_i \cap \partial M \cong C_i \cap \partial W$, for all i, to obtain a closed $\mathscr{P}$-manifold N and a map $(f \cup g)\colon N \to X$. Now $(N \times [0,1], (f \cup g) \circ pr_N)$ is a $\mathscr{P}$-bordism between $(N, j \circ (f \cup g))$ and (M, f) in (X, A). The cases $\ker j_* \subset \operatorname{im} i_*$ and $\ker i_* \subset \operatorname{im} \partial$ are obvious.

Excision axiom. Let U be an open subset of X such that $\overline{U} \subset \mathring{A}$, then the inclusion $i\colon (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

$$i_*\colon \mathcal{P}_n(X - U, A - U) \xrightarrow{\cong} \mathcal{P}_n(X, A).$$

We first show that i_* is epic: Let $(M, f, A_i, B_I, \phi_I, f_i)$ be a $\mathscr{P}$-manifold in (X, A). We are looking for a smooth submanifold $N \subset M$ such that $f^{-1}(X - A) \subset N$ and $f^{-1}(U) \cap N = \emptyset$. In addition, N should respect the local product structures in the sense that $\phi_I(A_I \cap N) = P_I \times C_I$ for some $C_I \subset B_I$. It then follows that N inherits a $\mathscr{P}$-structure of M by restricting the A_i's and the ϕ_I's to N. Now $(N, F|_N)$ defines an element in $\mathcal{P}_n(X - U, A - U)$, and

$$(M \times [0,1], f \circ pr_M, A_i \times [0,1], B_I \times [0,1], \phi_I \times \mathrm{id}, f_i \circ pr_{B_i})$$

is a $\mathscr{P}$-bordism between $(N, i \circ f|_N)$ and (M, f) in (X, A).

The construction of N requires a preliminary observation. Until the end of this proof, we shall denote the 'inner' boundary $\overline{\partial A_i - \partial M}$ by δA_i, likewise for comparable sets. If $\delta A_i \times [0,1]$ is a collar neighborhood for $\delta A_i \subset A_i$, say $\delta A_i \times \{0\} = \delta A_i$, we set

$$A_i^t = A_i - (\delta A_i \times [0, t))$$

for all $0 \le t \le 1$ (see Figure 2.1). One now observes that for all i there exist collar neighborhoods $\delta A_i \times [0,1]$ such that for any two arbitrary sequences of numbers $0 \le t_1, \ldots, t_k \le 1$ and $0 \le t'_1, \ldots, t'_k \le 1$

$$\left(M, f, A_i^{t_i}\right) \text{ and } \left(M, f, A_i^{t'_i}\right)$$

become $\mathscr{P}$-bordant $\mathscr{P}$-manifolds in (X, A), by suitable restrictions.

Let us turn to the construction of N. We set $Q = f^{-1}(X - \mathring{A})$ and $R = f_I^{-1}(\overline{U})$. One can show (see [Con79, Lemma 3.1]) that there exists an n-dimensional submanifold $N_0 \subset M$, closed as a subset, such that $Q \subset N_0$ and $R \cap N_0 = \emptyset$. Of course, N_0 does not have to respect the local product structures. Therefore we shall modify N_0 as follows. Set

$$B_i' = pr_{B_i}\left(\phi_i\left(N_0 \cap A_i\right)\right)$$

and consider the *saturation of the* P_i*-fibers*

$$N_1 = \bigcup_{i=1}^{k}\left(\bigcup_{b \in B_i'} \phi_i^{-1}\left(P_i \times \{b\}\right)\right).$$

Due to the condition $\cap_{i \in I} A_i \cong P_I \times B_I$, the set N_1 respects the local product structures. In addition, as f locally factors over B_I, one concludes $f(N_0) = f(N_1)$. Now N_1 is the union of manifolds modeled on $\mathbb{H}_1 \cap \mathbb{H}_2$. The non-smooth points of N_1 only occur on $C_1 := \cup_{i=1}^{k} \delta A_i$. With respect to the metric induced by the collar neighborhoods, let U_1 be an open $1/k$-neighborhood of C_1. One finds a smooth submanifold $N_1' \subset M$ such that $N_1' - U_1 = N_1 - U_1$. In view of continuity, N_1' can be chosen such that furthermore $Q \subset N_1'$ and $R \cap N_1' = \emptyset$. Note that N_1' respects the local product structures except on U_1.

We now replace A_i by $A_i^{1/k}$ for all i and repeat the above procedure. The saturations of the P_i-fibers with respect to the $A_i^{1/k}$'s yield an $N_2 \subset M$. One merely has to perform this saturation step on U_1. Again, non-smooth points only appear on $\cup_{i=1}^{k} \delta A_i^{1/k}$. Hence the non-smooth points of N_2 occur on an open $1/k$-neighborhood U_2 of

$$\begin{aligned} C_2 :&= C_1 \cap \left(\bigcup_{i=1}^{k} \delta A_i^{1/k}\right) \\ &= \bigcup_{1 \le i,j \le k} \left(\delta A_i \cap \delta A_j^{1/k}\right). \end{aligned}$$

In a similar way as above, we find a smooth submanifold $N_2' \subset M$ which satisfies $N_2' - U_2 = N_2 - U_2$ and $Q \subset N_2'$, $R \cap N_2' = \emptyset$.

Proceeding in this fashion $k-2$ more times, in each step replacing $A_i^{t/k}$ by $A_i^{(t+1)/k}$, we obtain a smooth submanifold $N_k' \subset M$ which satisfies the local

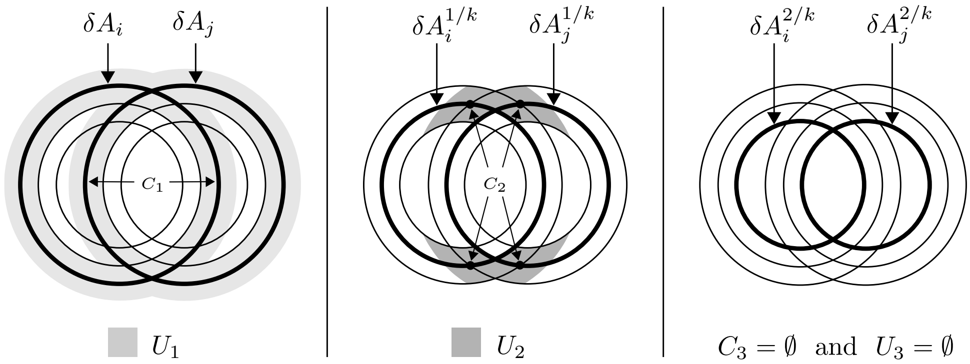

Figure 2.1: Saturation and missing product structures after smoothing

product structures except on an open $1/k$-neighborhood U_k of

$$C_k := \bigcup_{1\le i_1,\dots,i_k\le k} \left(\delta A_{i_1} \cap \delta A_{i_2}^{1/k} \cap \cdots \cap \delta A_{i_k}^{(k-1)/k}\right).$$

But now C_k is empty. In fact, on the one hand, for $1 \le i_1,\dots,i_k \le k$ pairwise disjoint, the interiors of A_{i_1}, $A_{i_2}^{1/k},\dots,A_{i_k}^{(k-1)/k}$ cover M. This implies that, for example δA_{i_1} lies in the interior of $A_{i_2}^{1/k} \cup \cdots \cup A_{i_k}^{(k-1)/k}$. On the other hand, if an index i_j occurs twice, one has $\delta A_{i_j}^{s/k} \cap \delta A_{i_j}^{t/k} = \emptyset$ because then necessarily $s \neq t$. One concludes that $C_k = \emptyset$, and thus we may take N'_k to be N (see Figure 2.1).

Similarly one sees that i_* is monic: Let (M, f) be an n-dimensional $\mathscr{P}$-manifold in $(X - U, A - U)$ and $i_*[M, f] = 0$. Then there exists a zero $\mathscr{P}$-bordism (W, g) for $(M, i \circ f)$ in (X, A). As above we can find an $(n+1)$-dimensional submanifold N which respects the local product structures, and for which we have $g(N) \cap U = \emptyset$ and $g^{-1}(X - \mathring{A}) \subset N$. It follows that $(N, g|_N)$ is a zero $\mathscr{P}$-bordism for (M, f) in $(X - U, A - U)$. $\square$

2.3 Singular $\mathscr{P}$-Bordism

Once and for all we assume that P_0 denotes a point. Let $\mathscr{P} = (P_1, \dots, P_k)$ be a finite family of smooth closed manifolds and $\mathscr{P}_0 = \mathscr{P} \cup \{P_0\}$.

Definition 2.3.1. An n-dimensional singular *$\mathscr{P}$-manifold* is a smooth manifold M such that ∂M is an $(n-1)$-dimensional $\mathscr{P}_0$-manifold. A singular $\mathscr{P}$-manifold M is called *closed* if the P_0-subset of ∂M is empty.

In the sequel let M be a singular $\mathscr{P}$-manifold and

$$\partial M = \left(\partial M, (A_i)_{0 \le i \le k}, (B_I, \phi_I)_{I \subset \{0,\dots,k\}}\right).$$

Definition 2.3.2. Let X be a space and $A \subset X$. An n-dimensional singular *$\mathscr{P}$-manifold in* (X, A) is a pair (M, f) where M is a singular $\mathscr{P}$-manifold and $f\colon (M, A_0) \to (X, A)$ is a continuous map such that $f|_{\partial M}$ is a $\mathscr{P}_0$-manifold in X.

Definition 2.3.3. An n-dimensional singular $\mathscr{P}$-manifold (M, f) in (X, A) is said to *singular $\mathscr{P}$-bord* if

- there exists a zero $\mathscr{P}_0$-bordism $(N, g, (C_i)_{0 \le i \le k})$ for $(\partial M, f|_{\partial M})$ in X such that $g(C_0) \subset A$,
- there exists an ordinary zero bordism (W, h) for the closed manifold $(M, f) \cup_{\partial M = \partial N} (-N, g)$ in X.

Such a *zero singular $\mathscr{P}$-bordism* is denoted by (N, g, W, h).

The disjoint union of two singular $\mathscr{P}$-manifolds in (X, A) is again a singular $\mathscr{P}$-manifold. We say that two singular $\mathscr{P}$-manifolds (M, f) and (N, g) in (X, A) are singular $\mathscr{P}$-bordant if $(M, f) \dot\cup (-N, g)$ singular $\mathscr{P}$-bords.

Lemma 2.3.4. *Singular $\mathscr{P}$-bordism defines an equivalence relation.*

Proof. To prove reflexivity, let (M, f) be a singular $\mathscr{P}$-manifold in (X, A). As in the proof of Lemma 2.2.4, $(\partial M \times [0,1], f|_{\partial M} \circ pr_{\partial M})$ can be considered as a $\mathscr{P}_0$-bordism between $(\partial M, f|_{\partial M})$ and itself. Then, after straightening the angle, $(\partial M \times [0,1], f|_{\partial M} \circ pr_{\partial M}, M \times [0,1], f \circ pr_M)$ is a singular $\mathscr{P}$-bordism between (M, f) and itself.

The symmetry relation is obvious.

To prove transitivity, let (M, f) and (N, g) resp. (N, g) and (O, h) be singular $\mathscr{P}$-bordant singular $\mathscr{P}$-manifolds in (X, A), say via (V, v, W, w) resp. (X, x, Y, y). Again, as in the proof of Lemma 2.2.4, we can glue the singular $\mathscr{P}_0$-bordisms (V, v) and (X, x) to obtain a $\mathscr{P}_0$-bordism $(V \cup X, v \cup x)$

between $(\partial M, f|_{\partial M})$ and $(\partial O, h|_{\partial O})$. Furthermore, gluing (W, w) and (Y, y) along (N, g) yields, after straightening the angle, a zero bordism for

$$(M, f) \cup (V \cup X, v \cup x) \cup (O, h).$$

□

We denote by $\Omega_n^{\mathscr{P}}(X, A)$ the set of all singular $\mathscr{P}$-bordism classes of n-dimensional singular $\mathscr{P}$-manifolds in (X, A). Via the disjoint union it becomes an Abelian group with zero element the singular $\mathscr{P}$-bordism class which singular $\mathscr{P}$-bords. A map $g\colon (X, A) \to (Y, B)$ induces a group homomorphism $\Omega_n^{\mathscr{P}}(X, A) \to \Omega_n^{\mathscr{P}}(Y, B)$ by $[M, f] \mapsto [M, g \circ f]$.

In order to define a boundary map $\Omega_n^{\mathscr{P}}(X, A) \to \Omega_{n-1}^{\mathscr{P}}(A, \emptyset)$, we first prove the following

Lemma 2.3.5. *Consider a $\mathscr{P}_0$-manifold*

$$\left(N, (A_j)_{0 \leq j \leq k}, (B_I, \phi_I)_{I \subset \{0, \dots, k\}}\right),$$

and let $0 \leq i \leq k$ and $\mathscr{R} = \mathscr{P}_0 - \{P_i\}$.

Then ∂B_i inherits the structure of an $\mathscr{R}$-manifold.

Proof. Let $I \subset \{1, \dots, \hat{\imath}, \dots, k\}$. There are manifolds $C_I \subset B_{\{i\} \cup I}$ such that

$$\phi_{\{i\} \cup I}(\partial A_i \cap A_I) = P_i \times P_I \times C_I.$$

We then have an inclusion $\phi_i^{\{i\} \cup I}\colon P_I \times C_I \hookrightarrow \partial B_i$ (see point four of Definition 2.2.1), and we observe that

$$\bigcup_{j \neq i} \phi_i^{\{i,j\}}(P_j \times C_j) = \partial B_i.$$

The induced $\mathscr{R}$-structure on ∂B_i is now defined by setting

$$\left(\partial B_i, \left(\phi_i^{\{i,j\}}\left(P_j \times C_j\right)\right)_{j \neq i}, \left(C_I, \left(\phi_i^{\{i\} \cup I}\right)^{-1}\Big|_{\operatorname{im} \phi_i^{\{i\} \cup I}}\right)_{I \subset \{1, \dots, \hat{\imath}, \dots, k\}}\right).$$

□

Let (M, f) be a singular $\mathscr{P}$-manifold in (X, A) with $\mathscr{P}_0$-manifold $(\partial M, A_i)$. The lemma above shows that $\partial A_0 \cong \partial B_0$ inherits the structure of a $\mathscr{P}$-manifold. This implies that $(\partial A_0, f|_{\partial A_0})$ becomes a $\mathscr{P}$-manifold in A and, hence, $(A_0, f|_{A_0})$ becomes a singular $\mathscr{P}$-manifold in A. It is denoted by $\tilde{\partial}(M, f)$. The boundary map $\tilde{\partial}\colon \Omega_n^{\mathscr{P}}(X, A) \to \Omega_{n-1}^{\mathscr{P}}(A)$ is then defined by $[M, f] \mapsto [\tilde{\partial}(M, f)]$.

Proposition 2.3.6. Singular $\mathscr{P}$-bordism

$$\Omega_*^{\mathscr{P}}(X, A) = \bigoplus_{n \geq 0} \Omega_n^{\mathscr{P}}(X, A)$$

is a homology theory.

The homology theory $\Omega_*^{\mathscr{P}}(\_)$ is a smooth description of $MG(\mathscr{P})_*(\_)$, the homology theory which was introduced in [Baa73].

Proof. The homotopy and exactness axiom could be verified using the methods of Proposition 2.2.5 and Lemma 2.3.4.

To prove the excision axiom, we may use Proposition 2.3.7 below which relates the bordism spanned by $\mathscr{P}$ and singular $\mathscr{P}$-bordism by means of the long exact sequence mentioned in the introduction. Omitting the 'G', let $\Omega_*(X, A)$ denote the ordinary bordism groups of manifolds equipped with the tangential structure G. We consider the map, induced by the inclusion $(X - U, A - U) \hookrightarrow (X, A)$, between exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & \mathcal{P}_n(X-U, A-U) \arrow[r, "\iota"] \arrow[d, "\cong"] & \Omega_n(X-U, A-U) \arrow[r, "\pi"] \arrow[d, "\cong"] & \Omega_n^{\mathscr{P}}(X-U, A-U) \arrow[r, "\delta"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\delta"] & \mathcal{P}_n(X, A) \arrow[r, "\iota"] & \Omega_n(X, A) \arrow[r, "\pi"] & \Omega_n^{\mathscr{P}}(X, A) \arrow[r, "\delta"] & \cdots .
\end{tikzcd}$$

We have the indicated isomorphisms since $\mathcal{P}_*(\_)$ and $\Omega_*(\_)$ satisfy the excision axiom. It now follows from the five lemma that also i_* is an isomorphism. $\square$

Let us reveal the relationship between the *bordism spanned by $\mathscr{P}$* on the one hand and *singular $\mathscr{P}$-bordism* on the other. We have the following three natural transformation. First there is a map

$$\iota\colon \mathcal{P}_n(X, A) \to \Omega_n(X, A),$$

defined by forgetting the $\mathscr{P}$-structure. Next we may interpret a manifold $(M, \partial M)$ in (X, A) as a singular $\mathscr{P}$-manifold in (X, A) by setting $A_0 = \partial M$. Hence one obtains a map

$$\pi\colon \Omega_n(X, A) \to \Omega_n^{\mathscr{P}}(X, A).$$

Furthermore, there is a boundary map

$$\delta\colon \Omega_n^{\mathscr{P}}(X, A) \to \mathcal{P}_{n-1}(X, A),\ [M, f] \mapsto \left[\partial M - \mathring{A}_0, f|_{\partial M - \mathring{A}_0}\right],$$

where $\mathring{A}_0$ is understood with respect to $A_0 \subset \partial M$.

Proposition 2.3.7. *The sequence*

$$\cdots \to \mathcal{P}_n(X, A) \xrightarrow{\iota} \Omega_n(X, A) \xrightarrow{\pi} \Omega_n^{\mathscr{P}}(X, A) \xrightarrow{\delta} \mathcal{P}_{n-1}(X, A) \to \cdots$$

is exact. It is called the generalized Bockstein exact sequence*.*

Proof. Although it is proved using a straightforward gluing procedure, we shall give a detailed proof of Proposition 2.3.7. As noted above, the verification of the excision property for singular $\mathscr{P}$-bordism relies on it. For the sake of convenience, we will omit signs.

- $\pi \circ \iota = 0$: Let (M, f) be a $\mathscr{P}$-manifold in (X, A). Its structure as a singular $\mathscr{P}$-manifold is given by $A_0 = \partial M$. After straightening the angle, $\partial(M \times [0,1]) = (\partial M \times [0,1]) \cup (M \times \{0,1\})$ becomes a smooth closed manifold. On $(\partial M \times [0,1]) \cup (M \times \{1\})$ we define a $\mathscr{P}_0$-structure as follows: For $1 \le i \le k$ we may extend the given P_i-parts on $M \times \{1\}$ a bit to $(\partial M \times [0,1]) \cup (M \times \{1\})$. More precisely, we first choose an outer collar neighborhood $\partial M \times [0, \epsilon]$ of $M \times \{1\}$ in $\partial(M \times [0,1])$. If A_i denotes the P_i-part of M, we get an induced P_i-part $\tilde{A}_i$ of ∂M. We then take

$$\left(\left(\tilde{A}_i \times [0, \epsilon]\right) \cup_{A_i \cap \partial M} A_i\right) \subset \partial(M \times [0,1])$$

as the P_i-part of $\partial(M \times [0,1])$. Its P_0-part is $\partial M \times [0,1]$. One concludes that

$$((\partial M \times [0,1]) \cup (M \times \{1\}), (f|_{\partial M} \circ pr_{\partial M}) \cup f, M \times [0,1], f \circ pr_M))$$

is a zero singular $\mathscr{P}$-bordism for $(M \times \{0\}, f)$.

- $\delta \circ \pi = 0$: Let (M, f) be a manifold in (X, A). Then its structure as a singular $\mathscr{P}$-manifold is given by $A_0 = \partial M$. This implies $\mathring{A}_0 = \partial M$, hence $(\delta \circ \pi)[M, f] = 0$.

- $\iota \circ \delta = 0$: Let (M, f) be a singular $\mathscr{P}$-manifold in (X, A). Then $(\partial M, f|_{\partial M})$ is a zero bordism for $(\partial M - \mathring{A}_0, f|_{\partial M - \mathring{A}_0})$ as $f(\mathring{A}_0) \subset A$.

- $\ker \pi \subset \operatorname{im} \iota$: Let $[M, f] \in \ker \pi$. Then there exists a zero singular $\mathscr{P}$-bordism (N, g, W, h) for (M, f), regarded as a singular $\mathscr{P}$-manifold in (X, A). Let A_0 denote the P_0-part of the $\mathscr{P}_0$-manifold N. Now $(N - \mathring{A}_0, g|_{N - \mathring{A}_0})$ is a smooth $\mathscr{P}$-manifold in (X, A). In addition, (W, h) is a bordism between (M, f) and $(N - \mathring{A}_0, g|_{N-\mathring{A}_0})$ in (X, A) as $\partial W - (M \cup (N - \mathring{A}_0)) = \mathring{A}_0$ and $h(\mathring{A}_0) \subset A$.

- $\ker \iota \subset \operatorname{im} \delta$: Let $[M, f] \in \ker \iota$. Then there exists a zero bordism (W, h) for (M, f), regarded as an ordinary manifold in (X, A). Now (W, h) can be considered as a singular $\mathscr{P}$-manifold in (X, A) if we define a $\mathscr{P}_0$-structure on ∂W as follows. As above, we may extend the given $\mathscr{P}$-structure on M a bit to ∂W. This yields P_i-parts on ∂W for $1 \le i \le k$. As the P_0-part we take $A_0 = \partial W - \mathring{M}$. Then one has $\delta[W, g] = [M, f]$.

- $\ker \delta \subset \operatorname{im} \pi$: Let $[M, f] \in \ker \delta$. Then there exists a zero $\mathscr{P}$-bordism (N, g) for $(\partial M - \mathring{A}_0, f|_{\partial M - \mathring{A}_0})$, where A_0 denotes the P_0-part of ∂M. We glue (M, f) and (N, g) along $\partial M - \mathring{A}_0$ and obtain $(M \cup N, f \cup g)$, a smooth manifold with boundary in (X, A). We want to show that $\pi[M \cup N, f \cup g] = [M, f]$.

  Consider
  $$W = \partial\left((M \cup N) \times [0,1]\right) - \left(\left(\mathring{M} \times \{0\}\right) \cup \left(\left(M \mathring{\cup} N\right) \times \{1\}\right)\right).$$

  After straightening the angle, this becomes a smooth manifold which we equip with a $\mathscr{P}_0$-structure as follows: For $1 \le i \le k$ let us extend the given P_i-parts on $N \times \{0\}$ and $\partial M \times \{0\}$ a bit to W, and take $\partial(M \cup N) \times [0, 1]$ as P_0-part. Then
  $$(W, (f \cup g) \circ pr_{M \cup N}|_W, (M \cup N) \times [0,1], (f \cup g) \circ pr_{M \cup N})$$
  is a singular $\mathscr{P}$-bordism between $(M \times \{0\}, f)$ and $((M \cup N) \times \{1\}, f \cup g)$, bearing in mind that the latter singular $\mathscr{P}$-structure is given by taking $\partial(M \cup N)$ as the P_0-part (see Figure 2.2).

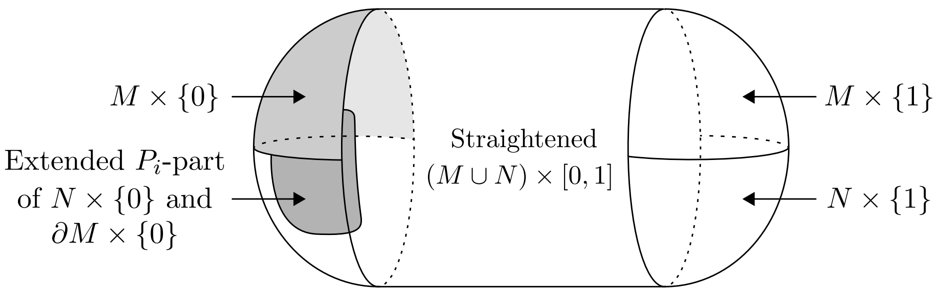

Figure 2.2: A singular $\mathscr{P}$-bordism

□

For later use, we note the following variant of the generalized Bockstein exact sequence. Let Q denote a closed manifold of dimension l, and let M denote a closed singular $(\mathscr{P} \cup \{Q\})$-manifold in X whose Q-part of its boundary is $Q \times B$. By Lemma 2.3.5, we see that B becomes a closed singular $\mathscr{P}$-manifold in X, denoted by $\delta_Q M$. Now the sequence

$$\cdots \to \Omega_n^{\mathscr{P}}(X) \xrightarrow{\times Q} \Omega_{n+l}^{\mathscr{P}}(X) \xrightarrow{\pi} \Omega_{n+l}^{\mathscr{P}\cup\{Q\}}(X) \xrightarrow{\delta_Q} \Omega_{n-1}^{\mathscr{P}}(X) \to \cdots \quad (2.3.1)$$

is exact (see also [Baa73, Theorem 3.2]).

Remark 2.3.8. According to Brown's representability theorem, the homology theories $\mathcal{P}_*(\_)$, $\Omega_*(\_)$ resp. $\Omega_*^{\mathscr{P}}(\_)$ are associated to spectra $\mathcal{P}$, MG resp. $MG^{\mathscr{P}}$. In addition, $\iota\colon \mathcal{P}_*(\_) \to \Omega_*(\_)$ and $\pi\colon \Omega_*(\_) \to \Omega_*^{\mathscr{P}}(\_)$ correspond to spectrum maps $\iota\colon \mathcal{P} \to MG$ and $\pi\colon MG \to MG^{\mathscr{P}}$. It seems to be a reasonable aim to show that

$$\mathcal{P} \xrightarrow{\iota} MG \xrightarrow{\pi} MG^{\mathscr{P}}$$

is homotopic to a cofiber sequence. This can be proved under certain assumptions. In general one is faced with *lim^1-problems* (see [Baa73, p. 297-299] for a similar discussion), which we do not want to address. However, by the methods of Section 2.5, one could show that in the situations treated in the next chapters, we in fact encounter cofiber sequences.

Let us finally note that everything mentioned so far immediately generalizes to possibly *infinite* families $\mathscr{P}$. In fact, we denote the family of all finite subsets of $\mathscr{P}$ by $\mathfrak{F}$. The bordism spanned by $\mathscr{F} \in \mathfrak{F}$ is denoted by $\mathcal{F}_*(\_)$. By taking inclusions, $\mathfrak{F}$ becomes a directed set, and thus we can form the direct limit

$$\mathcal{P}_*(\_) = \lim_{\mathscr{F} \in \mathfrak{F}} \mathcal{F}_*(\_).$$

Since the direct limit preserves exactness, $\mathcal{P}_*(\_)$ is again a homology theory. The same obviously works for singular $\mathscr{P}$-bordism. After these general constructions, we turn in the next section to the computation of the coefficients $\mathcal{P}_*$ and $\Omega_*^{\mathscr{P}}$.

2.4 Computation of Coefficients

From now on we will assume that the Cartesian product of manifolds (equipped with a tangential structure G) induces a ring structure on Ω_*. It is apparent that this holds for oriented and spin bordism. In addition, for the remainder of this section we fix a *regular sequence* $\mathscr{P} = (P_1, P_2, \dots)$ of closed manifolds. This means by definition that

$$\Omega_*/\left([P_1], \dots, [P_{i-1}]\right) \xrightarrow{\times [P_i]} \Omega_*/\left([P_1], \dots, [P_{i-1}]\right) \tag{2.4.1}$$

is injective for all $i \geq 1$, where $([P_1], \dots, [P_{i-1}])$ denotes the ideal generated by $[P_1], \dots, [P_{i-1}]$. One can show (see [CW11, Proposition 2.7.1.]) that in our situation any permutation of $P_1, P_2, \dots$ is again a regular sequence.

Recall the natural transformation of homology theories $\iota_* \colon \mathcal{P}_*(\_) \to \Omega_*(\_)$ defined by forgetting the $\mathscr{P}$-structure. The following proposition is the crucial step in determining the coefficients.

Proposition 2.4.1. *The map $\iota_* \colon \mathcal{P}_* \to \Omega_*$ is injective.*

Proof. Let $\mathfrak{R}_k$ denote the family of all subsets of $\mathscr{P}$ consisting of k elements. For $\mathscr{R} \in \mathfrak{R}_k$ we have the bordism spanned by $\mathscr{R}$, denoted by $\mathcal{R}_*(\_)$, and the forgetful map $\iota_*^{\mathcal{R}} \colon \mathcal{R}_*(\_) \to \Omega_*(\_)$. We shall prove the following statement by induction on k from which Proposition 2.4.1 follows immediately:

$$\text{Let } \mathscr{R} \in \mathfrak{R}_k, \text{ then } \iota_*^{\mathcal{R}} \colon \mathcal{R}_* \to \Omega_* \text{ is injective.}$$

For the initial step we consider $\mathscr{R} \in \mathfrak{R}_1$. In this case a closed $\mathscr{R}$-manifold M is diffeomorphic to $P_l \times B_l$ for $\{P_l\} = \mathscr{R}$. If $\iota_*^{\mathcal{R}}[M] = 0$ then $[B_l] = 0$ in

Ω_*, because $[P_l]$ is not a zero divisor. Hence $B_l = \partial W$ for some manifold W. Now the P_l-manifold $P_l \times W$ establishes a zero $\mathscr{R}$-bordism for M.

