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Report 2004-04 Januar 2004

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Abstract Modal Kleene algebras are Kleene algebras with forward and backward modal operators defined via domain and codomain operations. The paper investigates the algebraic structure of modal operators. It studies and compares different notions of termination in this class, including an algebraic correspondence proof of Löb's formula from modal logic. It gives calculational proofs of two fundamental statements from rewriting theory that involve termination: Bachmair's and Dershowitz's well-founded union theorem and Newman's lemma. These results are also of general interest for the termination analysis of programs and state transition systems.

1 Introduction

Kleene algebras are fundamental structures in computer science with applications ranging from program development and analysis to rewriting theory and concurrency control. Initially conceived as algebras of regular events [13], they have recently been extended to variants with infinite iteration [5] and abstract domain and codomain operations [7]. The second extension leads to modal algebras. Forward and backward boxes and diamonds are definable "semantically" from domain and codomain operations. Their symmetries can be expressed by dualities and Galois connections.

We propose modal Kleene algebras as a useful tool for termination analysis, both for the investigation and comparison of different notions of termination and for specifications and proofs that involve this concept. Modal Kleene algebra allows a simple and calculational style of reasoning that is also well-suited for mechanization. Induction with respect to "external" measures is avoided in favour of "internal" fixed-point reasoning and contraction law. Proofs can often be written in a point-free style in the algebra of modal operators. This introduces a new level of abstraction and conciseness.

Our main results are as follows. First, we provide a point-wise lifting from modal Kleene algebras to the algebras of modal operators. These are both lattice-ordered monoids with Boolean retracts and again Kleene algebras. Consequently, the Boolean and Kleenean laws are available for point-free modal reasoning.

Second, we investigate a notion of termination in modal Kleene algebra that arises by abstraction from set-theoretic relations (c.f. [8]). It roughly says that a relation does *not* terminate, if some set is contained in its image under the relation, thus providing a basis for infinite iteration. We compare this notion with two alternatives. The first one models termination as absence of proper infinite iteration. We show that this notion is not equivalent to the previous one, even under natural additional assumptions. It turns out that the notion of termination induced by modal Kleene algebra is the more natural and more useful one. The second alternative arises in modal logic as Löb's formula [4]. Building on previous results for general modal algebras [10], we show that the notions from modal logics and modal Kleene algebra are essentially equivalent. Viewing Kleene algebra as an algebraic semantics for modal logics, we thus obtain a simple calculational correspondence proof for a second-order frame property. Note however, that the star operation of Kleene algebra is usually not available in classical modal logic.

Third, we continue our research on abstract rewriting in Kleene algebra [18,19]. We prove Bachmair's and Dershowitz's well-founded union theorem [2] and a variant of Newman's lemma (cf. [1]) in modal Kleene algebra. These proofs are simpler than previous results in related structures [16,8]. Moreover, modal Kleene algebra provides an algebraic semantics to the usual rewrite diagrams; the algebraic proofs immediately reflect their diagrammatic counterparts. Together with our earlier results this shows that a large part of abstract rewriting is indeed conveniently modeled by modal Kleene algebra.

2 Kleene Algebra

A *semiring* is a structure $(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition from the left and right and zero is a left and right annihilator, that is, $a0 = 0 = 0a$ for all $a \in K$ (the operation symbol $\cdot$ is omitted here and in the sequel). The semiring is *idempotent* if it satisfies $a + a = a$ for all $a \in K$. Then K has a *natural ordering* $\leq$ defined for all $a, b \in K$ by $a \leq b$ iff $a + b = b$. It induces a semilattice with $+$ as join and 0 as the least element; addition and multiplication are isotone w.r.t. the natural ordering.

