ADAPTIVE MULTILEVEL TECHNIQUES FOR MIXED FINITE ELEMENT DISCRETIZATIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS*

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Abstract. We consider mixed finite element discretizations of linear second-order elliptic boundary value problems with respect to an adaptively generated hierarchy of possibly highly nonuniform simplicial triangulations. By a well-known postprocessing technique the discrete problem is equivalent to a modified nonconforming discretization which is solved by preconditioned CG iterations using a multilevel preconditioner in the spirit of Bramble, Pasciak, and Xu designed for standard nonconforming approximations. Local refinement of the triangulations is based on an a posteriori error estimator which can be easily derived from superconvergence results. The performance of the preconditioner and the error estimator is illustrated by several numerical examples.

Key words. mixed finite elements, multilevel preconditioned CG iterations, a posteriori error estimator

AMS subject classifications. 65N30, 65N55

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1. Introduction. In this work, we are concerned with adaptive multilevel techniques for the efficient solution of mixed finite element discretizations of linear second-order elliptic boundary value problems. In recent years, mixed finite element methods have been increasingly used in applications, in particular, for such problems where instead of the primal variable, its gradient is of major interest. As examples we mention the flux in stationary flow problems or neutron diffusion and the current in semiconductor device simulation (cf., e.g., [4], [14], [15], [23], [28], [39], [45], and [48]). An excellent treatment of mixed methods and further references can be found in the monography of Brezzi and Fortin [13].

Mixed discretization gives rise to linear systems associated with saddle point problems whose characteristic feature is a symmetric but indefinite coefficient matrix. Since the systems typically become large for discretized partial differential equations, there is a need for fast iterative solvers. We note that preconditioned iterative methods for saddle point problems have been considered by Bank, Welfert, and Yserentant [8] based on a modification of Uzawa's method leading to an outer/inner iterative scheme and by Rusten and Winther [47] relying on the minimum residual method. Moreover, there are several approaches using domain decomposition techniques and related multilevel Schwarz iterations (cf., e.g., Cowsar [16], Ewing and Wang [24, 25, 26], Mathew [35, 36], and Vassilevski and Wang [50]). A further important aspect is to increase efficiency by using adaptively generated triangulations. In contrast to the existing concepts for standard conforming finite element discretizations as realized, for example, in the finite element codes PLTMG [5] and KASKADE [20, 21], not much work has been done concerning local refinement of the triangulations in mixed discretizations. There is some work by Ewing, Lazarov, Russell, and Vassilevski

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[22] in the case of quadrilateral mixed elements, but the emphasis is more on the appropriate treatment of the slave nodes than on efficient and reliable indicators for local refinement. Recently Braess and Verfürth [11] suggested an a posteriori residual-based error estimator for Raviart–Thomas finite elements. It is the purpose of this paper to develop a fully adaptive algorithm for mixed discretizations based on the lowest-order Raviart–Thomas elements featuring a multilevel iterative solver and an a posteriori error estimator as indicators for local refinement. The paper is organized as follows.

In section 2 we will present the mixed discretization and a postprocessing technique due to Fraeijs de Veubeke [27] and Arnold and Brezzi [1]. This technique is based on the elimination of the continuity constraints for the normal components of the flux on the interelement boundaries from the conforming Raviart–Thomas ansatz space. Instead, the continuity constraints are taken care of by appropriate Lagrangian multipliers resulting in an extended saddle point problem. Static condensation of the flux leads to a linear system which is equivalent to a modified nonconforming approach involving the lowest-order Crouzeix–Raviart elements augmented by cubic bubble functions. Section 3 is devoted to the numerical solution of that nonconforming discretization by a multilevel preconditioned CG iteration using a BPX-type preconditioner. This preconditioner has been designed by the authors [30, 29] for standard nonconforming approaches and is closely related to that of Oswald [42]. By an application of Nepomnyaschikh's fictitious domain lemma [37, 38] it can be verified that the spectral condition number of the preconditioned stiffness matrix behaves like $O(1)$. The error estimator investigated by Braess and Verfürth [11] controls the error of the primal variable in a weighted discrete L^2-norm which tends to the H^1-norm for $h \to 0$. In contrast to [11], we present in section 4 an a posteriori error estimator in terms of the L^2-norm which can be derived from a superconvergence result for mixed discretizations due to Arnold and Brezzi [1]. It will be shown that the error estimator is equivalent to a weighted sum of the squares of the jumps of the approximation of the primal variable across the interelement boundaries. Finally, in section 5 some numerical results are given illustrating both the performance of the preconditioner and the error estimator.

2. Mixed discretization and postprocessing. We consider linear, second-order elliptic boundary value problems of the form

$$\begin{aligned} -\mathrm{div}(a \cdot \nabla u) + b \cdot u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω stands for a bounded, polygonal domain in the Euclidean space $\mathbb{R}^2$ with boundary Γ and f is a given function in $L^2(\Omega)$. We further assume that $a = (a_{ij})_{i,j=1}^2$ is a symmetric 2×2 matrix-valued function with $a_{ij} \in L^\infty(\Omega)$ and b is a function in $L^\infty(\Omega)$ satisfying

$$\begin{aligned} \alpha_0 \cdot |\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x) \cdot \xi_i \xi_j \le \alpha_1 \cdot |\xi|^2, \quad \xi \in \mathbb{R}^2, \quad 0 < \alpha_0 \le \alpha_1, \\ 0 \le \beta_0 \le b(x) \le \beta_1, \end{aligned} \tag{2.2}$$

for almost all $x \in \Omega$. We note that only for simplicity have we chosen homogeneous Dirichlet boundary conditions in (2.1). Other boundary conditions of Neumann type or mixed boundary conditions can be treated as well. Introducing the Hilbert space

$$H(\mathrm{div};\Omega) = \left\{ \underline{q} \in \left(L^2(\Omega)\right)^2 \mid \quad \mathrm{div}(\underline{q}) \in L^2(\Omega) \right\}$$

and the flux

$$\underline{j} = -a\nabla u$$

as an additional unknown, the standard mixed formulation of (2.1) is given as follows: find $(\underline{j}, u) \in H(\text{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a(\underline{j}, \underline{q}) + b(\underline{q}, u) &= 0, \quad \underline{q} \in H(\text{div};\Omega), \\ b(\underline{j}, v) - d(u, v) &= -(f, v)_0, \quad v \in L^2(\Omega), \end{aligned} \tag{2.3}$$

where the bilinear forms $a : H(\text{div};\Omega)\times H(\text{div};\Omega) \longmapsto \mathbb{R}$, $b : H(\text{div};\Omega)\times L^2(\Omega) \longmapsto \mathbb{R}$, and $d : L^2(\Omega) \times L^2(\Omega) \longmapsto \mathbb{R}$ are given by

$$\begin{aligned} a(\underline{j}, \underline{q}) &:= \int_\Omega c\underline{j} \cdot \underline{q}\, dx, \quad \underline{j}, \underline{q} \in H(\text{div};\Omega),\ c := a^{-1}, \\ b(\underline{q}, v) &:= -\int_\Omega \text{div}\underline{q} \cdot v\, dx, \quad \underline{q} \in H(\text{div};\Omega),\ v \in L^2(\Omega), \\ d(u, v) &:= \int_\Omega bu \cdot v\, dx, \quad u, v \in L^2(\Omega), \end{aligned}$$

and $(\cdot,\cdot)_0$ stands for the usual L^2 inner product. Note that under the above assumption on the data of the problem the existence and uniqueness of a solution to (2.3) is well established (cf., e.g., [13]). For the mixed discretization of (2.3) we suppose that a regular simplicial triangulation $\mathcal{T}_h$ of Ω is given. In particular, for an element $K \in \mathcal{T}_h$ we refer to e_i, $1 \leq i \leq 3$, as its edges and we denote by $\mathcal{E}_h$ the set of edges of $\mathcal{T}_h$ and by $\mathcal{E}_h^0 := \mathcal{E}_h \cap \Omega$, $\mathcal{E}_h^\Gamma := \mathcal{E}_h \cap \Gamma$ the subsets of interior and boundary edges, respectively. Further, for $D \subseteq \Omega$ we refer to $|D|$ as the measure of D and we denote by $P_k(D)$, $k \geq 0$, the linear space of polynomials of degree $\leq k$ on D. Then, a conforming approximation of the flux space $H(\text{div};\Omega)$ is given by $V_h := RT_0(\Omega, \mathcal{T}_h)$, where

$$RT_0(\Omega, \mathcal{T}_h) := \left\{\underline{q}_h \in H(\text{div};\Omega) \quad \underline{q}_h|_K \in RT_0(K), K \in \mathcal{T}_h\right\}$$

and $RT_0(K)$ stands for the lowest-order Raviart–Thomas element

$$RT_0(K) := (P_0(K))^2 + \underline{x} \cdot P_0(K).$$

Note that any $\underline{q}_h \in RT_0(K)$ is uniquely determined by its normal components $\underline{n}\cdot\underline{q}_h|_{e_i}$ on the edges e_i, $1 \leq i \leq 3$, of $K \in \mathcal{T}_h$, where $\underline{n}$ denotes the outer normal vector of K. In particular, the conformity of the approximation is guaranteed by specifying the basis in such a way that continuity of the normal components

$$\underline{n} \cdot \underline{q}_h \Big|_{e\cap K} = - \underline{n} \cdot \underline{q}_h \Big|_{e\cap K'}, \quad K \cap K' = e \in \mathcal{E}_h^0, \tag{2.4}$$

is satisfied across interelement boundaries. Consequently, we have $\text{dim}V_h = n_h$, where $n_h = \#\mathcal{E}_h$. Observing $\text{div}V_h = W_h := W_0(\Omega;\mathcal{T}_h)$, where

$$W_k(\Omega;\mathcal{T}_h) := \left\{ v_h \in L^2(\Omega) \mid v_h|_K \in P_k(K), K \in \mathcal{T}_h \right\}, \quad k \in \mathbb{N},$$

the standard mixed discretization of (2.3) is given by the following: find $(\underline{j}_h, u_h) \in V_h \times W_h$ such that

$$\begin{aligned} a(\underline{j}_h, \underline{q}_h) + b(\underline{q}_h, u_h) &= 0, \quad \underline{q}_h \in V_h, \\ b(\underline{j}_h, v_h) - d(u_h, v_h) &= -(f, v_h)_0, \quad v_h \in W_h. \end{aligned} \tag{2.5}$$

For $D \subseteq \Omega$ we denote by $(\cdot,\cdot)_{k,D}$, $k \geq 0$, the standard inner products and by $\|\cdot\|_{k,D}$ the associated norms on the Sobolev spaces $H^k(D)$ and $\left(H^k(D)\right)^2$, respectively. For simplicity, the lower index D will be omitted if $D = \Omega$. Then, it is well known that assuming $u \in H^2(\Omega)$ and $\underline{j} \in \left(H^1(\Omega)\right)^2$, the following a priori error estimates hold true:

$$\begin{aligned}\|u - u_h\|_0 &\leq C \cdot h \cdot \|u\|_2,\\ \|\underline{j} - \underline{j}_h\|_0 &\leq C \cdot h \cdot \|\underline{j}\|_1,\end{aligned}$$

where h stands, as usual, for the maximum diameter of the elements of $\mathcal{T}_h$ and C is a positive constant independent of h, u, and $\underline{j}$ (cf., e.g., [1, Thm. 1.1]).