We next turn to the induction step. Assume that $\iota_*^{\mathcal{S}} \colon \mathcal{S}_* \to \Omega_*$ is injective for all $\mathscr{S} \in \mathfrak{R}_{k-1}$. Let $\mathscr{R} = \{P_{i_1}, \ldots, P_{i_k}\} \in \mathfrak{R}_k$ and let

$$\big(M, (A_{i_j})_{1 \le j \le k}, (B_I, \phi_I)_{I \subset \{i_1, \ldots, i_k\}}\big)$$

be a closed $\mathscr{R}$-manifold. One has to show that $[M] = 0$ in $\mathcal{R}_*$ if $\iota_*^{\mathcal{R}}[M] = 0$. Our strategy will be to apply surgery to the submanifolds $\partial A_{i_j} \subset M$ in order to obtain an $\mathscr{R}$-bordism between M and some $\mathscr{R}$-manifold consisting of a disjoint union of P_{i_j}-manifolds. Afterwards, an almost purely algebraic consideration yields the claim.

For all $1 \le j \le k$ we have diffeomorphisms

$$\phi_{i_j} \colon \partial A_{i_j} \to P_{i_j} \times \partial B_{i_j}.$$

The induction hypothesis becomes applicable by means of Lemma 2.3.5. Namely, if $\mathscr{S} = \mathscr{R} - \{P_{i_j}\}$, i.e. $\mathscr{S} \in \mathfrak{R}_{k-1}$, then ∂B_{i_j} inherits the structure of an $\mathscr{S}$-manifold.

In the first surgery step we consider ∂A_{i_1}. Since $[P_{i_1}]$ is not a zero divisor, $[\partial B_{i_1}]$ vanishes in Ω_*. Hence, by the induction hypothesis, ∂B_{i_1} is zero $\mathscr{S}$-bordant with $\mathscr{S} = \mathscr{R} - \{P_{i_1}\}$, i.e. there exists an $\mathscr{S}$-manifold (N, C_i) with $\partial N = \partial B_{i_1}$. By abuse of notation, we shall use the indices $\{1, \ldots, k\}$ instead of $\{i_1 \ldots, i_k\}$ from now on. Fix bicollar neighborhoods $\partial A_i \times [-1, 1]$ of $\partial A_i \subset M$, say $\partial A_i \times \{-1\} \subset A_i$, such that $(\partial A_i \times [-1, 1]) \cap A_j$ is a bicollar neighborhood for $(\partial A_i) \cap A_j$ in A_j for all j.

We now attach $P_1 \times N \times [-1, 1]$ to $M \times [0, 1]$ by identifying

$$\begin{aligned} &(x, y, 1) \in (\partial A_1 \times [-1, 1] \times [0, 1]) \subset (M \times [0, 1]) \ \text{ with} \\ &(\phi_1(x), y) \in (P_1 \times \partial N) \times [-1, 1] \end{aligned}$$

to obtain a manifold W, admitting a tangential structure G which restricts on $M \times \{0\} \subset \partial W$ to the given one on M. It is well-known that the corners of W can be smoothened in a canonical way.

Let us equip W with the structure of an $\mathscr{R}$-manifold (see Figure 2.3). On $M \times [0, 1]$ we simply set $A_i' = A_i \times [0, 1]$. On $P_1 \times N \times [-1, 1]$ we define $A_1'' = P_1 \times N \times [-1, 0]$ and, for $i > 1$, $A_i'' = P_1 \times C_i \times [-1, 1]$. We now take $\hat{A}_i = A_i' \cup A_i''$ as the P_i-part of W for all i. Note that this $\mathscr{R}$-structure on W induces the given one on $M \times \{0\}$.

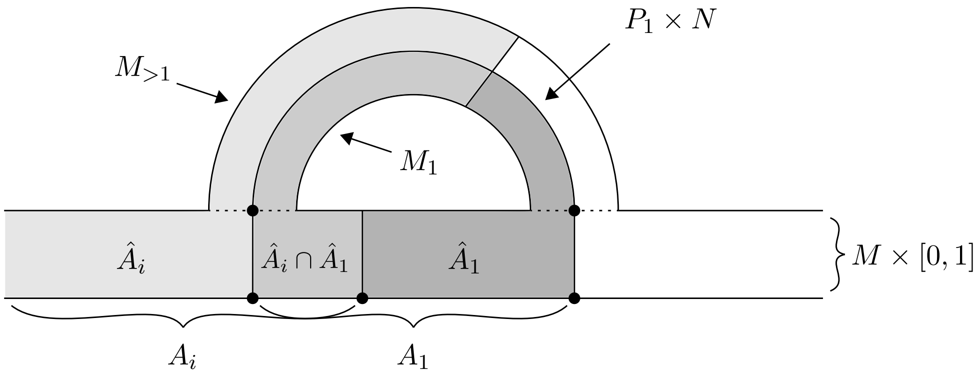

Figure 2.3: $\mathscr{R}$-structure of W

By construction, W is an $\mathscr{R}$-bordism between M and the disjoint union of

- $M_1 := A_1 \cup (P_1 \times N)$ glued along $\phi_1(\partial A_1) = \partial(P_1 \times N)$,
- $M_{>1} := (M - \mathring{A}_1) \cup (P_1 \times N)$ glued along $\phi_1(\partial(M - \mathring{A}_1)) = \partial(P_1 \times N)$.

The trace of the bordism induces an $\mathscr{R}$-structure on $M_{>1}$ with an empty P_1-part. A priori, W induces an $\mathscr{R}$-structure on M_1 with non-empty P_i-parts for *all* $i \geq 1$. However, M_1 is completely covered by the P_1-part $P_1 \times (B_1 \cup N)$. The following lemma shows that we can ignore redundant subsets A_i, more precisely:

Lemma 2.4.2. *Let* $(M, (A_i)_{1 \leq i \leq k})$ *denote a closed* $\mathscr{R}$*-manifold, and let* $1 \leq j \leq k$ *and* $M = \cup_{i=1, i \neq j}^{k} A_i$.

Then $(M, (A_i)_{1 \leq i \leq k})$ *is* $\mathscr{R}$*-bordant to* $(M, (A'_i)_{1 \leq i \leq k})$ *where* $A'_i = A_i$ *for* $i \neq j$ *and* $A'_j = \emptyset$.

Proof. Let A_j be diffeomorphic to $P_j \times B_j$. Choose a collar neighborhood $\partial B_j \times [-1, 0]$, say $\partial B_j \times \{0\} = \partial B_j$, such that for the induced collar neighborhood $\partial A_j \times [-1, 0]$ it is true that $(\partial A_j \times [-1, 0]) \cap A_i$ is a collar neighborhood for $(\partial A_j) \cap A_i$ in A_i for all i. Let $\gamma\colon [-1, 0] \to [-1, 0] \times [0, 0{,}5]$ be a smooth injective convex curve with $\gamma(t) = (t, 0{,}5)$ for $t < -0{,}9$ and $\gamma(t) = (0, -t)$ for $t > -0{,}1$. We now define the desired $\mathscr{R}$-bordism (see Figure 2.4) by

$$M \times [0, 1] = \bigcup_{i=1}^{k} \overline{A_i},$$

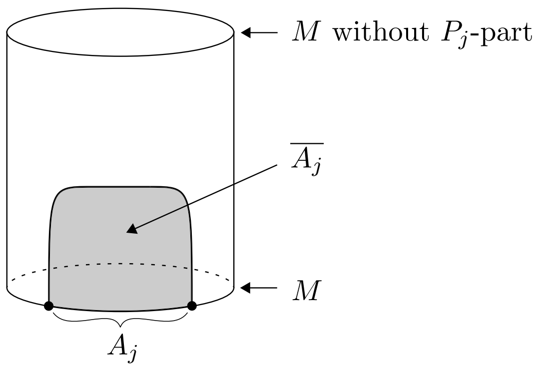

Figure 2.4: Redundant A_j-part

where we set $\overline{A_i} = A_i \times [0,1]$ for $i \neq j$ and $\overline{A_j}$ equal to

$$\left\{ (x,s) \in A_j \times [0,1] \,|\, s \leq \begin{cases} 0{,}5 & \text{if } x \in A_j - (\partial A_j \times [-1,0]) \\ \gamma(t)_2 & \text{if } x = (y,t) \in \partial A_j \times [-1,0) \\ \max\limits_{\{t \,|\, \gamma(t)_1 = 0\}} \gamma(t)_2 & \text{if } x \in \partial A_j \times \{0\} \end{cases} \right\}.$$

□

Applying this statement $(k-1)$-times to M_1, it follows that M_1 becomes a P_1-manifold. As noted above, any permutation of $P_1, P_2, \ldots, P_k$ is again a regular sequence. We can therefore repeat the above surgery procedure applied to the $\mathscr{R}$-manifold $M_{>1}$. This yields an $\mathscr{R}$-bordism between $M_{>1}$ on the one hand and a P_2-manifold M_2 resp. an $\mathscr{R}$-manifold $M_{>2}$ with empty P_1- and P_2-parts on the other. In this fashion we obtain an $\mathscr{R}$-bordism between M and a disjoint union

$$(P_1 \times Q_1) \,\dot{\cup} \cdots \dot{\cup}\, (P_k \times Q_k) \tag{2.4.2}$$

where each $P_i \times Q_i$ is as a $\mathscr{R}$-manifold a P_i-manifold.

To complete the induction step, we have to show that if $\iota_*^{\mathcal{R}}[M] = 0$ in Ω_* then $[M] = 0$ in $\mathcal{R}_*$. In Ω_* we observe the following: Since M is zero bordant, it follows that for all $1 \leq j \leq k$

$$\sum_i [P_i \times Q_i] = 0 \mod ([P_1], \ldots, \widehat{[P_j]}, \ldots, [P_k]),$$

hence $[P_j \times Q_j] \in ([P_1], \ldots, \widehat{[P_j]}, \ldots, [P_k])$. Regularity then implies that $[Q_j]$ lies in $([P_1], \ldots, \widehat{[P_j]}, \ldots, [P_k])$. It follows that for all $1 \leq s, t \leq k$ there exists a closed manifold Q_{st} such that M is bordant to $\sum_{s,t} P_{\{s,t\}} \times Q_{st}$. Next, for all $1 \leq j, l \leq k$ we consider

$$\sum_{s,t} [P_{\{s,t\}} \times Q_{st}] = 0 \quad \mathrm{mod}\ ([P_1], \ldots, \widehat{[P_j]}, \ldots, \widehat{[P_l]}, \ldots, [P_k]),$$

and conclude $[P_{\{s,t\}} \times Q_{st}] \in ([P_1], \ldots, \widehat{[P_j]}, \ldots, \widehat{[P_l]}, \ldots, [P_k])$ for $\{s,t\} = \{j,l\}$. Again, by regularity one has $[Q_{st}] \in ([P_1], \ldots, \widehat{[P_j]}, \ldots, \widehat{[P_l]}, \ldots, [P_k])$. Proceeding in this fashion, we find a closed manifold Q such that M is bordant to $P_1 \times \cdots \times P_k \times Q$.

We now take into account the P_i-factors in (2.4.2). It follows from the above procedure that there exist closed manifolds R_i such that M is $\mathscr{R}$-bordant to

$$(\overline{P_1} \times \cdots \times P_k \times R_1) \dot{\cup} \cdots \dot{\cup} (P_1 \times \cdots \times \overline{P_k} \times R_k), \tag{2.4.3}$$

where $\overline{P_1} \times \cdots \times P_k \times R_1$ denotes the P_1-*manifold* $P_1 \times \cdots \times P_k \times R_1$ etc. It is not difficult to see that the specification of the P_i's in (2.4.3) is immaterial. In fact, $\overline{P_i} \times P_j$ is $\mathscr{R}$-bordant to $P_i \times \overline{P_j}$ via the $\mathscr{R}$-bordism $[0,1] \times P_i \times P_j$ with $\left[0, \frac{2}{3}\right] \times P_i \times P_j$ as the P_i-part and $\left[\frac{1}{3}, 1\right] \times P_i \times P_j$ as the P_j-part.

We conclude that M is $\mathscr{R}$-bordant to, say, $\overline{P_1} \times \cdots \times P_k \times Q$. Now, since $[P_1 \times \cdots \times P_k]$ is not a zero divisor, it follows that Q is zero bordant, i.e. $Q = \partial R$ for some R. The P_1-manifold $\overline{P_1} \times \cdots \times P_k \times R$ is the required zero $\mathscr{R}$-bordism for M. This finishes the proof of the induction step. $\square$

Remark 2.4.3. Generalizing the argument in the next-to-last paragraph, that $\overline{P_i} \times P_j$ is $\mathscr{R}$-bordant to $P_i \times \overline{P_j}$, we conclude the following. Let $\mathscr{P}$ be a family of closed manifolds (regularity is immaterial). Then there exists a natural ring structure on $\mathcal{P}_*$. Namely, let $M = A_1 \cup \cdots \cup A_n$ and $N = A'_1 \cup \cdots \cup A'_n$ denote two closed $\mathscr{P}$-manifolds and define a $\mathscr{P}$-structure on $M \times N$ as follows: Set $A_i^M = A_i \times N$ for all i. The above argument ensures that $A_1^M \cup \cdots \cup A_n^M$ is $\mathscr{P}$-bordant to $A_1^N \cup \cdots \cup A_n^N$, where $A_i^N = M \times A'_i$. However, we note that there is no unit map $S^0 \to \mathcal{P}$.

2.5 Identification of Homology Theories

The aim of this section is to identify, after inverting two, singular homology and connective real K-theory with appropriate singular bordism theories -

compatible with the orientation maps. More precisely, we will find a sequence $\mathscr{P}^u$ of closed oriented manifolds and an equivalence $\Psi\colon \Omega_*^{\mathscr{P}^u}(\_) \to H_*(\_)$ such that

$$\begin{tikzcd} \Omega_*^{SO}(\_) \arrow[r, "U"] \arrow[d, "\pi"] & H_*(\_) \\ \Omega_*^{\mathscr{P}^u}(\_) \arrow[ur, dashed, "\cong", "\Psi"'] & \end{tikzcd}$$

commutes. A corresponding result holds in the case of connective real K-theory for a sequence $\mathscr{P}^\alpha$ of closed spin manifolds. Of course, whenever we talk about singular $\mathscr{P}^u$- resp. $\mathscr{P}^\alpha$-bordism, we always mean *oriented* resp. *spin* singular $\mathscr{P}^u$ resp. $\mathscr{P}^\alpha$-bordism.

Recall again that homology theories correspond to spectra and natural transformations between homology theories correspond to maps between spectra. In this way, the orientation maps can be obtained as follows. Let MSO resp. $MSpin$ denote the oriented resp. spin Thom spectrum. That is, MSO resp. $MSpin$ are built up by the Thom spaces of the universal oriented resp. spin bundles.

One can choose compatible orientation classes in the integer cohomology of the Thom spaces of the universal oriented bundles. This leads to a map $u\colon MSO \to H\mathbb{Z}$, $H\mathbb{Z}$ denoting the integer Eilenberg-MacLane spectrum, which corresponds to the orientation $U\colon \Omega_*^{SO}(\_) \to H_*(\_)$.

Likewise, in [ABS64] it is shown that vector bundles with a spin structure admit orientations with respect to real K-theory. This can be used to construct a map $MSpin \to KO$ where KO denotes the real K-theory spectrum. Let ko denote the connective cover of KO, i.e. ko_i vanishes for $i < 0$ and there is a map $per\colon ko \to KO$, called the periodization map, which induces isomorphisms on non-negative homotopy groups. Since $MSpin$ also has trivial negative homotopy groups, obstruction theory then implies that there exists an induced map $\alpha\colon MSpin \to ko$, corresponding to $A\colon \Omega_*^{Spin}(\_) \to ko_*(\_)$.

Let us start with the identification of singular homology. By results of Averbuh [Ave59] and Milnor [Mil60], the oriented bordism Ω_*^{SO} contains no odd torsion. Moreover, one can show (see [Nov60]) that Ω_*^{SO} modulo torsion is a polynomial algebra. One finds closed oriented manifolds $Q_1, Q_2, \ldots$, $\dim Q_i = 4i$, such that

$$\Omega_*^{SO}/\text{torsion} \cong \mathbb{Z}[[Q_1],[Q_2],\ldots] \tag{2.5.1}$$

(see also [Sto68, p. 180]). For example, we may take complex projective spaces and so-called Milnor manifolds, hypersurfaces of degree $(1,1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$, as generators. Another list of generators - in analogy to the spin case below - is given by total spaces of $\mathbb{C}P^2$-bundles (see [Füh08]).

Since Ω_*^{SO} contains no odd torsion, we conclude

$$\Omega_*^{SO} \otimes \mathbb{Z}\left[\tfrac{1}{2}\right] \cong \mathbb{Z}\left[\tfrac{1}{2}\right][[Q_1],[Q_2],\ldots]. \tag{2.5.2}$$

For the remainder of this section we consider all spectra and groups after inverting 2.

We now set $\mathscr{P}^u = (Q_1, Q_2, \ldots)$. In view of (2.5.2), $\mathscr{P}^u$ is a regular sequence. Propositions 2.3.7 and 2.4.1 then imply that

$$\Omega_*^{\mathscr{P}^u} = \Omega_*^{SO}/([Q_1],[Q_2],\ldots),$$

and $\pi_*\colon \Omega_*^{SO} \to \Omega_*^{\mathscr{P}^u}$ is given by the obvious projection. That is, on coefficients, π can be identified with u. We have to show that this holds in general, which can be verified by the identification of the corresponding spectra. Let $MSO^{\mathscr{P}^u}$ denote the spectrum associated to singular $\mathscr{P}^u$-bordism.

Proposition 2.5.1. *There is a homotopy equivalence $\Psi\colon MSO^{\mathscr{P}^u} \xrightarrow{\simeq} H\mathbb{Z}$ such that $u \simeq \Psi \circ \pi$ as maps $MSO \to H\mathbb{Z}$.*

Proof. Let $\mathcal{C}$ denote the homotopy fiber of $\pi\colon MSO \to MSO^{\mathscr{P}^u}$, and consider the extension problem

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\iota"] & MSO \arrow[r, "u"] \arrow[d, "\pi"] & H\mathbb{Z}. \\ & MSO^{\mathscr{P}^u} \arrow[ur, dashed] & \end{tikzcd}$$

Since Ω_1^{SO} and $\Omega_2^{\mathscr{P}^u}$ vanish, we conclude by using the long exact homotopy sequence associated to $\mathcal{C} \xrightarrow{\iota} MSO \xrightarrow{\pi} MSO^{\mathscr{P}^u}$ that also $\mathcal{C}_1$ vanishes. This implies that $\widetilde{H}^0(\mathcal{C})$ is trivial, and therefore the composition $\mathcal{C} \xrightarrow{\iota} MSO \xrightarrow{u} H\mathbb{Z}$ is null homotopic. By obstruction theory, it follows that there exists a solution (even unique up to homotopy) $\Psi\colon MSO^{\mathscr{P}^u} \to H\mathbb{Z}$. It is clear that Ψ induces isomorphisms on homotopy groups. Hence, according to Whitehead's theorem, Ψ is a homotopy equivalence. □

We turn to the identification of connective real K-theory. Since the inclusion $MSpin \to MSO$ is a homotopy equivalence after inverting 2, there exist closed spin manifolds K,$R_2, R_3, \ldots$, $\dim K = 4$, $\dim R_i = 4i$, such that

$$\Omega_*^{Spin} \cong \mathbb{Z}\left[\tfrac{1}{2}\right][[K], [R_2], [R_3], \ldots].$$

By means of Bott periodicity, one can show that the coefficient ring of the real periodic K-theory spectrum KO is given by

$$KO_* \cong \mathbb{Z}[\eta, \omega, \mu, \mu^{-1}]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\mu),$$

where $deg(\eta) = 1$, $deg(\omega) = 4$ and $deg(\mu) = 8$ (see e.g. [LM89, p. 63 and Chapter 3, § 10]). With 2 inverted this implies that $KO_* \cong \mathbb{Z}\left[\frac{1}{2}\right][\omega, \omega^{-1}]$ and $ko_* \cong \mathbb{Z}\left[\frac{1}{2}\right][\omega]$.

The Atiyah-Bott-Shapiro orientation $\alpha_*\colon \Omega_*^{Spin} \to ko_*$ is now given as follows. In [KS93, Section 4] it is shown that R_i can be chosen as the total space of an $\mathbb{H}P^2$-bundle, for all $i \geq 2$. Moreover, these $\mathbb{H}P^2$-bundles admit positive scalar curvature metrics. By means of the index-theoretic interpretation of α_* (see Subsection 4.3.1), one concludes $\alpha_*[R_i] = 0$ for all $i \geq 2$. In addition, K can be chosen to be a K3 surface of signature 16, which implies that α_* maps $[K]$, properly oriented, to ω.

We now consider the *regular* sequence $\mathscr{P}^\alpha = (R_2, R_3, \ldots)$. Let $MSpin^{\mathscr{P}^\alpha}$ denote the spectrum associated to singular $\mathscr{P}^\alpha$-bordism. As above, $\pi\colon MSpin \to MSpin^{\mathscr{P}^\alpha}$ induces the natural projection

$$\begin{aligned}\Omega_*^{Spin} &\to \Omega_*^{Spin}/([R_2], [R_3], \ldots)\\ &= \mathbb{Z}\left[\tfrac{1}{2}\right][\pi_*[K]].\end{aligned}$$

In other words, on coefficients, π can again be identified with α. The proof of the corresponding proposition requires an additional argument, however.

Proposition 2.5.2. *There is a homotopy equivalence $\Phi\colon MSpin^{\mathscr{P}^\alpha} \xrightarrow{\simeq} ko$ such that $\alpha \simeq \Phi \circ \pi$ as maps $MSpin \to ko$.*

Remark 2.5.3. At this point we recover a geometric description of $ko_*(\_)\left[\frac{1}{2}\right]$.

Proof of Proposition 2.5.2. The idea is that we can combine the isomorphism on the level of coefficients with the Conner-Floyd theorem to obtain the desired spectrum map. A similar argument can be found in [Lan76, p. 597]. In order to proceed in this way, one has to turn to periodic theories first.

We shall start with a general remark. Let X be a ring spectrum and $a \in X_*$. One can consider the ordinary ring theoretic localization $X_*[a^{-1}]$. This is a flat X_*-module, which implies that $X_*(\_) \otimes_{X_*} X_*[a^{-1}]$ is a homology theory. The associated spectrum is denoted by $X[a^{-1}]$. The natural map $loc_*\colon X_* \to X_*[a^{-1}]$ induces a spectrum map $loc\colon X \to X[a^{-1}]$. It follows that $X[a^{-1}]$ inherits the structure of an X-module spectrum.

Let us set $k = \pi_*[K]$. We now consider the $MSpin$-module spectrum $MSpin^{\mathscr{P}\alpha}[k^{-1}]$, having the coefficients

$$MSpin^{\mathscr{P}\alpha}[k^{-1}]_* \cong \mathbb{Z}\left[\tfrac{1}{2}\right][k, k^{-1}].$$

It follows that there is a unique Ω_*^{Spin}-module isomorphism

$$\phi\colon MSpin^{\mathscr{P}\alpha}[k^{-1}]_* \xrightarrow{\cong} KO_* \tag{2.5.3}$$

such that

$$per_* \circ \alpha_* = \phi \circ loc_* \circ \pi_* \tag{2.5.4}$$

as maps $\Omega_*^{Spin} \to KO_*$.

In [HH92, Theorem 1] it is shown that the natural map

$$\mu\colon \Omega_*^{Spin}(X) \otimes_{\Omega_*^{Spin}} KO_* \to KO_*(X),$$

induced by the $MSpin$-module structure of KO, is an isomorphism. Let us note that after inverting 2, this statement is just the real Connor-Floyd theorem (see [CF66]). Consider now

$$\begin{tikzcd} \Omega_*^{Spin}(X) \otimes_{\Omega_*^{Spin}} KO_* \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\otimes\phi^{-1}"] & KO_*(X) \\ \Omega_*^{Spin}(X) \otimes_{\Omega_*^{Spin}} MSpin^{\mathscr{P}\alpha}[k^{-1}]_* \arrow[r, "\eta"] & MSpin^{\mathscr{P}\alpha}[k^{-1}]_*(X), \end{tikzcd} \tag{2.5.5}$$

where η is induced by the $MSpin$-module structure of $MSpin^{\mathscr{P}\alpha}[k^{-1}]$. It follows that the natural transformation of homology theories

$$\eta \circ (\mathrm{id} \otimes \phi^{-1}) \circ \mu^{-1}\colon KO_*(\_) \to MSpin^{\mathscr{P}\alpha}[k^{-1}]_*(\_)$$

is an equivalence. This implies that there exists a homotopy equivalence $KO \simeq MSpin^{\mathscr{P}\alpha}[k^{-1}]$. Since $MSpin^{\mathscr{P}\alpha}$ is a connective spectrum and the periodization map $per\colon ko \to KO$ induces isomorphisms on non-negative homotopy groups, we conclude that the lifting problem

$$\begin{tikzcd} & & ko \arrow[d] \\ MSpin^{\mathscr{P}\alpha} \arrow[r] \arrow[urr, dashed] & MSpin^{\mathscr{P}\alpha}[k^{-1}] \arrow[r, "\simeq"] & KO \end{tikzcd}$$

has a unique solution $\Phi\colon MSpin^{\mathscr{P}^\alpha} \to ko$ (up to homotopy), which clearly induces isomorphisms on homotopy groups. Note that the equality (2.5.4) implies $\alpha \simeq \Phi \circ \pi$ as maps $MSpin \to ko$. □

2.6 Concluding Remarks

Proposition 2.5.2 could also be proved from the index-theoretic point of view. Namely, as indicated in Chapter 4, one can consider the Dirac operator on singular $\mathscr{P}^\alpha$-manifolds in order to obtain a natural transformation

$$\alpha^{\mathscr{P}^\alpha} : \Omega_*^{\mathscr{P}^\alpha}(\_) \to ko_*(\_),$$

which generalizes the corresponding index map for *closed* spin manifolds. Then it is apparent that

$$\begin{tikzcd} \Omega_*^{Spin}(\_) \arrow[r, "A"] \arrow[d, "\pi"] & ko_*(\_) \\ \Omega_*^{\mathscr{P}^\alpha}(\_) \arrow[ur, dashed, "\alpha^{\mathscr{P}^\alpha}"'] & \end{tikzcd}$$

commutes. In order to show that $\alpha^{\mathscr{P}^\alpha}$ is an equivalence, we certainly need the results of Section 2.4.

Let $\widehat{MSO}$ resp. $\widehat{MSpin}$ denote the homotopy fibers associated to the orientation maps $u\colon MSO \to H\mathbb{Z}$ resp. $\alpha\colon MSpin \to ko$. In a similar way as above, we could find a geometric description of $\widehat{MSO}$ resp. $\widehat{MSpin}$ by $\mathscr{P}^u$- resp. $\mathscr{P}^\alpha$-manifolds, that is, $\mathcal{P}_*^u(\_) = \widehat{MSO}_*(\_)$ and $\mathcal{P}_*^\alpha(\_) = \widehat{MSpin}_*(\_)$.

Chapter 3

Positive Scalar Curvature on Closed Manifolds

3.1 The Surgery Lemma and its Implications

This chapter deals with the existence problem of positive scalar curvature metrics on non-simply connected manifolds. As a preliminary point, and in order to emphasize the importance of the Surgery Lemma, let us ask in general which manifolds actually admit metrics of positive scalar curvature.

First, basic spaces in geometry naturally come equipped with positive scalar curvature metrics, including 'round' spheres or projective spaces with the Fubini-Study metric. More generally, symmetric spaces of compact type carry metrics of positive scalar curvature.

With these standard examples at hand, new spaces of positive scalar curvature can be constructed as follows. Scalar curvature behaves nicely with respect to products. Namely, if g_0 resp. g_1 are metrics on M resp. N, then

$$scal_{g_0 \times g_1} = scal_{g_0} + scal_{g_1}.$$

In addition, for $\epsilon > 0$ one has $scal_{\epsilon g} = \epsilon^{-1} scal_g$. We conclude that if M and N are compact, and N admits a positive scalar curvature metric, then so does $M \times N$. This observation can be generalized to 'twisted' products as follows. Let $F \to E \to B$ denote a fiber bundle of closed manifolds. If the associated structure group acts by isometries on the fibers, and the fiber F admits a metric of positive scalar curvature, then so does the total space E.

This is a consequence of the O'Neill formulas (see e.g. [Bes87, Theorem 9.59 and Proposition 9.70d]).

We therefore see that based on certain standard spaces, there are several manifolds admitting positive scalar curvature metrics. However, these examples of course cover only a small part of the wild world of manifolds.

The Surgery Lemma is now the crucial tool to construct positive scalar curvature metrics on manifolds not having such a 'symmetric flavor'. We recall the basic surgery process. Let M be a closed manifold of dimension n, and assume that $\phi\colon S^k \times D^{n-k} \hookrightarrow M$ is an embedding. Then one considers

$$N := \left(M - \phi\left(S^k \times \mathring{D}^{n-k}\right)\right) \cup_{\phi|_{S^k \times S^{n-k-1}}} \left(D^{k+1} \times S^{n-k-1}\right).$$

One says that N is obtained from M by performing surgery in dimension k, or equivalently, codimension $n-k$. This procedure clearly breaks diverse kinds of symmetries of M. We note that the surgery process induces a bordism between M and N.

Theorem 3.1.1 (Surgery Lemma, [GL80],[SY79]). *Let M be closed manifold which admits a positive scalar curvature metric. If a closed manifold N is obtained from M by surgery in codimension greater or equal than three, then also N admits a positive scalar curvature metric.*

For example, if two closed manifolds of dimension greater or equal than three both admit positive scalar curvature metrics, then so does their connected sum.

The idea of the proof of the Surgery Lemma can be sketched as follows. Let $\phi\colon S^k \times D^{n-k} \hookrightarrow M$ be the characteristic embedding of the surgery process. Assume that $S^{n-k-1} \times [0,\epsilon)$ is a collar neighborhood for S^{n-k-1} in D^{n-k}. One now shows that the positive scalar curvature metric on $\phi(S^k \times S^{n-k-1} \times [0,\epsilon))$, induced by M, can be extended to a positive scalar curvature metric on $D^{k+1} \times S^{n-k-1}$, where $\phi(S^k \times S^{n-k-1} \times [0,\epsilon))$ is considered as a collar neighborhood of $S^k \times S^{n-k-1}$ in $D^{k+1} \times S^{n-k-1}$. We see that the assumption placed on the codimension makes sense since it implies that S^{n-k-1} admits a positive scalar curvature metric.