A *Kleene algebra* [13] is a structure $(K, ^*)$ such that K is an idempotent semiring and the *star* * satisfies, for $a, b, c \in K$, the *unfold* and *induction laws*

$$1 + aa^* \leq a^*, \qquad (*\text{-}1) \qquad\qquad b + ac \leq c \Rightarrow a^*b \leq c, \qquad (*\text{-}3)$$
$$1 + a^*a \leq a^*, \qquad (*\text{-}2) \qquad\qquad b + ca \leq c \Rightarrow ba^* \leq c. \qquad (*\text{-}4)$$

Therefore, a^* is the least pre-fixpoint and the least fixpoint of the mappings $\lambda x.ax + b$ and $\lambda x.xa + b$. The star is isotone with respect to the natural ordering.

Models of KA are for instance the set-theoretic relations under set union, relational composition and reflexive transitive closure, the sets of regular languages (regular events) over some finite alphabet, the algebra of path sets in a directed graph under path concatenation and the algebra of imperative programs with non-deterministic choice, composition and iteration.

A *Boolean algebra* is a complemented distributive lattice. By overloading, we usually write $+$ and $\cdot$ also for the Boolean join and meet operation and use 0 and 1 for the least and greatest elements of the lattice. The symbol $\neg$ denotes the operation of complementation. We will consistently use the letters $a, b, c \ldots$ for semiring elements and $p, q, r, \ldots$ for Boolean elements.

A *test semiring* is a two-sorted structure (K, B), where K is an idempotent semiring and $B \subseteq K$ is a Boolean algebra embedded into K such that the B operations coincide with the restrictions of the K operations to B. In particular, $p \leq 1$ for all $p \in B$. But in general, B is only a subalgebra of the subalgebra of all elements below 1 in K. We call elements of B *tests* and write $\mathsf{test}(K)$ instead of B for the algebra of tests. We will also use relative complement $p - q = p \sqcap \neg q$ and implication $p \to q = \neg p + q$ with their standard laws.

A *Kleene algebra with tests* [14] is a test semiring (K, B) such that K is a KA. For all $p \in \mathsf{test}(K)$ we have that $p^* = 1$.

3 Modal Kleene Algebra

Let a semiring element a describe an action or abstract program and a test p a proposition or assertion. Then pa describes a restricted program that acts like a when the initial state satisfies p and aborts otherwise. Symmetrically, ap describes a restriction of a in its possible final states. We now introduce an abstract domain operator $\ulcorner$ that assigns to a the test that describes precisely its enabling states.

An *semiring with domain* [7] (a $\ulcorner$-semiring) is a structure $(K, \ulcorner)$, where K is an idempotent semiring and the *domain operation* $\ulcorner\colon K \to \mathsf{test}(K)$ satisfies for all $a, b \in K$ and $p \in \mathsf{test}(K)$

$$a \le (\ulcorner a)a, \ \text{(d1)} \quad \ulcorner(pa) \le p, \ \text{(d2)} \quad \ulcorner(a \ulcorner b) \le \ulcorner(ab). \ \text{(d3)}$$

If K is a KA, we speak of a *KA with domain*, briefly $\ulcorner$*-KA*. To explain (d1) and (d2) we note that their conjunction is equivalent to each of

$$\ulcorner a \le p \Leftrightarrow a \le pa, \qquad \text{(llp)} \qquad\qquad \ulcorner a \le p \Leftrightarrow \neg pa \le 0, \qquad \text{(gla)}$$

which constitute elimination laws for $\ulcorner$. (llp) says that $\ulcorner a$ is the least left preserver of a. (gla) says that $\neg\ulcorner a$ is the greatest left annihilator of a. Both properties obviously characterize domain in set-theoretic relations. (d3) states that the domain of ab is not determined by the inner structure of b or its codomain; information about $\ulcorner b$ in interaction with a suffices.

Many natural properties follow from the axioms. Domain is uniquely defined. It is strict ($\ulcorner a = 0 \Leftrightarrow a = 0$), additive ($\ulcorner(a+b) = \ulcorner a + \ulcorner b$), isotone ($a \le b \Rightarrow \ulcorner a \le \ulcorner b$), local ($\ulcorner(ab) = \ulcorner(a\ulcorner b)$) and stable on tests ($\ulcorner p = p$). It satisfies an import/export law ($\ulcorner(pa) = p\ulcorner a$), and an induction law ($\ulcorner(ap) \le p \Rightarrow \ulcorner(a^*p) \le p$). Finally, domain commutes with all existing suprema. See [7] for further information.