We further observe that the algebraic formulation of (2.5) gives rise to a linear system with coefficient matrix

$$\begin{matrix} A & B^T \\ B & -D \end{matrix}$$

which is symmetric but indefinite. There exist several efficient iterative solvers for such systems, for example, those proposed by Bank, Welfert, and Yserentant [8], Cowsar [16], Ewing and Wang [24, 25, 26], Mathew [35], Rusten and Winther [47], and Vassilevski and Wang [50]. However, we will follow an idea suggested by Fraeijs de Veubeke [27] and further analyzed by Arnold and Brezzi in [1] (cf. also [13]). Eliminating the continuity constraints (2.4) from V_h results in the nonconforming Raviart–Thomas space $\hat{V}_h := RT_0^{-1}(\Omega;\mathcal{T}_h)$, where

$$RT_0^{-1}(\Omega;\mathcal{T}_h) := \left\{ \underline{q}_h \in \left(L^2(\Omega)\right)^2 \,\middle|\, \underline{q}_h|_K \in RT_0(K), K \in \mathcal{T}_h \right\}.$$

Since there are now two basic vector fields associated with each $e \in \mathcal{E}_h^0$, we have $\hat{n}_h := \dim \hat{V}_h = n_h + \#\mathcal{E}_h^0$. Instead, the continuity constraints are taken care of by Lagrangian multipliers living in $M_h := M_0(\mathcal{E}_h)$, where

$$M(\mathcal{E}_h) := \quad \mu_h \in L^2(\mathcal{E}_h) \,|\quad \mu_h|_e \in P_0(e), e \in \mathcal{E}_h \Big\}$$

and

$$M_0(\mathcal{E}_h) := \quad \mu_h \in M(\mathcal{E}_h) \,\Big|\quad \mu_h|_e = 0, e \in \mathcal{E}_h^\Gamma \Big\}.$$

Then, the nonconforming mixed discretization of (2.3) is to find $(\underline{j}_h, u_h, \lambda_h) \in \hat{V}_h \times W_h \times M_h$ such that

$$\begin{aligned}\hat{a}(\underline{j}_h, \underline{q}_h) + \hat{b}(\underline{q}_h, u_h) + c(\lambda_h, \underline{q}_h) &= 0, \quad \underline{q}_h \in \hat{V}_h,\\ \hat{b}(\underline{j}_h, v_h) - d(u_h, v_h) &= -(f, v_h)_0, \quad v_h \in W_h,\\ c(\mu_h, \underline{j}_h) &= 0, \quad \mu_h \in M_h,\end{aligned} \tag{2.6}$$

where $\hat{a} : \hat{V}_h \times \hat{V}_h \longmapsto \mathbb{R}$, $\hat{b} : \hat{V}_h \times W_h \longmapsto \mathbb{R}$, and $c : M_h \times \hat{V}_h \longmapsto \mathbb{R}$ are given by

$$\begin{aligned}\hat{a}(\underline{j}_h, \underline{q}_h) &:= \sum_{K \in \mathcal{T}_h} \int_K c\underline{j}_h \cdot \underline{q}_h \, dx, \quad \underline{j}_h, \underline{q}_h \in \hat{V}_h,\\ \hat{b}(\underline{q}_h, v_h) &:= -\sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}\underline{q}_h \cdot v_h \, dx, \quad \underline{q}_h \in \hat{V}_h,\ v_h \in W_h,\\ c(\mu_h, \underline{q}_h) &:= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu_h \underline{n} \cdot \underline{q}_h \, d\sigma, \quad \mu_h \in M_h,\ \underline{q}_h \in \hat{V}_h.\end{aligned}$$

As shown in [1] the above multiplier technique has two significant advantages. The first one is some sort of a superconvergence result concerning the approximation of the solution u in (2.1) in the L^2-norm while the second one is related to the specific structure of (2.6) and has an important impact on the efficiency of the solution process. To begin with the first one, we denote by Π_h the L^2 projection onto M_h. Then, it is easy to see that there exists a unique $\hat{u}_h \in W_1(\Omega; \mathcal{T}_h)$ such that

$$\Pi_h \hat{u}_h = \lambda_h$$

(cf. [1, Lem. 2.1]). The function $\hat{u}_h$ represents a nonconforming interpolation of λ_h which can be shown to provide a more accurate approximation of u in the L^2-norm. In particular, if $u \in H^2(\Omega)$ and $f \in H^1(\Omega)$, then there exists a constant $c > 0$ independent of h, u, and $\underline{j}$ such that

$$\|u - \hat{u}_h\|_0 \leq c \cdot h^2 \cdot (\|u\|_2 + \|f\|_1) \tag{2.7}$$

(cf. [13, Chap. 5, Thm. 3.1]). The preceding result will be used for the construction of a local a posteriori error estimator to be developed in section 4.

As far as the efficient solution of (2.6) is concerned we note that the algebraic formulation leads to a linear system with a coefficient matrix of the form

$$\begin{pmatrix} \hat{A} & \hat{B}^T & C^T \\ \hat{B} & -D & 0 \\ C & 0 & 0 \end{pmatrix}.$$

In particular, $\hat{A}$ stands for a block-diagonal matrix, each block being a 3×3 matrix corresponding to an element $K \in \mathcal{T}_h$. Hence, $\hat{A}$ is easily invertible which suggests block elimination of the unknown flux (also known as static condensation) resulting in a 2×2 block system with a symmetric, positive definite coefficient matrix. This linear system is equivalent to a modified nonconforming approximation involving the lowest-order Crouzeix–Raviart elements augmented by cubic bubble functions. Denoting by m_e the midpoint of an edge $e \in \mathcal{E}_h$ we introduce

$$\begin{aligned} CR_h := \ & v_h \in L^2(\Omega) \,|\, \quad v_h|_K \in P_1(K), \quad K \in \mathcal{T}_h, \\ & v_h|_K(m_e) = v_h|_{K'}(m_e), \quad e = K \cap K' \in \mathcal{E}_h^0, \\ & v_h(m_e) = 0, \quad e \in \mathcal{E}_h^\Gamma \big\}, \\ B_h := \ & v_h \in L^2(\Omega) \,|\, \quad v_h|_K \in P_3(K), \ v_h|_{\partial K} = 0, \quad K \in \mathcal{T}_h \big\}, \end{aligned}$$

and we set

$$N_h = CR_h \oplus B_h.$$

Note that $\dim CR_h = \#\mathcal{E}_h^0 = \dim M_h$ and $\dim B_h = \#\mathcal{T}_h$. Further, we denote by P_h and $\hat{P}_c$ the L^2 projections onto W_h and $\hat{V}_h$, the latter with respect to the weighted L^2 inner product $(\cdot,\cdot)_{0,c} = (c\cdot,\cdot)_0$. As shown in [1, Lem. 2.3 and Lem. 2.4], there exists a unique $\Psi_h \in N_h$ such that

$$P_h \Psi_h = u_h, \quad \Pi_h \Psi_h = \lambda_h. \tag{2.8}$$

Originally, Lemma 2.4 in [1] is only proved for $b \equiv 0$, but the result can be easily generalized for functions $b \geq 0$. Due to (2.6) and (2.8) we obtain

$$\underline{j}_h = -\hat{P}_c(a \nabla \Psi_h). \tag{2.9}$$

Moreover, Ψ_h is the unique solution of the variational problem

$$a_{N_h}(\Psi_h, \eta_h) = (P_h f, \eta_h)_0, \quad \eta_h \in N_h, \tag{2.10}$$

where the bilinear form $a_{N_h} : N_h \times N_h \longmapsto \mathbb{R}$ is given by

$$a_{N_h}(\Psi_h, \eta_h) := \sum_{K \in \mathcal{T}_h} \int_K \hat{P}_c(a\nabla\Psi_h) \cdot \nabla\eta_h + bP_h\Psi_h \cdot P_h\eta_h \quad dx, \quad \Psi_h, \eta_h \in N_h.$$

We will solve (2.10) numerically by preconditioned CG iterations using a multilevel preconditioner of BPX-type. The construction of that preconditioner will be dealt with in the following section.

3. Iterative solution by multilevel preconditioned CG iterations. We assume a hierarchy $(\mathcal{T}_k)_{k=0}^j$ of possibly highly nonuniform triangulations of Ω obtained by the refinement process due to Bank, Sherman, and Weiser [6] based on regular refinements (partitioned into four congruent subtriangles) and irregular refinements (bisection). For a detailed description, including the refinement rules, we refer to [5] and [18]. We remark that the refinement rules are such that each $K \in \mathcal{T}_k$, $1 \le k \le j$, is geometrically similar either to an element of $\mathcal{T}_0$ or to an irregular refinement of a triangle in $\mathcal{T}_0$. Consequently, there exist constants $0 < \kappa_0 \le \kappa_1$ depending only on the local geometry of $\mathcal{T}_0$ such that for all $K \in \mathcal{T}_k$, $0 \le k \le j$, and its edges $e \subset \partial K$

$$\kappa_0 |e|^2 \le |K| \le \kappa_1 |e|^2. \tag{3.1}$$

Moreover, the refinement rules imply the property of local quasi uniformity; i.e., there exists a constant $\kappa_2 > 0$ depending only on the local geometry of $\mathcal{T}_0$ such that for all $K, K' \in \mathcal{T}_k$, $K \cap K' \neq \emptyset$, $0 \le k \le j$,

$$h_K \le \kappa_2 h_{K'},$$

where $h_K := \text{diam} K$.

We consider the modified nonconforming approximation (2.10) on the highest level j

$$a_{N_j}(\Psi_j, \eta_j) = (P_{h_j} f, \eta_j)_0, \quad \eta_j \in N_j := N_{h_j}, \tag{3.2}$$

and we attempt to solve (3.2) by preconditioned CG iterations. The preconditioner will be constructed by means of the natural splitting of N_j into the standard nonconforming part $CR_j := CR_{h_j}$ and the "bubble" part $B_j := B_{h_j}$ and a further multilevel preconditioning of BPX-type for the nonconforming part. For that purpose we introduce the bilinear form $a_{CR_j} : CR_j \times CR_j \longmapsto \mathbb{R}$

$$a_{CR_j}(u_j^{CR}, v_j^{CR}) := \sum_{K \in \mathcal{T}_j} a|_K(u_j^{CR}, v_j^{CR}), \quad u_j^{CR}, v_j^{CR} \in CR_j, \tag{3.3}$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \longmapsto \mathbb{R}$ is the standard bilinear form associated with the primal variational formulation (2.1)

$$a(u, v) := \int_\Omega (a\nabla u \cdot \nabla v + bu \cdot v)\, dx, \quad u, v \in H_0^1(\Omega). \tag{3.4}$$

In what follows we will refer to $A : H_0^1(\Omega) \longmapsto H_0^1(\Omega)$ as the operator associated with the bilinear form a.