A generalization of the Surgery Lemma is given by Gajer (see [Gaj87, p. 180]). In the situation of Theorem 3.1.1, he showed that the positive scalar curvature metric on M can even be extended to the bordism between M and N which corresponds to the surgery process.

The Surgery Lemma mostly applies together with methods from the proof of the s-cobordism theorem. In this regard, one can show the following statement which turns out to be very useful.

Theorem 3.1.2 ([Sto95, Theorem 3.3]). *Let W be bordism between two closed n-dimensional manifolds M and N. Assume that n is greater or equal than five and that the inclusion $N \hookrightarrow W$ is a two equivalence, i.e. the induced map on the i-th homotopy group is an isomorphism for $i = 0, 1$ and is surjective for $i = 2$. Then any positive scalar curvature metric on M extends to a positive scalar curvature metric on W.*

In Chapters 4 and 5 we will need the following statement which generalizes Theorem 3.1.2 to the case that M and N are allowed to have a common boundary.

Corollary 3.1.3. *Let M and N denote compact n-dimensional manifolds with the same boundary, and let W be compact manifold such that $\partial W = M \cup_{\partial M = \partial N} (-N)$. Assume that n is greater or equal than five and that the inclusion $N \hookrightarrow W$ is a two equivalence. Then any positive scalar curvature metric on M extends to a positive scalar curvature metric on W.*

This is an immediate consequence of the proof of Theorem 3.1.2 since the surgery process does not affect the boundary of M.

As explained in the introduction, with the Surgery Lemma at hand the question of positive scalar curvature is solved for closed simply connected manifolds of dimension greater or equal than five. In order to treat the case of non-simply connected manifolds, we recall that for any discrete group π there is an associated classifying space $B\pi$, which can assumed to be a CW complex. One has $\pi_1(B\pi) = \pi$ and $\pi_i(B\pi) = 0$ for $i \geq 2$, implying that $B\pi$ is unique up to homotopy equivalence. In addition, for a locally finite CW complex X with $\pi_1(X) = \pi$, the universal cover of X is classified by a map $X \to B\pi$, which is unique up to homotopy and induces an isomorphism on fundamental groups.

An important consequence of Theorem 3.1.2 is the Bordism Theorem from the introduction (see [RS94, Theorem 3.3]).

Theorem 3.1.4 (Bordism Theorem). *Let M be a closed connected manifold of dimension n greater or equal than five with fundamental group π. Furthermore, let $B\pi$ be the classifying space of π, and let $f\colon M \to B\pi$ be the classifying map of the universal cover of M. Then the following hold:*

1. *If M admits a spin structure, then M admits a positive scalar curvature metric if and only if $[M, f] \in {}^{+}\Omega_n^{Spin}(B\pi)$.*
2. *If M is orientable and totally non-spin, i.e. its universal cover does not admit a spin structure, then M admits a positive scalar curvature metric if and only if $[M, f] \in {}^{+}\Omega_n^{SO}(B\pi)$.*

Proof. For the spin case let (W, h) be a spin bordism between (M, f) and (N, g) where N admits a positive scalar curvature metric. Now W is in particular oriented, and h restricted to M induces an isomorphism on fundamental groups. By means of Morse theory, one can show that W has the homotopy type of a finite CW complex. Hence $\pi_1(W)$ is finitely generated. We conclude that we may assume by surgery that $h_*\colon \pi_1(W) \to \pi_1(B\pi)$ is an isomorphism. Since W is also spin, one also tends to kill the elements in the kernel of $h_*\colon \pi_2(W) \to \pi_2(B\pi)$ by surgery (see e.g. the proof of Theorem 1.5 in [RS95]). Then, since $f\colon M \to B\pi$ is a two equivalence, it would follow that also the inclusion $M \hookrightarrow W$ is a two equivalence, and we could apply Theorem 3.1.2 to equip M with a positive scalar curvature metric. However, the second homotopy group of a compact manifold is in general *not* finitely generated.

Instead one proceeds as follows. Consider M as a subspace of W. Then the pair (W, M) is 1-connected. The relative Hurewicz theorem (see e.g. [Bre93, Theorem 10.7]) implies that the Hurewicz homomorphism induces an isomorphism between the orbit space of the $\pi_1(M)$-action on $\pi_2(W, M)$ and $H_2(W, M)$. Since $H_2(W, M)$ is a finitely generated Abelian group, it follows that $\pi_2(W, M)$ is finitely generated as a $\pi_1(M)$-module. By means of the long exact homotopy sequence of the pair (W, M),

$$\cdots \to \pi_2(M) \xrightarrow{i_*} \pi_2(W) \to \pi_2(W, M) \xrightarrow{\partial} \pi_1(M) \xrightarrow{\cong} \pi_1(W) \to \cdots,$$

we see that the cokernel of i_* is isomorphic to $\pi_2(W, M)$. Now, killing an embedded two sphere by surgery, always annihilates the whole $\pi_1(M)$-orbit of this sphere. Hence we achieve by surgery that $i_*\colon \pi_2(M) \to \pi_2(W)$ becomes surjective.

In the case of orientable, totally non-spin manifolds, one has to refine this argument slightly. We first state a

Lemma 3.1.5. *Let $\xi \to \Sigma$ be a vector bundle of rank n greater or equal than three over some surface Σ (or any other two dimensional complex). Then ξ is trivial if and only if its first and second Stiefel-Whitney classes vanish.*

Proof. The obstructions against a section in the associated n-frame bundle vanish (see [MS74, p. 140, 143]). □

Now let (W, h) be an oriented bordism between (M, f) and (N, g) where M is totally non-spin and N admits a positive scalar curvature metric. As above, we see that the cokernel of $\pi_2(M) \to \pi_2(W)$ is a finitely generated $\pi_1(M)$-module. However, its elements may be represented by two dimensional spheres having non-trivial normal bundles.

Let $p\colon \widetilde{M} \to M$ be the covering map, and let $\tau\colon M \to BSO$ resp. $\tilde{\tau}\colon \widetilde{M} \to BSO$ be the classifying maps of the stable tangent bundles. Then we have $\tau \circ p \simeq \tilde{\tau}$. Now, since the universal cover $\widetilde{M}$ is not spin, and $\widetilde{M}$ is by definition simply connected, one concludes (see [LM89, Corollary 2.11]) that there exists at least one embedded two dimensional sphere in $\widetilde{M}$ with non-trivial normal bundle. The normal bundle of such a sphere has vanishing first Stiefel-Whitney class. Then Lemma 3.1.5 implies that the second Stiefel-Whitney class of this bundle does *not* vanish. It follows that $\tilde{\tau}_*\colon \pi_2(\widetilde{M}) \to \pi_2(BSO)$ is surjective, and hence so is $\tau_*\colon \pi_2(M) \to \pi_2(BSO)$. We conclude that there exists an embedded two dimensional sphere also in M, say S^2_*, with non-trivial normal bundle. Then, by taking finitely many connected sums, in W, of S^2_* and generators of the cokernel of $\pi_2(M) \to \pi_2(W)$, we achieve that $\pi_2(M) \to \pi_2(W)$ becomes surjective. □

Remark 3.1.6. In contrast to the treatment of simply connected manifolds, the so-called twisted case occurs. By this one understands manifolds which do not admit spin structures but their universal covers do. To the twisted case we also include unoriented manifolds. In order to handle the twisted case, Stolz introduced so-called γ-structures (see [Sto, Definition 2.6]). By means of these γ-structures one can formulate and prove a corresponding twisted Bordism Theorem (see also [RS94, p. 249]).

As announced in the introduction, we want to complete the proof of

Theorem 3.1.7 (Homology Theorem)**.** *Under the same assumptions as in the Bordism Theorem, the following hold:*

1. *If M admits a spin structure, then M admits a positive scalar curvature metric if and only if $A[M, f] \in ko^+_n(B\pi)$.*
2. *If M is orientable and totally non-spin, then M admits a positive scalar curvature metric if and only if $U[M, f] \in H^+_n(B\pi)$.*

Remark 3.1.8. The treatment of a twisted version of this statement is more complicated. It involves the notion of *parametrized homology theories*, which are e.g. discussed in [MS06]. Let B denote a fix space. A parametrized homology theory over B associates an Abelian group to a space X which comes along with a map $X \to B$. In case B is a point, one recovers an underlying ordinary homology theory. For orientable, non-spin manifolds whose universal cover *is* spin, it turns out that one has to consider parametrized homology theories over $K(\mathbb{Z}_2, 2)$; in the unoriented case in addition over $K(\mathbb{Z}_2, 1)$. The formulation and the 2-local proof of the twisted Homology Theorem is the subject of the doctoral thesis of F. Hebestreit [Heb].

We note that the discussion of the unoriented case is from the outset purely 2-local. In addition, after inverting 2, $K(\mathbb{Z}_2, 2)$ is homotopy equivalent to a point. This suggests that with 2 inverted the twisted Homology Theorem can be deduced from the untwisted spin case, which we handle in the sequel.

3.2 The Homology Theorem

Consider the orientation maps

$$A\colon \Omega_n^{Spin}(X) \to ko_n(X) \quad \text{and} \quad U\colon \Omega_n^{SO}(X) \to H_n(X).$$

In order to deduce the Homology Theorem from the Bordism Theorem, one has to show

Theorem 3.2.1. $\ker A \subset {}^+\Omega_*^{Spin}(B\pi)$ *and* $\ker U \subset {}^+\Omega_*^{SO}(B\pi)$.

In fact, assume that $A[M, f]$ lies in $ko_n^+(B\pi)$. Then there exists an element $[N, g] \in {}^+\Omega_n^{Spin}(B\pi)$ such that $A[M, f] = A[N, g]$. This means that $[M, f] - [N, g]$ lies in the kernel of A. According to Theorem 3.2.1, this element is then represented by a closed spin manifold (O, h) in $B\pi$ admitting a positive scalar curvature metric. We conclude that (M, f) is bordant to $(N, g) \dot\cup (O, h)$. By means of the Bordism Theorem, one can now extend the positive scalar curvature metric on $N \dot\cup O$ to M. The orientable, totally non-spin case works analogously.

As explained in the introduction, it remains to give a proof for the 2 inverted part. In the previous chapter we saw that with 2 inverted A resp. U can be identified with the projection $\Omega_*^{Spin}(B\pi) \to \Omega_*^{\mathscr{P}^\alpha}(B\pi)$ resp. $\Omega_*^{SO}(B\pi) \to \Omega_*^{\mathscr{P}^u}(B\pi)$. By the Bockstein exact sequence, or by the very definition of singular $\mathscr{P}$-bordism, we may now assume that elements in the kernel of A resp. U are represented by $\mathscr{P}^\alpha$- resp. $\mathscr{P}^u$-manifolds.

Let us call a family of manifolds locally finite if it contains only finitely many elements of dimension n for all numbers n. The main statement is now

Theorem 3.2.2. *Let $\mathscr{P} = (P_i)_{i\in I}$ be a locally finite family of closed manifolds admitting positive scalar curvature metrics. Then a closed $\mathscr{P}$-manifold, considered as a smooth manifold with additional structure, also admits a positive scalar curvature metric.*

We note that neither orientability and spin structures nor regularity (see 2.4.1) are needed. Before proving Theorem 3.2.2, we first complete the proof of Theorem 3.2.1 with 2 inverted. We have to show that the elements of the locally finite families $\mathscr{P}^u$ and $\mathscr{P}^\alpha$ admit positive scalar curvature metrics.

Proof. As mentioned in Section 2.5, with 2 inverted the kernel of $u_*\colon \Omega_*^{SO} \to H\mathbb{Z}_*$ is generated by projective spaces and hypersurfaces of degree $(1,1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$. In [GL80] it is explained why these manifolds admit positive scalar curvature metrics: The standard Fubini-Study metric on $\mathbb{C}\mathrm{P}^n$ is of positive scalar curvature. Hypersurfaces of degree $(1,1)$ in $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m$ are projective space bundles over projective spaces, and the associated structure group acts by isometries on the fibers. The O'Neill formulas now guarantee that these total spaces can be equipped with positive scalar curvature metrics.

In [KS93, Section 4] it is proved that with 2 inverted the kernel of $\alpha_*\colon \Omega_*^{Spin} \to ko_*$ is generated by total spaces of $\mathbb{H}P^2$-bundles, once again with isometric actions of the structure group. $\square$

Before starting the proof of Theorem 3.2.2, we begin with some preliminary remarks. First, we remind the reader that a *collar metric* on a manifold M with boundary is a collar neighborhood $\partial M \times [0,1]$ together with a metric on M which restricts to $g \times dt^2$ on $\partial M \times [0,1]$, where g is some metric on ∂M and dt^2 is the standard metric on $[0,1]$. All metrics are assumed to be of this form.

Furthermore, we recall that two positive scalar curvature metrics g_0 and g_1 on a closed manifold M are called

- *isotopic* if g_0 and g_1 lie in the same path component of $\mathcal{R}^+(M)$, the space of positive scalar curvature metrics on M equipped with the C^∞ topology,
- *concordant* if there exists a positive scalar curvature metric H on $M \times [0,1]$ such that $H|_{M\times\{0\}} = g_0$ and $H|_{M\times\{1\}} = g_1$.

It is well known fact that isotopy implies concordance (see e.g. [Weh03, Lemma 4.3]. The proof proceeds as follows. If $\gamma\colon [0,1] \to \mathcal{R}^+(M)$ is an isotopy between g_0 and g_1, then the metric on $M \times [0,K]$ which is given by $\gamma(s/K) \times dt^2$ at a point on $M \times \{s\}$, $s \in [0,K]$, is of positive scalar curvature. This metric can be pulled back to a positive scalar curvature metric on $M \times [0,1]$.

We now fix an arbitrary family $(g_i)_{i\in I}$ of positive scalar curvature metrics on $(P_i)_{i\in I}$. The crucial step in the proof of Theorem 3.2.2 is a simple concordance argument which can easily be demonstrated in the case of a $\mathscr{P}$-manifold M consisting of two P_i-parts, i.e.

$$\left(M, (A_i)_{1\le i\le 2}, (B_I, \phi_I)_{I\subset\{1,2\}}\right).$$

According to the definition of a $\mathscr{P}$-manifold, there is an inclusion map $\phi_2^{\{1,2\}}\colon P_1 \times B_{\{1,2\}} \hookrightarrow B_2$ such that

$$\begin{tikzcd} A_1 \cap A_2 \arrow[r, hook] \arrow[d, "\phi_{\{1,2\}}"] & A_2 \arrow[d, "\phi_2"] \\ P_2 \times P_1 \times B_{\{1,2\}} \arrow[r, hook, "\mathrm{id}\times\phi_2^{\{1,2\}}"] & P_2 \times B_2 \end{tikzcd}$$

commutes. We see that there exist submanifolds $Q \subset \partial B_{\{1,2\}}$ and $B_2' \subset B_2$ such that $\phi_{\{1,2\}}(\partial A_1) = P_2 \times P_1 \times Q$ and $\phi_2(A_2') = P_2 \times B_2'$, where $A_2' := A_2 - \mathring{A}_1$. Moreover, by a slight abuse of notation,

$$\phi_2^{\{1,2\}}\colon P_1 \times Q \to \partial B_2'$$

is a diffeomorphism.

Let g_1 and g_2 be positive scalar curvature metrics on P_1 and P_2, respectively. We equip M with a positive scalar curvature metric as follows. First, choose a metric h on Q and extend ${\phi_2^{\{1,2\}}}_*(g_1 \times h)$ to a metric h_2 on B_2'. Then there exists $\epsilon_2 > 0$ such that $G_2 := (\epsilon_2 g_2) \times h_2$ is a positive scalar curvature metric on $\phi_2(A_2')$. Next, choose a collar neighborhood $\partial B \times [-1,0]$ of ∂B

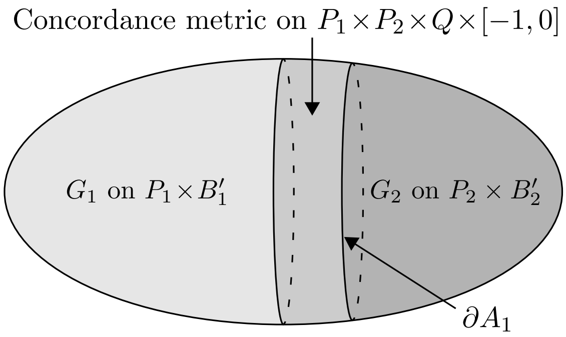

Figure 3.1: Construction of a positive scalar curvature metric on $A_1 \cup A_2$

and extend ${\phi_1^{\{1,2\}}}_*((\epsilon_2 g_2) \times h)$ to a metric h_1 on B_1 such that h_1 restricted to $\partial B \times [-1,0]$ is given by ${\phi_1^{\{1,2\}}}_*((\epsilon_2 g_2) \times h) \times dt^2$. We now find $\epsilon_1 > 0$ such that $G_1 := (\epsilon_1 g_1) \times h_1$ is a positive scalar curvature metric on $\phi_1(A_1)$. Being collar metrics, $\phi_1^*(G_1)$ and $\phi_2^*(G_2)$ restricted to ∂A_1 are of positive scalar curvature. It is obvious that they are isotopic, thus concordant. To obtain the desired positive scalar curvature metric on M, we use the concordance metric on $\partial A_1 \times [-1,0] \subset A_1$ to join $\phi_1^*(G_1)$ restricted to $(A_1 - (\partial A_1 \times [-1,0])$ and $\phi_2^*(G_2)$ on A_2' (see Figure 3.1, we set $B_1' = B_1 - (\partial B_1 \times [-1,0])$).

The main work in the proof of Theorem 3.2.2 for $\mathscr{P}$-manifolds consisting of more P_i-parts merely lies in the fact that metrics on several submanifolds have to be chosen in a compatible way.

For simplicity we omit diffeomorphisms in the sequel. Let M denote an arbitrary $\mathscr{P}$-manifold. Since $\mathscr{P}$ is locally finite, M is covered by finitely many P_i-subsets, say

$$M = A_1 \cup \cdots \cup A_k.$$

As above, fix bicollar neighborhoods $\partial A_i \times [-1,1]$ of $\partial A_i \subset M$, say $\partial A_i \times \{-1\} \subset A_i$, such that $(\partial A_i \times [-1,1]) \cap A_j$ is a bicollar neighborhood for $(\partial A_i) \cap A_j$ in A_j for all j. Define a new covering by $A_1' := A_1$ and, for $i > 1$, $A_i' := A_i - (\cup_{j<i} \mathring{A}_j)$. Then one has $A_i' = P_i \times B_i'$ for appropriate $B_i' \subset B_i$.

We set $A^j = \cup_{i=j}^k A_i'$. Note that A^j is modeled on $\mathbb{H}_1^n \cap \cdots \cap \mathbb{H}_{j-1}^n$, for all $1 < j \leq k$. In addition, A^j inherits collar neighborhoods $(\partial A_i \cap A^j) \times [0,1]$ for all $i < j$. By a collar metric on A^j we understand a metric which

- extends to a smooth metric on M,

- restricts for all $i < j$ to a product metric on $(\partial A_i \cap A^j) \times [0,1]$ with the standard metric on the second factor.

The same notation is used for other manifolds modeled on intersections of half spaces, like A'_j, B'_j and Q^j_I (defined below). All metrics on these manifolds are assumed to be collar metrics.

For all $1 \leq j \leq k$ and

$$I = \{i_1, \dots, i_s\},\ 1 \leq i_1 < \cdots < i_s \leq j-1 \tag{3.2.1}$$

there exists a manifold Q^j_I such that

$$\left(\bigcap_{i\in I} \partial A_i\right) \cap A'_j = P_{i_1} \times \cdots \times P_{i_s} \times P_j \times Q^j_I. \tag{3.2.2}$$

With this notation we have $Q^i_\emptyset = B'_i$. Since $(\cap_{i=1}^{k-1} \partial A_i)$ lies in A'_k, we note that

$$\left(\bigcap_{i=1}^{k-1} \partial A_i\right) = P_1 \times \cdots \times P_k \times Q^k_{\{1,\dots,k-1\}}.$$

The manifold $Q := Q^k_{\{1,\dots,k-1\}}$ can be described as some kind of 'deepest' point of M (see Figure 3.2). Recall that $(g_i)_{i\in I}$ denotes an arbitrary family of positive scalar curvature metrics on $(P_i)_{i\in I}$. We now need the above concordance argument in the following form.

Lemma 3.2.3. *Assume that there exist $1 \leq j < k$ and a positive scalar curvature metric G^{j+1} on A^{j+1}. One finds an $R \subset \partial B'_j$ such that $\partial A_j \cap A^{j+1} = P_j \times R$. In fact,*

$$R = \bigcup_{i=j+1}^{k} P_i \times Q^i_j$$

(see Figure 3.2). Assume further that $G^{j+1}|_{P_j\times R} = g_j \times h_j|_R$ for some metric h_j on B'_j. Then there exists an extension of G^{j+1} to a positive scalar curvature metric G^j on A^j.

Proof. There exists $\epsilon_j > 0$ such that $(\epsilon_j g_j) \times h_j$ is a positive scalar curvature metric on A'_j. Since G^{j+1} and $(\epsilon_j g_j) \times h_j$ are collar metrics, $g_j \times h_j|_R$ and $(\epsilon_j g_j) \times h_j|_R$ are positive scalar curvature metrics. It is obvious that they

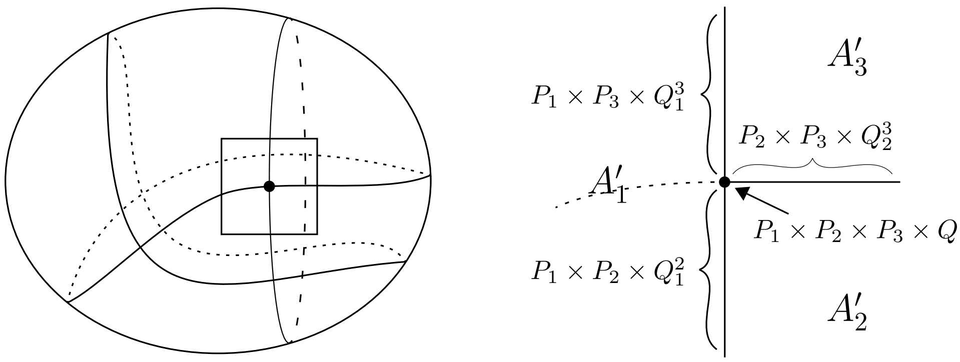

Figure 3.2: $\mathscr{P}$-manifold and deepest point neighborhood

are isotopic, hence concordant. Denote by G the concordance metric on $(P_j \times R) \times [-1, 0]$. We then define a positive scalar curvature metric on

$$A^j = A^{j+1} \cup (P_j \times R \times [-1, 0]) \cup (A'_j - (P_j \times R \times [-1, 0]))$$

by

$$G^{j+1} \cup G \cup \Big((\epsilon_j g_j) \times h_j|_{B'_j - (R \times [-1,0])} \Big) .$$

□

Let us describe our strategy of how to construct a positive scalar curvature metric on M in terms of a $\mathscr{P}$-manifold with three P_i-parts (see Figure 3.2). Take a metric h on Q, and recall that we chose arbitrary positive scalar curvature metrics g_i on P_i, for $1 \leq i \leq 3$. We then:

1. Extend $g_2 \times h$ to a metric h_1^3 on Q_1^3 and extend $g_1 \times h$ to a metric h_2^3 on Q_2^3.

2. Extend $(g_1 \times h_1^3) \cup (g_2 \times h_2^3)$ to a metric h_3 on B'_3.

3. Choose $\epsilon_3 > 0$ such that $G^3 := (\epsilon_3 g_3) \times h_3$ is a positive scalar curvature metric on A'_3.

4. Extend $(\epsilon_3 g_3) \times h$ to a metric h_1^2 on Q_1^2 and $(g_1 \times h_1^2) \cup ((\epsilon_3 g_3) \times h_2^3)$ to a metric h_2 on B'_2.

5. Apply Lemma 3.2.3 to extend G^3 to a positive scalar curvature metric G^2 on $A'_2 \cup A'_3$. Observe that Lemma 3.2.3 gives us a metric f_1 on $P_2 \times Q_1^2$ such that $G^2|_{P_1\times(P_2\times Q_1^2)} = g_1 \times f_1$.
6. Extend $((\epsilon_3 g_3) \times h_1^3) \cup f_1$ to a metric h_1 on B'_1.
7. Apply Lemma 3.2.3 to extend G^2 to a positive scalar curvature metric G^1 on $A'_1 \cup A'_2 \cup A'_3$.

Let us turn back to the general case. For the construction below, it is very helpful to keep Figure 3.2 in mind. Let $1 \le j \le k$ and $1 \le r < j$. One verifies that

$$\partial A^j = \bigcup_{r=1}^{j-1} \left(\partial A_r \cap A^j\right)$$

and

$$\partial A_r \cap A^j = P_r \times \left(\bigcup_{i=j}^{k} P_i \times Q_r^i\right).$$

The following statement is then proved by induction on j, starting with $j = k$ and ending with $j = 1$, from which Theorem 3.2.2 follows immediately.

Lemma 3.2.4. *There exists a positive scalar curvature metric G^j on A^j with compatible product structures on ∂A^j which means that for all $1 \le r < j$ there are metrics f_r^j on $\cup_{i=j}^k P_i \times Q_r^i$ such that*

$$G^j|_{\partial A_r \cap A^j} = g_r \times f_r^j, \tag{3.2.3}$$

where g_r are the fixed positive scalar curvature metrics from above.

Proof. For the initial step, which corresponds to the steps (1) - (3) above, one has to consider A'_k. We first define a metric h_k on B'_k. Observe that for $J \subsetneq \{1, \ldots, k-1\}$

$$\partial Q_J^k = \bigcup_{\{S\subset\{1,\ldots,k-1\}\,|\,J\subset S,\ |S|=|J|+1\}} P_{S-J} \times Q_S^k. \tag{3.2.4}$$

Step by step, starting with $|J| = k-2$ and ending with $J = \emptyset$, we extend the metric h on the deepest point Q to metrics h_J^k on Q_J^k such that

$$h_J^k|_{(P_r\times Q^k_{J\cup\{r\}})\subset\partial Q_J^k} = g_r \times h^k_{J\cup\{r\}}$$

for $\{r\} = I - J$.

In this way one obtains a metric $h_k := h^k_\emptyset$ on $B'_k = Q^k_\emptyset$ such that for all $1 \le r < k$

$$h_k|_{(P_r \times Q^k_r) \subset \partial B'_k} = g_r \times h^k_r. \tag{3.2.5}$$

Now choose an $\epsilon_k > 0$ such that $G^k := (\epsilon_k g_k) \times h_k$ is a positive scalar curvature metric on A'_k. Set $f^k_r := (\epsilon_k g_k) \times h^k_r$. By means of (3.2.5) the condition (3.2.3) is satisfied. This proves the initial step.

We turn to the induction step which corresponds to the steps (4) - (5) resp. (6) - (7) above. For $1 \le j < k$ let G^{j+1} be a positive scalar curvature metric on A^{j+1} such that (3.2.3) is satisfied. We first define a metric on

$$\partial B'_j = \left(\bigcup_{r=1}^{j-1} P_r \times Q^j_r \right) \cup \left(\bigcup_{i=j+1}^{k} P_i \times Q^i_j \right)$$

as follows. As above one finds metrics h^j_r on Q^j_r and we consider $g_r \times h^j_r$ on $P_r \times Q^j_r$ for all $r \le j-1$. According to the induction hypothesis, there is a metric f^{j+1}_j on $\cup^k_{i=j+1} P_i \times Q^i_j$. Now extend

$$\left(\bigcup_{r=1}^{j-1} g_r \times h^j_r \right) \cup f^{j+1}_j$$

to a metric h_j on B'_j.

Finally we apply Lemma 3.2.3 to obtain a positive scalar curvature metric G^j on A^j. We note that the concordance metric in Lemma 3.2.3 does not alter the g_r factor on $P_r \times P_j \times Q^j_r$. Hence, for all $r \le j-1$, there is an induced metric f_r on $P_j \times Q^j_r$ such that $G^j|_{P_r \times (P_j \times Q^j_r)} = g_r \times f_r$. By means of the induction hypothesis, one verifies that the condition (3.2.3) is satisfied with $f^j_r := f_r \cup f^{j+1}_r$ on $\cup^k_{i=j} P_i \times Q^i_r$. $\square$

Chapter 4

Positive Scalar Curvature on Singular $\mathscr{P}$-Manifolds

4.1 Introduction

According to the fundamental theorems of Gromov-Lawson [GL80, Corollary C] in the non-spin case and Stolz [Sto92, Theorem A] in the spin case, the question of the existence of positive scalar curvature metrics on closed simply connected manifolds of dimension greater or equal than five is completely solved. The previous chapter was concerned with this question in the presence of non-trivial fundamental groups. We have completed the proof of the Homology Theorem. In doing so, we constructed on $\mathscr{P}^u$- and $\mathscr{P}^\alpha$-manifolds positive scalar curvature metrics. In other words, we encountered closed singular $\mathscr{P}^u$- and $\mathscr{P}^\alpha$-manifolds which admit positive scalar curvature metrics on their boundaries.

We have the following general definition. Assume that $\mathscr{P}$ is an arbitrary locally finite family of closed manifolds equipped with positive scalar curvature metrics.

Definition 4.1.1. Let M be a closed singular $\mathscr{P}$-manifold, and let g be a positive scalar curvature metric on ∂M obtained by Theorem 3.2.2. We say that M admits a *positive scalar curvature metric* if g can be extended to a positive scalar curvature metric on M.