A codomain operation $\urcorner$ can easily be defined as a domain operation in the opposite semiring. As usual in algebra, opposition just swaps the order of multiplication. We call a semiring K with domain and codomain also a *modal semiring*; if K in addition is a KA, we call it a *modal KA*.

Let K be a modal semiring. We introduce forward and backward diamond operators via abstract preimage and image.

$$|a\rangle p = \ulcorner(ap), \qquad (1) \qquad\qquad \langle a| p = (pa)\urcorner, \qquad (2)$$

for all $a \in K$ and $p \in \mathsf{test}(K)$. It follows that diamond operators are strict additive mappings (or *hemimorphisms*) on the algebra of tests. Hence $(\mathsf{test}(K), |a\rangle)$ and $(\mathsf{test}(K), \langle a|)$ are Boolean algebras with operators à la Jónsson and Tarski [12]. These structures are called *modal algebras* in [10].

Duality with respect to opposition transforms forward diamonds into backward diamonds and vice versa. If follows that they satisfy an *exchange law*. For all $a \in K$ and $p, q \in \mathsf{test}(K)$,

$$|a\rangle p \le \neg q \Leftrightarrow \langle a| q \le \neg p. \qquad (3)$$

Duality with respect to complementation transforms diamonds into boxes and vice versa, for instance $|a]p = \neg|a\rangle\neg p$ and $|a\rangle p = \neg|a]\neg p$. It follows that diamonds and boxes are lower and upper adjoints of Galois connections:

$$|a\rangle p \le q \Leftrightarrow p \le [a|q, \qquad (4) \qquad\qquad \langle a|p \le q \Leftrightarrow p \le |a]q, \qquad (5)$$

for all $a \in K$ and $p, q \in \mathsf{test}(K)$. The Galois connections are useful as theorem generators and the dualities as theorem transformers.

In the sequel, when the direction of diamonds and boxes does not matter, we will use the notation $\langle a\rangle$ and $[a]$.

If $\mathsf{test}(K)$ is complete then $\ulcorner$ always exists; moreover, since it commutes with all suprema, it has a unique upper adjoint which is $\urcorner$. So in this case, the modal algebra is completely characterized by the domain axioms and the Galois connection. If $\mathsf{test}(K)$ is not complete, this need not be the case.

The Galois connections have some interesting consequences. In particular diamonds (boxes) commute with all existing suprema (infima) of the test algebra.

For a test p we have

$$\langle p \rangle q = pq, \qquad [p]q = p \to q.$$

Hence, $\langle 1 \rangle = [1]$ is the identity function on tests. Moreover, $\langle 0 \rangle p = 0$ and $[0]p = 1$.

Finally, we note that in a KA with converse ${}^{\smile}$ we have

$$|a^{\smile}\rangle = \langle a|, \qquad |a^{\smile}] = [a|.$$

4 Algebras of Modal Operators

Modal semirings have a much richer structure than plain modal algebras. We will now show that the algebra of modal operators that arises from a pointwise lifting is a lattice-ordered monoid that contains an idempotent semiring or a variant of a KA as a retract. This abstraction allows a more succinct pointfree style of reasoning.

A *lattice-ordered monoid* is a structure $(K, +, \sqcap, \cdot, 1)$ such that $(K, +, \sqcap)$ is a lattice, $(K, \cdot, 1)$ is a monoid and left and right multiplication are additive. These structures have extensively been studied in [3]. If the lattice reduct of the monoid is distributive (Boolean), we call the respective structure a *d-monoid* (a *b-monoid*).