Further, we define the bilinear form $a_{B_j} : B_j \times B_j \longmapsto \mathbb{R}$ by

$$a_{B_j}(w_j^B, z_j^B) := \sum_{K \in \mathcal{T}_j} \int_K a\hat{P}_{\mathrm{Id}}(\nabla w_j^B) \cdot \hat{P}_{\mathrm{Id}}(\nabla z_j^B) + bP_{h_j}(w_j^B) \cdot P_{h_j}(z_j^B) \quad dx$$

for all $w_j^B, z_j^B \in B_j$. Denoting by A_{D_j}, $D_j \in \{N_j, CR_j, B_j\}$, the operators associated with a_{D_j}, we will prove the spectral equivalence of A_{N_j} and $A_{CR_j} + A_{B_j}$. To this end we need the following technical lemmas.

LEMMA 3.1. *For all $u_j^{CR} \in CR_j$ and $K \in \mathcal{T}_j$ there holds*

$$\|P_{h_j} u_j^{CR}\|_{0,K}^2 \geq \|u_j^{CR}\|_{0,K}^2 - \frac{h_K^2}{12} \|\nabla u_j^{CR}\|_{0,K}^2. \tag{3.5}$$

Proof. For the reference triangle $\hat{K}$ with vertices $(0,0)$, $(1,0)$, and $(0,1)$ it is easy to establish

$$\|v\|_{0,\hat{K}}^2 \leq \|P_{h_j} v\|_{0,\hat{K}}^2 + \frac{1}{12} \|\nabla v\|_{0,\hat{K}}^2, \quad v \in P_1(\hat{K}).$$

Equation (3.5) can be deduced by the affine equivalence of the Crouzeix–Raviart elements. □

LEMMA 3.2. *For all $w_j^B \in B_j$ and $K \in \mathcal{T}_j$ there holds*

$$\|P_{h_j} w_j^B\|_{0,K}^2 \leq \frac{1}{12} \left(\frac{\kappa_1}{\kappa_0}\right)^2 h_K^2 \|\hat{P}_{\mathrm{Id}}(\nabla w_j^B)\|_{0,K}^2. \tag{3.6}$$

Proof. Since $w_j^B|_K = \alpha \lambda_1 \lambda_2 \lambda_3$, $\alpha \in \mathbb{R}$, where λ_i, $1 \leq i \leq 3$, are the barycentric coordinates of K, we have $P_{h_j} w_j^B|_K = \frac{\alpha}{60}$ and thus

$$\|P_{h_j} w_j^B\|_{0,K}^2 = \frac{\alpha^2}{3600} |K|. \tag{3.7}$$

Denoting by $\underline{\tau}_K^i$, $1 \leq i \leq 3$, the local basis of $\hat{V}_h$ and by $(\mathbf{\hat{A}_{Id}})_\mathbf{K}$, $K \in \mathcal{T}_h$, the matrix representation of $\hat{a}|_K$ in case $c = \mathrm{Id}$ we find

$$\|\hat{P}_{\mathrm{Id}}(\nabla w_j^B)\|_{0,K}^2 = \mathbf{b^T} {(\mathbf{\hat{A}_{Id}})_\mathbf{K}}^{-1} \mathbf{b}, \tag{3.8}$$

where $\mathbf{b} = (b_1, b_2, b_3)^T$, $b_i = (\nabla w_j^B, \underline{\tau}_K^i)_{0,K}$, $1 \leq i \leq 3$. Observing $\underline{\tau}_K^i = (2|K|)^{-1}|e_i|(\underline{x} - \underline{a}_i)$, where $\underline{a}_i$ stands for the vertex opposite to e_i, by Green's formula

$$b_i = -\left(w_j^B, \mathrm{div} \underline{\tau}_K^i\right)_{0,K} = -\frac{\alpha}{60}|e_i|, \quad 1 \leq i \leq 3. \tag{3.9}$$

If we consider the reference triangle $\hat{K}$, where the vertices are given by $(0,0)$, $(1,0)$, and $(0,1)$, we obtain

$$\frac{1}{6}\mathbf{Id} \leq (\mathbf{\hat{A}_{Id}})_{\mathbf{\hat{K}}} \leq \frac{1}{2}\mathbf{Id},$$

where "$\leq$" refers to the usual partial order on the set of symmetric, positive definite matrices. Moreover, taking advantage of the affine equivalence of the Raviart–Thomas elements it is easy to show that

$$\frac{1}{48}\kappa_1^{-2}|K|\mathbf{Id} \leq (\mathbf{\hat{A}_{Id}})_\mathbf{K} \leq \frac{1}{4}\kappa_0^{-2}|K|\mathbf{Id}. \tag{3.10}$$

Using (3.1), (3.9), and (3.10) in (3.8) and observing (3.7) it follows that

$$h_K^2\|\hat{P}_{\mathrm{Id}}(\nabla w_j^B)\|_{0,K}^2 \geq 12\left(\frac{\alpha}{60}\cdot\frac{\kappa_0}{\kappa_1}\right)^2|K| = 12\left(\frac{\kappa_0}{\kappa_1}\right)^2\cdot\|P_{h_j}w_j^B\|_{0,K}^2. \qquad \square$$

We assume a and b to be locally constant, i.e., $a_{ij}|_K = \text{const.}$, $1 \leq i,j \leq 2$, $b_K = b|_K = \text{const.}$, $K \in \mathcal{T}_j$, and we denote by $\alpha_{0,K}$ and $\alpha_{1,K}$ the lower and upper bounds arising in (2.2) when restricting a to K. We further suppose that a and b are such that

$$\min_{K\in\mathcal{T}_j}\left(4\alpha_{0,K} - h_K^2 b_K\left(\frac{\kappa_0}{\kappa_1}\right)^2\right) \geq 0. \tag{3.11}$$

Note that for simplicity only we have chosen the strong inequality (3.11). All results can be extended to the more general case that a constant $c > 0$, independent of K, exists such that for all $K \in \mathcal{T}_j$ $\alpha_{0,K} - ch_K^2 b_K \geq 0$ holds. Under assumption (3.11) the following holds.

THEOREM 3.3. *Under assumption* (3.11) *there exist constants* $0 < c_0 \leq c_1$ *depending on the local bounds* $\alpha_{l,K}$, $l \in \{0,1\}$, $K \in \mathcal{T}_j$, *such that for all* $\psi_j \in N_j$ *with* $\psi_j = u_j^{CR} + w_j^B$, $u_j^{CR} \in CR_j$, $w_j^B \in B_j$,

$$\begin{aligned} a_{N_j}(\psi_j,\psi_j) &\geq c_0\left(a_{CR_j}(u_j^{CR},u_j^{CR}) + a_{B_j}(w_j^B,w_j^B)\right), \\ a_{N_j}(\psi_j,\psi_j) &\leq c_1\left(a_{CR_j}(u_j^{CR},u_j^{CR}) + a_{B_j}(w_j^B,w_j^B)\right). \end{aligned} \tag{3.12}$$

Proof. For the proof of the preceding result we use the following lemma which can easily be established.

LEMMA 3.4. *For all* $\psi_i \in N_j$ *and* $K \in \mathcal{T}_j$ *there hold*

$$(\hat{P}_c a\nabla\psi_j,\nabla\psi_j)_{0,K} \leq \frac{\alpha_{1,K}}{\alpha_{0,K}}\left((a\nabla u_j^{CR},\nabla u_j^{CR})_{0,K} + (a\hat{P}_{\mathrm{Id}}(\nabla w_j^B),\hat{P}_{\mathrm{Id}}(\nabla w_j^B))_{0,K}\right), \tag{3.13a}$$

$$(\hat{P}_c a\nabla\psi_j,\nabla\psi_j)_{0,K} \geq \alpha_{0,K}\left(\|\nabla u_j^{CR}\|_{0,K}^2 + \|\hat{P}_{\mathrm{Id}}(\nabla w_j^B)\|_{0,K}^2\right). \tag{3.13b}$$

Proof. Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \left(\hat{P}_c a\nabla\psi_j,\nabla\psi_j\right)_{0,K}^2 &= \left(\hat{P}_c a\nabla\psi_j,\hat{P}_{\mathrm{Id}}\nabla\psi_j\right)_{0,K}^2 \\ &\leq \left(\hat{P}_c a\nabla\psi_j,\hat{P}_c a\nabla\psi_j\right)_{0,K}\left(\hat{P}_{\mathrm{Id}}\nabla\psi_j,\hat{P}_{\mathrm{Id}}\nabla\psi_j\right)_{0,K} \\ &\leq \alpha_{1,K}\left(c\hat{P}_c a\nabla\psi_j,\hat{P}_c a\nabla\psi_j\right)_{0,K}\left(\hat{P}_{\mathrm{Id}}\nabla\psi_j,\hat{P}_{\mathrm{Id}}\nabla\psi_j\right)_{0,K}. \end{aligned}$$

Using $\hat{P}_{\mathrm{Id}}(\nabla u_j^{CR}) = \nabla u_j^{CR}$ as well as the orthogonality $(\nabla u_j^{CR},\nabla w_j^B)_{0,K} = 0$, we get $(\hat{P}_{\mathrm{Id}}\nabla\psi_j,\hat{P}_{\mathrm{Id}}\nabla\psi_j)_{0,K} = \|\nabla u_j^{CR}\|_{0,K}^2 + \|\hat{P}_{\mathrm{Id}}(\nabla w_j^B)\|_{0,K}^2$. Observing $c\cdot a = \mathrm{Id}$ we obtain

$$\begin{aligned} \left(\hat{P}_c a\nabla\psi_j,\nabla\psi_j\right)_{0,K} &\leq \alpha_{1,K}\left(\hat{P}_{\mathrm{Id}}\nabla\psi_j,\hat{P}_{\mathrm{Id}}\nabla\psi_j\right)_{0,K} \\ &\leq \tfrac{\alpha_{1,K}}{\alpha_{0,K}}\left((a\nabla u_j^{CR},\nabla u_j^{CR})_{0,K} + (a\hat{P}_{\mathrm{Id}}(\nabla w_j^B),\hat{P}_{\mathrm{Id}}(\nabla w_j^B))_{0,K}\right). \end{aligned}$$

The following inequality deduces (3.13b):

$$
\begin{aligned}
\left(\hat{P}_{\rm Id}\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)^2_{0,K} &= \left(\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)^2_{0,K} = \left(c\hat{P}_c a\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)^2_{0,K}\\
&\le \left(\hat{P}_{\rm Id}\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)_{0,K}\left(c\hat{P}_c a\nabla\psi_j,c\hat{P}_c a\nabla\psi_j\right)_{0,K}\\
&\le \alpha_{0,K}^{-1}\left(\hat{P}_{\rm Id}\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)_{0,K}\left(c\hat{P}_c a\nabla\psi_j,\hat{P}_c a\nabla\psi_j\right)_{0,K}\\
&\le \alpha_{0,K}^{-1}\left(\hat{P}_{\rm Id}\nabla\psi_j,\hat{P}_{\rm Id}\nabla\psi_j\right)_{0,K}\left(\hat{P}_c a\nabla\psi_j,\nabla\psi_j\right)_{0,K}. \qquad \square
\end{aligned}
$$