In the sequel we will show that this definition depends only on the chosen positive scalar curvature metrics on the elements of $\mathscr{P}$, and not on further choices made in the proof Theorem 3.2.2.

Our aim is to address the question of positive scalar curvature on singular manifolds, in the sense of Definition 4.1.1. This problem is not only of theoretical interest. Singular homology and connective real K-theory admit a description by singular manifolds, at least after inverting 2. This means that our notion of positive scalar curvature for singular manifolds yields an a priori definition of positivity for these homology theories; avoiding the detour over ordinary bordism classes of closed manifolds which admit positive scalar curvature metrics. This will be important in the next chapter. In this chapter we study the question of positive scalar curvature on simply connected singular $\mathscr{P}^u$- and $\mathscr{P}^\alpha$-manifolds.

As a preliminary point we want to show that the positive scalar curvature metrics obtained by Theorem 3.2.2 are canonical with respect to $\mathscr{P}$-bordism in the following sense.

Proposition 4.1.2. *Let $\mathscr{P} = (P_i)_{i \in I}$ be a locally finite family of closed manifolds with fixed positive scalar curvature metrics $(g_i)_{i\in I}$. Equip two closed $\mathscr{P}$-manifolds M and N with the positive scalar curvature metrics obtained by Theorem 3.2.2. If W is a $\mathscr{P}$-bordism between M and N, then W admits a positive scalar curvature metric which extends the given metrics on M and N.*

On $\mathscr{P}$-manifolds we will therefore speak of the *canonical positive scalar curvature metric with respect to $(g_i)_{i\in I}$*.

Proof. Let W be a zero $\mathscr{P}$-bordism for a $\mathscr{P}$-manifold M. We will show that a positive scalar curvature metric g on M, obtained by Theorem 3.2.2, can be extended to a positive scalar curvature metric G on W.

It is straightforward to construct a positive scalar curvature metric on W using the proof of Theorem 3.2.2. We note the following: Let $1 \le j \le k$ and $I \subset \{1, \dots, j-1\}$, and let $\hat{Q}^j_I$ denote the corresponding Q^j_I-manifolds of W. This means that

$$(P_I \times P_j \times \hat{Q}^j_I) \cap \partial W = P_I \times P_j \times Q^j_I.$$

If we apply the proof of Lemma 3.2.4 to W, one additional term, namely Q^k_J, occurs in (3.2.4), i.e. for $J \subsetneq \{1, \dots, k-1\}$ we have

$$\partial \hat{Q}^k_J = Q^k_J \cup \bigcup_{\{S \subset \{1,\dots,k-1\} \,|\, J \subset S,\ |S|=|J|+1\}} P_{S-J} \times \hat{Q}^k_S.$$

For the deepest points one has $\partial\hat{Q} = Q$. There is the metric h on Q from the construction of the positive scalar curvature metric on M, and we extend h to some metric $\hat{h}$ on $\hat{Q}$. The extension procedure after (3.2.4) now looks as follows. Again there are the metrics h^k_J on Q^k_J from the construction of the positive scalar curvature metric on M, and we extend stepwise

$$h^k_J \cup \bigcup_{\{S\subset\{1,\ldots,k-1\}\,|\,J\subset S,\ |S|=|J|+1\}} g_{S-J} \times \hat{h}^k_S$$

to metrics $\hat{h}^j_J$ on $\hat{Q}^j_J$.

In the induction step we proceed analogously and obtain a positive scalar curvature metric G on W.

The positive scalar curvature metric $G|_{\partial W}$ may not coincide with g because in the construction of G we chose different scalings $(\hat{\epsilon}_i)_{i\in I}$ than in the construction of g. However, $G|_{\partial W}$ and g are concordant so that $M \times [0, 1]$ and W glued along $M \times \{1\} \cong \partial W$ can be equipped with a positive scalar curvature metric which extends g on $M \times \{0\}$ and G on W. We shall show this concordance in the case of two singularities.

The situation is presented in Figure 4.1 where we use the notations of Figure 3.1 for M (the thick left bar), and the corresponding sets in the description of W (the light gray part) are equipped with a '^' (we replace $[-1, 0]$ by $[0, 1]$). By the construction of G and g, we have

$$g|_{P_i\times B'_i} = (\epsilon_i g_i) \times h_i \quad \text{and} \quad G|_{(P_i\times\hat{B}'_i)\cap\partial W} = (\hat{\epsilon}_i g_i) \times h_i$$

for $i = 1, 2$. These metrics are concordant, hence $(P_1 \times B'_1 \cup P_2 \times B'_2) \times [0, 1]$ (the dark gray part) can be equipped with a positive scalar curvature metric which extends the given restrictions of g and G. We can assume that at a point on $P_i \times B'_i \times \{s\}$, $s \in [0, 1]$, this concordance metric is given by $((1 - s)\epsilon_i + s\hat{\epsilon}_i)g_i \times h_i \times dt^2$.

It remains to construct a positive scalar curvature metric on $P_1 \times P_2 \times Q \times [0, 1]^2$ (the blank square) which extends the given metric on its boundary. This will be a consequence of the following lemma which is a generalization of the statement that isotopy implies concordance. We proceed similarly to the proof of [Weh03, Lemma 4.3].

Recall that $\mathfrak{R}(N)$ denotes the space of all metrics on some manifold N, equipped with the C^∞ topology. The scalar curvature is a continuous function on $\mathfrak{R}(N)$, and the subspace of positive scalar curvature metrics on N, $\mathfrak{R}^+(N)$, is an open subset of $\mathfrak{R}(N)$.

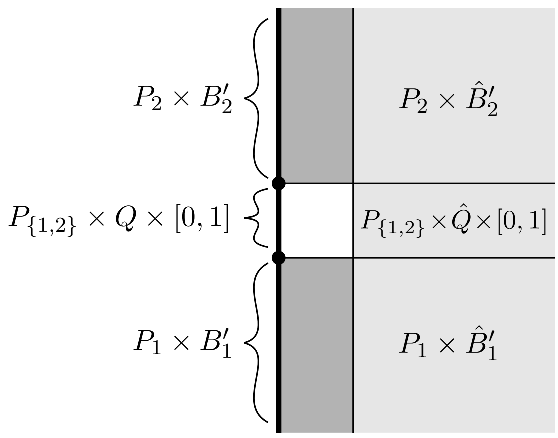

Figure 4.1: Positive scalar curvature metric on $(M \times [0,1]) \cup W$

Lemma 4.1.3. *Let $f\colon [0,1]^2 \to \mathfrak{R}^+(N)$ be a continuous map such that $h := f(x,y) \times dt_x^2 \times dt_y^2$ is a positive scalar curvature metric in a neighborhood of $N \times \partial([0,1]^2)$. Then there exists a positive scalar curvature metric on $N \times [0,1]^2$ which extends h restricted to $N \times \partial([0,1]^2)$.*

Proof. We shall show that there is a number $K > 0$ such that

$$h^K := f\left(\frac{(x,y)}{K}\right) \times dt_x^2 \times dt_y^2$$

is a positive scalar curvature metric on $N \times [0,K]^2$, where $(x,y) \in [0,K]^2$. Consider then the diffeomorphisms $\phi_t\colon N \times [0,1]^2 \to N \times [0, 1 + t(K-1)]^2$, $(n,s) \mapsto (n, (1+t(K-1))s)$, for all $t \in [0,1]$. The family

$$h_t := \phi_t^*\left(f\left(\frac{(x,y)}{1+t(K-1)}\right) \times dt_x^2 \times dt_y^2\right),$$

$(x,y) \in [0, 1+t(K-1)]^2$, defines an isotopy between h and $\phi_1^*(h^K)$, all metrics restricted to $N \times \partial([0,1]^2)$. By gluing the corresponding concordance metric on $N \times \partial([0,1]^2) \times [0,1]$ and $\phi_1^*(h^K)$, we obtain the desired positive scalar curvature metric on $N \times [0,1]^2$.

We may assume that f extends to a neighborhood $U \subset \mathbb{R}^2$ of $[0,1]^2$. Choose $\epsilon > 0$ such that for all $\delta \in [-\epsilon, \epsilon]^2$ the point $\nu + \lambda\delta$ lies in U, for all $\nu \in [0,1]^2$

and $\lambda \in [0,1]$. On $Z := N \times [-\epsilon,\epsilon]^2$ we consider the metric $h_{\nu,\lambda}$ where $h_{\nu,\lambda}$ at a point on $N \times \{\delta\}$, $\delta \in [-\epsilon,\epsilon]^2$, is given by

$$f(\nu+\lambda\delta) \times dt_x^2 \times dt_y^2,$$

$\nu \in [0,1]^2$ and $\lambda \in [0,1]$. Since $h_{\nu,0}$ is a positive scalar curvature metric, and $\mathfrak{R}^+(Z)$ is open in $\mathfrak{R}(Z)$, there exist $\delta_\nu > 0$ and $\lambda_\nu > 0$ such that $h_{\nu',\lambda'}$ is a positive scalar curvature metric for all $(\nu',\lambda') \in B_{\delta_\nu}(\nu) \times (0,\lambda_\nu)$, where $B_{\delta_\nu}(\nu)$ denotes the open ball of radius δ_ν centered at ν. By compactness we can assume that $[0,1]^2$ is covered by finitely many balls $B_{\delta_{\nu_i}}(\nu_i)$, for some $\nu_i \in [0,1]^2$. Now let $\Lambda := \min(\lambda_{\nu_i})$. Then $h_{\nu,\lambda}$ is a positive scalar curvature metric on Z for all $(\nu,\lambda) \in [0,1]^2 \times [0,\Lambda]$.

We now choose $K \geq \Lambda^{-1}$. Any point of $N \times [0,K]^2$ is contained in some $Z_d := N \times (d + [-\epsilon,\epsilon]^2)$ with $d \in [0,K]^2$. But now the map

$$(p,x) \mapsto (p, x-d), \quad p \in N \text{ and } x \in d + [\epsilon,\epsilon]^2,$$

is an isometry between $(Z_d, h^K|_{Z_d})$ and $(Z, h_{d/K,1/K})$, and the latter Riemannian manifold is of positive scalar curvature by the choice of K. This finishes the proof of Lemma 4.1.3. □

Now let us equip $P_1 \times P_2 \times Q \times [0,1]^2$ with a positive scalar curvature metric. We can assume that at a point on $P_1 \times P_2 \times Q \times \{0\} \times \{t\}$, $t \in [0,1]$, the concordance metric of the restriction of g is given by $(t + (1-t)\epsilon_1)g_1 \times \epsilon_2 g_2 \times dt^2$. Analogously, at a point on $P_1 \times P_2 \times Q \times \{1\} \times \{t\}$, $t \in [0,1]$, the concordance metric of the restriction of G is given by $(t + (1-t)\hat{\epsilon}_1)g_1 \times \hat{\epsilon}_2 g_2 \times dt^2$.

The positive scalar curvature metric on $((M \times [0,1]) \cup W) - P_1 \times P_2 \times (0,1)^2$ constructed above induces a metric $\mathfrak{g}$ on $\partial(P_1 \times P_2 \times Q \times [0,1]^2)$ which can now be described as follows. Let $(s,t) \in \partial([0,1]^2)$, and let h denote the metric on Q. Then there is a function

$$f\colon \partial([0,1]^2) \to \mathfrak{R}^+(P_1 \times P_2 \times Q)$$

such that at a point on $P_1 \times P_2 \times Q \times \{(s,t)\}$ the metric $\mathfrak{g}$ is given by

$$f_1(s,t)g_1 \times f_2(s)g_2 \times h \times dt^2$$

where $f_1(s,t) := t + (1-t)((1-s)\epsilon_1 + s\hat{\epsilon}_1)$, $f_2(s) := (1-s)\epsilon_2 + s\hat{\epsilon}_2$ and dt^2 is either the standard metric on $[0,1] \times \{0,1\}$ or on $\{0,1\} \times [0,1]$. Since

$$f_1(s,t) \leq \max(1,\epsilon_1,\hat{\epsilon}_1) \quad \text{and} \quad f_2(s) \leq \max(\epsilon_2,\hat{\epsilon}_2)$$

for all $(s,t) \in [0,1]^2$, it follows that f can be extended to a map

$$[0,1]^2 \to \mathfrak{R}^+(P_1 \times P_2 \times Q).$$

We can now apply Lemma 4.1.3 to equip $P_1 \times P_2 \times Q \times [0,1]^2$ with a positive scalar curvature metric which extends the given metric on the ambient manifold. As in the proof of Theorem 3.2.2, for several singularities Proposition 4.1.2 now follows by induction. □

Corollary 4.1.4. *Let $\mathscr{P} = (P_i)_{i\in I}$ and $(g_i)_{i\in I}$ be as above. Then the concordance class of the positive scalar curvature metric on a $\mathscr{P}$-manifold M, obtained by Theorem 3.2.2, does not depend on the choices made in the proof of Theorem 3.2.2.*

Proof. Let $M \times \{0\}$ and $M \times \{1\}$ be equipped with - possibly different - positive scalar curvature metrics, obtained by Theorem 3.2.2. Then $M \times [0,1]$ can be considered as a $\mathscr{P}$-bordism on which these metrics extend to positive scalar curvature metrics. □

We conclude that Definition 4.1.1 does not depend on the choices made in the proof of Theorem 3.2.2, apart from the positive scalar metrics $(g_i)_{i\in I}$.

4.2 Non-Spin Manifolds

Unless otherwise stated, we assume that all singular manifolds are closed in the sense of Definition 2.3.1. Recall the following fundamental result of Gromov and Lawson.

Theorem 4.2.1 ([GL80, Corollary C]). *Let M be a closed simply connected manifold of dimension greater or equal than five which does not admit a spin structure. Then M admits a metric of positive scalar curvature.*

We remind the reader that $\mathscr{P}^u = (Q_1, Q_2, \ldots)$ denotes a sequence of generators of Ω_*^{SO} modulo torsion which admit positive scalar curvature metrics. For the remainder of this section, we fix an *arbitrary* sequence $(g_1, g_2, \ldots)$ of positive scalar curvature metrics on $(Q_1, Q_2, \ldots)$. The aim of this section is to prove the following singular version of Theorem 4.2.1.

Theorem 4.2.2. *Let M be a simply connected singular $\mathscr{P}^u$-manifold of dimension greater or equal than five which does not admit a spin structure. Then M admits a metric of positive scalar curvature.*

Remark 4.2.3. We note that we neither assume that ∂M is simply connected nor that ∂M does not admit a spin structure.

The main geometric arguments come from the proof of Theorem 4.2.1 and Proposition 4.1.2. In addition, we need

Lemma 4.2.4. *The sequence $\mathscr{P}^u$ of polynomial generators of Ω_*^{SO} modulo torsion is - even without inverting* 2 *- a regular sequence in Ω_*^{SO}.*

Proof. The multiplicative structure of Ω_*^{SO} is described in [Pen82]. We first need some preliminaries. According to Theorem 2.1 loc. cit., there exist elements $y_i \in H_i(MSO;\mathbb{Z}_2)$ such that $H_*(MSO;\mathbb{Z}_2) = \mathbb{Z}_2[y_2, y_3, \ldots]$. One introduces the subalgebra

$$C = \mathbb{Z}_2[y_4, y_5, \ldots, y_n, \ldots],\ n \neq 2 \text{ and } n \neq 2^j - 1.$$

Let ∂ denote the differential on C which is dual to the action of the first Steenrod square on cohomology. It turns out that the kernel of ∂ is the subalgebra of $H_*(MSO;\mathbb{Z}_2)$ consisting of all primitive elements, with respect to the comodule structure of $H_*(MSO;\mathbb{Z}_2)$ over the dual Steenrod algebra. The Hurewicz homomorphism is then a map $Q\colon \Omega_*^{SO} \to \ker \partial$. Furthermore, the image of ∂ is an ideal in the kernel of ∂.

Recall again that Ω_*^{SO} modulo torsion is a polynomial algebra over the integers with certain generators $\hat{u}_{4i} \in \Omega_{4i}^{SO}$, $i \geq 1$. Above we mentioned that we may take $\hat{u}_{4i} = [Q_i]$. Theorem 3.4 in [Pen82] states that there is a group isomorphism

$$\Omega_*^{SO} \cong \mathbb{Z}[\hat{u}_4, \ldots, \hat{u}_{4i}, \ldots] \oplus \operatorname{im} \partial,$$

and multiplication in Ω_*^{SO} is given by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, Q(a_1, 0)b_2 + Q(a_2, 0)b_1 + b_1 b_2).$$

In addition, $Q(\hat{u}_{4i}, 0) = u_{4i}$ where we set

$$u_{4i} = \begin{cases} y_{4i} & \text{if } i = 2^j \\ y_{2_i}^2 & \text{otherwise} \end{cases}.$$

Now let us turn to the proof of the regularity of the sequence $Q_1, Q_2, \ldots$. Let $[a_2, b_2] \in \Omega_*^{SO}/((\hat{u}_4, 0), \ldots, (\hat{u}_{4(n-1)}, 0))$. Then we have

$$\begin{aligned} [Q_n] \cdot [a_2, b_2] &= (\hat{u}_{4n}, 0) \cdot [a_2, b_2] \\ &= [\hat{u}_{4n} a_2, Q(\hat{u}_{4n}, 0) b_2] \\ &= [\hat{u}_{4n} a_2, u_{4n} b_2]. \end{aligned}$$

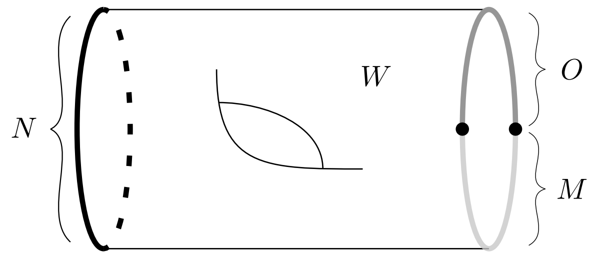

Figure 4.2: Extending a positive scalar curvature metric to M

Assume that $[Q_n] \cdot [a_2, b_2]$ vanishes. Since $\mathbb{Z}[\hat{u}_4, \dots, \hat{u}_{4i}, \dots]$ is a polynomial ring, we first conclude that a_2 lies in the ideal spanned by $\hat{u}_4, \dots, \hat{u}_{4(n-1)}$.

Second we see that

$$u_{4n} b_2 = \sum_{i=1}^{n-1} u_{4i} c_i \tag{4.2.1}$$

for appropriate $c_i \in \operatorname{im} \partial$. Consider (4.2.1) as an equality in the polynomial algebra C. It follows from the definition of the u_{4i}'s that u_{4n} is relatively prime to each $u_4, \dots, u_{4(n-1)}$. We deduce that $c_i = u_{4n} d_i$ for appropriate $d_i \in \operatorname{im} \partial$. Since C is an integral domain, one now concludes that $b_2 = \sum_{i=1}^{n-1} u_{4i} d_i$. Summarizing, we have shown that (a_2, b_2) lies in the ideal generated by $(\hat{u}_4, 0), \dots, (\hat{u}_{4(n-1)}, 0)$. $\square$

Proof of Theorem 4.2.2. Let M be a simply connected singular non-spin $\mathscr{P}^u$-manifold of dimension greater or equal than five. By Lemma 4.2.4 Proposition 2.4.1 applies, and the Bockstein exact sequence (see Proposition 2.3.7) then implies that $\pi\colon \Omega_*^{SO} \to \Omega_*^{\mathscr{P}^u}$ is surjective. It follows that M is singular $\mathscr{P}^u$-bordant to a closed manifold N. Since Ω_*^{SO} is generated by manifolds of positive scalar curvature (see [GL80]), we may assume that N comes equipped with a positive scalar curvature metric. Now let W be a singular $\mathscr{P}^u$-bordism between M and N. Then we have

$$\partial W = N \,\dot{\cup}\, (M \cup_{\partial M} (-O))$$

where O is a zero $\mathscr{P}^u$-bordism for ∂M (see Figure 4.2). According to Proposition 4.1.2, the canonical positive scalar curvature metric on ∂M extends to some positive scalar curvature metric on O.

We briefly recall the arguments from [GL80] to make $M \hookrightarrow W$ into a two equivalence. Clearly, we may assume that W is connected. Since W is oriented, we can kill $\pi_1(W)$ by surgery. The classifying map of the stable tangent bundle of W induces a homomorphism $\tau\colon \pi_2(W) \to \pi_2(BSO) \cong \mathbb{Z}_2$, and elements in the kernel of τ can again be killed by surgery. Note that $\pi_2(W) = H_2(W)$ is finitely generated (compare the proof of the Bordism Theorem). The tangent bundle of W restricted to M splits as the sum of the tangent bundle of M and the trivial bundle. This implies that the classifying map of the stable tangent bundle of M induces the map

$$\pi_2(M) \xrightarrow{incl_*} \pi_2(W) \xrightarrow{\tau} \pi_2(BSO).$$

Since M does not admit a spin structure, this composition is surjective, and τ is an isomorphism. We conclude that $incl_*$ is surjective as well. That is, the inclusion $M \hookrightarrow W$ is a two equivalence, and we may apply Corollary 3.1.3 to extend the positive scalar curvature metric on $N \cup O$ to a positive scalar curvature metric on M. $\square$

4.3 Spin Manifolds

4.3.1 Survey on the Closed Case

In contrast to the non-spin case, a closed simply connected *spin* manifold of dimension greater or equal than five might *not* admit a positive scalar curvature metric. As explained below, one encounters index-theoretic obstructions. A reference for the following remarks is [LM89].

Atiyah and Singer defined on any Riemannian spin manifold (M, g) a first order elliptic differential operator, the so-called *Dirac operator* $D\!\!\!/$ (see [AS63]). Lichnerowicz found the following Weizenböck-type formula

$$D\!\!\!/^2 = \nabla^*\nabla + \frac{scal_g(M)}{4}$$

(see [Lic63]). Here, $\nabla\colon \mathcal{C}^\infty(\$\!\!\!/) \to \mathcal{C}^\infty(T^*M \otimes \$\!\!\!/)$ is the covariant derivative of the spinor bundle $\$\!\!\!/$, and ∇^* is its adjoint with respect to the Hilbert space structure on $\mathcal{C}^\infty(\$\!\!\!/)$ resp. $\mathcal{C}^\infty(T^*M \otimes \$\!\!\!/)$ induced by g.

This observation establishes a very important relationship between the index of Dirac operators and positive scalar curvature metrics: If $scal_g(M)$ is strictly positive, then $D\!\!\!/$ is a positive operator, that is, the index of $D\!\!\!/$ vanishes.

The index of the (graded) Dirac operator can be computed by the Atiyah-Singer index theorem, which tells us that it coincides with the $\hat{A}$-genus of M, a certain characteristic number defined by means of the Pontrjagin classes of M. The $\hat{A}$-genus is therefore an appropriate obstruction against a positive scalar curvature metric on M in case the dimension of M is divisible by four.

Hitchin generalized the arguments to arbitrary dimensions (see [Hit74]). He considered families of operators to define a refined version of the Dirac operator. More precisely, one has a Dirac operator, by abuse of notation again denoted by $D\!\!\!/$, acting on the sections of a canonical associated Clifford bundle, and $D\!\!\!/$ commutes with the action of the Clifford algebra of M. This implies that the index of $D\!\!\!/$ is a module over the Clifford algebra. Using the Atiyah-Bott-Shapiro isomorphism (see [ABS64]), which relates equivalence classes of Clifford modules to KO-theory, one obtains for any n-dimensional spin manifold M the *Clifford index* $\alpha(M) \in ko_n$. According to a family version of the Atiyah-Singer index theorem, the Clifford index only depends on the spin bordism type of M. The resulting map

$$\alpha\colon \Omega_*^{Spin} \to ko_* \tag{4.3.1}$$

is the index-theoretic interpretation of the Atiyah-Bott-Shapiro orientation $MSpin \to ko$.

Stolz showed that in the simply connected case the vanishing of $\alpha(M)$ is also *sufficient* for the existence of a positive scalar curvature metric. The spin counterpart of Theorem 4.2.1 is therefore

Theorem 4.3.1 ([Sto92, Theorem A]). *Let M be a closed simply connected spin manifold of dimension greater or equal than five. Then M admits a positive scalar curvature metric if and only if $\alpha(M) = 0$.*

4.3.2 Singular Manifolds

As in the case of non-spin manifolds, it is our intention to prove a corresponding relative version of Theorem 4.3.1. However, it turns out that the treatment of the spin case is more complicated, and we only obtain partial results. Our first aim shall be to define a singular version of the index map (4.3.1).

For the moment let us consider an arbitrary compact spin manifold M of dimension n admitting a positive scalar curvature metric g on its boundary. On M we shall define the Dirac operator as follows. Attach the cylinder

$\mathbb{R}_{\geq 0} \times \partial M$ to M to obtain a spin manifold $\widetilde{M}$. Now extend g to an arbitrary metric in the interior of M and to $dt^2 \times g$ on the cylinder. In this way, $\widetilde{M}$ becomes a complete spin manifold. On $\widetilde{M}$ we consider the Clifford linear Dirac operator $\not{D}$.

The obstruction argument from the closed case carries over. Theorem 3.2 in [GL83] implies that $\not{D}$ is a Fredholm operator. If g can be extended to a positive scalar curvature metric in the interior of M, the corresponding metric on $\widetilde{M}$ is strictly positive, hence the Lichnerowicz argument implies that the index of $\not{D}$ vanishes.

We will need the singular bordism invariance of this index. For this reason we briefly introduce a relative bordism group R_n as follows. Elements in R_n consists of equivalence classes of pairs (M, g) where M denotes a compact spin manifold and g a positive scalar curvature metric on ∂M. An element bounds if

- there exists a spin zero bordism N for ∂M,
- g can be extended to a positive scalar curvature metric on N,
- $M \cup_{\partial M = \partial N} (-N)$ is spin zero bordant.

Remark 4.3.2. If M is simply connected one may use Corollary 3.1.3 to show that g extends to a positive scalar curvature metric on M if $[M, g] = 0$ in R_n. To treat the case of non-trivial fundamental groups, Hajduk introduced groups $R_n(\pi)$ (see [Haj91]). Hajduk (see Theorem 7.1 loc. cit.) and Stolz (see [Sto, Theorem 1.1]) showed that g extends to a positive scalar curvature metric on M if $[M, g] = 0$ in $R_n(\pi)$.

Now it is a consequence of the Clifford linear version of Gromov and Lawson's relative index theorem (see [GL83, Theorem 4.18]) that $\not{D}$ induces a homomorphism

$$\theta \colon R_n \to ko_n. \tag{4.3.2}$$

This is proved by Bunke in the more general situation taking non-trivial fundamental groups into account. In the simply connected case the index map in [Bun95, Theorem 1.17] reduces to (4.3.2).

We now turn to singular $\mathscr{P}^\alpha$-manifolds. Recall that $\mathscr{P}^\alpha = (R_2, R_3, \ldots)$ denotes a sequence of closed spin manifolds which admit positive scalar curvature metrics and which generate $\ker(\alpha \colon \Omega_*^{Spin} \to ko_*)$ with 2 inverted. We first consider an *arbitrary* sequence of positive scalar curvature metrics

$\mathfrak{g} = (g_2, g_3, \ldots)$ on $(R_2, R_3, \ldots)$. We equip the boundary of some singular $\mathscr{P}^\alpha$-manifold M with the associated canonical positive scalar curvature metric. If M is zero singular $\mathscr{P}^\alpha$-bordant, we have a pair (N, W) where N is a zero $\mathscr{P}^\alpha$-bordism of ∂M and W a zero bordism for $M \cup_{\partial M = \partial N} (-N)$. According to Proposition 4.1.2, the canonical positive scalar curvature metric on ∂M extends to a positive scalar curvature metric on N, hence one obtains a well-defined map $\phi\colon \Omega_*^{\mathscr{P}^\alpha} \to R_*$. We then define the singular index map as

$$\alpha^{(\mathscr{P}^\alpha, \mathfrak{g})}\colon \Omega_*^{\mathscr{P}^\alpha} \xrightarrow{\phi} R_* \xrightarrow{\theta} ko_*. \tag{4.3.3}$$

We note that the composition of $\pi\colon \Omega_*^{Spin} \to \Omega_*^{\mathscr{P}^\alpha}$ with $\alpha^{(\mathscr{P}^\alpha, \mathfrak{g})}$ coincides with (4.3.1). As long as the family $\mathfrak{g}$ can be chosen arbitrarily, we will write $\alpha^{\mathscr{P}^\alpha}$ instead of $\alpha^{(\mathscr{P}^\alpha, \mathfrak{g})}$.

As noted above, $\alpha^{\mathscr{P}^\alpha}$ is an obstruction against positive scalar curvature metrics on singular $\mathscr{P}^\alpha$-manifolds.

Corollary 4.3.3. *Let M denote a singular $\mathscr{P}^\alpha$-manifold admitting a positive scalar curvature metric. Then $\alpha^{\mathscr{P}^\alpha}(M) = 0$.*

This is the 'only if' part of our

Conjecture 4.3.4. *Let M be a simply connected singular $\mathscr{P}^\alpha$-manifold of dimension greater or equal than five. Then M admits a positive scalar curvature metric if and only if $\alpha^{\mathscr{P}^\alpha}(M) = 0$.*

For the remainder of this chapter the 'if' part of Conjecture 4.3.4 is proved in several cases. Because of the complicated multiplicative structure of the spin bordism ring, we are not able not prove Conjecture 4.3.4 completely.