We will use the pointwise ordering between functions $f, g : \mathsf{test}(K) \to \mathsf{test}(K)$ given by

$$f \leq g \Leftrightarrow \forall p \,.\, f p \leq g\, p. \tag{6}$$

Let $\langle K \rangle$ be the sets of all mappings $\lambda x. \langle a \rangle x$ with $a \in K$ on some domain or codomain semiring K. We define addition (or join), meet and multiplication on $\langle K \rangle$ pointwise by

$$\begin{aligned}
(\langle a \rangle + \langle b \rangle) p &= \langle a \rangle p + \langle b \rangle p, \\
(\langle a \rangle \sqcap \langle b \rangle) p &= (\langle a \rangle p)(\langle b \rangle p), \\
(\langle a \rangle \cdot \langle b \rangle) p &= \langle a \rangle \langle b \rangle p.
\end{aligned}$$

Then the structures $(\langle K \rangle, +, \sqcap, \cdot, \langle 0 \rangle, \langle 1 \rangle)$ are d-monoids. Dually, with addition, meet and multiplication defined by

$$\begin{aligned}
([a] + [b])(p) &= ([a]p)([b]p), \\
([a] \sqcap [b])(p) &= [a]p + [b]p, \\
([a] \cdot [b])(p) &= [a][b]p,
\end{aligned}$$

the structures $([K], +, \sqcap, \cdot, [0], [1])$ are d-monoids In both cases the pointwise ordering coincides with the natural semiring ordering which also is the lattice ordering. Using both mappings $\lambda x.[a]x$ and $\lambda x.\langle a \rangle x$, we can extend the d-monoids to b-monoids, defining

$$(\neg \langle a \rangle)(p) = [a] \neg p, \qquad (\neg [a])(p) = \langle a \rangle \neg p.$$

We will also use the pointwise liftings of $-$ and $\to$ to the operator level.

In the case of a $\ulcorner$-KA, the algebras of operators can be extended to KAs because of the following unfold and induction laws at the operator level (cf. [7]).

$$|1\rangle + |a\rangle|a^*\rangle \leq |a^*\rangle, \qquad |1\rangle + |a^*\rangle|a\rangle \leq |a^*\rangle, \tag{7}$$

$$|b\rangle + |a\rangle|c\rangle \leq |c\rangle \Rightarrow |a^*\rangle|b\rangle \leq |c\rangle. \tag{8}$$

These laws for the "inner star" induce an "outer star" $|a\rangle^*$ that coincides with $|a^*\rangle$ and turns the algebra of boxes into a left KA. Analogous laws hold for the backward modal operators. They imply the star fixpoint laws

$$|a^*\rangle = |1\rangle + |a\rangle|a^*\rangle \;, \qquad |a^*] = |1] \sqcap |a]|a^*] \;. \tag{9}$$

Many properties of modal operators can now be presented much more succinctly in the respective algebra of operators. First, the test-level Galois connections can be lifted to operators $f, g : \mathsf{test}(K) \to \mathsf{test}(K)$:

$$|a\rangle f \le g \Leftrightarrow f \le [a|g, \qquad \langle a|f \le g \Leftrightarrow f \le |a]g, \tag{10}$$

for all $a \in K$. From this we immediately get the cancellation laws

$$|a\rangle[a| \le \langle 1\rangle \le [a||a\rangle, \qquad \langle a||a] \le \langle 1\rangle \le |a]\langle a|, \tag{11}$$
$$f|a] \le |b\rangle \Leftrightarrow f \le |b\rangle\langle a|. \tag{12}$$

The latter is proved as follows:

$$f|a] \le |b\rangle \Rightarrow f|a]\langle a| \le |b\rangle\langle a| \Rightarrow f \le |b\rangle\langle a| \Rightarrow f|a] \le |b\rangle\langle a||a] \Rightarrow f|a] \le |b\rangle.$$

The first step uses isotonicity, the second one cancellation (11) and neutrality of $\langle 1\rangle$, the third one isotonicity and the fourth one cancellation (11) again.