On the other hand, in view of $\|P_{h_j}u_j^{CR}\|^2_{0,k} \le \|u_j^{CR}\|^2_{0,k}$ we have

$$
(3.14a) \qquad (P_{hj}b\psi_j,\psi_j)_{0,K} \le 2\left((bu_j^{CR},u_j^{CR})_{0,K} + (bP_{h_j}w_j^B,P_{h_j}w_j^B)_{0,K}\right).
$$

Combining (3.13a) and (3.14a) gives the upper bound in (3.12) with $c_1 = \max(\max_{K\in\mathcal{T}_0}\frac{\alpha_{1,K}}{\alpha_{0,K}},2)$. Further, by Young's inequality, $0<\epsilon<1$, and (3.5), (3.6)

$$
\begin{aligned}
&(P_{hj}b\psi_j,\psi_j)_{0,K}\\
&\quad\ge b_K\left(\|P_{h_j}u_j^{CR}\|^2_{0,K} + \|P_{h_j}w_j^B\|^2_{0,K} - 2\|P_{h_j}u_j^{CR}\|_{0,K}\cdot\|P_{h_j}w_j^B\|_{0,K}\right)\\
&\quad\ge b_K\left((1-\epsilon)\|P_{h_j}u_j^{CR}\|^2_{0,K} + (1-\tfrac{1}{\epsilon})\|P_{h_j}w_j^B\|^2_{0,K}\right)\\
(3.14b)\qquad &\quad\ge (1-\epsilon)\left((bu_j^{CR},u_j^{CR})_{0,K} + (bP_{h_j}w_j^B,P_{h_j}w_j^B)_{0,K}\right)\\
&\qquad -(1-\epsilon)\frac{h_K^2b_K}{12}\|\nabla u_j^{CR}\|^2_{0,K} - \left(\tfrac{1}{\epsilon}-\epsilon\right)\left(\tfrac{\kappa_1}{\kappa_0}\right)^2\frac{h_K^2b_K}{12}\|\hat{P}_{\rm Id}(\nabla w_j^B))\|_{0,K}\\
&\quad\ge (1-\epsilon)\left((bu_j^{CR},u_j^{CR})_{0,K} + (bP_{h_j}w_j^B,P_{h_j}w_j^B)_{0,K}\right)\\
&\qquad -\left(\tfrac{1}{\epsilon}-\epsilon\right)\left(\tfrac{\kappa_1}{\kappa_0}\right)^2\frac{h_K^2b_K}{12}\left(\|\hat{P}_{\rm Id}(\nabla w_j^B))\|_{0,K} + \|\nabla u_j^{CR}\|^2_{0,K}\right).
\end{aligned}
$$

Consequently, using (3.13b), (3.14b), (3.11), and $\epsilon=\frac{1}{2}$

$$
\begin{aligned}
a_{N_j}|_K(\psi_j,\psi_j) \ge{}& \frac{1}{2}\frac{\alpha_{0,K}}{\alpha_{1,K}}\left((a\nabla u_j^{CR},\nabla u_j^{CR})_{0,K} + (a\hat{P}_{\rm Id}(\nabla w_j^B),\hat{P}_{\rm Id}(\nabla w_j^B))_{0,K}\right)\\
&+\frac{1}{2}\left((bu_j^{CR},u_j^{CR})_{0,K} + (bP_{h_j}w_j^B,P_{h_j}w_j^B)_{0,K}\right)
\end{aligned}
$$

which yields the lower bound in (3.12) with $c_0 = \frac{1}{2}\min_{K\in\mathcal{T}_0}\frac{\alpha_{0,K}}{\alpha_{1,K}}$. $\square$

We note that the bilinear form a_{B_j} gives rise to a diagonal matrix which thus can be easily used in the preconditioning process. On the other hand, the bilinear form a_{CR_j} corresponds to the standard nonconforming approximation of (2.1) by the lowest-order Crouzeix–Raviart elements. Multilevel preconditioners for such nonconforming finite element discretizations have been developed by Oswald [42, 43], Zhang [56], and the authors [30, 29]. Here we will use a BPX-type preconditioner based on the use of a pseudointerpolant which allows us to identify CR_j with a closed subspace of the standard conforming ansatz space with respect to the next higher level. More precisely, we denote by $\mathcal{T}_{j+1}$ the triangulation obtained from $\mathcal{T}_j$ by regular refinement of all elements $K\in\mathcal{T}_j$, and we refer to $S_k\subset H_0^1(\Omega)$, $0\le k\le j+1$, as the standard conforming ansatz space generated by continuous, piecewise linear finite elements with respect to the triangulation $\mathcal{T}_k$. Denoting by $\mathcal{N}_{j+1}^0$ the set of interior vertices of $\mathcal{T}_{j+1}$ and recalling that the midpoints m_e of interior edges $e \in \mathcal{E}_j^0$ correspond to vertices $p \in \mathcal{N}_{j+1}^0$, we define a mapping $P_j^{CR} : CR_j \longmapsto S_{j+1}$ by

$$(P_j^{CR} u_j)(p) = \begin{cases} u_j^{CR}(p) & \text{if } p = m_e, \\ \nu_p^{-1} \sum_{i=1}^{\nu_p} u_j^{CR}(m_{e_i}^p) & \text{if } p \neq m_e, \end{cases} \tag{3.15}$$

where $m_{e_i}^p$, $1 \leq i \leq \nu_p$, are the midpoints of those interior edges $e \in \mathcal{E}_j^0$ having $p \in \mathcal{N}_{j+1}^0$ as a common vertex. We note that this pseudointerpolant has been originally proposed by Cowsar [16] in the framework of related domain decomposition techniques. The following result will lay the basis for the construction of the multilevel preconditioner.

LEMMA 3.5. *Let P_j^{CR} be the pseudointerpolant given by* (3.15)*. Then, there exist constants* $0 < \delta_0 \leq \delta_1$ *depending only on the local geometry of $\mathcal{T}_0$ such that for all* $u_j \in CR_j$

$$\delta_0 a_{CR_j}(u_j, u_j) \leq a(P_j^{CR} u_j, P_j^{CR} u_j) \leq \delta_1 a_{CR_j}(u_j, u_j). \tag{3.16}$$

Proof. The assertion follows by arguing literally in the same way as in [16, Thm. 2] and taking advantage of the local quasi uniformity of the triangulations. □

It follows from (3.16) that $\tilde{S}_{j+1} := P_j^{CR} CR_j$ represents a closed subspace of S_{j+1} being isomorphic to CR_j. Based on this observation we may now use the well-known BPX preconditioner for conforming discretizations with respect to the hierarchy $(S_k)_{k=0}^{j+1}$ of finite element spaces (cf., e.g., [10], [12], [17], [44], [53], [55], and [56]). We remark that for a nonvanishing Helmholtz term in (2.1) the initial triangulation $\mathcal{T}_0$ should be chosen in such a way that the magnitude of the coefficients of the principal part of the elliptic operator is not dominated by the magnitude of the Helmholtz coefficient times the square of the maximal diameter of the elements in $\mathcal{T}_0$ (cf., e.g., [44], [54]).

Denoting by $\Gamma_k := \{\phi_1^{(k)}, \ldots, \phi_{n_k}^{(k)}\}$, $n_k := \dim S_k$, the set of nodal basis functions of S_k, $0 \leq k \leq j+1$, the BPX preconditioner is based on the following structuring of the nodal bases of varying index k:

$$\Phi_0 := \Gamma_0, \quad \Phi_k := \Gamma_k \setminus \Gamma_{k-1}, \quad 1 \leq k \leq j+1.$$

We introduce the Hilbert space

$$V := V_0 \times \prod_{k=1}^{j+1} \prod_{\phi \in \Phi_k} V_\phi, \quad V_0 := S_0, \quad V_\phi := \mathrm{span}\{\phi\}, \tag{3.17}$$

equipped with the inner product

$$(\overline{u}, \overline{v})_V := (u_0, v_0)_0 + \sum_{k=1}^{j+1} \sum_{\phi \in \Phi_k} (u_\phi, v_\phi)_0, \quad \overline{u}, \overline{v} \in V,$$

where $\overline{u} = (u_0, (u_\phi)_{\phi \in \Phi_1}, \ldots, (u_\phi)_{\phi \in \Phi_{j+1}})$, and we consider the bilinear form $\tilde{b} : V \times V \longmapsto \mathbb{R}$ given by

$$\tilde{b}(\overline{u}, \overline{v}) := a(u_0, v_0) + \sum_{k=1}^{j+1} \sum_{\phi \in \Phi_k} a(u_\phi, v_\phi), \quad \overline{u}, \overline{v} \in V, \tag{3.18}$$

denoting by $\tilde{B} : V \longmapsto V$ the operator associated with $\tilde{b}$. We further define a mapping $R_V : V \longmapsto S_{j+1}$ by

$$R_V \overline{u} := u_0 + \sum_{k=1}^{j+1} \sum_{\phi \in \Phi_k} u_\phi, \quad \overline{u} \in V, \tag{3.19}$$

and refer to R_V^* as its adjoint in the sense that $(R_V \overline{u}, v)_0 = (\overline{u}, R_V^* v)_V$, $\overline{u} \in V$, $v \in S_{j+1}$. Then, the BPX-preconditioner is given by

$$C = R_V \tilde{B}^{-1} R_V^* \tag{3.20}$$

satisfying

$$\gamma_0 a(u, u) \le a(CAu, u) \le \gamma_1 a(u, u), \quad u \in S_{j+1}, \tag{3.21}$$

with constants $0 < \gamma_0 \le \gamma_1$ depending only on the local geometry of $\mathcal{T}_0$ and on the bounds for the data a, b in (2.2).

The condition number estimates (3.21) have been established by various authors (cf. [10], [17], [41]). They can be derived using the powerful Dryja–Widlund theory [19] of additive Schwarz iterations. Another approach due to Oswald [44] is based on Nepomnyaschikh's fictitious domain lemma.

LEMMA 3.6. *Let S and V be two Hilbert spaces with inner products $(\cdot,\cdot)_S$ and $(\cdot,\cdot)_V$ and consider bilinear forms $a_S : S \times S \longmapsto \mathbb{R}$ and $\tilde{b} : V \times V \longmapsto \mathbb{R}$ generated by symmetric, positive definite operators $A_S : S \longmapsto S$ and $\tilde{B} : V \longmapsto V$. Assume that there exists a linear operator $R : V \longmapsto S$, some (not necessarily linear) operator $T : S \longmapsto V$, and constants $0 < c_0 \le c_1$ such that*

$$R \cdot Tu = u, \quad u \in S, \tag{3.22a}$$

$$a_S(Rv, Rv) \le c_1 \tilde{b}(v, v), \quad v \in V, \tag{3.22b}$$

$$c_0 \tilde{b}(Tu, Tu) \le a_S(u, u), \quad u \in S. \tag{3.22c}$$

Then, there holds

$$c_0 a_S(u, u) \le a_S(R\tilde{B}^{-1} R^* Au, u) \le c_1 a_S(u, u), \quad u \in S,$$

where $R^* : S \longmapsto V$ is the adjoint of R in the sense that $(Rv, u)_S = (v, R^* u)_V$, $v \in V$, $u \in S$.