We need

Proposition 4.3.5. *Let M be a singular $\mathscr{P}^\alpha$-manifold equipped with the canonical positive scalar curvature metric g on its boundary. If there exists a spin zero bordism N for ∂M such that $g|_{\partial M = \partial N}$ extends to a positive scalar curvature metric on N, then Conjecture 4.3.4 is true for M.*

Proof. Let $W = M \cup_{\partial M = \partial N} (-N)$. Then we have

$$\alpha(W) = \alpha^{\mathscr{P}^\alpha}(M) - \alpha^{\mathscr{P}^\alpha}(N).$$

We must show that M admits a positive scalar curvature metric if $\alpha^{\mathscr{P}^\alpha}(M) = 0$. Since N admits a positive scalar curvature metric, we have

$\alpha^{\mathscr{P}^\alpha}(N) = 0$. Hence, if $\alpha^{\mathscr{P}^\alpha}(M) = 0$ then also $\alpha(W) = 0$. Killing the elements of $\pi_1(W)$ by surgery, we obtain a spin bordism V between W and a simply connected spin manifold W'. Then $\alpha(W') = 0$, and according to Theorem 4.3.1, W' admits a metric of positive scalar curvature. By surgery we may assume once more that the inclusion $M \hookrightarrow V$ is a two equivalence. We can now apply Corollary 3.1.3 to extend the positive scalar curvature metric on $W' \dot{\cup} N$ to one on M. □

An immediate consequence is

Corollary 4.3.6. *Let M be a singular $\mathscr{P}^\alpha$-manifold such that $[M]$ lies in the image of $\pi\colon \Omega_*^{Spin} \to \Omega_*^{\mathscr{P}^\alpha}$, i.e. M is singular $\mathscr{P}^\alpha$-bordant to a closed manifold, then Conjecture 4.3.4 holds for M.*

Proof. If M is singular $\mathscr{P}^\alpha$-bordant to a closed manifold, then there exists a zero $\mathscr{P}^\alpha$-bordism N for ∂M. By Proposition 4.1.2 the canonical positive scalar curvature metric on ∂M extends to N. □

The crucial difficulties in the spin case arise from the fact that

$$\pi\colon \Omega_*^{Spin} \to \Omega_*^{\mathscr{P}^\alpha} \tag{4.3.4}$$

is not surjective in general - in contrast to the oriented case. The reader may have noticed that $\mathscr{P}^u$ was assumed to be a sequence of polynomial generators of Ω_*^{SO} modulo torsion, and $\mathscr{P}^\alpha$ is merely any sequence of polynomial generators of Ω_*^{Spin} with 2 inverted. However, it turns out that we never achieve that (4.3.4) becomes surjective, even if $\mathscr{P}^\alpha$ is chosen with caution.

In order to give an example of a singular $\mathscr{P}^\alpha$-manifold which is not singular $\mathscr{P}^\alpha$-bordant to a closed manifold, we consider characteristic numbers for spin bordism. Besides the Stiefel-Whitney numbers, which we shall call $\mathbb{Z}_2$-characteristic numbers, so-called KO-characteristic numbers are crucial. First, we have to know that an n-dimensional closed spin manifold M has a spin orientation class $[M] \in KO_n(M)$ (see [ABS64]). Second, for $j \geq 0$ one can define KO-Pontrjagin classes $\pi^j \in KO^0(BSpin)$ (see [ABP66]). Now let $J = (j_1, \dots, j_k)$ denote a sequence of integers, set $\pi^J = j_1 \cdots j_k$, and let $f\colon M \to BSpin$ be the classifying map of the stable tangent bundle of M. In the same way as for other characteristic numbers, the Kronecker pairing

$$\pi^J(M) := \left\langle f^*(\pi^J), [M] \right\rangle \in KO_n$$

yields the desired KO-characteristic numbers for M. We recall at this point

$$KO_* \cong \mathbb{Z}[\eta, \omega, \mu, \mu^{-1}]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\mu)$$

where $deg(\eta) = 1$, $deg(\omega) = 4$ and $deg(\mu) = 8$. Let us note that $\pi^0(M)$ coincides with the image of $\alpha(M)$ under the periodization map.

Structure results about spin bordism are given in the fundamental work [ABP67]. Corollary 2.4 loc. cit. states that M is spin zero bordant if and only if all its $\mathbb{Z}_2$- and KO-characteristic numbers vanish.

Example 4.3.7. Consider the 13-dimensional spin manifold $R_3 \times S^1$, where R_3 is any polynomial generator of Ω_*^{Spin} (with 2 inverted) of dimension 12, and S^1 is equipped with its non-trivial spin structure, i.e. $[S^1] \neq 0$ in Ω_1^{Spin}. Recall that KO_n is trivial for $n \equiv 3, 5, 6, 7 \mod 8$. Hence all KO-characteristic numbers of $R_3 \times S^1$ vanish for dimension reasons. Since S^1 has vanishing $\mathbb{Z}_2$-characteristic numbers, so does $R_3 \times S^1$. One concludes that $R_3 \times S^1$ is spin zero bordant, say via W. Then W defines a singular R_3-manifold. Now, $[W]$ does not lie in the image of $\pi\colon \Omega_*^{Spin} \to \Omega_*^{R_3}$ because $\delta_{R_3}[W] = [S^1]$ does not vanish in Ω_1^{Spin} (compare the long exact sequence (2.3.1)).

We conclude that R_i is in general a zero divisor, and the sequence $R_2, R_3, \ldots$ is far from being regular. One may say that singular $\mathscr{P}^\alpha$-manifolds appear in a greater diversity than singular $\mathscr{P}^u$-manifolds, which are always singular $\mathscr{P}^u$-bordant to closed manifolds (as a consequence of Lemma 4.2.4).

In other words, the treatment of singular $\mathscr{P}^\alpha$-manifolds turns out to be more complicated than the treatment of singular $\mathscr{P}^u$-manifolds. Even for manifolds with a single R_i-singularity the problem is non-trivial, and a fortiori for several singularities. Therefore, for the remainder of this chapter, we will concentrate on $\mathscr{P}^\alpha$-manifolds M with a single R_i-part on its boundary. That is, ignoring indices, we assume from now on that M is a compact spin manifold with $\partial M = R \times B$, for some $R \in \mathscr{P}^\alpha$ and some spin manifold B.

Since Ω_*^{Spin} modulo torsion is the subalgebra of some polynomial algebra (see [Sto66, Theorem 1]), Ω_*^{Spin} modulo torsion is an integral domain. In addition, R is of infinite order in Ω_*^{Spin}. It follows that $R \times B$ can be zero bordant only if B is a torsion element.

The natural map $MSpin \to MSO$ is a homotopy equivalence after inverting 2, and Ω_*^{SO} contains no odd torsion. This implies that the spin bordism ring Ω_*^{Spin} contains no odd torsion either. The remaining 2-primary torsion is described in [ABP67]. To explain these results, we recall the spin bordism groups up to dimension eight. For the remainder of this chapter let S^1 be equipped with its non-trivial spin structure. Then $\Omega_1^{Spin} \cong \mathbb{Z}_2$ is generated by S^1, $\Omega_2^{Spin} \cong \mathbb{Z}_2$ is generated by $S^1 \times S^1$, $\Omega_4^{Spin} \cong \mathbb{Z}$ is generated by a K3 surface, denoted above by K, $\Omega_8^{Spin} \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the quaternionic projective space $\mathbb{H}P^2$ and the Bott manifold B (recall that $\alpha(B)$ in $ko_8 \cong \mathbb{Z}$ is a generator). The bordism groups Ω_3^{Spin}, Ω_5^{Spin}, Ω_6^{Spin} and Ω_7^{Spin} are trivial.

In [ABP67] it is shown (see also [Sto68, p. 339])

Theorem 4.3.8. *Let $J = (j_1, \ldots, j_k)$ denote a sequence of integers with $k \geq 1$ and $j_i \geq 2$, set $n(J) = j_1 + \cdots + j_k$. There exist characteristic classes $z_i \in H^*(BSpin; \mathbb{Z}_2)$ such that the following hold:*

For J above with $n(J)$ even there are spin manifolds M_J of $\dim 4n(J)$ and of infinite order in Ω_*^{Spin} such that $\pi^J(M_J)$ is odd, as a multiple of the KO_* generators. All further KO-characteristic numbers of M_J and all $z_i(M_J)$ (z_i evaluated on the $H\mathbb{Z}_2$-fundamental class of M_J) vanish.

For J above with $n(J)$ odd there are spin manifolds N_J of $\dim 4n(J) - 2$ and of order two in Ω_*^{Spin}, and spin manifolds M_J of $\dim 4n(J)$ and of infinite order in Ω_*^{Spin}, such that $\pi^J(N_J)$ and $\pi^J(M_J)$ are odd, as a multiple of the KO_* generators. All further KO-characteristic numbers of N_J, M_J and all $z_i(N_J)$, $z_i(M_J)$ vanish.

In addition, there are spin manifolds Z_i of $\dim |z_i|$ and of order two in Ω_*^{Spin} with vanishing KO-characteristic numbers such that $z_i(Z_j)$ is non-trivial if and only if $i = j$.

A list of additive generators for $\Omega_*^{Spin} \otimes \mathbb{Z}_2$ is then given by

1. $[M_J] \times [B]^k \times [S^1]^i$, $k \geq 0$, $0 \leq i \leq 2$, $n(J)$ *even,*
2. $[M_J] \times [K] \times [B]^k$, $k \geq 0$, $n(J)$ *even,*
3. $[Z_i]$,
4. $[N_J]$, $n(J)$ *odd,*
5. $[M_J] \times [B]^k$, $k \geq 0$, $n(J)$ *odd,*
6. $([M_J] \times [K])/4 \times [B]^k \times [S^1]^i$, $k \geq 0$, $0 \leq i \leq 2$, $n(J)$ *odd.*

We saw that B has to be 2-torsion if $R \times B$ is zero bordant. In the list above, we detect the torsion elements under the items 1 (with $i \neq 0$), 3, 4 and 6 (with $i \neq 0$). It follows that B is bordant to a spin manifold of the form

$$(S^1 \times C) \dot\cup D$$

where $\alpha(S^1 \times C) \neq 0$ (C is some spin manifold which we do not specify further) and D is a sum consisting of appropriate Z_i's, N_J's and $S^1 \times X$ with $\alpha(S^1 \times X) = 0$ (X is some spin manifold which we do not specify further). Let us note that $\alpha(D) = 0$ since the α-invariant vanishes for all Z_i, N_J and $S^1 \times X$.

The following extension problem will be solved in the next two subsections.

Theorem 4.3.9. *Assume that $R \times D$ and $R \times S^1 \times C$ are zero bordant.*

1. *Let g be an arbitrary positive scalar curvature metric on R, and let h^D be a metric on D such that $g \times h^D$ is of positive scalar curvature. Then there exists a zero bordism $\hat{D}$ for $R \times D$ such that $g \times h^D$ extends to a positive scalar curvature metric on $\hat{D}$.*
2. *There exist a metric h^C on $S^1 \times C$, a zero bordism $\hat{C}$ for $R \times S^1 \times C$ and a positive scalar curvature metric g on R such that $g \times h^C$ is a positive scalar curvature metric on $R \times S^1 \times C$ which extends to one on $\hat{C}$.*

For a single singularity, Conjecture 4.3.4 can now be deduced from Theorem 4.3.9, with a specific positive scalar curvature metric on the singularity.

Corollary 4.3.10. *Let M be a simply connected singular R-manifold of dimension greater or equal than five. Then there exists a positive scalar curvature metric g on R such that M admits a positive scalar curvature metric if and only if $\alpha^{(R,g)}(M) = 0$.*

Before proving Corollary 4.3.10, we mention one result concerning the ring structure of Ω_*^{Spin}. Let R_* be the ring

$$\mathbb{Z}[x_{4i}, Y_{8j+2}, \theta_1 \,|\, i \geq 1, j \geq 1]$$

modulo the relations

$$2\theta_1 = 2Y_t = Y_t Y_r = \theta_1^3 = 0,\ Y_{8j+2}x_4 = x_{8j+4}\theta_1^2,\ Y_{8j+2}x_{8l+4} = Y_{8l+2}x_{8j+4}.$$

In addition, let I_* denote the subset of Ω_*^{Spin} consisting of all classes for which all KO-characteristic numbers vanish. By the Cartan formula, $\pi^i(M \times N) = \sum_{j+k=i} \pi^j(M)\pi^k(N)$, one sees that I_* is an ideal in Ω_*^{Spin}. The ring structure of Ω_*^{Spin}/I_* is determined in [ABP67, Theorem 2.10] (see also [Sto68, p. 344]).

Theorem 4.3.11. *The ring Ω_*^{Spin}/I_* is isomorphic to the subring of R_* generated by the Y's, θ_1, the R_{8k}'s and $2R_{8k+4}$'s.*

Proof of Corollary 4.3.10. Let $\partial M = R \times B$ and recall that B is bordant to a disjoint union $(S^1 \times C) \dot{\cup} D$. We first want to show that $R \times S^1 \times C$ and $R \times D$ themselves have to be zero bordant. Note that M is a bordism between $R \times S^1 \times C$ and $R \times D$. Hence it suffices to show that all $\mathbb{Z}_2$- and KO-characteristic numbers of $R \times S^1 \times C$ vanish.

The $\mathbb{Z}_2$-characteristic numbers of S^1 vanish, and in turn those of $R \times S^1 \times C$.

If $\dim R \equiv 4 \mod 8$, then the KO-characteristic numbers of $R \times S^1$ vanish for dimension reasons, and in turn those of $R \times S^1 \times C$.

Let $\dim R \equiv 0 \mod 8$. The projection of the spin bordism class of a closed spin manifold X in Ω_*^{Spin}/I_* is denoted by $\overline{X}$. By means of Theorem 4.3.11 we conclude the following: If $\overline{R}$ is odd, as a multiple of the Ω_*^{Spin}/I_*-generators, then $\overline{R}$ is not a zero divisor. Since $\overline{R \times S^1 \times C} = \overline{R \times D}$, it would follow that $\overline{S^1 \times C} = \overline{D}$ which implies that the KO-characteristic numbers of $S^1 \times C$ and D coincide; in particular $\alpha(S^1 \times C) = 0$ which we excluded above, however. If $\overline{R}$ is even, then $\overline{R \times S^1 \times C}$ is zero, implying that all KO-characteristic numbers of $R \times S^1 \times C$ vanish.

We conclude that $R \times S^1 \times C$ and $R \times D$ are zero bordant. According to Theorem 4.3.9, there exist zero bordisms $\hat{C}$ and $\hat{D}$ for $R \times S^1 \times C$ and $R \times D$, respectively, and a positive scalar curvature metric g on R such that $g \times h^C$ and $g \times h^D$ extend to positive scalar curvature metrics on these zero bordisms.

Now let M be as in Corollary 4.3.10. Equip $\partial M = R \times B$ with the canonical positive scalar curvature metric with respect to g, i.e. choose a metric h^B on B such that $g \times h^B$ is of positive scalar curvature. Take any bordism W between B and $(S^1 \times C) \dot{\cup} D$. It follows that

$$(R \times W) \cup_{R \times ((S^1 \times C) \dot{\cup} D)} (-(\hat{C} \dot{\cup} \hat{D}))$$

is a zero bordism for ∂M, and $g \times h^B$ extends to a positive scalar curvature metric on this zero bordism. We can now use Proposition 4.3.5 to finish the proof of Corollary 4.3.10. □

Remark 4.3.12. We assumed that R is a polynomial generator of $\ker \alpha \subset \Omega_*^{Spin}$ with 2 inverted. One may ask for which R there actually exist singular R-manifolds that do not lie in the image of $\pi\colon \Omega_*^{Spin} \to \Omega_*^R$. An answer would involve a detailed analysis of the KO- and $\mathbb{Z}_2$-characteristic classes of the specifically chosen R, which we do want to address at this point. We note the following. One could replace any R by $R \dot\cup R$, then $R \times B$ bounds for any torsion element B. In contrast, even if we choose R such that $[R]$ is not divisible by 2 in Ω_*^{Spin}, we would not prevent $[R]$ from being a zero divisor (see Example 4.3.7).

4.3.3 First Extension Problem

We begin with a general remark. To a map of spectra $A \to X$, one can associate its mapping cone or homotopy cofiber $X \cup CA$, which comes along with a natural map $X \to X \cup CA$ (see [Swi02, Chapter 8]). This assignment is functorial in the following sense. If $A \to X$ and $B \to Y$ denote maps between spectra such that

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

commutes, then one gets an induced map $f\colon X \cup CA \to Y \cup CB$. Likewise, one obtains an induced map $g\colon B \cup CA \to Y \cup CX$, and the homotopy cofibers of f and g coincides, i.e. there is a natural homotopy equivalence

$$(Y \cup CB) \cup C(X \cup CA) \simeq (Y \cup CX) \cup C(B \cup CA). \tag{4.3.5}$$

An analogue statement holds for homotopy *fibers*.

Let P denote a closed spin manifold. We remind the reader that the Thom-Pontrjagin construction yields an isomorphism $\pi_*(MSpin) \cong \Omega_*^{Spin}$. Let $\phi\colon S^{\dim P} \to MSpin$ correspond to P. According to [Baa73, Theorem 4.2], $MSpin^P$ is homotopy equivalent to the homotopy cofiber of

$$\omega\colon S^{\dim P} \wedge MSpin \xrightarrow{\phi\wedge \mathrm{id}} MSpin \wedge MSpin \xrightarrow{mult} MSpin, \tag{4.3.6}$$

where *mult* denotes the multiplication of the ring spectrum $MSpin$. We note that the associated long exact sequence is given by (2.3.1).

We obtain a further spectrum ko^P as the homotopy cofiber of

$$\eta\colon S^{\dim P}\wedge ko \xrightarrow{\phi\wedge\mathrm{id}} MSpin\wedge ko \xrightarrow{\alpha\wedge\mathrm{id}} ko\wedge ko \xrightarrow{mult'} ko, \tag{4.3.7}$$

where $mult'$ denotes the multiplication of the ring spectrum ko.

Since α is a map of ring spectra, these cofiber sequences induce a map $MSpin^P \to ko^P$ and, on coefficients, an 'index' map

$$\overline{\alpha}\colon \Omega_*^P \to ko_*^P.$$

The main result of this subsection is

Proposition 4.3.13. *Let N denote a compact simply connected singular P-manifold of dimension greater or equal than five with $\partial N = P\times Q$. Assume that P admits a positive scalar curvature metric g, and let h^Q be a metric on Q such that $g\times h^Q$ is a positive scalar curvature metric on ∂N.*

If $\overline{\alpha}[N]$ vanishes, then $g\times h^Q$ extends to a positive scalar curvature metric on N.

Let us note that $\overline{\alpha}$ does not define an obstruction against positive scalar curvature metrics. As a consequence of the second part of Theorem 4.3.9, the singular R_3-manifold W in Example 4.3.7 admits a positive scalar curvature metric. However, it follows from Diagram 4.3.8 below and $\alpha(S^1)\neq 0$ that $\overline{\alpha}[W]$ does not vanish.

Before verifying Proposition 4.3.13, we give the

Proof of Theorem 4.3.9, Part 1. Recall that $R\times D$ is assumed to be zero bordant, and $\alpha(D)=0$. We will use Proposition 4.3.13 with $P=R$ and $Q=D$. There is a map between exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_*^{Spin} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \Omega_*^R \arrow[r, "\delta_R"] \arrow[d, "\overline{\alpha}"] & \Omega_{*-\dim R-1}^{Spin} \arrow[r] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r] & ko_* \arrow[r, "\tilde{\pi}"] & ko_*^R \arrow[r, "\tilde{\delta}_R"] & ko_{*-\dim R-1} \arrow[r] & \cdots .
\end{tikzcd} \tag{4.3.8}$$

Let V be any zero bordism for $R\times D$. We may consider V as a singular R-manifold. Since $(\alpha\circ\delta_R)[V]=\alpha[D]$ vanishes, it follows that $\overline{\alpha}[V]$ lies in the kernel of $\tilde{\delta}_R$, hence in the image of $\tilde{\pi}$. As $\alpha\colon \Omega_*^{Spin}\to ko_*$ is surjective, there exists $[W]\in\Omega_n^{Spin}$ such that $(\tilde{\pi}\circ\alpha)[W]=\overline{\alpha}[V]$. This implies that $[V]-\pi[W]$

lies in the kernel of $\overline{\alpha}$. We note that taking the connected sum of V and $(-W)$ leads to a singular R-bordism between $V \dot{\cup} (-W)$ and $V \sharp (-W)$. In addition, after surgery we may assume that $V \sharp (-W)$ is simply connected (again, this surgery process corresponds to a singular R-bordism). We now have $\overline{\alpha}[V \sharp (-W)] = 0$. According to Proposition 4.3.13, the product of an arbitrary positive scalar curvature metric on R with a metric h^d on D extends to a positive scalar curvature metric on $V \sharp (-W)$. We can therefore take $V \sharp (-W)$ as $\hat{D}$ to prove Theorem 4.3.9. □

The proof of Proposition 4.3.13 heavily depends on a result of Stolz (see Theorem 4.3.14 below), we shall briefly recall the associated construction presented in [Sto92]. Stolz' proof of Theorem 4.3.1 as well as his proof of the 2-local part of the Homology Theorem depends on it. In addition, we note that in [KS93, Section 4] this construction yields the generators of $\ker(\alpha\colon \Omega_*^{Spin} \to ko_*)$ with 2 inverted, which we used in the proof of the Homology Theorem.

The projective symplectic group $G = PSp(3)$ acts on $\mathbb{H}P^2$ with isotropy group $H = P(Sp(2) \times Sp(1))$. Hence we obtain a fiber bundle

$$\mathbb{H}P^2 = G/H \hookrightarrow BH \xrightarrow{p} BG.$$

Let $f\colon X \to BG$ denote a closed spin manifold in BG. Consider its pullback $f^*(BH)$ in BH composed with the projection $BH \xrightarrow{pr} pt$, i.e.

$$\begin{tikzcd} f^*(BH) \arrow[r] \arrow[d] & BH \arrow[r, "pr"] \arrow[d, "pr"] & pt \\ X \arrow[r, "f"] & BG. & \end{tikzcd}$$

Then $f^*(BH)$ is the total space of an $\mathbb{H}P^2$-bundle. It turns out that the (unique) spin structure of $\mathbb{H}P^2$ induces a spin structure on the tangent bundle along the fibers over BH. Now it is not difficult to see that there is an induced map on bordism groups

$$\psi\colon \Omega_{*-8}^{Spin}(BG) \to \Omega_*^{Spin}, \quad [X, f] \mapsto [f^*(BH)]. \tag{4.3.9}$$

By means of Thom-Pontrjagin theory and a homotopy theoretic interpretation of transfer maps (see [BG75]), one obtains a good understanding of ψ. In fact, using results of [Boa66] (also carried out in [Füh08, Proposition 2.22]), Stolz showed that ψ is induced by a spectrum map

$$T\colon MSpin \wedge \Sigma^8 BG_+ \to MSpin, \tag{4.3.10}$$

here we have the suspension isomorphisms $\widetilde{\Omega}_{*-8}(BG_+) \cong \widetilde{\Omega}_*(\Sigma^8 BG_+)$ in mind. Let $\widehat{MSpin}$ denote the homotopy fiber of $\alpha\colon MSpin \to ko$. One can show (see [Sto92, Proposition 1.1]) that $\alpha \circ T$ is null homotopic, hence T lifts to a map $\widehat{T}\colon MSpin \wedge \Sigma^8 BG_+ \to \widehat{MSpin}$. Using techniques from the Adams spectral sequence and splitting results of $MSpin$-module spectra, Stolz proved [Sto94, Proposition 8.3]

Theorem 4.3.14. *The map $\widehat{T}$ is a 2-local split surjection of spectra.*

As mentioned above, in [KS93, Section 4] it is shown that *with* 2 *inverted* $\widehat{T}_*$ is a surjection, i.e. the sequence

$$\Omega^{Spin}_{*-8}(BG)\left[\tfrac{1}{2}\right] \xrightarrow{\psi} \Omega^{Spin}_{*}\left[\tfrac{1}{2}\right] \xrightarrow{\alpha} ko_*\left[\tfrac{1}{2}\right] \tag{4.3.11}$$

is exact.

The construction of ψ also works in the singular case: Let (S, f) be a singular P-manifold in BG, say $\partial S = P \times B$. By definition there is a map $\tilde{f}\colon B \to BG$ such that $f|_{\partial S} = \tilde{f} \circ pr_B$. That is, we have $\partial(f^*(BH)) = P \times \tilde{f}^*(BH)$ and we obtain a map

$$\psi^P\colon \Omega^P_{*-8}(BG) \to \Omega^P_*, \quad [S, f] \mapsto [f^*(BH)].$$

Proof of Proposition 4.3.13. The first step will be homotopy-theoretic by nature. Taking homotopy fibers, $\omega\colon S^{\dim P} \wedge MSpin \to MSpin$ (see 4.3.6) induces a map $\hat{\omega}\colon S^{\dim P} \wedge \widehat{MSpin} \to \widehat{MSpin}$. In the following diagram, the left part commutes, and one obtains a map $\widehat{T}^P$ between homotopy cofibers:

$$\begin{tikzcd}
\Sigma^{\dim P} MSpin \wedge \Sigma^8 BG_+ \arrow[r, "\omega\wedge\mathrm{id}"] \arrow[d, "\Sigma^n(\widehat{T})"] & MSpin \wedge \Sigma^8 BG_+ \arrow[r, "\pi\wedge\mathrm{id}"] \arrow[d, "\widehat{T}"] & MSpin^P \wedge \Sigma^8 BG_+ \arrow[d, dashed, "\widehat{T}^P"] \\
\Sigma^{\dim P}\widehat{MSpin} \arrow[r, "\hat{\omega}"] & \widehat{MSpin} \arrow[r] & \widehat{MSpin^P}.
\end{tikzcd}$$

Consider this diagram now localized at 2. Since taking homotopy cofibers is functorial, it follows that $\widehat{T}^P$, which is induced by the split surjections $\Sigma^{\dim P}(\widehat{T})$ and $\widehat{T}$, is split surjective as well. In addition, the induced map on homotopy groups of the composition of $\widehat{T}^P$ with the inclusion $\widehat{MSpin^P} \to MSpin^P$ is just given by ψ^P. Now observe that $\widehat{MSpin^P}$ coincides with the homotopy fiber of $\overline{\alpha}\colon MSpin^P \to ko^P$ (homotopy fiber version of 4.3.5). We therefore proved, 2-locally,

Lemma 4.3.15. $\ker\left(\overline{\alpha}\colon \Omega_*^P \to ko_*^P\right) \subset \operatorname{im}\left(\psi^P\colon \Omega_{*-8}^P(BG) \to \Omega_*^P\right)$.

With 2 inverted, Lemma 4.3.15 can easily be shown as follows. We consider the following diagram with 2 inverted

$$\begin{tikzcd}
\Omega_{*-8}^{Spin}(BG) \arrow[r, "\psi"] \arrow[d, "\pi'"] & \Omega_*^{Spin} \arrow[r, "\alpha"] \arrow[d, "\pi"] & ko_* \arrow[d, "\tilde{\pi}"] \\
\Omega_{*-8}^P(BG) \arrow[r, "\psi^P"] & \Omega_*^P \arrow[r, "\overline{\alpha}"] & ko_*^P .
\end{tikzcd}$$

Let y be in the kernel of $\overline{\alpha}$. First, π is surjective because $[P]$ is not a zero divisor in Ω_*^{Spin} (see Propositions 2.3.7 and 2.4.1). Hence there exists $x \in \Omega_*^{Spin}$ such that $\pi(x) = y$. The cofiber sequence associated to (4.3.7) induces the long exact sequence

$$\cdots \to ko_{*-\dim P} \xrightarrow{\eta} ko_* \xrightarrow{\tilde{\pi}} ko_*^P \xrightarrow{\tilde{\delta}_P} ko_{*-\dim P-1} \to \cdots .$$

Since P admits a positive scalar curvature metric, $\alpha(P) = 0$, and η is the zero map. It follows that $\tilde{\pi}$ is injective. Since $\overline{\alpha}(\pi(x)) = \overline{\alpha}(y)$ vanishes, the element x lies in the kernel of α. According to (4.3.11), the first row is exact. Then there exists $u \in \Omega_{*-8}^{Spin}(BG)$ such that $\psi(u) = x$, and $\psi^P(\pi'(u)) = y$. We conclude that Lemma 4.3.15 holds without localizing.

We can now finish the proof of Proposition 4.3.13. Recall that N denotes a compact simply connected singular P-manifold with $\overline{\alpha}(N) = 0$. Lemma 4.3.15 implies that $[N]$ lies in the image of ψ^P. Let (S, f) be a singular P-manifold in BG, $\partial S = P \times B$, such that $\psi^P[S, f] = [N]$. We noted above that by the definition of a singular P-manifold in BG, there is a map $\tilde{f}\colon B \to BG$ such that $f|_{\partial S} = \tilde{f} \circ pr_B$, and hence $\partial(f^*(BH)) = P \times \tilde{f}^*(BH)$.

We assumed that $g \times h^Q$ is a positive scalar curvature metric on $P \times Q$. Choose a metric h^B on B and extend $g \times h^B$ on ∂S to a metric h^S on S. Equip $\mathbb{H}P^2$ with a positive scalar curvature metric. By shrinking the fibers, the typical fiber bundle metric $g^{f^*(BH)}$ on $f^*(BH)$ is of positive scalar curvature. Note that there is an induced metric $h^{\tilde{f}^*(BH)}$ on $\tilde{f}^*(BH)$ such that

$$g^{f^*(BH)}|_{\partial(f^*(BH)) = P \times \tilde{f}^*(BH)} = g \times h^{\tilde{f}^*(BH)}.$$

Take any singular P-bordism W between $f^*(BH)$ and N. Then W comprises a bordism V between $\tilde{f}^*(BH)$ and Q (see Figure 4.3). Extend the given metrics $h^{\tilde{f}^*(BH)}$ and h^Q to a metric h^V on V. For $\epsilon > 0$ small enough,

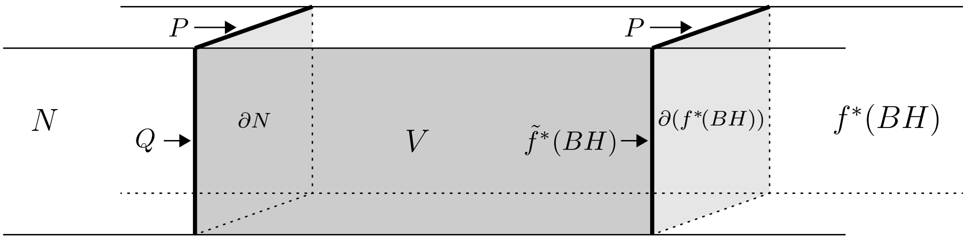

Figure 4.3: Boundary of the singular P-bordism W

$(\epsilon g) \times h^V$ is a positive scalar curvature metric on $P \times V$. By using concordance metrics, we see that

$$f^*(BH) \cup_{\partial(f^*(BH))=P\times\tilde{f}^*(BH)} (-P \times V)$$

is a zero bordism for ∂N on which $g \times h^Q$ extends to a positive scalar curvature metric. Finally, we apply Corollary 3.1.3 to extend $g \times h^Q$ to a positive scalar curvature metric on N.