Semiring expressions inside of operators can be decomposed by the laws

$$\langle a+b\rangle = \langle a\rangle + \langle b\rangle, \qquad |ab\rangle = |a\rangle|b\rangle, \qquad \langle ab| = \langle b|\langle a|,$$
$$[a+b] = [a] \sqcap [b], \qquad |ab] = |a]|b], \qquad [ab| = [b|[a|.$$

Note that the decomposition with respect to multiplication is covariant for forward modalities and contravariant for backward modalities. This results from the symmetry between domain and codomain via opposition. The decomposition can be used to transform expressions into normal form and to reason entirely at the level of modal algebra in the sense of [10].

Diamonds are isotone, that is, $a \le b$ implies $\langle a\rangle \le \langle b\rangle$. Dually, boxes are antitone, that is, $a \le b$ implies $[b] \le [a]$.

5 Termination in Modal Kleene Algebra

We now abstract a notion of termination for modal semirings from set-theoretic relations. A similar characterization has been used, for instance, in [10] for related structures. A set-theoretic relation $R \subseteq A \times A$ on a set A is well-founded if there are no infinitely descending R-chains, that is, no infinite chains $x_0, x_1, \ldots$ such that $(x_{i+1}, x_i) \in R$. It is Noetherian, if there are no infinitely ascending R-chains, that is, no infinite chains $x_0, x_1, \ldots$ such that $(x_i, x_{i+1}) \in R$. Thus R is *not* well-founded if there is a non-empty set $P \subseteq A$ (denoting the infinite chain) such that for all $x \in P$ there exists some $y \in P$ with $(y, x) \in R$. Equivalently, therefore, P is contained in the image of P under R, that is, $P \subseteq (PR)^{\urcorner}$. Consequently, if R *is* well-founded, then only the empty set may satisfy this condition.

Abstracting to a modal semiring K we say that a is *well-founded* if

$$p \le \langle a|p \Rightarrow p \le 0. \tag{13}$$

for all $p \in \mathsf{test}(K)$. Dually, a is *Noetherian*, if for all $p \in \mathsf{test}(K)$,

$$p \le |a\rangle p \Rightarrow p \le 0 \tag{14}$$

Note that by duality w.r.t. complementation a is Noetherian iff, for all $p \in \mathsf{test}(K)$,

$$|a]p \le p \Rightarrow 1 \le p\ . \tag{15}$$

The set of Noetherian elements in K is denoted by $\mathcal{N}(K)$. A ${}^{\ulcorner}$-semiring K is *Noetherian* if $K = \mathcal{N}(K)$.

Let us look at these definitions from another angle. According to the standard definition, a relation R on a set A is well-founded iff every non-empty subset of A has an R-minimal element. In a $\ulcorner$- semiring K the minimal part of $p \in \mathsf{test}(K)$ w.r.t. some $a \in K$ can algebraically be characterized as $p - \langle a|p$, i.e., as the set of points that have no a-predecessor in p. So, by contraposition, the well-foundedness condition holds iff for all $p \in \mathsf{test}(K)$

$$p - \langle a|p \;\le\; 0 \;\Rightarrow\; p \le 0,$$

which by simple Boolean algebra can be transformed into (13).

We now state abstract algebraic variants of some simple and well-known properties of well-founded and Noetherian relations. We restrict our attention to Noethericity, whence to $\ulcorner$-semirings. Analogous well-foundedness properties hold automatically in the dual algebra. For algebraic proofs of these properties see [7].

Lemma 1. *Let K be a $\ulcorner$-semiring. Let $a, b \in K$, $p \in \mathsf{test}(K)$ and $0 \neq 1$.*

(i) $0 \in \mathcal{N}(K)$*.*

(ii) $p \notin \mathcal{N}(K)$*, if* $p \neq 0$ *and in particular* $1 \notin \mathcal{N}(K)$*.*

(iii) $b \in \mathcal{N}(K)$ *and* $a \le b$ *imply* $a \in \mathcal{N}(K)$*.*

(iv) $a \in \mathcal{N}(K)$ *implies* $a \sqcap 1 \le 0$*, that is, a is irreflexive.*

(v) $a \not\le 0$ *and* $a \in \mathcal{N}(K)$ *imply* $a \not\le aa$*, that is a is not dense.*