Proof. See, e.g., [38]. □

In the framework of the BPX preconditioner $S = S_{j+1}$ with a_S being the bilinear form in (3.4) while V, $\tilde{b}$, and $R = R_V$ are given by (3.17), (3.18), and (3.19), respectively, the estimate (3.22b) is usually established by means of a strengthened Cauchy–Schwarz inequality. Further, $T = T_S$ is an appropriately chosen decomposition operator such that the P. L. Lions-type estimate (3.22c) holds true (cf., e.g., [44, Chap. 4]).

Now, returning to the nonconforming approximation we define $I_{j+1}^S : S_{j+1} \longmapsto CR_j$ by $\left(I_{j+1}^S u_{j+1}\right)(m_e) = u_{j+1}(m_e)$, $u_{j+1} \in S_{j+1}$. Note that in view of (3.15) the operator $I_{j+1}^S P_j^{CR}$ corresponds to the identity on CR_j. Then, with C as in (3.20) the operator

$$C_{NC} = I_{j+1}^S C (I_{j+1}^S)^* \tag{3.23}$$

is an appropriate BPX preconditioner for the nonconforming discretization of (2.1). In particular, we have the following.

THEOREM 3.7. *Let* C_{NC} *be given by* (3.23). *Then, there exist positive constants* η_0, η_1 *depending only on the local geometry of* $\mathcal{T}_0$ *and on the bounds for the coefficients* a, b *in* (2.2) *such that for all* $u \in CR_j$

$$\eta_0 a_{CR_j}(u,u) \leq a_{CR_j}(C_{NC} A_{CR_j} u, u) \leq \eta_1 a_{CR_j}(u,u).$$

Proof. In view of the fictitious domain lemma we choose $S = CR_j$, a_{CR_j} as in (3.3) and V and $\tilde{b}$ according to (3.17), (3.18). Furthermore, we specify $R : V \longmapsto CR_j$ by $R = I^S_{j+1} R_V$ with R_V from (3.19) and $T : CR_j \longmapsto V$ by $T = T_S P_j^{CR}$ with T_S as the decomposition operator in the conforming setting. Obviously

$$RTu = I^S_{j+1} R_V T_S P_j^{CR} u = I^S_{j+1} P_j^{CR} u = u, \quad u \in CR_j. \tag{3.24a}$$

Moreover, using the obvious inequality

$$a_{CR_j}(I^S_{j+1} u_{j+1}, I^S_{j+1} u_{j+1}) \leq \kappa a(u_{j+1}, u_{j+1}), \quad u_{j+1} \in S_{j+1},$$

and (3.22b), for all $v \in V$ we have

$$\begin{aligned} a_{CR_j}(Rv, Rv) &= a_{CR_j}(I^S_{j+1} R_V v, I^S_{j+1} R_V v) \\ &\leq \kappa a(R_V v, R_V v) \leq \kappa \gamma_1 \tilde{b}(v,v). \end{aligned} \tag{3.24b}$$

Finally, using again (3.16) and (3.22c) for $T = T_S P_j^{CR}$, for all $u \in CR_j$ we get

$$\begin{aligned} \delta_1^{-1} \gamma_0 \tilde{b}(Tu, Tu) &= \delta_1^{-1} \gamma_0 \tilde{b}(T_S P_j^{CR} u, T_S P_j^{CR} u) \\ &\leq \delta_1^{-1} a_{CR_j}(P_j^{CR} u, P_j^{CR} u) \leq a_{CR_j}(u,u). \end{aligned} \tag{3.24c}$$

In terms of (3.24a–c) we have verified the hypotheses of the fictitious domain lemma which gives the assertion. ☐

4. A posteriori error estimation. Efficient and reliable error estimators for the total error, providing indicators for local refinement of the triangulations, are an indispensable tool for efficient adaptive algorithms. Concerning the finite element solution of elliptic boundary value problems we mention the pioneering work done by Babuška and Rheinboldt [2, 3] which has been extended by, among others, Bank and Weiser [7], Johnson and Hansbo [34], and Deuflhard, Leinen, and Yserentant [18] to derive element-oriented and edge-oriented local error estimators for standard conforming approximations. We remark that these concepts have been adapted to nonconforming discretizations by the authors in [29, 30, 31]. The basic idea is to discretize the defect problem for the available approximation with respect to a finite element space of higher accuracy. For a detailed representation of the different concepts and further references we refer to the monographs of Johnson [33], Szabó and Babuška [49], and Zienkiewicz and Taylor [57] (cf. also the recent survey articles by Bornemann, Erdmann, and Kornhuber [9] and Verfürth [51, 52]).

Pioneering work concerning error estimation for mixed finite elements was recently done by Braess and Verfürth [11]. They investigate discrete weighted norms and suggest a residual-based error estimator. In contrast to this work we consider a pressure-based estimator for the natural norm. This section is devoted to the derivation of an error estimator for the L^2-norm of the total error in the primal variable u based on the superconvergence result (2.7). As we shall see that this estimator does not require the solution of an additional defect problem and hence is much cheaper than the estimators mentioned above. We note, however, that an error estimator for the total error in the flux based on the solution of localized defect problems has been developed by the authors in [32]. In the standard conforming case, error estimators obtained by some postprocessing of the approximation are well known [58] and further analyzed by Rodriguez [46]. They are based on the idea that the smoothed recovered gradient gives a better approximation. In contrast to this approach, we start with a better finite element solution and prove that it is equivalent to an average of the original one.

We suppose that $\tilde{\psi}_h \in N_h$ is an approximation of the solution $\psi_h \in N_h$ of (2.10) obtained, for example, by the multilevel iterative solution process described in the preceding section. Then, in view of (2.8) and (2.9), we get an approximation $(\tilde{\underline{j}}_h, \tilde{u}_h, \tilde{\lambda}_h) \in \hat{V}_h \times W_h \times M_h$ of the unique solution $(\underline{j}_h, u_h, \lambda_h) \in \hat{V}_h \times W_h \times M_h$ of (2.6) by means of

$$\tilde{u}_h := P_h \tilde{\psi}_h, \quad \tilde{\lambda}_h := \Pi_h \tilde{\psi}_h, \tag{4.1}$$

and

$$\tilde{\underline{j}}_h := -\hat{P}_c(a \nabla \tilde{\psi}_h).$$

Note that, in general, $\tilde{\underline{j}}_h$ is in contrast to $\underline{j}_h$ not contained in $H(\mathrm{div};\Omega)$. Further, we denote by $\hat{\tilde{u}}_h \in CR_h$ the nonconforming extension of $\tilde{\lambda}_h$.

In light of the superconvergence result (2.7) we assume the existence of a constant $0 \leq \beta < 1$ such that

$$\|u - \hat{u}_h\|_0 \leq \beta \|u - u_h\|_0. \tag{4.2}$$

In other words, (4.2) states that the nonconforming extension $\hat{u}_h$ of λ_h does provide a better approximation of the primal variable u than the piecewise constant approximation u_h.

It is easy to see that (4.2) yields

$$\begin{aligned} \|u - \tilde{u}_h\|_0 &\leq (1-\beta)^{-1} \Big(\|\tilde{u}_h - \hat{\tilde{u}}_h\|_0 + \big(\beta \|u_h - \tilde{u}_h\|_0 + \|\hat{u}_h - \hat{\tilde{u}}_h\|_0 \big) \Big), \\ \|u - \tilde{u}_h\|_0 &\geq (1+\beta)^{-1} \Big(\|\tilde{u}_h - \hat{\tilde{u}}_h\|_0 - \big(\beta \|u_h - \tilde{u}_h\|_0 + \|\hat{u}_h - \hat{\tilde{u}}_h\|_0 \big) \Big). \end{aligned} \tag{4.3}$$

Observing (2.8) and (4.1), we have

$$\|u_h - \tilde{u}_h\|_0 = \|P_h(\psi_h - \tilde{\psi}_h)\|_0 \leq \|\psi_h - \tilde{\psi}_h\|_0, \tag{4.4}$$

$$\begin{aligned} \|\hat{u}_h - \hat{\tilde{u}}_h\|_0 &= \Big(\sum_{K \in \mathcal{T}_h} \tfrac{1}{3} |K| \sum_{i=1}^{3} \big| \big(\lambda_h - \tilde{\lambda}_h \big)_{|e_i} \big|^2 \Big)^{1/2} \\ &= \Big(\sum_{K \in \mathcal{T}_h} \tfrac{1}{3} |K| \sum_{i=1}^{3} \big| \big(\psi_h - \tilde{\psi}_h \big)(m_{e_i}) \big|^2 \Big)^{1/2} \\ &\leq \sqrt{\tfrac{10}{3}} \|\psi_h - \tilde{\psi}_h\|_0. \end{aligned} \tag{4.5}$$

Using (4.4), (4.5) in (4.3) we get

$$\|u-\tilde{u}_h\|_0 \le (1-\beta)^{-1}\|\tilde{u}_h-\hat{\tilde{u}}_h\|_0 + \sqrt{\tfrac{10}{3}\tfrac{1+\beta}{1-\beta}}\|\psi_h-\tilde{\psi}_h\|_0,$$

$$\|u-\tilde{u}_h\|_0 \ge (1+\beta)^{-1}\|\tilde{u}_h-\hat{\tilde{u}}_h\|_0 - \sqrt{\tfrac{10}{3}}\|\psi_h-\tilde{\psi}_h\|_0.$$

We note that $\|\psi_h-\tilde{\psi}_h\|_0$ represents the L^2-norm of the iteration error whose actual size can be controlled by the iterative solution process. Therefore, the term $\|\tilde{u}_h-\hat{\tilde{u}}_h\|_0$ provides an efficient and reliable error estimator for the L^2-norm of the total error whose local contributions $\|\tilde{u}_h-\hat{\tilde{u}}_h\|_{0,K}$, $K\in\mathcal{T}_h$, can be used as indicators for local refinement of $\mathcal{T}_h$. Moreover, the estimator can be cheaply computed, since it only requires the evaluation of the available approximations $\tilde{u}_h\in W_h$ and $\tilde{\lambda}_h\in M_h$.