□

4.3.4 Second Extension Problem

Let us first show that if $R \times S^1 \times C$ is zero bordant, then either $R \times S^1$ or $S^1 \times C$ has to be zero bordant. Clearly, all $\mathbb{Z}_2$-characteristic number of $R \times S^1$ and $S^1 \times C$ are zero. We must therefore show that either the KO-characteristic numbers of $R \times S^1$ vanish or those of $S^1 \times C$. If $\dim R \equiv 4 \mod 8$, then all KO-characteristic numbers of $R \times S^1$ vanish for dimension reasons. Let $\dim R \equiv 0 \mod 8$. If $\overline{R}$ (see the proof of Corollary 4.3.10) is odd in Ω_*^{Spin}/I_*, then it is not a zero divisor. Hence $\overline{S^1 \times C} = 0$ which implies that the KO-characteristic numbers of $S^1 \times C$ vanish. If $\overline{R}$ is even, then $\overline{R \times S^1} = 0$, and $R \times S^1$ has vanishing KO-characteristic numbers.

If $S^1 \times C$ is zero bordant, say via E, we are clearly done: Let g be an arbitrary positive scalar curvature metric on R. We then choose a metric h on E such that $g \times h$ is a positive scalar curvature metric on $R \times E$. We conclude that Theorem 4.3.9 holds with $\hat{C} = R \times E$. It therefore remains to treat the case when $R \times S^1$ is zero bordant.

The question of positive scalar curvature on manifolds having an S^1-factor on their boundaries is treated in [Bot01]. Loco cit. a different point of view is adopted, namely, these manifolds are considered as spin manifolds with S^1-singularity.

It turns out that complex K-theory is the appropriate index theory when dealing with spin manifolds with S^1-singularity. Let $k_* = \mathbb{Z}[u]$, $deg(u) = 2$, denote the coefficients of connective complex K-theory.

Theorem 4.3.16 ([Bot01, Theorem 1.1]). *There is a map*

$$\alpha^{S^1} : \Omega_*^{S^1} \to k_*$$

such that the following holds: Let N denote a simply connected singular S^1-manifold with $\partial N = P \times S^1$. Assume that P is non-empty, simply connected and of dimension greater or equal than five.

Then N admits a positive scalar curvature metric, which restricts on ∂N to a product of some positive scalar curvature metric g on P and the standard metric dt^2 on S^1, if and only if $\alpha^{S^1}(N) = 0$.

Homotopy-theoretically, α^{S^1} is defined as follows (see [Bot01, p. 687]). Botvinnik showed that $MSpin^{S^1} \simeq MSpin \wedge \Sigma^{-2}\mathbb{C}P^2$ and $ko \wedge \Sigma^{-2}\mathbb{C}P^2 \simeq k$, where k denotes the connective complex K-theory spectrum. Then α^{S^1} is given by

$$\alpha \wedge \mathrm{id} \colon MSpin \wedge \Sigma^{-2}\mathbb{C}P^2 \to ko \wedge \Sigma^{-2}\mathbb{C}P^2.$$

Index-theoretically, $\alpha^{S^1}(N)$ can be computed in the following way (see [Bot01, Remark 5.10]). Consider S^1 as a spin^c manifold and the unit disc D^2 as a spin^c zero bordism for S^1. Then α^{S^1} coincides with the index of the usual Dirac operator on the closed spin^c manifold

$$N \cup_{\partial N = \partial(P \times D^2)} \left(-P \times D^2\right).$$

An immediate consequence of Theorem 4.3.16 is that in case the dimension of N is odd, N admits a metric of positive scalar curvature. However, since $\dim R \equiv 0 \mod 4$ we are interested in the case $\dim N \equiv 2 \mod 4$.

Proposition 4.3.17. *Let N be a simply connected singular S^1-manifold of dimension $n \equiv 2 \mod 4$, greater or equal than five, with $\partial N = P \times S^1$. Assume that P admits a positive scalar curvature metric.*

Then $\alpha^{S^1}(N) = 0$, i.e. N admits a positive scalar curvature metric (in the sense of Theorem 4.3.16).

Proof. We consider the index map $\beta\colon \Omega_*^{Spin^c} \to k_*$ induced by the usual Dirac operator on spin^c-manifolds. Let us note that the index maps α and β are compatible in the sense that

$$\begin{array}{ccc} \Omega_n^{Spin} & \xrightarrow{\alpha} & ko_n \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle c} \\ \Omega_n^{Spin^c} & \xrightarrow{\beta} & k_n \end{array}$$

commutes; here f associates to a spin manifold the underlying spin^c manifold, and c denotes the complexification map. Choose some positive scalar curvature metric $\hat{g}$ on P. We now want to consider the spin^c analogue of the singular index map (4.3.3). We apply the same procedure as in the spin case and obtain a natural transformation

$$\beta^{(P,\hat{g})}\colon \Omega_*^{Spin^c,P} \to k_*.$$

There is the corresponding commutative diagram for singular P-manifolds, i.e.

$$\begin{array}{ccc} \Omega_n^{P} & \xrightarrow{\alpha^{(P,\hat{g})}} & ko_n \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle c} \\ \Omega_n^{Spin^c,P} & \xrightarrow{\beta^{(P,\hat{g})}} & k_n. \end{array}$$

We remark that for $n \equiv 2 \mod 4$ one has $ko_n = \mathbb{Z}_2$ or $ko_n = 0$, and $k_n = \mathbb{Z}$, which implies that $c \circ \alpha^{(P,\hat{g})} = 0$.

Now let N be as in the proposition. We first consider N as a manifold with S^1-singularity. Above we noted that $\alpha^{S^1}(N)$ is given by the usual index of the closed spin^c manifold $N \cup (-P \times D^2)$. Now it is trivially true that $N \cup (-P \times D^2)$, considered as a spin^c singular P-manifold, is spin^c singular P-bordant to N, itself considered as a spin^c singular P-manifold. Namely, $P \times D^2$ is a zero spin^c P-bordism for ∂N, and

$$\big(N \cup \big(-P \times D^2\big)\big) \times [0,1]$$

is a zero spin^c bordism for $(N \cup (-P \times D^2)) \times \{0,1\}$ (compare Definition

2.3.3). We therefore have

$$\begin{aligned}
\alpha^{S^1}[N] &= \beta\left[N \cup \left(-P \times D^2\right)\right] \\
&= \left(\beta^{(P,\hat{g})} \circ \pi\right)\left[N \cup \left(-P \times D^2\right)\right] \\
&= \left(\beta^{(P,\hat{g})} \circ f\right)[N] \\
&= \left(c \circ \alpha^{(P,\hat{g})}\right)[N]
\end{aligned}$$

where in the third and fourth row N is considered as a spin singular P-manifold, and $\pi\colon \Omega_*^{Spin^c} \to \Omega_*^{Spin^c,P}$ as usual denotes the projection. It follows that $\alpha^{S^1}(N) = 0$ if $n \equiv 2 \mod 4$. $\square$

One concludes that Theorem 4.3.16 yields a positive scalar curvature metric on N which restricts on $\partial N = P \times S^1$ to $g \times dt^2$ where g is *any* positive scalar curvature metric on P. We do not claim that g coincides with $\hat{g}$.

We can now finish the proof Theorem 4.3.9. Recall that $R \times S^1$ is assumed to be zero bordant, say via V. According to Proposition 4.3.17, with $P = R$ and $N = V$, there exists a positive scalar curvature metric g on R such that $g \times dt^2$ extends to one on V. Choose a metric k on C such that $g \times dt^2 \times k$ is of positive scalar curvature. We then take $V \times C$ as $\hat{C}$ and $dt^2 \times k$ as h^C to prove the second part of Theorem 4.3.9.

Chapter 5

Positive Homology

5.1 Introduction

This chapter is concerned with the question of positive scalar curvature on singular manifolds which are not simply connected. We will restrict our attention to orientable, totally non-spin manifolds. As in the case of closed manifolds, one will again have to study the singular homology of the fundamental groups of the manifolds in question.

Let $\mathscr{P}^u$ denote further on a sequence of polynomial generators of Ω_*^{SO} modulo torsion. At first sight, it seems that the question of positive scalar curvature on non-simply connected closed singular $\mathscr{P}^u$-manifolds is more complicated than the corresponding question for ordinary closed manifolds, closed singular $\mathscr{P}^u$-manifolds being more general objects than ordinary closed manifolds. However, in view of the study of singular homology groups, it actually seems to be more natural to consider singular $\mathscr{P}^u$-manifolds because singular homology admits a description by singular $\mathscr{P}^u$-bordism, at least after inverting 2.

Although we will almost exclusively dealing with the sequence $\mathscr{P}^u$, or actually with a sequence $\mathscr{P}^{u'}$ of p-local polynomial generators of Ω_*^{SO}, p an odd prime, we begin with the following general situation. Let $\mathscr{P}$ denote an arbitrary locally finite family of closed oriented manifolds equipped with positive scalar curvature metrics. As in the proof of Proposition 2.5.1, one obtains a natural transformation

$$U^{\mathscr{P}} : \Omega_*^{\mathscr{P}}(\_) \to H_*(\_) \tag{5.1.1}$$

such that the projection $\pi\colon \Omega_*^{SO}(\_) \to \Omega_*^{\mathscr{P}}(\_)$ composed with $U^{\mathscr{P}}$ coincides with the orientation map $U\colon \Omega_*^{SO}(\_) \to H_*(\_)$ (the regularity of the sequence is immaterial). In other words, singular $\mathscr{P}$-manifolds represent singular homology classes. Now let X denote an arbitrary space. We have the following basic

Definition 5.1.1. A class $\sigma \in H_*(X)$ is called *positive* if σ is representable by a closed singular $\mathscr{P}$-manifold in X which admits a positive scalar curvature metric. The subgroup of all positive classes is denoted by $H_*^{\oplus}(X)$.

We remind the reader that a positive scalar curvature metric on a singular $\mathscr{P}$-manifold is assumed to be canonical on its boundary (see Definition 4.1.1). Although omitted in this definition, $H_*^{\oplus}(X)$ clearly depends on $\mathscr{P}$ and the chosen positive scalar curvature metrics.

For the remainder of this section, we will now restrict our attention to $\mathscr{P} = \mathscr{P}^u$, a family of polynomial generators of Ω_*^{SO} modulo torsion, equipped with arbitrary positive scalar curvature metrics. We emphasize the difference between $H_*^{\oplus}(X)$ and $H_*^{+}(X)$, the latter being defined in the introduction as the image of $U\colon \Omega_*^{SO}(X) \to H_*(X)$ restricted to ${}^{+}\Omega_*^{SO}(X)$. It is apparent that $H_*^{+}(X)$ is a subgroup of $H_*^{\oplus}(X)$. But $H_*^{\oplus}(X)$ will presumably contain more elements than $H_*^{+}(X)$. In fact, with 2 inverted, we know that all homology classes are representable by singular $\mathscr{P}^u$-manifolds. Let us note also that all homology classes are representable by singular $\mathscr{P}^u$-manifolds 2-locally. In this case, the orientation map $u\colon MSO \to H\mathbb{Z}$ is split surjective, hence U is surjective and in turn $U^{\mathscr{P}}$ as well.

Our point of view is that $H_*^{\oplus}(X)$ seems to be a more natural object than $H_*^{+}(X)$ since $H_*^{\oplus}(X)$ checks *all* homology classes for positivity while $H_*^{+}(X)$ only considers homology classes which are representable by closed manifolds. This will become clearer in the next section (compare Remark 5.2.3).

We turn to the question of positive scalar curvature on non-simply connected $\mathscr{P}^u$-manifolds. In analogy to the closed case we define

$${}^{+}\Omega_n^{\mathscr{P}^u}(X) = \big\{[N,g] \in \Omega_n^{\mathscr{P}^u}(X) \,|\, N \text{ admits a positive scalar curvature metric}\big\}.$$

That is, we have $H_*^{\oplus}(X) = \operatorname{im} U^{\mathscr{P}^u}|_{{}^{+}\Omega_n^{\mathscr{P}^u}(X)}$. The corresponding Bordism and Homology Theorem read as follows.

Theorem 5.1.2. *Let M be a connected orientable totally non-spin closed singular $\mathscr{P}^u$-manifold of dimension n greater or equal than five with fundamental group π. Furthermore, let $f\colon M \to B\pi$ denote the classifying map of the universal cover of M. Then the following hold:*

1. *The singular manifold M admits a positive scalar curvature metric if and only if* $[M,f] \in {}^{+}\Omega_n^{\mathscr{P}^u}(B\pi)$.
2. *The singular manifold M admits a positive scalar curvature metric if and only if* $U^{\mathscr{P}^u}[M,f] \in H_n^{\oplus}(B\pi)$.

We mention the second statement of Theorem 5.1.2 only for the sake of completeness. Below we are actually interested in p-local considerations, p an odd prime, and in this case the two statements coincide.

Proof. The first statement can be deduced from Corollary 3.1.3 by means of the established methods. Let us note that it obviously holds for an arbitrary locally finite family $\mathscr{P}$ of closed oriented manifolds equipped with positive scalar curvature metrics.

To prove the second part, we have to show, as above, that $\ker U^{\mathscr{P}^u}$ lies in ${}^{+}\Omega_n^{\mathscr{P}^u}(B\pi)$. After inverting 2 the map $U^{\mathscr{P}^u}$ is an isomorphism, so we may work 2-locally. The claim will be a consequence of

Lemma 5.1.3. *The map $\pi\colon MSO \to MSO^{\mathscr{P}^u}$ is a 2-local split surjection.*

Proof. Recall from the proof of Lemma 4.2.4 that on homotopy groups, π can be identified with the projection

$$\pi\colon \Omega_*^{SO} \cong \mathbb{Z}[\hat{u}_4, \ldots, \hat{u}_{4i}, \ldots] \oplus \operatorname{im}\partial \to \operatorname{im}\partial = \Omega_*^{\mathscr{P}^u}.$$

Moreover, we see that this surjection splits. Since MSO localized at 2 is a graded Eilenberg-MacLane spectrum, it follows that π is a 2-local split surjection (see [Rud98, Lemma 7.2]). $\square$

This means in particular that $\pi\colon \Omega_n^{SO}(B\pi) \to \Omega_n^{\mathscr{P}^u}(B\pi)$ is 2-locally surjective. Hence, for all x which lie in the kernel of $U^{\mathscr{P}^u}\colon \Omega_n^{\mathscr{P}^u}(B\pi) \to H_n(B\pi)$ there exists $y \in \Omega_n^{SO}(B\pi)$ such that $\pi(y) = x$. But now y lies in the kernel of $U\colon \Omega_n^{SO}(B\pi) \to H_n(B\pi)$, which is in turn generated by manifolds admitting positive scalar curvature metrics (compare the proof of the 2-local part of the Homology Theorem). $\square$

Theorem 5.1.2 raises the question of how much of $H_*(B\pi)$ is exhausted by $H_*^{\oplus}(B\pi)$. A subgroup of $H_*(B\pi)$, for which it is difficult to decide whether its elements are positive, consists of *toral classes.* A class in $H_*(B\pi)$ is called a *toral generator* if it is representable by a map $T^k \to B\pi$, where

$T^k = S^1 \times \cdots \times S^1$ as usual denotes the standard k-torus. Let $H_*^{\mathrm{toral}}(B\pi)$ denote the subgroup of $H_*(B\pi)$ generated by toral generators. The elements of $H_*^{\mathrm{toral}}(B\pi)$ are called toral classes.

We define a complement to the toral classes as follows.

Definition 5.1.4. For all $k \leq r$, the cup product induces a map $\bigwedge^k H^1(B\pi;\mathbb{Z}_p) \to H^k(B\pi;\mathbb{Z}_p)$ whose image consists of *toral cohomology classes.* A homology class in $H_k(B\pi;\mathbb{Z})$ is called *atoral* if its projection in $H_k(B\pi;\mathbb{Z}_p)$ is annihilated by these toral cohomology classes. The subgroup of atoral classes is denoted by $H_*^{\mathrm{atoral}}(B\pi)$.

This definition is similar to [BR05, Definition 2.1] where the notion of an atoral class is introduced for elements of $\Omega_*^{SO}(B\pi)$.

In the next section we will study the homology of elementary Abelian p-groups, p an odd prime. We will show that here atoral classes are positive. This statement also yields consequences for positive scalar curvature metrics on *closed* manifolds.

5.2 Homology of Elementary Abelian p-Groups

We fix an odd prime p. Let $\mathbb{Z}_p^r = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ denote the elementary Abelian p-group of rank $r \geq 1$. We remind the reader that Ω_*^{SO} localized at p is a polynomial algebra having one generator in each dimension divisible by four. Our aim of this section is to prove

Theorem 5.2.1. *There exists a family $\mathscr{P}^{u'}$ of p-local polynomial generators of Ω_*^{SO}, equipped with specific positive scalar curvature metrics $\mathfrak{g}$, such that*

$$H_*^{atoral}(B\mathbb{Z}_p^r) \subset H_*^{\oplus}(B\mathbb{Z}_p^r)$$

where '$\oplus$' is understood with respect to $\mathscr{P}^{u'}$ and $\mathfrak{g}$.

One can show that Theorem 5.2.1 also holds for $p = 2$, where it is a consequence of [BR02, Theorem 5.8].

Theorem 5.2.1 together with Theorem 5.1.2 immediately implies

Corollary 5.2.2. *Let M be a connected orientable totally non-spin closed singular $\mathscr{P}^{u'}$-manifold of dimension greater or equal than five with fundamental group $\mathbb{Z}_p^r$, and let $f\colon M \to B\mathbb{Z}_p^r$ be the classifying map of the universal cover of M.*

Then M admits a positive scalar curvature metric if $f_*[M] \in H_*^{atoral}(B\mathbb{Z}_p^r)$. In particular, if the dimension of M is greater than r, then M admits a positive scalar curvature metric.

Special cases of closed singular $\mathscr{P}^{u'}$-manifolds are of course ordinary closed manifolds. For those, Corollary 5.2.2 was proven before by Botvinnik and Rosenberg in [BR02, Theorem 5.8]. An important component in the proof of [BR02, Theorem 5.8] is the case $r = 2$. This was shown in [BG97], based on an eta invariant calculation, and, independently, in [Sch97] using the truth of the Segal Conjecture, among other things.

We shall start the proof of Theorem 5.2.1 by considering the case $r = 1$. One shows that

$$H_n(B\mathbb{Z}_p) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_p & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, the lens spaces

$$L^{2k-1} = S^{2k-1}/\mathbb{Z}_p \overset{\iota}{\hookrightarrow} L^\infty = B\mathbb{Z}_p, \quad k \geq 1,$$

serve as generators, where the action of $\mathbb{Z}_p$ on $S^{2k-1} \subset \mathbb{C}^k$ is generated by $z \mapsto exp((2\pi i)/p)z$. For $k > 1$ the standard metric on S^{2k-1} induces a positive scalar curvature metric on the lens space L^{2k-1}. Furthermore, we certainly have $H_*^{\mathrm{atoral}}(B\mathbb{Z}_p) = H_{*>1}(B\mathbb{Z}_p)$. It follows that

$$H_*^{\mathrm{atoral}}(B\mathbb{Z}_p) = H_*^{\oplus}(B\mathbb{Z}_p).$$

The proof of Theorem 5.2.1 for $r = 2$ is carried out in the next subsection. It turns out that the case of arbitrary r then easily follows by induction.

5.2.1 Groups of Rank 2

Being Eilenberg-MacLane spaces, $B(\mathbb{Z}_p \times \mathbb{Z}_p)$ and $B\mathbb{Z}_p \times B\mathbb{Z}_p$ are homotopy equivalent. The homology of $B\mathbb{Z}_p \times B\mathbb{Z}_p$ can now be computed by the Künneth formula, i.e. the split exact sequence

$$0 \to [H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}_p)]_n \xrightarrow{\times} H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p) \to [H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)]_{n-1} \to 0 \tag{5.2.1}$$

where the latter term denotes the torsion product of Abelian groups. The proof of Theorem 5.2.1 for even n is simple. Namely, in this case $[H_*(B\mathbb{Z}_p) *$

$H_*(B\mathbb{Z}_p)]_{n-1}$ vanishes. It follows that $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$ is generated by products of lens spaces $[L^i] \otimes [L^j]$, $i + j = n$. But $[L^i] \otimes [L^j]$ is atoral only if i or j is > 1, and in either case $[L^i] \otimes [L^j]$ is positive.

For the remainder of this subsection let n be odd. The group $H_1(B\mathbb{Z}_p \times B\mathbb{Z}_p)$ does not contain non-trivial atoral classes, therefore let $n > 1$. Then $H_n^{\mathrm{atoral}}(B\mathbb{Z}_p^2) = H_n(B\mathbb{Z}_p^2)$, we must therefore show that all elements of $H_n(B\mathbb{Z}_p^2)$ are positive. We will distinguish two cases. It turns out that for $1 < n \leq 2p - 1$ all elements in $H_n(B\mathbb{Z}_p^2)$ are representable by closed manifolds admitting positive scalar curvature metrics. This will be proved by rather elementary methods involving the ring structure of $H^*(B\mathbb{Z}_p; \mathbb{Z}_p)$ given by the cup product. To prove the claim for $n \geq 2p + 1$, we will construct singular $\mathscr{P}^{u'}$-manifolds of positive scalar curvature that generate $H_n(B\mathbb{Z}_p^2)$, where $\mathscr{P}^{u'}$ is a suitable sequence of p-local polynomial generators of Ω_*^{SO}. This involves results about the structure of $\Omega_*^{SO}(B\mathbb{Z}_p)$ from [CF64] and the construction of a product for singular manifolds.

Remark 5.2.3. If one works with $H_*^+(B\mathbb{Z}_p^r)$ instead of $H_*^{\oplus}(B\mathbb{Z}_p^r)$, a statement corresponding to Theorem 5.2.1 would be that the image of the subgroup of atoral bordism classes (see [BR05, Definition 2.1]) under the orientation map is contained in $H_*^+(B\mathbb{Z}_p^r)$ (which is shown in [BR02, Theorem 5.8]). The proof involves a study of the relatively complicated object $\mathrm{Tor}^*_{\Omega_*^{SO}}(\Omega_*^{SO}(B\mathbb{Z}_p), \Omega_*^{SO}(B\mathbb{Z}_p))$. By contrast, we can work in purely homological terms in order to show that all classes of $H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p) = \mathrm{Tor}_{\mathbb{Z}}(H_*(B\mathbb{Z}_p), H_*(B\mathbb{Z}_p))$ are '$\oplus$'-positive, that is, we only have to consider the torsion product of Abelian groups.

The Case $n \leq 2p - 1$.

We fix an $n = 2m + 1$ for some $m \geq 1$. Let e_n denote an arbitrary generator of $H_n(B\mathbb{Z}_p)$. We observe that $1 \times e_n$ and $e_n \times 1$ lie in $H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p)$. To obtain further elements in $H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p)$, we consider the map

$$\Delta_l \colon \mathbb{Z}_p \xrightarrow{\Delta} \mathbb{Z}_p \times \mathbb{Z}_p \xrightarrow{\cdot l \times \mathrm{id}} \mathbb{Z}_p \times \mathbb{Z}_p$$

where Δ denotes the diagonal map and $\cdot l$ denotes multiplication by a number l. Then Δ_l induces a map in homology whose image also lies in $H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p)$. One verifies that the dimensions of the $\mathbb{Z}_p$-vector spaces $[H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}_p)]_n$ and $[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)]_{n-1}$ are two and m, respectively. Our aim is to show that the elements

$$1 \times e_n, e_n \times 1, \Delta_1(e_n), \Delta_2(e_n) \ldots, \Delta_m(e_n) \tag{5.2.2}$$

form a basis for the $m+2$-dimensional $\mathbb{Z}_p$-vector space $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$, provided that $n \leq 2p-1$. For this we will determine the representation of these elements in another basis of $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$, which we will obtain by considering elements coming from the $\mathbb{Z}_p$-homology of $B\mathbb{Z}_p \times B\mathbb{Z}_p$.

To start with recall that associated to the short exact sequences $0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ resp. $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ there are Bockstein homomorphisms

$$\beta\colon H_*(\_;\mathbb{Z}_p) \to H_{*-1}(\_;\mathbb{Z}_p) \quad \text{resp.} \quad \beta'\colon H_*(\_;\mathbb{Z}_p) \to H_{*-1}(\_;\mathbb{Z}),$$

and analogously for cohomology with increasing degree. One computes $H^i(B\mathbb{Z}_p;\mathbb{Z}_p) \cong \mathbb{Z}_p$ for all $i \geq 0$. As a ring, $H^*(B\mathbb{Z}_p;\mathbb{Z}_p)$ is isomorphic to the tensor product of an exterior and polynomial algebra. More precisely, let us choose a generator $x \in H^1(B\mathbb{Z}_p;\mathbb{Z}_p)$. Then $y := \beta(x)$ is non-trivial in $H^2(B\mathbb{Z}_p;\mathbb{Z}_p)$ and there is a ring isomorphism

$$H^*(B\mathbb{Z}_p;\mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}(x) \otimes \mathbb{Z}_p[y]$$

(see e.g. [Hat02, Example 3E.2]). Let $\tilde{e}_i \in H_i(B\mathbb{Z}_p;\mathbb{Z}_p)$ be the dual of $xy^{\frac{i-1}{2}}$ for i odd resp. the dual of $y^{\frac{i}{2}}$ for i even. One then computes $\beta(\tilde{e}_i) = 0$ for i odd and $\beta(\tilde{e}_i) = \tilde{e}_{i-1}$ for i even. For $1 \leq k \leq m$ we set

$$\tilde{e}(n,k) := \tilde{e}_{2k-1} \times \tilde{e}_{n-2k+1} + \tilde{e}_{2k} \times \tilde{e}_{n-2k} \in H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p;\mathbb{Z}_p).$$

It is apparent that $\tilde{e}(n,1),\ldots,\tilde{e}(n,m)$ are linearly independent in $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p;\mathbb{Z}_p)$. By means of the derivation property of the Bockstein homomorphism, it follows that $\beta(\tilde{e}(n,k)) = 0$, for all $1 \leq k \leq m$. Consider now the map between the Bockstein exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot p"] & H_*(B\mathbb{Z}_p^2;\mathbb{Z}) \arrow[r] \arrow[d] & H_*(B\mathbb{Z}_p^2;\mathbb{Z}_p) \arrow[r, "\beta'"] \arrow[d, "="] & H_{*-1}(B\mathbb{Z}_p^2;\mathbb{Z}) \arrow[r, "\cdot p"] \arrow[d, "pr"] & \cdots \\
\cdots \arrow[r, "\cdot p"] & H_*(B\mathbb{Z}_p^2;\mathbb{Z}_{p^2}) \arrow[r] & H_*(B\mathbb{Z}_p^2;\mathbb{Z}_p) \arrow[r, "\beta"] & H_{*-1}(B\mathbb{Z}_p^2;\mathbb{Z}_p) \arrow[r, "\cdot p"] & \cdots .
\end{tikzcd}$$

Since pr is injective for $* > 1$, $\tilde{e}(n,k)$ also lies in the kernel of β'. Hence $\tilde{e}(n,k)$ is the reduction of an integral class $e(n,k) \in H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p;\mathbb{Z})$.

We conclude that the $m+2$ linearly independent elements

$$(1 \times e_n, e(n,1), \ldots, e(n,m), e_n \times 1) \tag{5.2.3}$$

form a basis for $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p; \mathbb{Z})$, where e_n denotes an integral homology class whose mod p reduction is $\tilde{e}_n$. We shall express the elements (5.2.2) with respect to this basis.