(vi) $a \in \mathcal{N}(K)$ *iff* $a^+ \in \mathcal{N}(K)$*, for K a $\ulcorner$-KA.*

(vii) $a^* \notin \mathcal{N}(K)$*, for K a KA with domain.*

(viii) $a + b \in \mathcal{N}(K)$ *implies* $a \in \mathcal{N}(K)$ *and* $b \in \mathcal{N}(K)$*.*

In general, $a \in \mathcal{N}(K)$ and $b \in \mathcal{N}(K)$ do not imply $a + b \in \mathcal{N}(K)$, so that $\mathcal{N}(K)$ is not a semilattice-ideal. A trivial counterexample is given by the relations $a = \{(0, 1)\}$ and $b = \{(1, 0)\}$. In Section 9 we will present commutativity conditions that enforce this implication.

Proposition 2. *The class of Noetherian $\ulcorner$-semirings is a quasi-variety.*

Proof. Let K denote the class of $\ulcorner$-semirings that satisfy (14). Since the class of $\ulcorner$-semirings is a variety and (14) is a universal Horn sentence, K is closed under subalgebras and direct products. It remains to show that K is not closed under homomorphisms. In [10], this has been shown for the class of modal algebras that satisfy (14). Since the modal algebras are a subclass of the $\ulcorner$-semirings, also K is not closed under homomorphisms. □

6 Termination in Modal Logics

We now investigate two alternative equational characterizations of Noethericity. The first one uses the star. The second one is without the star. It holds for the special case of a *transitive* Kleenean element a i.e., when $aa \le a$.

Let K be a $\ulcorner$-semiring or a $\ulcorner$-KA, respectively. Consider the equations

$$|a\rangle \le |a\rangle^+(|1\rangle - |a\rangle), \quad (16) \qquad\qquad |a\rangle \le |a\rangle(|1\rangle - |a\rangle). \quad (17)$$

The equation (17) is a translation of Löb's formula from modal logic (cf. [4]) which expresses well-foundedness in Kripke structures. We say that a is *pre-Löbian* if it satisfies (16). We say that a is *Löbian* if it satisfies (17). The sets of pre-Löbian and Löbian elements of K are denoted by $p\mathcal{L}(K)$ and $\mathcal{L}(K)$, resp.

In the relational model, Löb's formula states that a is transitive and that there are no infinite a-chains. We will now relate Löb's formula and Noethericity.

Theorem 3. *Let T be the set of transitive elements of a KA K with codomain.*
(i) $\mathcal{L}(K) \subseteq \mathcal{N}(K)$.
(ii) $p\mathcal{L}(K) \subseteq \mathcal{N}(K)$.
(iii) $\mathcal{N}(K) \subseteq p\mathcal{L}(K)$.
(iv) $\mathcal{N}(T) \subseteq \mathcal{L}(T)$.

Properties (i) and (iv) already hold in $\ulcorner$-semirings. A calculational proof of (iii) based on [10] can be found in [7].

Corollary 4. *The class of $\ulcorner$-semirings K in which $\mathcal{N}(K)$ comprises all transitive elements of K is a variety.*

The calculational translation between the Löb-formula and our definition of Noethericity is quite interesting for the correspondence theory of modal logic. In this view, our property of Noethericity expresses a frame property, which is part of semantics, whereas the Löb formula stands for a modal formula, which is part of syntax. In modal semirings, we are able to express syntax and semantics in one and the same formalism. Moreover, while the traditional proof of the correspondence uses model-theoretic semantic arguments based on infinite chains, the algebraic proof is entirely calculational and avoids infinity. This is quite beneficial for instance for mechanization.

7 Termination via Infinite Iteration

Cohen has extended KA with an $^\omega$ operator for modeling infinite iteration [5]; he has also shown applications in concurrency control. In [19], this algebra has been used for calculating proofs of theorems from abstract rewriting that use simple termination assumptions.