At the moment the error estimator depends on $\tilde{u}_h$ and $\hat{\tilde{u}}_h$. If the original system (2.5) is solved, $\hat{\tilde{u}}_h$ is not available without any additional computational amount. Therefore, we are interested in the investigation of an error estimator which can be evaluated by means of $\tilde{u}_h$. The rest of this section will be devoted to showing that the introduced estimator is equivalent to a weighted sum of the squares of the jumps of $\tilde{u}_h$ across the edges $e\in\mathcal{E}_h$. For that purpose we introduce the jump and the average of piecewise continuous functions v along edges $e\in\mathcal{E}_h$. In particular, for $e\in\mathcal{E}_h^0$ we denote by K_{in} and K_{out} the two adjacent triangles and by $\underline{n}_e$ the unit normal outward from K_{in}. On the other hand, for $e\in\mathcal{E}_h^\Gamma$ we refer to $\underline{n}_e$ as the usual outward normal. Then, we define the average $[v]_A$ of v on $e\in\mathcal{E}_h$ and the jump $[v]_J$ of v on $e\in\mathcal{E}_h$ according to

$$[v]_A(x) = \begin{cases} \frac{1}{2}\left(v|_{K_{in}}(x)+v|_{K_{out}}(x)\right), & x\in e=K_{in}\cap K_{out}\in\mathcal{E}_h^0,\\ \frac{1}{2}v|_K(x), & x\in e=\partial K\cap\mathcal{E}_h^\Gamma,\end{cases}$$

$$[v]_J(x) = \begin{cases} \left(v|_{K_{in}}(x)-v|_{K_{out}}(x)\right), & x\in e=K_{in}\cap K_{out}\in\mathcal{E}_h^0,\\ v|_K(x), & x\in e=\partial K\cap\mathcal{E}_h^\Gamma.\end{cases}$$

It is easy to see that for piecewise continuous functions u, v the following hold:

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} u|_K\cdot v|_K\,d\sigma = \sum_{e\in\mathcal{E}_h}\int_e \left(u|_{K_{in}}\cdot v|_{K_{in}} + u|_{K_{out}}\cdot v|_{K_{out}}\right)d\sigma, \tag{4.6a}$$

$$\sum_{e\in\mathcal{E}_h}\int_e \left(u|_{K_{in}}\cdot v|_{K_{in}} - u|_{K_{out}}\cdot v|_{K_{out}}\right)d\sigma = \sum_{e\in\mathcal{E}_h}\int_e \left([u]_A\cdot[v]_J + [u]_J\cdot[v]_A\right)d\sigma. \tag{4.6b}$$

Further, we observe that for vector fields $\underline{q}$ the quantity $[\underline{n}\cdot\underline{q}]_J$ is independent of the choice of K_{in} and K_{out}.

In terms of the averages $[\underline{n}_e\cdot\underline{q}_h]_A$ and the jumps $[\underline{n}_e\cdot\underline{q}_h]_J$ we may decompose the nonconforming Raviart–Thomas space $\hat{V}_h$ into the sum

$$\hat{V}_h = \hat{V}_h^A + \hat{V}_H^J,$$

where the subspaces $\hat{V}_h^A$ and $\hat{V}_H^J$ are given by

$$\hat{V}_h^A = \left\{\underline{q}_h\in\hat{V}_h \mid \quad [\underline{n}_e\cdot\underline{q}_h]_A|_e = 0,\, e\in\mathcal{E}_h^0\right\},$$

$$\hat{V}_h^J = \left\{\underline{q}_h\in\hat{V}_h \mid \quad [\underline{n}_e\cdot\underline{q}_h]_J|_e = 0,\, e\in\mathcal{E}_h^0\right\}.$$

Obviously, we have $\hat{V}_h^J = V_h$ and $\hat{V}_h^A \cap \hat{V}_h^J = \hat{V}_h^\Gamma := \{\underline{q}_h \in \hat{V}_h |\ \underline{n}_e \cdot \underline{q}_h|_e = 0,\ e \in \mathcal{E}_h^0\}$. As the main result of this section we will prove the following.

THEOREM 4.1. *Let* $(\underline{j}_h, u_h, \lambda_h) \in \hat{V}_h \times W_h \times M_h$ *be the unique solution of* (2.6) *and let* $\hat{u}_h \in CR_h$ *be the nonconforming extension of* λ_h. *Then, there exist constants* $0 < \sigma_0 \le \sigma_1$ *depending only on the shape regularity of* $\mathcal{T}_h$ *and the ellipticity constants in* (2.2) *such that*

$$\sigma_0 \left(\sum_{e\in\mathcal{E}_h} |e|^2 \left([u_h]_J|_e\right)^2 \right)^{1/2} \le \|u_h - \hat{u}_h\|_0 \le \sigma_1 \left(\sum_{e\in\mathcal{E}_h} |e|^2 \left([u_h]_J|_e\right)^2 \right)^{1/2}. \tag{4.7}$$

The proof of the preceding result will be provided in several steps. First, due to the shape regularity of $\mathcal{T}_h$ we have the following.

LEMMA 4.2. *Under the assumptions of Theorem* 4.1 *the following hold:*

$$\begin{aligned} &\tfrac{1}{3}\kappa_0 \sum_{e\in\mathcal{E}_h} |e|^2 \left(2\left([u_h]_A|_e - \lambda_h|_e\right)^2 + \tfrac{1}{2}\left([u_h]_J|_e\right)^2 \right) \le \|u_h - \hat{u}_h\|_0^2, \\ &\tfrac{1}{3}\kappa_1 \sum_{e\in\mathcal{E}_h} |e|^2 \left(2\left([u_h]_A|_e - \lambda_h|_e\right)^2 + \tfrac{1}{2}\left([u_h]_J|_e\right)^2 \right) \ge \|u_h - \hat{u}_h\|_0^2. \end{aligned} \tag{4.8}$$

Proof. By straightforward computation

$$\begin{aligned} \|u_h - \hat{u}_h\|_0^2 &= \sum_{K\in\mathcal{T}_h} \|u_h - \hat{u}_h\|_{0,K}^2 = \tfrac{1}{3} \sum_{K\in\mathcal{T}_h} |K| \sum_{i=1}^{3} \left(u_h|_K - \hat{u}_h(m_{e_i})\right)^2 \\ &= \tfrac{1}{3} \sum_{e\in\mathcal{E}_h^0} \left(|K_{in}| \left(u_h|_{K_{in}} - \lambda_h|_e\right)^2 + |K_{out}| \left(u_h|_{K_{out}} - \lambda_h|_e\right)^2 \right) \\ &\quad + \tfrac{1}{3} \sum_{e\in\mathcal{E}_h^\Gamma} |K| \left(u_h|_K\right)^2 \end{aligned}$$

which easily gives (4.8) by taking advantage of (3.1). ∎

As a direct consequence of Lemma 4.2 we obtain the lower bound in (4.7) with $\sigma_0 = \sqrt{\frac{\kappa_0}{6}}$. However, the proof of the upper bound is more elaborate. In view of (4.8) it is sufficient to show that

$$\sum_{e\in\mathcal{E}_h} |e|^2 \left(\left([u_h]_A - \lambda_h\right)|_e\right)^2 \le c \sum_{e\in\mathcal{E}_h} |e|^2 \left([u_h]_J|_e\right)^2 \tag{4.9}$$

holds true with an appropriate positive constant c. As a first step in this direction we will establish the following relationship between λ_h and the averages and jumps of u_h.

LEMMA 4.3. *Under the assumptions of Theorem* 4.1 *for all* $\underline{q}_h \in \hat{V}_h$ *there holds*

$$\sum_{e\in\mathcal{E}_h} |e| \left(\left([u_h]_A|_e - \lambda_h|_e\right) \cdot [\underline{n}_e \cdot \underline{q}_h]_J|_e + \left([u_h]_J|_e\right) \cdot [\underline{n}_e \cdot (\underline{q}_h - P_c \underline{q}_h)]_A|_e \right) = 0, \tag{4.10}$$

where P_c *denotes the projection onto* V_h *with respect to the weighted* L^2 *inner product* $(\cdot,\cdot)_{0,c}$.

Proof. We denote by ϕ_h the unique element in B_h satisfying

$$\int_K \phi_h \, dx = \int_K u_h \, dx, \quad K \in \mathcal{T}_h.$$

In view of (2.5) we thus have

$$\sum_{K\in\mathcal{T}_h}\left(\int\limits_K c\underline{j}_h\cdot\underline{q}_h\,dx-\int\limits_K \mathrm{div}\underline{q}_h\cdot\phi_h\,dx\right)=0,\quad \underline{q}_h\in V_h.$$

By Green's formula, observing $\phi_h|_{\partial K}=0$,

$$\int\limits_K \mathrm{div}\underline{q}_h\cdot\phi_h\,dx=-\int\limits_K \underline{q}_h\cdot\nabla\phi_h\,dx$$

and hence

$$\int\limits_\Omega c\quad \underline{j}_h+a\nabla\phi_h\quad\cdot\underline{q}_h\,dx=0,\quad \underline{q}_h\in V_h,$$

which shows that

$$\underline{j}_h=-P_c(a\nabla\phi_h).$$

Consequently, for $\underline{q}_h\in\hat{V}_h$

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K c\underline{j}_h\cdot\underline{q}_h\,dx&=-\sum_{K\in\mathcal{T}_h}\int_K cP_c(a\nabla\phi_h)\cdot\underline{q}_h\,dx\\&=-\sum_{K\in\mathcal{T}_h}\int_K\nabla\phi_h\cdot P_c(\underline{q}_h)\,dx=\sum_{K\in\mathcal{T}_h}\int_K\phi_h\cdot\mathrm{div}P_c(\underline{q}_h)\,dx\\&=\sum_{K\in\mathcal{T}_h}\int_K u_h\cdot\mathrm{div}P_c(\underline{q}_h)\,dx=\sum_{K\in\mathcal{T}_h}\int_{\partial K}u_h\cdot\underline{n}\cdot P_c(\underline{q}_h)\,d\sigma.\end{aligned}$$

It follows from (2.6) that

$$\sum_{K\in\mathcal{T}_h}\int\limits_{\partial K}\quad u_h\cdot\underline{n}\cdot\quad\underline{q}_h-P_c(\underline{q}_h)\quad-\lambda_h\cdot\underline{n}\cdot\underline{q}_h\quad d\sigma=0,\quad \underline{q}_h\in\hat{V}_h,$$

which by (4.6b) is clearly equivalent to the assertion. ▯

For a particular choice of $\underline{q}_h\in\hat{V}_h$ in Lemma 4.3 we obtain an explicit representation of λ_h on $e\in\mathcal{E}_h^0$. We choose $\underline{q}_h=\underline{\tau}_e$, where

$$\underline{\tau}_e=\frac{1}{2}\left(\underline{\tau}_e^{K_{in}}+\underline{\tau}_e^{K_{out}}\right.\tag{4.11}$$

and $\underline{\tau}_e^{K_{in}}$, $\underline{\tau}_e^{K_{out}}$ are the standard basis vector fields in $\hat{V}_h$ with support in K_{in} and K_{out}, respectively, given by

$$\underline{n}\cdot\underline{\tau}_e^K|_{e'}=\delta_{e,e'},\quad e'\subset\partial K, K\in\{K_{in},K_{out}\}.$$

COROLLARY 4.4. *Let the assumptions of Lemma 4.2 be satisfied and let $\underline{\tau}_e\in\hat{V}_h$, $e\in\mathcal{E}_h^0$, be given by (4.11). Then, there holds*

$$\lambda_h|_e=[u_h]_A|_e-\sum_{e'\in\mathcal{E}_h}\frac{|e'|}{|e|}[u_h]_J|_{e'}[\underline{n}_{e'}\cdot P_c(\underline{\tau}_e)]_A|_{e'}.$$

Proof. Observing $[\underline{n}_{e'} \cdot \underline{\tau}_e]_A|_{e'} = 0$, $e' \in \mathcal{E}_h$, and $[\underline{n}_{e'} \cdot \underline{\tau}_e]_J|_{e'} = \delta_{e,e'}$, $e' \in \mathcal{E}_h$, the assertion is a direct consequence of (4.10). ▯

Moreover, with regard to (4.9) we get the following.