Lemma 5.2.4. *As above, let $\cdot l\colon \mathbb{Z}_p \to \mathbb{Z}_p$ denote multiplication by a number l, as well as the induced map on classifying spaces. Then the induced map in homology $\cdot l_*\colon H_i(B\mathbb{Z}_p; \mathbb{Z}_p) \to H_i(B\mathbb{Z}_p; \mathbb{Z}_p)$ is given by*

$$\tilde{e}_i \mapsto \begin{cases} l^{\frac{i+1}{2}} \cdot \tilde{e}_i & \text{for } i \text{ odd} \\ l^{\frac{i}{2}} \cdot \tilde{e}_i & \text{for } i \text{ even} \end{cases}.$$

Proof. On $H_1(B\mathbb{Z}_p; \mathbb{Z}_p)$ one recovers the given map $\cdot l\colon \mathbb{Z}_p \to \mathbb{Z}_p$. By the naturality of the Bockstein homomorphism β, we see that $\cdot l$ also induces multiplication by l on $H_2(B\mathbb{Z}_p; \mathbb{Z}_p)$. Recall that $H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}(x) \otimes \mathbb{Z}_p[y]$. By duality, $\cdot l^*(x) = l \cdot x$ and $\cdot l^*(y) = l \cdot y$. It follows that

$$\cdot l^*\colon H^i(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^i(B\mathbb{Z}_p; \mathbb{Z}_p)$$

is given by

$$xy^{\frac{i-1}{2}} \mapsto l^{\frac{i+1}{2}} \cdot xy^{\frac{i-1}{2}} \text{ for } i \text{ odd} \quad \text{and} \quad y^{\frac{i}{2}} \mapsto l^{\frac{i}{2}} \cdot y^{\frac{i}{2}} \text{ for } i \text{ even.}$$

Again by duality, the same holds in homology. □

The ring structure of $H^*(B\mathbb{Z}_p; \mathbb{Z}_p)$ also implies that the diagonal map $\Delta\colon \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ induces the following map in $\mathbb{Z}_p$-homology

$$\tilde{e}_n \mapsto 1 \times \tilde{e}_n + \tilde{e}_1 \times \tilde{e}_{n-1} + \cdots + \tilde{e}_{n-1} \times \tilde{e}_1 + \tilde{e}_n \times 1.$$

Together with Lemma 5.2.4, we now see that $\Delta_l\colon \mathbb{Z}_p \xrightarrow{\Delta} \mathbb{Z}_p \times \mathbb{Z}_p \xrightarrow{\cdot l \times \mathrm{id}} \mathbb{Z}_p \times \mathbb{Z}_p$ induces the following map in $\mathbb{Z}_p$-homology

$$\begin{aligned} \tilde{e}_n \mapsto\ & l^0 1 \times \tilde{e}_n + l\tilde{e}_1 \times \tilde{e}_{n-1} + l\tilde{e}_2 \times \tilde{e}_{n-2} + l^2\tilde{e}_3 \times \tilde{e}_{n-3} + \cdots + l^{m+1}\tilde{e}_n \times 1 \\ &= 1 \times \tilde{e}_n + l\tilde{e}(n,1) + l^2\tilde{e}(n,2) + \cdots + l^m\tilde{e}(n,m) + l^{m+1}\tilde{e}_n \times 1. \end{aligned}$$

The coordinates of $\Delta_l(e_n) \in H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p; \mathbb{Z})$ with respect to the basis (5.2.3) are therefore $(1, l^1, l^2, \ldots, l^{m+1})$. Of course, the coordinates of $1 \times e_n$ resp. $e_n \times 1$ are $(1, 0, \ldots, 0)$ resp. $(0, \ldots, 0, 1)$. We now consider the matrix

$$\begin{pmatrix} 1 \times e_n \\ \Delta_1(e_n) \\ \Delta_2(e_n) \\ \vdots \\ \Delta_m(e_n) \\ e_n \times 1 \end{pmatrix} = \begin{pmatrix} \begin{array}{|cccc|} \hline 1 & 0 & \cdots & 0 \\ 1^0 & 1^1 & \cdots & 1^m \\ 2^0 & 2^1 & \cdots & 2^m \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ m^0 & m^1 & \cdots & m^m \\ \hline \end{array} & \begin{array}{c} 0 \\ 1^{m+1} \\ 2^{m+1} \\ \vdots \\ m^{m+1} \end{array} \\ \begin{array}{cccc} 0 & 0 & \cdots & 0 \end{array} & 1 \end{pmatrix}.$$

The framed part is known as the Vandermonde matrix. Its determinant is given by

$$\prod_{0 \leq s < t \leq m} t - s.$$

This value is not zero modulo p for $m \leq p-1$ or, equivalently, $n \leq 2p-1$. Elementary linear algebra implies that the $m+2$ positive classes $1 \times e_n, \Delta_1(e_n), \Delta_2(e_n), \ldots, \Delta_m(e_n), e_n \times 1$ also form a basis for $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$, for odd n with $1 < n \leq 2p-1$.

The Case $n \geq 2p+1$.

We continue to write $n = 2m+1$. Recall from above that $1 \times e_n$, $e_n \times 1$ together with $e(n,k)$, $k = 1, \ldots, m$, form a vector space basis for $H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$. Obviously, $1 \times e_n$ and $e_n \times 1$ lie in $H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p)$. In the sequel our aim will be to represent the elements $e(n,k)$ by singular manifolds of positive scalar curvature. The map

$$[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)]_{n-1} \to H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p), \quad e_{2k-1} * e_{n-2k} \mapsto e(n,k),$$

$k = 1, \ldots, m$, is a splitting for the right map in the Künneth formula (5.2.1). Therefore we might also say that our task is to prove

$$[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)]_{n-1} \subset H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p). \tag{5.2.4}$$

To show (5.2.4), we may work p-locally, and that is what we shall do. Let $\mathscr{P}^{u'}$ denote an, initially arbitrary, sequence of p-local polynomial generators of Ω_*^{SO}, and let $\hat{p}$ be the closed oriented manifold consisting of p copies of a point. In addition, let us set $\mathscr{P}_p^{u'} = \mathscr{P}^{u'} \cup \{\hat{p}\}$. In Chapter 2 we identified, after inverting 2, singular $\mathscr{P}^u$-bordism with singular homology with integer coefficients. Localized at an odd prime p, we may use $\mathscr{P}^{u'}$ instead of $\mathscr{P}^u$, i.e.

$$\Omega_*^{\mathscr{P}^{u'}}(\_) = H_*(\_;\mathbb{Z}).$$

One can also show
$$\Omega_*^{\mathscr{P}_p^{u'}}(\_) = H_*(\_;\mathbb{Z}_p)$$
since the solution of the related extension problem (as in Proposition 2.5.1) induces an isomorphism on coefficients, p-locally.

Recall the exact sequence (2.3.1): With $Q = \hat{p}$ we have

$$\cdots \to \Omega_*^{\mathscr{P}^{u'}}(\_) \xrightarrow{\times\hat{p}} \Omega_*^{\mathscr{P}^{u'}}(\_) \xrightarrow{\pi} \Omega_*^{\mathscr{P}_p^{u'}}(\_) \xrightarrow{\delta_{\hat{p}}} \Omega_{*-1}^{\mathscr{P}^{u'}}(\_) \to \cdots, \tag{5.2.5}$$

where we remind the reader that $\delta_{\hat{p}}$ associates to a closed singular $\mathscr{P}_p^{u'}$-manifold, having $\hat{p}\times B$ as the $\hat{p}$-part of its boundary, the closed singular $\mathscr{P}^{u'}$-manifold B. We observe that this sequence is just given by the Bockstein exact sequence

$$\cdots \to H_*(\_;\mathbb{Z}) \xrightarrow{\cdot p} H_*(\_;\mathbb{Z}) \xrightarrow{pr} H_*(\_;\mathbb{Z}_p) \xrightarrow{\beta'} H_{*-1}(\_;\mathbb{Z}) \to \cdots .$$

In the sequel, when talking about manifolds or bordisms, we will generally omit maps to $B\mathbb{Z}_p$ or $B\mathbb{Z}_p \times B\mathbb{Z}_p$. Recall that

$$pr(e(n,k)) = \tilde{e}_{2k-1} \times \tilde{e}_{n-2k+1} + \tilde{e}_{2k} \times \tilde{e}_{n-2k}.$$

We represent e_{2k-1} by a lens space L^{2k-1}. Since $[\hat{p} \times L^{2k-1}]$ vanishes in $H_{2k-1}(B\mathbb{Z}_p)$, there exists a zero singular $\mathscr{P}^{u'}$-bordism W^{2k} for $\hat{p} \times L^{2k-1}$, that is, W^{2k} is a bordism between some closed $\mathscr{P}^{u'}$-manifold and $\hat{p} \times L^{2k-1}$. We consider now W^{2k} as a closed singular $\mathscr{P}_p^{u'}$-manifold. As $\delta_{\hat{p}}[W^{2k}] = [L^{2k-1}]$ one concludes that W^{2k} represents $\tilde{e}_{2k}$.

Now $W^{2k} \times L^{n-2k}$ inherits a singular $\mathscr{P}_p^{u'}$-structure from W^{2k}, and $\delta_{\hat{p}}[W^{2k} \times L^{n-2k}] = [L^{2k-1} \times L^{n-2k}]$. Analogously, $\delta_{\hat{p}}[L^{2k-1} \times W^{n-2k+1}] = -[L^{2k-1} \times L^{n-2k}]$. We note that the proof of $\ker \delta_{\hat{p}} \subset \operatorname{im}\pi$ in (5.2.5) proceeds as follows. If M is a singular $\mathscr{P}_p^{u'}$-manifold and $\delta_{\hat{p}}M$ is zero singular $\mathscr{P}^{u'}$-bordant via V, then $M \cup_{\delta_{\hat{p}}M} V$ defines a closed singular $\mathscr{P}^{u'}$-manifold which is, when considered as a closed singular $\mathscr{P}_p^{u'}$-manifold, singular $\mathscr{P}_p^{u'}$-bordant to M. We now conclude that $e(n,k)$ is represented by the closed singular $\mathscr{P}^{u'}$-manifold

$$\left(L^{2k-1} \times W^{n-2k+1}\right) \cup \left(W^{2k} \times L^{n-2k}\right), \tag{5.2.6}$$

where the union is taken along the common $\hat{p}$-part of the boundaries, $L^{2k-1} \times \hat{p} \times L^{n-2k}$. The expression (5.2.6) (at least its non-singular version) is known as the *Toda bracket* or *Massey product*, and is denoted by $\langle L^{2k-1}, \hat{p}, L^{n-2k}\rangle$

(see [Ale72]). Let us note that $[\langle L^{2k-1}, \hat{p}, L^{n-2k}\rangle] \in H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p)$ does not depend on the chosen zero singular $\mathscr{P}^{u'}$-bordisms since $\beta'\colon H_{2*}(B\mathbb{Z}_p;\mathbb{Z}_p) \to H_{2*-1}(B\mathbb{Z}_p;\mathbb{Z})$ is an isomorphism. We have proven (recall $n = 2m+1$)

Proposition 5.2.5. *The Toda brackets* $[\langle L^{2k-1}, \hat{p}, L^{n-2k}\rangle]$, $k = 1, \ldots, m$, *form a basis for* $[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)]_{n-1} \subset H_n(B\mathbb{Z}_p^2)$.

According to Proposition 5.2.5, for the proof of Theorem 5.2.1 we have to show that $\langle L^{2k-1}, \hat{p}, L^{n-2k}\rangle$ admits a positive scalar curvature metric, for all $k = 1, \ldots, m$. This is simple for $k \neq 1$ and m. Namely, in this case L^{2k-1} and $L^{n-2k} = L^{2m+1-2k}$ admit positive scalar curvature metrics. The concordance argument illustrated in Figure 3.1 (see also [BR02, Theorem 3.5]) then implies that the same is true for $\langle L^{2k-1}, \hat{p}, L^{n-2k}\rangle$. We restate this result as

Corollary 5.2.6. *For all* $k \neq 1$ *and* m,

$$H_{2k-1}(B\mathbb{Z}_p) * H_{n-2k}(B\mathbb{Z}_p) \subset H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p).$$

The proof of the corresponding statement for $k = 1$ and m requires more work. The following notation is convenient.

Definition 5.2.7. A *positive scalar curvature metric* on a closed singular $\mathscr{P}_p^{u'}$-manifold M is assumed to restrict on the $\mathscr{P}^{u'}$-manifold $\partial M - int(\hat{p} \times \delta_{\hat{p}} M)$ to the canonical positive scalar curvature metric and on $\hat{p} \times \delta_{\hat{p}} M$ to p copies of some positive scalar curvature metric on $\delta_{\hat{p}} M$.

We remind the reader that one example of a closed singular $\mathscr{P}_p^{u'}$-manifold M is of course a bordism between $\hat{p} \times \delta_{\hat{p}} M$ and some closed $\mathscr{P}^{u'}$-manifold.

In order to prove Corollary 5.2.6 for $k = 1$ and m, one tries to find a bordism W^{2m} between $\hat{p} \times L^{2m-1}$ and some singular $\mathscr{P}^{u'}$-manifold such that W^{2m} admits a positive scalar curvature metric (in the sense of Definition 5.2.7). However, we are not bound at the specific generator L^{2m-1} of $H_{2m-1}(B\mathbb{Z}_p)$. One might as well work with some other generator $\mathcal{L}$ of $H_{2m-1}(B\mathbb{Z}_p)$, possibly represented by a singular $\mathscr{P}^{u'}$-manifold, and a zero singular $\mathscr{P}^{u'}$-bordism for $\hat{p} \times \mathcal{L}$ admitting a positive scalar curvature metric (see Figure 5.1; we set $p = 4$, the singular $\mathbb{Z}_2$-manifold 'closed interval' stands for the singular $\mathscr{P}^{u'}$-manifold $\mathcal{L}$, the zero singular $\mathscr{P}^{u'}$-bordism for $\hat{p} \times \mathcal{L}$ is denoted by $\mathfrak{W}$, the zero bordism for $\hat{p} \times S^1$ is denoted by W, the boundary of $\langle \mathcal{L}, \hat{p}, S^1\rangle$ is here the $\mathbb{Z}_2$-manifold $\mathbb{Z}_2 \times (T^2 \sharp T^2)$).

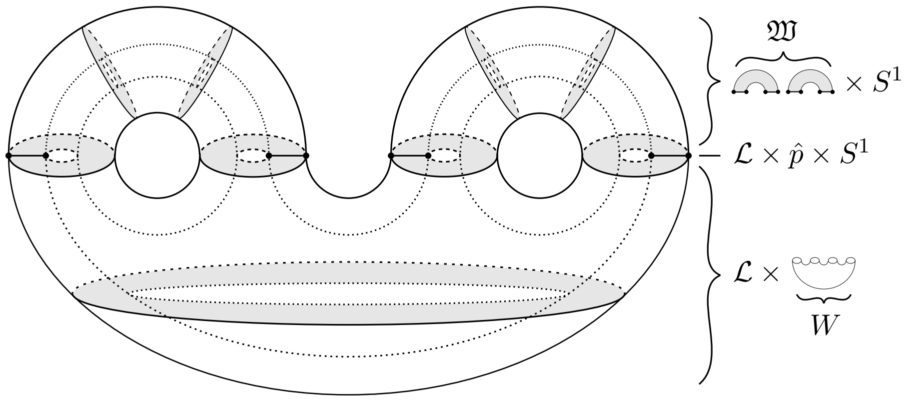

Figure 5.1: Toda bracket $\langle \mathcal{L}, \hat{p}, S^1 \rangle$

Recall that we treat the case $n \geq 2p+1$, or equivalently, $m \geq p$. Since $\beta' \colon H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p) \to H_{2m-1}(B\mathbb{Z}_p;\mathbb{Z})$ is an isomorphism, the following Theorem proves that

$$H_{2k-1}(B\mathbb{Z}_p) * H_{n-2k}(B\mathbb{Z}_p) \subset H_n^{\oplus}(B\mathbb{Z}_p \times B\mathbb{Z}_p),$$

for $k = 1$ and m, and finishes the proof of Theorem 5.2.1 for $r = 2$.

Theorem 5.2.8. *For all $m \geq p$ there exists a closed singular $\mathscr{P}_p^{u'}$-manifold $\mathfrak{W}^{2m}$ of positive scalar curvature in $B\mathbb{Z}_p$, representing a non-trivial element in $H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p)$.*

The proof of this statement is the most intricate part of the whole proof of Theorem 5.2.1. We distinguish the cases $m = p^i$ and $m \neq p^i$, $m > p$, for some $i \geq 1$. The first case involves results about the structure of $\Omega_*^{SO}(B\mathbb{Z}_p)$ from [CF64]. The second case will follow from the first one, using product structures for singular bordism and the H-space structure of $B\mathbb{Z}_p$; here we will need the fact that we may work with an arbitrary singular $\mathscr{P}_p^{u'}$-manifold as a generator of $H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p)$, and not only with a bordism between p copies of a lens space and some $\mathscr{P}^{u'}$-manifold.

Proof of Theorem 5.2.8 for $m = p^i$, $i \geq 1$. We need the following view on $\Omega_*^{SO}(B\mathbb{Z}_p)$ which is used in [CF64, Section 42]. Elements in $\Omega_*^{SO}(B\mathbb{Z}_p)$ are interpreted loc. cit. as closed oriented manifolds M admitting an orientation preserving fixed point free diffeomorphism T of period p. Such an element (T, M) *bords* if there exists a compact oriented manifold W admitting an orientation preserving fixed point free diffeomorphism S of period p such that $(S|_{\partial W}, \partial W)$ is oriented and equivariantly diffeomorphic to (T, M). The correspondence to the actual definition of $\Omega_*^{SO}(B\mathbb{Z}_p)$ is as follows. The natural projection $M \to M/T$ defines a principle $\mathbb{Z}_p$-bundle, hence we obtain a map $M/T \to B\mathbb{Z}_p$. Vice versa, a map $N \to B\mathbb{Z}_p$ induces a principle $\mathbb{Z}_p$-bundle $\hat{N} \to N$, meaning that $\hat{N}$ admits an orientation preserving fixed point free diffeomorphism of period p.

To avoid confusion, we note that the index k now appears in another context. We recall a construction in [CF64, Section 42]. Consider $S^3 \subset \mathbb{C}^2$ and $S^{2p-1} \subset \mathbb{C}^p$. As usual, $T^k = S^1 \times \cdots \times S^1$ denotes the k-torus. For all $k \geq 0$ one defines closed oriented manifolds L^{2k} as quotients $(S^3)^k/T^k$ where $(t_1, \ldots, t_k) \in T^k$ acts on $((z_1, w_1), (z_2, w_2), \ldots, (z_k, w_k)) \in (S^3)^k$ by

$$((t_1 z_1, t_1, w_1), (t_1^{-1} t_2 z_2, t_2 w_2), \ldots, (t_{k-1}^{-1} t_k z_k, t_k w_k)).$$

If v lies in $(S^3)^k$, we denote this expression by $\sigma(t_1, \ldots, t_k; v)$. One now obtains an action of T^k on $(S^3)^k \times S^{2p-1}$ where $(t_1, \ldots, t_k) \in T^k$ acts on $(v, (x_1, x_2 \ldots, x_p)) \in (S^3)^k \times S^{2p-1}$ by

$$(\sigma(t_1, \ldots, t_k; v), (x_1, t_k^{-1} x_2, \ldots, t_k^{-p+1} x_p)).$$

There is an induced $(2p-1)$-sphere bundle

$$\Big(\big(S^3\big)^k \times S^{2p-1}\Big)/T^k \to L^{2k}.$$

Dividing out the obvious $U(1)$-action on the fibers, one obtains an associated $\mathbb{C}P^{p-1}$-bundle, whose total space is denoted by $M^{2p+2k-2}$. We now fix k. As explained in [CF64, Section 42], there is a diffeomorphism $\overline{T}\colon M^{2p+2k} \to M^{2p+2k}$ of period p whose fixed point set separates into components

$$M^{2p+2k-2}, -M^{2p+2k-4}, \ldots, (-1)^k M^{2p-2}, (-1)^{k+1} \hat{p}.$$

It turns out that all normal bundles of these components are trivial. Cutting the interiors of the associated disk-bundles, one obtains a manifold with boundary in $B\mathbb{Z}_p$, which is denoted by W^{2p+2k}. In the sequel, $[M^*]$ denote elements in Ω_*^{SO}, and we remind the reader that $\Omega_*^{SO}(B\mathbb{Z}_p)$ is a module over Ω_*^{SO}. Loco cit. (42.11)[1] it is proved that W^{2p+2k} provides in $\Omega_{2p+2k-1}^{SO}(B\mathbb{Z}_p)$ the equality

$$\begin{aligned}\big[M^{2p+2k-2}\big]\big[T, S^1\big] - \big[M^{2p+2k-4}\big]\big[T, S^3\big] + \cdots \\ \cdots + (-1)^k \big[M^{2p-2}\big]\big[T, S^{2k+1}\big] + (-1)^{k+1} p\big[T, S^{2p+2k-1}\big] = 0,\end{aligned}$$

[1] In (42.11) there is a typing error, it should read $\big[T, S^{2k+1}\big]$ instead of $[T, S^{2k-1}]$.

where all T are fixed point free diffeomorphism induced by $\overline{T}$. For $1 \leq i \leq k+1$ the elements $[T, S^{2i-1}]$ are standard lens spaces. The action of T on $S^{2p+2k-1}$ is generated by

$$(x_1, \ldots, x_{p-1}, z_1, \ldots, z_{k+1}) \mapsto (\varrho x_1, \ldots, \varrho^{p-1} x_{p-1}, \varrho z_1, \ldots, \varrho z_{k+1}),$$

with $\varrho = exp((2\pi i)/p)$. It follows that $(T, S^{2p+2k-1})$ is a so-called generalized lens space in $B\mathbb{Z}_p$; for which it is also true that it inherits a positive scalar curvature metric from the standard metric on $S^{2p+2k-1}$ and that $[T, S^{2p+2k-1}] \neq 0$ in $H_{2p+2k-1}(B\mathbb{Z}_p)$.

Lemma 5.2.9. *The manifold W^{2p+2k} admits a positive scalar curvature metric which restricts*

- *to the product of two positive scalar curvature metrics, on $M^{2p+2k-2i} \times (T, S^{2i-1})$ for all $2 \leq i \leq k+1$,*
- *to the product of some positive scalar curvature metric on $M^{2p+2k-2}$ and some metric on (T, S^1), on $M^{2p+2k-2} \times (T, S^1)$,*
- *to p copies of some positive scalar curvature metric on $(T, S^{2p+2k-1})$, on $\hat{p} \times (T, S^{2p+2k-1})$.*

Proof. We equip $\mathbb{C}P^{p-1}$ with a positive scalar curvature metric and the $\mathbb{C}P^{p-1}$-bundle M^{2p+2k} with the typical fiber bundle metric. By construction, the inclusions of the $\mathbb{C}P^{p-1}$-bundles $M^{2p+2k-2i} \subset M^{2p+2k}$ are induced by inclusions $L^{2k+2-2i} \subset L^{2k+2}$, for all $1 \leq i \leq k+1$. By shrinking the fibers, we can assume that the associated metrics on $M^{2p+2k-2i}$ are of positive scalar curvature for all $0 \leq i \leq k+1$. Having a positive scalar curvature metric $M^{2p+2k-2i}$, $1 \leq i \leq k+1$, we may assume arbitrary metrics on (T, S^{2i-1}). For $2 \leq i \leq k+1$, those could be of positive scalar curvature.

It remains to show that the induced metric on $(T, S^{2p+2k-1})$ is of positive scalar curvature. This follows from an argument in [GL80, Lemma 1]. Namely, let x be a point in some n-dimensional manifold admitting a positive scalar curvature metric g. Denote by $S^{n-1}(\epsilon)$ the normal sphere of radius r around x. Then, for ϵ small enough, $g|_{S^{n-1}(\epsilon)}$ is concordant to the standard metric on S^{n-1}. Therefore, the induced metric on $S^{2p+2k-1}$ can be assumed to be standard which implies that it induces a positive scalar curvature metric on $(T, S^{2p+2k-1})$. □

We interpret W^{2p+2k} as a bordism between $(-1)^k\hat{p} \times (T, S^{2p+2k-1})$ and

$$M^{2p+2k-2} \times \big(T, S^1\big) - M^{2p+2k-4} \times \big(T, S^3\big) + \cdots + (-1)^k M^{2p-2} \times \big(T, S^{2k+1}\big). \tag{5.2.7}$$

Recall that for the proof of Theorem 5.2.8 we are looking for a closed singular $\mathscr{P}_p^{u'}$-manifold $\mathfrak{W}^{2m}$ in $B\mathbb{Z}_p$ that is non-trivial in $H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p)$ and admits a positive scalar curvature metric. We set $k = m - p$. We next try to obtain $\mathfrak{W}^{2m}$ from W^{2p+2k} while preserving the positive scalar curvature metric.

We ask whether (5.2.7) can be regarded as a closed $\mathscr{P}^{u'}$-manifold. A priori this is probably not true, but we can do the following. Since the elements of $\mathscr{P}^{u'}$ generate Ω_*^{SO} localized at p, the manifolds $M^{2p+2l-2}$ are bordant, say via V_l, to some $\mathscr{P}^{u'}$-manifolds $\mathcal{P}_{2p+2l-2}$, for all $0 \le l \le k$. We join the bordisms $V_l \times (T, S^{2k-2l+1})$ to W^{2p+2k} for all $0 \le l \le k-1$, obtaining a bordism $\mathfrak{W}^{2p+2k}$ between $(-1)^k\hat{p} \times (T, S^{2p+2k-1})$ and

$$M^{2p+2k-2} \times \big(T, S^1\big) - \mathcal{P}_{2p+2k-4} \times \big(T, S^3\big) + \cdots + (-1)^k \mathcal{P}_{2p-2} \times \big(T, S^{2k+1}\big). \tag{5.2.8}$$

According to Lemma 5.2.9, the manifolds $(T, S^3), \dots, (T, S^{2k+1})$ come equipped with positive scalar curvature metrics. It follows that the positive scalar curvature metric on W^{2p+2k} can be extended to $\mathfrak{W}^{2p+2k}$, this positive scalar curvature metric restricts on $\partial\mathfrak{W}^{2p+2k}$ to product metrics, and on $\mathcal{P}_{2p+2k-4}, \dots, \mathcal{P}_{2p-2}$ we may assume canonical positive scalar curvatures metrics.

We cannot proceed analogously for $M^{2p+2k-2}$ because (T, S^1) does not admit a positive scalar curvature metric. Now, the crucial point is that for $m = p^i$, or equivalently $k = p^i - p$, the manifold $M^{2p+2k-2}$ is already one of the p-local polynomial generators of Ω_*^{SO}. This follows from [CF64, Theorem 42.9]. Namely, loc. cit. it is proved that for $t = 2p^i - 2$

$$s_{t/4}\left[M^t\right] \equiv p \mod p^2$$

where $s_{t/4}[M^t] \in \mathbb{Z}$ denotes a certain characteristic number. Its importance is due to the fact that if $s_{t/4}[M^t] = p$, then M^t would serve as one of the polynomial generators of Ω_*^{SO} modulo torsion, without localizing (see e.g. [Sto68, p. 180]). In our situation, we have

$$\begin{aligned} s_{t/4}\left[M^t\right] &= p + sp^2 \\ &= p(1+sp) \end{aligned}$$

for some number s. Since $(1 + sp)$ is p-locally invertible, it follows that M^t can be taken as one of the p-local polynomial generators of Ω_*^{SO}, for $t = 2p^i - 2$.

We therefore consider $m = p^i$, so that $M^{2p+2k-2} = M^{2p^i-2}$ serves as one of the p-local polynomial generators of Ω_*^{SO}. We conclude: The existence of the closed singular $\mathscr{P}_p^{u'}$-manifold $\mathfrak{W}^{2p^i}$ proves the first part of Theorem 5.2.8 for a *particular* sequence $\mathscr{P}^{u'} = (P_1, P_2, \ldots)$ of p-local polynomial generators of Ω_*^{SO}. Namely, one has to take $P_{(p^i-1)/2} = M^{2p^i-2}$ for all $i \geq 1$. In addition, we note that the positive scalar curvature metrics on $P_{(p^i-1)/2}$, which enter into the definition of '$\oplus$'-positivity, must also be chosen in a particular way (see Lemma 5.2.9). $\square$

Proof of Theorem 5.2.8 for $m \neq p^i$, $m > p$. This part involves the construction of a product for singular bordism. A priori, the Cartesian product of two singular manifolds does *not* admit the structure of a singular manifold. Under which conditions singular bordism admits a *sensible* multiplicative structure, and how it can be defined, has been considered by Mironov (see [Mir75, Mir78]). Further references are [Bot92, Chapter 2] and [Rud98, Chapter 8, §2].

We consider the following general situation where we assume that all manifolds are oriented and all diffeomorphisms are orientation preserving. Let P denote a closed manifold. One defines a closed singular P-manifold $\overline{P} = P \times P \times [0,1]$ where the P-structure on the boundary is given by distinguishing different P-factors, i.e. by the diffeomorphism

$$\begin{aligned} \phi_{\overline{P}} \colon \partial(P \times P \times [0,1]) = P \times P \times \{0,1\} &\to P \times \delta_P \overline{P}, \\ (x,y,0) \mapsto (x,y,0),\ (x,y,1) &\mapsto (y,x,1) \end{aligned}$$

where $\delta_P \overline{P} = P \times \{0,1\}$. In the sequel we shall drop the 'P' in δ_P. The class $[\overline{P}] \in \Omega^P_{2\dim P+1}$ plays the role of an obstruction against the existence of a multiplicative structure on $\Omega_*^P(\_)$. One can prove (see e.g. [Bot92, Theorem 2.2.2], [Rud98, Theorem 2.4])

Theorem 5.2.10. *If $[\overline{P}]$ vanishes in Ω_*^P, then there exists an admissible product structure on $\Omega_*^P(\_)$, i.e. a collection of maps*

$$\hat{\times}_{\{(X,A),(Y,B)\}} \colon \Omega_*^P(X,A) \otimes \Omega_*^P(Y,B) \to \Omega_*^P(X \times Y, A \times Y \cup X \times B)$$

satisfying the usual *axioms (see e.g. [Bot92, Definition 2.1.2]).*

In case the conditions of Theorem 5.2.10 are satisfied, the product of two closed singular P-manifolds M and N is defined as follows. Let Q denote a zero singular P-bordism for $\overline{P}$, i.e. a singular P-manifold whose singular boundary is given by $\overline{P}$. Below must know what this means in detail:

First, Q is a compact manifold. Then there are compact manifolds $\tilde{\partial} Q$ and A with the same boundary such that

$$\partial Q = \tilde{\partial} Q \cup_{\partial(\tilde{\partial} Q)=\partial A} (-A).$$

Furthermore, we have a compact manifold δQ, a diffeomorphism $\phi\colon A \to P \times \delta Q$ and a diffeomorphism $\psi\colon \tilde{\partial} Q \to \overline{P}$ such that the diagram

$$\begin{tikzcd} \partial(\tilde{\partial} Q) \arrow[r, "\partial\psi"] \arrow[d, "\partial\phi"] & \partial\overline{P} \arrow[d, "\phi_{\overline{P}}"] \\ P \times \partial(\delta Q) \arrow[r, "\mathrm{id}\times\psi'"] & P \times \delta\overline{P} \end{tikzcd} \tag{5.2.9}$$

commutes, here $\partial\psi$ and $\partial\phi$ indicate that the diffeomorphisms must be restricted to the boundary, and ψ' is some map.