Dually to the Kleene star, the omega operator is defined as a greatest postfixed point. An *ω-algebra* is a structure (K, ω) where K is a KA and

$$a^\omega \leq aa^\omega, \quad (18) \qquad\qquad c \leq ac + b \Rightarrow c \leq a^\omega + a^*b, \quad (19)$$

for all $a, b, c \in K$. Consequently, a^ω is also the greatest fixpoint of $\lambda x \,.\, ax$.

Like in Section 4, ofr a $\ulcorner$-KA K it seems interesting to lift (18) and (19) to operator algebras, similar to the laws (7), and (8) for the star. This is very simple for (18): for $a \in K$,

$$|a^\omega\rangle \leq |a\rangle |a^\omega\rangle. \tag{20}$$

However, as we will see below, there is no law corresponding to (8) and (19). The proof of (8) uses (llp) and works, since the star occurs at the left-hand sides of inequalities. There is no similar law that allows us to handle the omega, which occurs at right-hand sides of inequalities.

But instead, one can axiomatize the greatest fixpoint $\nu|a\rangle$ of $|a\rangle$ for $a \in K$ by

$$\nu|a\rangle \leq |a\rangle\, \nu|a\rangle, \quad (21) \qquad\qquad p \leq |a\rangle p + q \Rightarrow p \leq \nu|a\rangle + |a^*\rangle q. \quad (22)$$

If $\mathsf{test}(K)$ is complete then by the Knaster-Tarski theorem $\nu|a\rangle$ always exists, since $|a\rangle$ is isotone. In that case one can use a weaker axiomatization (see [10]) from which (22) follows by greatest fixpoint fusion.

It will turn out that $\nu|a\rangle$ is more suitable for termination analysis than a^ω. Since $|a\rangle p = \neg|a]\neg p$, existence of $\nu|a\rangle$ also implies existence of the least fixpoint $\mu|a]$ of $|a]$, since $\mu|a] = \neg\nu|a\rangle$. In the modal μ-calculus, $\mu|a]$ is known as the *halting predicate* (see, e.g., [11]). With the help of $\nu|a\rangle$ we can rephrase Noethericity more concisely as

$$a \in \mathcal{N}(K) \Leftrightarrow \nu|a\rangle = 0. \tag{23}$$

As an immediate consequence of this we obtain

Corollary 5. *Define, for fixed $q \in \mathsf{test}(K)$ and $a \in K$, the function $f : \mathsf{test}(K) \to \mathsf{test}(K)$ by $fp = q + |a\rangle p$. If $\nu|a\rangle$ exists and $a \in \mathcal{N}(K)$ then f has the unique fixpoint $|a^*\rangle q$.*

Proof. The star axioms imply that that the least fixpoint of f is $|a^*\rangle q$. But by the assumption and (22) this is also the greatest fixpoint of f so that all fixpoints coincide with it. □

A notion of guaranteed termination can easily be defined in ω-algebra as the absence of infinite iteration. We call a *ω-Noetherian* if $a^\omega \leq 0$, and denote by $\mathcal{N}_\omega(K)$ the set of all ω-Noetherian elements.

We now study the relation between Noethericity and ω-Noethericity. We call a $\ulcorner$-KA K *extensional*, if

$$|a\rangle \leq |b\rangle \Rightarrow a \leq b$$

holds for all $a, b \in K$. Note that the language model is not extensional. The following lemma shows that the relation between Noethericity and ω-Noethericity does not depend on extensionality. This is somewhat surprising, since set-theoretic relations are extensional and in the relational model the two notions coincide.

Lemma 6. *Let K be an ω-algebra with domain.*
(i) $\mathcal{N}(K) \subseteq \mathcal{N}_\omega(K)$.
(ii) $\mathcal{N}_\omega(K) \not\subseteq \mathcal{N}(K)$*, for K suitably chosen.*
(iii) $\mathcal{N}_\omega(K) \not\subseteq \mathcal{N}(K)$ *for extensional K suitably chosen.*
(iv) $\mathcal{N}_\omega(K) \subseteq \mathcal{N}(K)$ *for non-extensional K suitably chosen.*

Proof. (i) Let a be Noetherian. By isotonicity, for all $p \in \mathsf{test}(K)$,

$$|a^\omega\rangle p \leq |aa^\omega\rangle p = |a\rangle|a^\omega\rangle p.$$

Hence Noethericity of a implies that $|a^\omega\rangle p = 0$ for all $p \in \mathsf{test}(K)$. But, by strictness of domain, this is the case iff $a^\omega = 0$.