Corollary 4.5. *Under the assumptions of Lemma* 4.2 *the following hold:*

$$
\begin{aligned}
&\Big(\sum_{e \in \mathcal{E}_h} |e|^2 \left(([u_h]_A - \lambda_h)|_e \right)^2 \Big)^{1/2} \\
&\quad \le \Big(\sum_{e \in \mathcal{E}_h} |e|^2 \left([u_h]_J|_e \right)^2 \Big)^{1/2} \cdot \sup_{\underline{q}_h \in \hat{V}_h^A} \frac{\Big(\sum_{e \in \mathcal{E}_h} \left([\underline{n}_e \cdot (\underline{q}_h - P_c(\underline{q}_h))]_A\big|_e \right)^2 \Big)^{1/2}}{\Big(\sum_{e \in \mathcal{E}_h} \left([\underline{n}_e \cdot \underline{q}_h]_J\big|_e \right)^2 \Big)^{1/2}}.
\end{aligned}
\tag{4.12}
$$

Proof. Since for each $\mu_h \in M_h(\mathcal{E}_h)$ there exists a unique $\underline{q}_h \in \hat{V}_h^A$ satisfying $[\underline{n}_e \cdot \underline{q}_h]_J|_e = |e|\mu_h$, $e \in \mathcal{E}_h$, by means of (4.10) we get

$$
\begin{aligned}
\Big(\sum_{e \in \mathcal{E}_h} |e|^2 \left(([u_h]_A - \lambda_h)|_e \right)^2 \Big)^{1/2} &= \sup_{\mu_h \in M_h(\mathcal{E}_h)} \frac{\sum_{e \in \mathcal{E}_h} |e|^2 \, ([u_h]_A - \lambda_h)|_e \cdot \mu_h|_e}{\Big(\sum_{e \in \mathcal{E}_h} |e|^2 (\mu_h|_e)^2 \Big)^{1/2}} \\
&= \sup_{\underline{q}_h \in \hat{V}_h^A} \frac{\sum_{e \in \mathcal{E}_h} |e| \, ([u_h]_A - \lambda_h)|_e \cdot [\underline{n}_e \cdot \underline{q}_h]_J|_e}{\Big(\sum_{e \in \mathcal{E}_h} \left([\underline{n}_e \cdot \underline{q}_h]_J|_e \right)^2 \Big)^{1/2}} \\
&= \sup_{\underline{q}_h \in \hat{V}_h^A} \frac{\sum_{e \in \mathcal{E}_h} |e| \, [u_h]_J|_e \cdot [\underline{n}_e \cdot \left(P_c(\underline{q}_h) - \underline{q}_h \right)]_A|_e}{\Big(\sum_{e \in \mathcal{E}_h} \left([\underline{n}_e \cdot \underline{q}_h]_J|_e \right)^2 \Big)^{1/2}}
\end{aligned}
$$

which gives (4.12) by the Schwarz inequality. ▯

The preceding result tells us that for the proof of (4.9) we have to verify

$$
\sum_{e \in \mathcal{E}_h} \Big| [\underline{n}_e \cdot \big(P_c(\underline{q}_h) - \underline{q}_h \big)]_A|_e \Big|^2 \le c \sum_{e \in \mathcal{E}_h} \Big| [\underline{n}_e \cdot \underline{q}_h]_J|_e \Big|^2, \quad \underline{q}_h \in \hat{V}_h^A. \tag{4.13}
$$

Since (4.13) obviously holds true for $\underline{q}_h \in \hat{V}_h^\Gamma$, it is sufficient to show the following.

Lemma 4.6. *Let the assumptions of Lemma* 4.2 *be satisfied. Then, there holds*

$$
\sum_{e \in \mathcal{E}_h} \Big| [\underline{n}_e \cdot P_c(\underline{q}_h)]_A|_e \Big|^2 \le \frac{1}{2} \left(\frac{\alpha_1 \cdot c_1}{\alpha_0 \cdot c_0} \right)^2 \sum_{e \in \mathcal{E}_h^0} \Big| [\underline{n}_e \cdot \underline{q}_h]_J|_e \Big|^2, \quad \underline{q}_h \in \hat{V}_h^A \backslash \hat{V}_h^\Gamma. \tag{4.14}
$$

Proof. We refer to $\mathbf{A}$, $\hat{\mathbf{A}}$, and $\mathbf{P_c}$ as the matrix representations of the operators A, $\hat{A}$, and P_c. With respect to the standard basis of V_h and $\hat{V}_h$ we may identify $\underline{q}_h \in V_h$ and $\hat{\underline{q}}_h \in \hat{V}_h$ with vectors $\mathbf{q_h} = ((q_e)_{e \in \mathcal{E}_h})$ and $\hat{\mathbf{q}}_\mathbf{h} = ((q_e^{K_{in}}, q_e^{K_{out}})_{e \in \mathcal{E}_h^0}, (q_e^K)_{e \in \mathcal{E}_h^\Gamma})$, respectively. We remark that $\underline{q}_h \in \hat{V}_h^A \setminus \hat{V}_h^\Gamma$ if and only if $q_e^{K_{in}} = q_e^{K_{out}}$, $e \in \mathcal{E}_h^0$, and $q_e^K = 0$, $e \in \mathcal{E}_h^\Gamma$. It follows that for $\underline{q}_h \in \hat{V}_h^A \setminus \hat{V}_h^\Gamma$

$$
\sum_{e \in \mathcal{E}_h} \Big| [\underline{n}_e \cdot P_c(\underline{q}_h)]_A|_e \Big|^2 = \sum_{e \in \mathcal{E}_h} (P_c q_h)_e^2 = (\mathbf{P_c q_h})^\mathbf{T} \cdot (\mathbf{P_c q_h}), \tag{4.15}
$$

$$\sum_{e\in\mathcal{E}_h^0} \Big([\underline{n}_e\cdot \underline{q}_h]_J|_e\Big)^2 = 2\sum_{e\in\mathcal{E}_h^0}\Big((q_e^{K_{in}})^2+(q_e^{K_{out}})^2\Big) = 2\mathbf{q_h^T}\cdot\mathbf{q_h}. \tag{4.16}$$

Obviously

$$(\mathbf{P_c q_h})^{\mathbf{T}}\cdot(\mathbf{P_c q_h}) \le \rho(\mathbf{P_c}\cdot\mathbf{P_c^T})\mathbf{q_h^T}\cdot\mathbf{q_h}, \tag{4.17}$$

where $\rho(\mathbf{P_c}\cdot\mathbf{P_c^T})$ stands for the spectral radius of $\mathbf{P_c}\cdot\mathbf{P_c^T}$. Denoting by S the natural embedding of V_h into $\hat{V}_h$ and by $\mathbf{S}$ its matrix representation, it is easy to see that

$$\mathbf{P_c} = \mathbf{A^{-1}S^T\hat{A}},$$

whence

$$\begin{aligned}\rho(\mathbf{P_c}\cdot\mathbf{P_c^T}) &\le \sup_{\mathbf{q_h}\neq\mathbf{0}} \frac{\Big(\mathbf{\hat{A}SA^{-1}q_h}\Big)^{\mathbf{T}}\Big(\mathbf{\hat{A}SA^{-1}q_h}\Big)}{\mathbf{q_h^T q_h}}\\ &= \sup_{\mathbf{q_h}\neq\mathbf{0}} \frac{(\mathbf{Sq_h})^{\mathbf{T}}\mathbf{\hat{A}^2}(\mathbf{Sq_h})}{\mathbf{q_h^T A^2 q_h}}.\end{aligned} \tag{4.18}$$

We further refer to $\mathbf{A_K}$ and $\mathbf{\hat{A}_K}$ as the local stiffness matrices. Using (2.2) and (3.10), we get

$$c_0\alpha_1^{-1}|K|\mathrm{Id} \le \mathbf{D_K} \le c_1\alpha_0^{-1}|K|\mathrm{Id}, \quad \mathbf{D_K}\in\{\mathbf{A_K},\mathbf{\hat{A}_K}\},$$

with $c_0 = \frac{1}{48}\cdot\kappa_1^{-2}$ and $c_1 = \frac{1}{4}\cdot\kappa_0^{-2}$. Consequently, introducing the local vectors $(\mathbf{q_h})_K = (q_{e_1}, q_{e_2}, q_{e_3})$, $e_i\subset\partial K$, $1\le i\le 3$, it follows that

$$\begin{aligned}\mathbf{q_h^T A^2 q_h} &= \sum_{K\in\mathcal{T}_h}(\mathbf{q_h})_{\mathbf{K}}^{\mathbf{T}}\mathbf{A^2}(\mathbf{q_h})_{\mathbf{K}} \ge c_0^2\alpha_1^{-2}\sum_{K\in\mathcal{T}_h}|K|^2(\mathbf{q_h})_{\mathbf{K}}^{\mathbf{T}}(\mathbf{q_h})_{\mathbf{K}}\\ &= c_0^2\alpha_1^{-2}\Bigg(\sum_{e\in\mathcal{E}_h^0}\Big(|K_{in}|^2+|K_{out}|^2\Big)q_e^2 + \sum_{e\in\mathcal{E}_h^\Gamma}|K|^2q_e^2\Bigg),\end{aligned} \tag{4.19}$$

$$\begin{aligned}(\mathbf{Sq_h})^{\mathbf{T}}\mathbf{\hat{A}^2}(\mathbf{Sq_h}) &= \sum_{K\in\mathcal{T}_h}(\mathbf{Sq_h})_{\mathbf{K}}^{\mathbf{T}}\mathbf{\hat{A}^2}(\mathbf{Sq_h})_{\mathbf{K}}\\ &\le c_1^2\alpha_0^{-2}\sum_{K\in\mathcal{T}_h}|K|^2(\mathbf{Sq_h})_{\mathbf{K}}^{\mathbf{T}}(\mathbf{Sq_h})_{\mathbf{K}}\\ &= c_1^2\alpha_0^{-2}\Bigg(\sum_{e\in\mathcal{E}_h^0}\Big(|K_{in}|^2+|K_{out}|^2\Big)q_e^2 + \sum_{e\in\mathcal{E}_h^\Gamma}|K|^2q_e^2\Bigg).\end{aligned} \tag{4.20}$$

Using (4.19), (4.20) in (4.18) we find

$$\rho(\mathbf{P_c}\cdot\mathbf{P_c^T}) \le (\alpha_0^{-1}\alpha_1 c_0^{-1}c_1)^2$$

which gives (4.14) in view of (4.15), (4.16), and (4.17). □

Summarizing the preceding results it follows that the upper estimate in (4.7) holds true with $\sigma_1 = \sqrt{\frac{\kappa_1}{3}((\frac{c_1\alpha_1}{c_0\alpha_0})^2+\frac{1}{2})}$. Altogether the essential result of this section is proved. The following theorem is a simple consequence of Theorem 4.1.