Let $\partial N \times [0,1]$ be a collar neighborhood of ∂N. The singular P-structures of M, N and Q induce a diffeomorphism $\omega\colon \delta M \times \delta N \times \tilde{\partial} Q \to \partial M \times \partial N \times [0,1]$. We now set

$$M \hat{\times} N = (M \times N) \cup_\omega (-\delta M \times \delta N \times Q). \tag{5.2.10}$$

One shows that there is a diffeomorphism $\partial(M \hat{\times} N) \to P \times \delta(M \hat{\times} N)$, where

$$\delta(M \hat{\times} N) = (\delta M \times N) \cup (\delta M \times \delta N \times \delta Q) \cup (M \times \delta N),$$

which turns $M \hat{\times} N$ into a singular P-manifold (see [Rud98, p. 471]).

The construction (5.2.10) also applies to elements of $\Omega_*^P(\_)$ since we may assume that the maps to background spaces, restricted to $\partial M \times \partial N \times [0,1]$, factor through $\delta M \times \delta N$. Theorem 5.2.10 generalizes to singular $\mathscr{P}$-manifolds where $\mathscr{P} = (P_1, P_2, \ldots)$ as usual denotes a sequence of closed manifolds: If $[\overline{P}_i] = 0$ for all i, then $\Omega_*^{\mathscr{P}}(\_)$ admits an admissible product structure. One obtains $M \hat{\times} N$ from $M \times N$ by attaching zero singular $\mathscr{P}$-bordisms for $\overline{P}_i$ for all i.

We observe that for oriented bordism and $\mathscr{P} = \mathscr{P}_p^{u'}$ all obstructions to an admissible product structure vanish since for all $P \in \mathscr{P}_p^{u'}$ they lie in

$$\Omega_i^P \cong \Omega_i^{SO}/([P]), \quad i \equiv 1 \mod 4,$$

which is p-locally trivial.

The singular product structure in general depends on the specifically chosen zero singular P-bordisms Q. In our situation, however, as a consequence of the following observation (or by Lemma 5.2.12 below), the specific choices of Q are immaterial.

On $\Omega_*^{\mathscr{P}_p^{u'}}(\_) = H_*(\_;\mathbb{Z}_p)$ the ordinary cross product $\times$ and the product $\hat{\times}$ defined above coincide. Namely, since both products satisfy the usual axioms of a product on a homology theory, one can show (see [Rud98, Theorem 7.3]) that they are induced by, a priori different, ring spectrum maps

$$r_\times, r_{\hat{\times}} \colon H\mathbb{Z}_p \wedge H\mathbb{Z}_p \to H\mathbb{Z}_p,$$

where $H\mathbb{Z}_p$ denotes the $\mathbb{Z}_p$-Eilenberg-MacLane spectrum. However, two such maps are uniquely determined up to homotopy by their induced maps on homotopy groups. Moreover, since $r_\times$ and $r_{\hat{\times}}$ are ring spectrum maps, one concludes that both induced maps $\mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ are given by the sum, that is, they have to coincide.

For our purpose we now need

Proposition 5.2.11. *Let M and N denote two singular $\mathscr{P}_p^{u'}$-manifolds. If M admits a positive scalar curvature metric, then the singular $\mathscr{P}_p^{u'}$-bordism class of $M \hat{\times} N$ is represented by a singular $\mathscr{P}_p^{u'}$-manifold of positive scalar curvature.*

Proof. According to (5.2.10), $M \hat{\times} N$ is obtained from $M \times N$ by attaching $\delta_P M \times \delta_P N \times Q$ for all $P \in \mathscr{P}_p^{u'}$, Q being some zero singular P-bordism for $\overline{P}$. The product metric on $M \times N$, properly scaled, is of positive scalar curvature. We have to show that it extends to a positive scalar curvature metric on $\delta_P M \times \delta_P N \times Q$ which restricts on $\delta_P M \times \delta_P N \times P \times \delta_P Q$

- to p copies of some positive scalar curvature metric on $\delta_P M \times \delta_P N \times \delta_P Q$, for $P = \hat{p}$,
- to the canonical positive scalar curvature metric, i.e. to the product of the chosen positive scalar curvature metric on P with any metrics on $\delta_P M$, $\delta_P N$ and $\delta_P Q$, for $P \in \mathscr{P}^{u'}$.

The case $P = \hat{p}$ is easy. Namely, equip M with a positive scalar curvature metric. Then there is an induced positive scalar curvature metric on $\delta_P M$. It now follows that $\delta_P M \times \delta_P N \times Q$ can be equipped with positive scalar curvature metric which satisfies the conditions above.

Now let $P \in \mathscr{P}^{u'}$. The attachment process of $\delta_P M \times \delta_P N \times Q$ involves only P-parts, therefore we shall simply speak of P-manifolds and P-bordism. As above let us drop the 'P' in δ_P. We consider $2M = M \dot{\cup} M$. Since 2 is p-locally invertible, it is enough to show that the singular P-bordism class of $2M \hat{\times} N$ is represented by a singular P-manifold of positive scalar curvature. Let Q be a zero singular P-bordism for $\overline{P}$. We must show that the singular P-bordism class of

$$(2M \times N) \cup_\omega (-2\delta M \times \delta N \times Q) \tag{5.2.11}$$

is positive, $\omega\colon 2\delta M \times \delta N \times \tilde{\partial} Q \to 2\partial M \times \partial N \times [0,1]$ as above.

Let $\{a\}$ and $\{b\}$ denote positively oriented points. There is a particularly easy zero singular P-bordism for $2\overline{P} = \overline{P} \times \{a, b\}$, namely $\overline{P} \times [0,1]$ whose singular P-structure is given as follows. We take $\tilde{\partial}(\overline{P} \times [0,1]) := \overline{P} \times \{0,1\}$, keeping orientations in mind: The interval $[0,1]$ induces an orientation on its boundary, and we assume that $\{0\}$ (resp. $\{1\}$) is positively (resp. negatively) oriented. The diffeomorphism $\psi\colon \tilde{\partial}(\overline{P} \times [0,1]) \to 2\overline{P}$ (see above Diagram 5.2.9) is defined by

$$\begin{gathered} P \times P \times [0,1] \times \{0,1\} \to P \times P \times [0,1] \times \{a,b\}, \\ (x,y,t,0) \mapsto (x,y,t,a), \ (x,y,t,1) \mapsto (y,x,1-t,b). \end{gathered} \tag{5.2.12}$$

Since P is even-dimensional, (5.2.12) is orientation preserving. Furthermore, we define the P-part of $\partial(\overline{P} \times [0,1])$ by $P^2 \times \{0,1\} \times [0,1]$ with the diffeomorphism

$$\begin{gathered} \phi\colon P \times P \times \{0,1\} \times [0,1] \to P \times \delta(\overline{P} \times [0,1]), \\ (x,y,0,t) \mapsto (x,y,0,t), \ (x,y,1,t) \mapsto (y,x,1,t) \end{gathered}$$

where $\delta(\overline{P} \times [0,1]) = P \times \{0,1\} \times [0,1]$. One verifies that the corresponding Diagram 5.2.9 commutes, so that $\overline{P} \times [0,1]$ establishes a zero singular P-bordism for $2\overline{P}$.

The singular P-structures of M, N and $\overline{P} \times [0,1]$ induce a diffeomorphism

$$\omega'\colon \delta M \times \delta N \times \tilde{\partial}(\overline{P} \times [0,1]) \to 2\partial M \times \partial N \times [0,1]$$

and

$$(2M \times N) \cup_{\omega'} (-\delta M \times \delta N \times \overline{P} \times [0,1]) \tag{5.2.13}$$

becomes a closed singular P-manifold which clearly admits a positive scalar curvature metric.

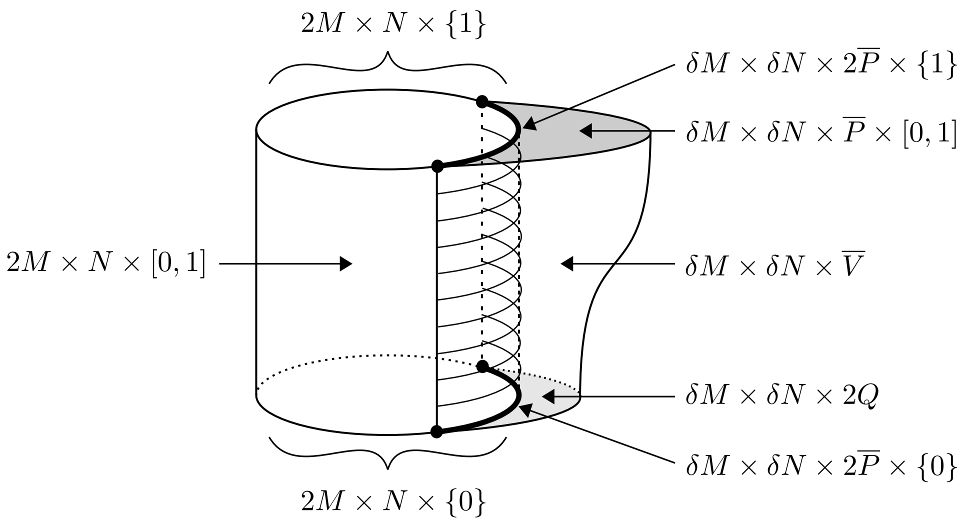

Figure 5.2: Singular P-bordism between (5.2.11) and (5.2.13)

Lemma 5.2.12. *The singular P-manifolds (5.2.11) and (5.2.13) are singular P-bordant.*

Proof. The closed singular P-manifold

$$(2Q) \cup_{\tilde{\partial}(2Q)=2\overline{P}\times\{0\}} (-2\overline{P} \times [0,1]) \cup_{2\overline{P}\times\{1\}=\tilde{\partial}(\overline{P}\times[0,1])} (-\overline{P} \times [0,1])$$

is denoted by V. Since $[V]$ lies in $\Omega^P_{2\dim P+2} = 0$, there exists a zero singular P-bordism $\overline{V}$ for V. Then

$$(2M \times N \times [0,1]) \cup_{\delta M\times\delta N\times 2\overline{P}\times[0,1]} (-\delta M \times \delta N \times \overline{V})$$

is a singular P-bordism between

$$\begin{aligned} &(2M \times N) \times \{0\} \cup_{\omega} (-\delta M \times \delta N \times 2Q) \\ \text{and} \quad &(2M \times N) \times \{1\} \cup_{\omega'} (-\delta M \times \delta N \times \overline{P} \times [0,1]) \end{aligned}$$

(see Figure 5.2). $\square$

It follows that the singular P-bordism class of (5.2.11) is positive. This finishes the proof of Proposition 5.2.11. $\square$

We now return to the proof of Theorem 5.2.8. It remains to handle the case $m \neq p^i$, $m > p$. Choose the unique i such that $p^i < m < p^{i+1}$. We will obtain $\mathfrak{W}^{2m}$ as a product of the closed singular $\mathscr{P}_p^{u'}$-manifold $\mathfrak{W}^{2p^i}$, which comes from the first part of the proof of Theorem 5.2.8, and an arbitrary generator of $H_{2(m-p^i)}(B\mathbb{Z}_p;\mathbb{Z}_p)$.

We observe that the map given by addition $A\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ induces an H-space structure on $B\mathbb{Z}_p$ and, in turn, a so-called Pontryagin product

$$\mu\colon H_*(B\mathbb{Z}_p;\mathbb{Z}_p) \times H_*(B\mathbb{Z}_p;\mathbb{Z}_p) \xrightarrow{\times} H_*(B\mathbb{Z}_p \times B\mathbb{Z}_p;\mathbb{Z}_p) \xrightarrow{A} H_*(B\mathbb{Z}_p;\mathbb{Z}_p).$$

It can be computed by means of the cup product structure of $H^*(B\mathbb{Z}_p)$. Recall that its even part is the polynomial algebra $\mathbb{Z}_p[\beta]$. It follows (see e.g. [Hat02, p. 290]) that the even part of $H_*(B\mathbb{Z}_p;\mathbb{Z}_p)$ is a divided polynomial algebra, denoted by $\Gamma_{\mathbb{Z}_p}[b]$ where b is dual to β. Multiplication in $\Gamma_{\mathbb{Z}_p}[b]$ is given as follows. Let b_j be dual to β^j, then

$$b_j b_l = \binom{j+l}{j} b_{j+l},$$

where the binomial coefficient is taken modulo p, of course. One can show that $\binom{s}{t}$ is divisible by p if and only if there is at least one digit in the base p extension of t which is greater than the corresponding digit in the expansion of s (this is known as Lucas' Theorem). In particular,

$$\binom{m}{p^i} \not\equiv 0 \mod p$$

for $p^i < m < p^{i+1}$. This implies that

$$H_{2p^i}(B\mathbb{Z}_p;\mathbb{Z}_p) \times H_{2(m-p^i)}(B\mathbb{Z}_p;\mathbb{Z}_p) \xrightarrow{\mu} H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p)$$

is non-trivial for $p^i < m < p^{i+1}$. Now let the closed singular $\mathscr{P}_p^{u'}$-manifold $N^{2(m-p^i)}$ be an arbitrary generator of $H_{2(m-p^i)}(B\mathbb{Z}_p;\mathbb{Z}_p)$. It follows that

$$\mu\left[\mathfrak{W}^{2p^i} \hat{\times} N^{2(m-p^i)}\right] \in H_{2m}(B\mathbb{Z}_p;\mathbb{Z}_p)$$

is a generator. According to Proposition 5.2.11, the singular P-manifold $\mathfrak{W}^{2p^i} \hat{\times} N^{2(m-p^i)}$ admits a positive scalar curvature metric.

We now see that it is important to work with general $\mathscr{P}_p^{u'}$-manifolds. Although $\mathfrak{W}^{2p^i}$ is, and $N^{2(m-p^i)}$ can assumed to be, a bordism between p copies of a lens space and some $\mathscr{P}^{u'}$-manifold, their product is not such a bordism. The boundary of the product of two connected manifolds with disconnected boundaries is connected. $\square$

5.2.2 Groups of Higher Rank

Before proving Theorem 5.2.1 for $r > 2$, we want to show that Toda brackets generate the torsion term in the Künneth formula

$$0 \to [H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}_p^r)]_n \xrightarrow{\times} H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p^r) \to [H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p^r)]_{n-1} \to 0. \tag{5.2.14}$$

For all $1 \leq i \leq n-2$, i odd, let $e_i \in H_i(B\mathbb{Z}_p)$ be a generator, the mod p reduction of e_i is denoted by $\tilde{e}_i$. In addition, for all $1 \leq i \leq n-2$, i odd, let $f_1^i, f_2^i, \ldots, f_{d(i)}^i$ be a basis for $H_{n-i-1}(B\mathbb{Z}_p^r)$, the mod p reduction of f_j^i is denoted by $\tilde{f}_j^i$. Although not used below, we note that $d(i) = \sum_{j=1}^{n-i-1}(-1)^{n-i-1-j}\binom{j+r-1}{r-1}$ (see [BR02, Proposition 5.2]). We have the map between the Bockstein exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot p"] & H_*(_;\mathbb{Z}) \arrow[r] \arrow[d] & H_*(_;\mathbb{Z}_p) \arrow[r, "\beta'"] \arrow[d, "="] & H_{*-1}(_;\mathbb{Z}) \arrow[r, "\cdot p"] \arrow[d, "pr"] & \cdots \\
\cdots \arrow[r, "\cdot p"] & H_*(_;\mathbb{Z}_{p^2}) \arrow[r] & H_*(_;\mathbb{Z}_p) \arrow[r, "\beta"] & H_{*-1}(_;\mathbb{Z}_p) \arrow[r, "\cdot p"] & \cdots .
\end{tikzcd}$$

As above, for all $1 \leq i \leq n-2$, i odd, let $\tilde{e}_{i+1} \in H_{i+1}(B\mathbb{Z}_p;\mathbb{Z}_p)$ such that $\beta(\tilde{e}_{i+1}) = \tilde{e}_i$. Similarly, since $H_*(B\mathbb{Z}_p^r;\mathbb{Z})$ contains only p torsion, multiplication by p is the zero map, and $\beta'\colon H_*(B\mathbb{Z}_p^r;\mathbb{Z}_p) \to H_{*-1}(B\mathbb{Z}_p^r;\mathbb{Z})$ is surjective for $* > 1$. We conclude that there exists $F_j^i \in H_{n-i}(B\mathbb{Z}_p^r;\mathbb{Z}_p)$ such that $\beta(F_j^i) = \tilde{f}_j^i$, for all $1 \leq i \leq n-2$, i odd, and $1 \leq j \leq d(i)$. We now define

$$\tilde{e}(n,i,j) := \tilde{e}_i \times F_j^i + \tilde{e}_{i+1} \times \tilde{f}_j^i \in H_n(B\mathbb{Z}_p^{r+1};\mathbb{Z}_p).$$

As above one shows that $\beta'(\tilde{e}(n,i,j))$ vanishes. This implies that $\tilde{e}(n,i,j)$ is the reduction of an integral class $e(n,i,j) \in H_n(B\mathbb{Z}_p^{r+1};\mathbb{Z})$. The classes

$$e(n,i,j), \quad 1 \leq i \leq n-2, i \text{ odd}, 1 \leq j \leq d(i),$$

are linearly independent and do not lie in the image of

$$[H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}_p^r)]_n \xrightarrow{\times} H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p^r).$$

It follows that

$$[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p^r)]_{n-1} \to H_n(B\mathbb{Z}_p \times B\mathbb{Z}_p^r), \quad e_i * f_j^i \mapsto e(n,i,j),$$

$1 \le i \le n-2, i$ odd$, 1 \le j \le d(i)$, is a splitting for the right map in (5.2.14). We note that the classes $e(n,i,j)$ are atoral. As above we can represent e_i, f^i_j resp. $\tilde{e}_i$, F^i_j by singular $\mathscr{P}^{u'}$- resp. $\mathscr{P}^{u'}_p$-manifolds in order to represent $e(n,i,j)$ by Toda brackets. The singular $\mathscr{P}^{u'}$-bordism class of such a Toda bracket depends on the chosen elements $F^i_j \in H_{n-i}(B\mathbb{Z}^r_p;\mathbb{Z}_p)$ since $\beta'\colon H_*(B\mathbb{Z}^r_p;\mathbb{Z}_p) \to H_{*-1}(B\mathbb{Z}^r_p;\mathbb{Z})$ is not injective. For our purpose the specific choice of the F^i_j's is immaterial, however. Summerizing, we have shown

Proposition 5.2.13. *The Toda brackets $\langle e_i, \hat{p}, f^i_j\rangle$, $1 \le i \le n-2, i$ odd$, 1 \le j \le d(i)$, form a basis for $[H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}^r_p)]_{n-1} \subset H_n(B\mathbb{Z}^{r+1}_p)$.*

Now let us turn to the proof of Theorem 5.2.1 for $r > 2$. We will show by induction on r that the following holds for all $r \ge 1$

$$H^{\mathrm{atoral}}_*(B\mathbb{Z}^r_p) \subset H^{\oplus}_*(B\mathbb{Z}^r_p). \tag{5.2.15}$$

We note that '$\oplus$' is understood with respect to the particular sequence $\mathscr{P}^{u'}$ and the particular positive scalar curvature metrics which were chosen in the proof of the statement for $r = 2$.

The cases $r = 1$ and 2 were proven above.

Fix $r > 2$ and assume that (5.2.15) holds for all $r' \le r$. Consider again the Künneth formula

$$0 \to [H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}^r_p)]_n \xrightarrow{\times} H_n(B\mathbb{Z}_p \times B\mathbb{Z}^r_p) \to [H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}^r_p)]_{n-1} \to 0.$$

Let $u \in H^{\mathrm{atoral}}_n(B\mathbb{Z}^{r+1}_p)$. By the splitting above we can write $u = v + w$ for some

$$v \in [H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}^r_p)]_n \quad \text{and} \quad w \in [H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}^r_p)]_{n-1}.$$

We shall show that v and w are positive. Let us first note that v is atoral since u and w are. We can write $v = \sum_{i+j=n} b_i \otimes c_j$ for some $b_i \in H_i(B\mathbb{Z}_p)$ and $c_j \in H_j(B\mathbb{Z}^r_p)$. If $i > 1$, then $b_i \otimes c_j$ is atoral and positive. We must therefore show that the atoral class $b_1 \otimes c_{n-1}$ is positive. But $b_1 \otimes c_{n-1}$ is atoral only if $c_{n-1} \in H_{n-1}(B\mathbb{Z}^r_p)$ is atoral. It now follows by the induction hypothesis that c_{n-1} is positive. This implies that $b_1 \otimes c_{n-1}$ is positive as well.

It remains to show that any $w \in [H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}^r_p)]_{n-1}$ is positive. We have $w = \langle x_1, \hat{p}, x\rangle$ for some $x_1 \in H_*(B\mathbb{Z}_p)$ and $x \in H_*(B\mathbb{Z}^r_p)$. Bearing the

Künneth formula for $B\mathbb{Z}_p^r = B\mathbb{Z}_p \times B\mathbb{Z}_p^{r-1}$ in mind, we can write $x = a + b$ for some

$$a \in H_*(B\mathbb{Z}_p) \otimes H_*(B\mathbb{Z}_p^{r-1}) \quad \text{and} \quad b \in H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p^{r-1}).$$

We shall show that $\langle x_1, \hat{p}, a\rangle$ and $\langle x_1, \hat{p}, b\rangle$ are positive. We have $a = c \otimes d$ for some $c \in H_*(B\mathbb{Z}_p)$ and $d \in H_*(B\mathbb{Z}_p^{r-1})$. By the construction of the Toda brackets (see [Ale72, Axiom 3, p. 198]), one sees that $\langle x_1, \hat{p}, c \otimes d\rangle = \langle x_1, \hat{p}, c\rangle \otimes d$. According to the induction hypothesis, or by the previous subsection, $\langle x_1, \hat{p}, c\rangle \in H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)$ is positive, and so is $\langle x_1, \hat{p}, c\rangle \otimes d$.

We have $b = \langle x_2, \hat{p}, x'\rangle$ for some $x_2 \in H_*(B\mathbb{Z}_p)$ and $x' \in H_*(B\mathbb{Z}_p^{r-1})$, and we must show that $\langle x_1, \hat{p}, \langle x_2, \hat{p}, x'\rangle\rangle$ is positive. Since any toral cohomology class evaluates non-trivially on some toral homology class, one concludes that there is a direct sum decomposition

$$H_*(B\mathbb{Z}_p^{r-1}) = H_*^{\mathrm{atoral}}(B\mathbb{Z}_p^{r-1}) \oplus H_*^{\mathrm{toral}}(B\mathbb{Z}_p^{r-1}).$$

According to the induction hypothesis, $H_*^{\mathrm{atoral}}(B\mathbb{Z}_p^{r-1})$ is a subgroup of $H_*^{\oplus}(B\mathbb{Z}_p^{r-1})$. We can therefore write $x' = y + z$ for some $y \in H_*^{\oplus}(B\mathbb{Z}_p^{r-1})$ and $z \in H_*^{\mathrm{toral}}(B\mathbb{Z}_p^{r-1})$. We shall show that $\langle x_1, \hat{p}, \langle x_2, \hat{p}, y\rangle\rangle$ and $\langle x_1, \hat{p}, \langle x_2, \hat{p}, z\rangle\rangle$ are positive.

In the first case we use the associativity property (see [Ale72, Axiom 6, p. 198])

$$-\langle\langle x_1, \hat{p}, x_2\rangle, \hat{p}, y\rangle + \langle x_1, \langle \hat{p}, x_2, \hat{p}\rangle, y\rangle + \langle x_1, \hat{p}, \langle x_2, \hat{p}, y\rangle\rangle = 0. \tag{5.2.16}$$

The element $\langle \hat{p}, x_2, \hat{p}\rangle$ defines a class in the even degrees of $H_*(B\mathbb{Z}_p)$. Hence it vanishes and so does the whole middle term in (5.2.16). We can therefore consider $\langle\langle x_1, \hat{p}, x_2\rangle, \hat{p}, y\rangle$. Since $\langle x_1, \hat{p}, x_2\rangle \in H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p)$ and y are positive, both classes can be represented by singular $\mathscr{P}^{u'}$-manifolds of positive scalar curvature. A modest generalization of the concordance argument illustrated in Figure 3.1 implies that the Toda bracket of these representatives also admits a positive scalar curvature metric.

In the second case we represent z as a linear combination of elements of the form $\phi\colon T^s \to B\mathbb{Z}_p^{r-1}$. Observe that each such ϕ contains a factor of the form $\iota\colon S^1 \to B\mathbb{Z}_p$, say

$$\phi = \phi' \times \iota\colon T^{s-1} \times S^1 \to B\mathbb{Z}_p^{r-2} \times B\mathbb{Z}_p.$$

We now have

$$\left\langle x_1, \hat{p}, \left\langle x_2, \hat{p}, \left[T^{s-1}\right] \otimes \left[S^1\right]\right\rangle\right\rangle = \left\langle x_1, \hat{p}, \left\langle x_2, \hat{p}, \left[T^{s-1}\right]\right\rangle\right\rangle \otimes \left[S^1\right]. \tag{5.2.17}$$

Since $\langle x_1, \hat{p}, \langle x_2, \hat{p}, [T^{s-1}]\rangle\rangle \in H_*(B\mathbb{Z}_p) * H_*(B\mathbb{Z}_p^{r-1})$ is atoral, it is positive by the induction hypothesis. It follows that (5.2.17) is positive as well. This completes the induction step and, in turn, the proof of Theorem 5.2.1.

5.3 Concluding Remarks

We proved that

$$H_*^{\mathrm{atoral}}(B\mathbb{Z}_p^r) \subset H_*^{\oplus}(B\mathbb{Z}_p^r). \tag{5.3.1}$$

Subsequently, there is the question of whether equality holds here. In this respect, we do not know any counterexamples, and we make the following

Conjecture 5.3.1.

$$H_*^{atoral}(B\mathbb{Z}_p^r) = H_*^{\oplus}(B\mathbb{Z}_p^r).$$

Let us note that for an elementary Abelian 2-group, the corresponding statement is false (see [Joa04]).

One could ask whether (5.3.1) or Conjecture 5.3.1 holds for more general spaces that are not necessarily aspherical. If a space X is simply connected, then $H_*^{\mathrm{atoral}}(X) = H_*(X)$. It is not difficult to show that all elements of $H_n(X)$, $n \neq 4$, are positive: According to the Hurewicz Theorem, all elements of $H_2(X)$ and $H_3(X)$ can be represented by spheres, which admit positive scalar curvature metrics. By using surgery, one can show that all classes of $H_n(X)$, $n \geq 4$, are representable by simply connected non-spin singular $\mathscr{P}^u$-manifolds. Those admit positive scalar curvature metrics by Theorem 4.2.2 provided that $n \geq 5$.

An equality as in Conjecture 5.3.1 is rare. Namely, for any n there is a map $T^n \to S^n$ of degree one. It follows that a homology class in $H_n(X)$ which is represented by a sphere is toral. Such a class is not atoral, and it lies in $H_n^{\oplus}(X)$ provided that $n > 1$.

It is an open question whether (5.3.1), and perhaps also Corollary 5.2.2, hold for arbitrary Abelian p-groups. In order to prove this statement, the methods of the previous section appear to be promising. In this direction, the crucial task may be to extend Conner and Floyd's results about the structure of $\Omega_*^{SO}(B\mathbb{Z}_p)$, which we used in the first part of the proof of Theorem 5.2.8, to the groups $\Omega_*^{SO}(B\mathbb{Z}_{p^s})$, $s > 1$.

We fixed an odd prime p and proved (5.3.1) for a particular sequence of p-local polynomial generators of Ω_*^{SO} with particular positive scalar curvature metrics. One might ask whether (5.3.1) holds for an arbitrary sequence of generators of Ω_*^{SO} modulo torsion with arbitrary positive scalar curvature metrics. In this case, (5.3.1) would be true for all primes simultaneously.

The first part of Theorem 5.1.2 clearly also holds in the spin case. By contrast, the spin version of the second statement of Theorem 5.1.2 seems difficult to prove because $\pi\colon \Omega_*^{Spin}(X) \to \Omega_*^{\mathscr{P}\alpha}(X)$ is not surjective, as it was the case in the oriented version. In fact, we saw in Chapter 4 that π is not even surjective in the case that X is a single point.

However, p-local computations as in the previous section should also be practicable in the spin case. Instead of positive homology one then has to work with the obvious notion of *positive connective real K-theory*. The study of how much of $ko_*(B\mathbb{Z}_p^r)$ is exhausted by $ko_*^{\oplus}(B\mathbb{Z}_p^r)$ might then yield a new proof of

Theorem 2.3 in [BR05]. *Let M be a connected closed spin manifold of dimension greater or equal than five with fundamental group $\mathbb{Z}_p^r$, and let $f\colon M \to B\mathbb{Z}_p^r$ be the classifying map of the universal cover of M. Assume that $\alpha(M)$ vanishes. Then M admits a positive scalar curvature metric if $[M, f] \in ko_*^{atoral}(B\mathbb{Z}_p^r)$.*

Loco cit. this is proved by showing that $[M, f]$ lies in $ko_*^{+}(B\mathbb{Z}_p^r)$. The advantage of $ko_*^{\oplus}(B\mathbb{Z}_p^r)$ over $ko_*^{+}(B\mathbb{Z}_p^r)$ is that the forthcoming computations will use the Künneth formula for connective real K-theory, instead of the more complicated Künneth formula for spin bordism.

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