THEOREM 4.7. *Let $\tilde{u}_h \in W_h$ be an approximation of the primal variable u obtained by an iterative solution process for* (2.5). *Then, there exist constants $0 < \tilde{\sigma}_0 \le \tilde{\sigma}_1$ and $0 < C_0 < C_1$ depending only on the shape regularity of T_0 and the ellipticity constants in* (2.2) *such that*

$$\|u - \tilde{u}_h\|_0 \le \tilde{\sigma}_1 \left(\sum_{e \in \mathcal{E}_h} |e|^2 \left([\tilde{u}_h]_J|_e\right)^2 \right)^{1/2} + C_1 \|u_h - \tilde{u}_h\|_0,$$

$$\|u - \tilde{u}_h\|_0 \ge \tilde{\sigma}_0 \left(\sum_{e \in \mathcal{E}_h} |e|^2 \left([\tilde{u}_h]_J|_e\right)^2 \right)^{1/2} - C_0 \|u_h - \tilde{u}_h\|_0,$$

with $\tilde{\sigma}_1 := \frac{\sigma_1}{1-\beta}$, $\tilde{\sigma}_0 := \frac{\sigma_0}{1+\beta}$, $C_1 := \frac{\sigma_1}{1-\beta}\sqrt{\frac{6}{\kappa_0}} + 1$ *and* $C_0 := \frac{\sigma_0}{1+\beta}\sqrt{\frac{6}{\kappa_0}} + 1$.

Proof. Due to (4.3), Theorem 4.1, and the triangle inequality we obtain

$$\|u - \tilde{u}_h\|_0 \le \tfrac{\sigma_1}{1-\beta} \left(\sum_{e \in \mathcal{E}_h} |e|^2 \left([u_h]_J|_e\right)^2 \right)^{1/2} + \|u_h - \tilde{u}_h\|_0,$$

$$\|u - \tilde{u}_h\|_0 \ge \tfrac{\sigma_0}{1+\beta} \left(\sum_{e \in \mathcal{E}_h} |e|^2 \left([u_h]_J|_e\right)^2 \right)^{1/2} - \|u_h - \tilde{u}_h\|_0.$$

It is easy to see that

$$|e|^2([w]_J|_e)^2 \le \frac{2}{\kappa_0} \|w\|^2_{0;T_1 \cup T_2}, \quad T_1 \cap T_2 = e, \; w \in W_h,$$

and hence the assertion is proved. ▯

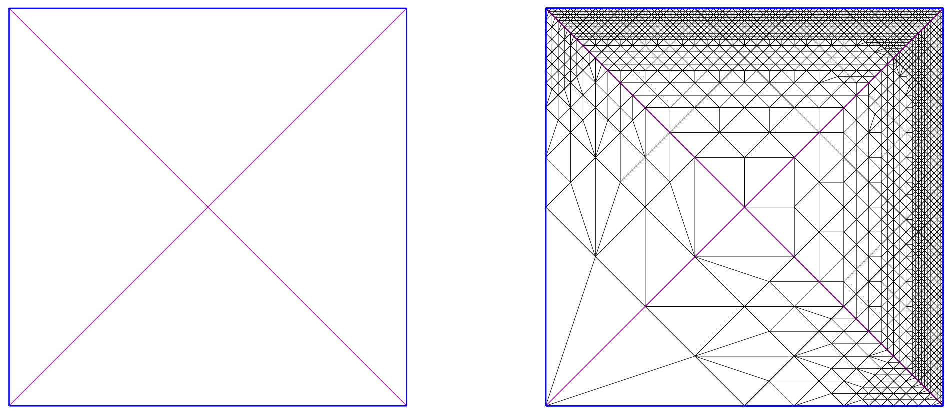
5. Numerical results. In this section, we will present the numerical results obtained by the application of the adaptive multilevel algorithm to some selected second-order elliptic boundary problems. In particular, we will illustrate the refinement process as well as the performance of both the multilevel preconditioner and the a posteriori error estimator. The following model problems from [5] and [21] have been chosen as test examples.

Problem 1. We take (2.1) with $a = 1$ and $b = 100$ on the unit square $\Omega = (0,1)^2$ with the right-hand side f and the Dirichlet boundary conditions according to the solution $u(x,y) = (2\cosh 10)^{-1}(\cosh(10x) + \cosh(10y))$ which has a boundary layer along the lines $x = 1$ and $y = 1$ (cf. Fig. 5.1).

Problem 2. We take (2.1) with the right-hand side $f \equiv 0$ and a hexagon Ω with corners $(\pm 1, 0)$, $(\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\pm\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The coefficients are chosen according to $b \equiv 0$ and $a(x,y)$ being piecewise constant with the values 1 and 100 on alternate triangles of the initial triangulation (cf. Fig. 5.2). The solution given by $u(x,y) = a^{-1}y(3x^2 - y)$ is continuous with a jump discontinuity of the first derivatives at the interfaces.

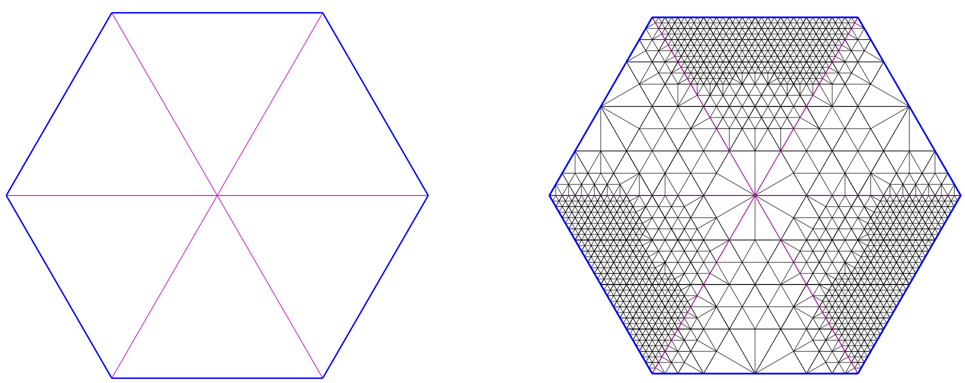
Starting from the initial coarse triangulations depicted in Figs. 5.1 and 5.2, on each refinement level l the discretized problems are solved by preconditioned CG iterations with a BPX-type preconditioner as described in section 3.

The iteration on level $l+1$ is stopped when the estimated iteration error ε_{l+1} is less then $\varepsilon_{l+1}^2 \le \mu\, \epsilon_l^2 \frac{N_l}{N_{l+1}}$, with the safety factor $\mu = 1.E-2$, ϵ_l denotes the estimated error on level l, and the number of nodes on level l and $l+1$ are given by N_l and N_{l+1}, respectively. Denoting by $(\tilde{j}_l, \tilde{u}_l, \tilde{\lambda}_l)$ the resulting approximation and by $\hat{\tilde{u}}_l$ the nonconforming extension of $\tilde{\lambda}_l$, for the local refinement of $\mathcal{T}_l$ the



Level 0, N = 8 Level 6, N = 5197

FIG. 5.1. *Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_6$ (Problem 1).*



Level 0, N = 12 Level 5, N = 3641

FIG. 5.2. *Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_5$ (Problem 2).*

elementwise error contributions $\epsilon_K^2 = \|\tilde{u}_l - \hat{\tilde{u}}_l\|_{0,K}^2$, $K \in \mathcal{T}_l$, and the weighted mean value $\bar{\epsilon}^2 = |\Omega|^{-1} \sum_{K \in \mathcal{T}_l} {}^2_K$ are computed. Then, an element $K \in \mathcal{T}_l$ is marked for refinement if $|K|^{-1}\epsilon_K^2 \geq \sigma\bar{\epsilon}^2$ where σ is a safety factor which is chosen as $\sigma = 0.95$. The interpolated values of the level l approximation are used as start iterates on the next refinement level. For the global refinement process we use $\bar{\epsilon}^2|\Omega| \leq \alpha\,\mathrm{tol}\,\|\tilde{\mathrm{u}}_\mathrm{l}\|_{0,\Omega}^2$ as stopping criteria, where α is a safety factor which is chosen as $\alpha = 0.95$ and tol is the required accuracy, $\mathrm{tol} = 2.\mathrm{E} - 3$.

Figures 5.1 and 5.2 represent the initial triangulations $\mathcal{T}_0$ and the final triangulations $\mathcal{T}_6$ and $\mathcal{T}_5$ for Problems 1 and 2, respectively. For Problem 1 we observe a pronounced refinement in the boundary layer (cf. Fig. 5.1). For Problem 2 there is a significant refinement in the areas where the diffusion coefficient is small with a sharp resolution of the interfaces between the areas of large and small diffusion coefficient (cf. Fig. 5.2).

The behavior of the a posteriori L^2 error estimator is illustrated in Fig. 5.3, where the ratio of the estimated error and the true error is shown as a function of the total

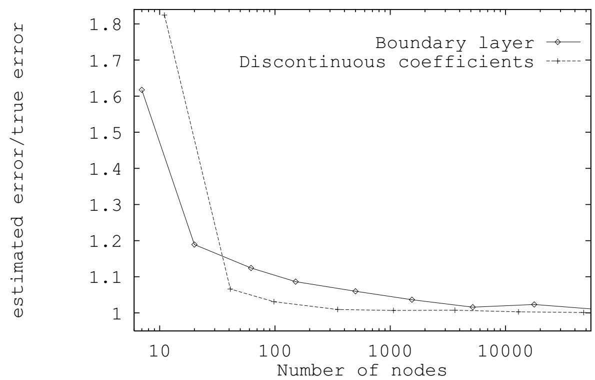


Fig. 5.3. *Error estimation for Problems* 1 *and* 2.

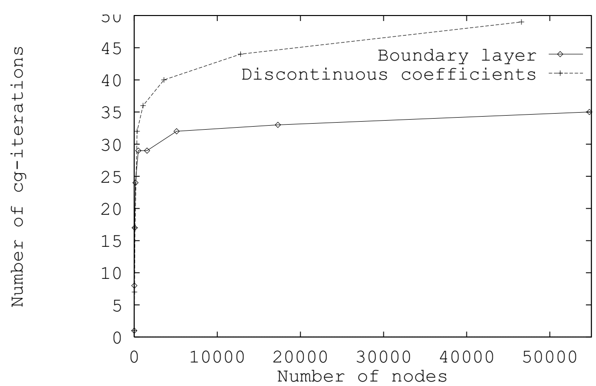


Fig. 5.4. *Preconditioner for Problems* 1 *and* 2.

number of nodes. The straight and the dashed lines refer to Problem 1 (boundary layer) and Problem 2 (discontinuous coefficients), respectively. In both cases we observe a slight overestimation at the very beginning of the refinement process, but the estimated error rapidly approaches the true error with increasing refinement level.

Finally, the performance of the preconditioner is depicted in Fig. 5.4 displaying the number of preconditioned CG iterations as a function of the total number of nodal points. Note that for an adequate representation of the performance we use zero as initial iterates on each refinement level and iterate until the relative iteration error is less than $\varepsilon = 1.E - 6$. In both cases, we observe an increase in the number of iterations at the beginning of the refinement process until we get into the asymptotic regime where the numerical results confirm the theoretically predicted $O(1)$ behavior.

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