(ii) In the language model we have $a^\omega = 0$ if $1 \sqcap a = 0$, but also $a \neq 0 \Rightarrow \forall p \,.\, |a\rangle p = p$.

(iii) We use an *atomic* KA, in which every element is the sum of *atoms*, i.e., minimal nonzero elements. The algebra has 4 atoms and hence 2^4 elements; it is order-isomorphic to the power set of the set of atoms under inclusion. The atoms of the test algebra are p and q, i.e., $1 = p + q$. The domain of an element x is the sum of all atomic tests t such that $tx \neq 0$.

Composition is given by a table for the atoms only; it extends to the other elements through disjunctivity, thus satisfying this axiom by construction. E.g., for atoms w, x, y, z we set $(w + x)(y + z) = wy + wz + xy + xz$. The composition table is

$\cdot$	p	q	a	b
p	p	0	a	0
q	0	q	0	b
a	0	a	0	0
b	b	0	0	0

The algebra is extensional. Moreover, it is easily checked that 0 is the only fixpoint of the function $\lambda x \,.\, (a + b)x$, so that $(a + b)^\omega = 0$. But $1 \leq |a + b\rangle 1$.

(iv) Consider the following KA K from Conway's book, p. 101 [6]. It is the first of his three-element examples. It consists of elements $0 < 1 < a$; the ordering defines the addition table. The only non-trivial relation in the multiplication table is $aa = a$. The star is defined by $a^* = a$ and $0^* = 1^* = 1$. We extend K to an ω-algebra by setting $0^\omega = 0$ and $1^\omega = a^\omega = a$. Moreover, we define a domain operation by $\ulcorner 0 = 0$ and $\ulcorner 1 = \ulcorner a = 1$.

Since $x^\omega = 0 \Leftrightarrow x = 0$ holds in K, that is, $\mathcal{N}_\omega(K) = \{0\}$, we have to verify $\mathcal{N}_\omega(K) \subseteq \mathcal{N}(K)$ only for the zero. But $0 \in \mathcal{N}(K)$ was already stated in Lemma 1(i). □

The following corollary shows that (19) cannot in general be lifted to (22).

Corollary 7. *There exists an $\ulcorner$-KA K such that $\nu|a\rangle \leq 0$, but $a^\omega > 0$ for some $a \in K$.*

Thus ω-algebra does not entirely capture the standard notion of termination.

8 Termination of Exhaustive Iteration

We now study the behaviour of the exhaustive finite iteration of an element $a \in K$, given by

$$\mathsf{exh}\, a = \mathsf{while}\, \ulcorner a\, \mathsf{do}\, a = a^* \neg\ulcorner a \ .$$

Then we can represent the set of points from which a terminal point can be reached via a-steps as

$$\ulcorner(\mathsf{exh}\, a) = \ulcorner(a^* \neg\ulcorner a) = |a^*\rangle \neg\ulcorner a. \tag{24}$$

Proposition 8. *If $a \in \mathcal{N}(K)$ then $\ulcorner(\mathsf{exh}\, a) = 1$, i.e., from* every *starting point a terminal point can be reached.*

Proof. We calculate a recursion equation for $\ulcorner(\mathsf{exh}\, a)$ as follows:

$$\begin{aligned}\ulcorner(\mathsf{exh}\, a) &= |a^*\rangle\neg\ulcorner a = (|1\rangle + |a\rangle|a^*\rangle)\neg\ulcorner a \\ &= \neg\ulcorner a + |a\rangle|a^*\rangle\neg\ulcorner a = \neg\ulcorner a + |a\rangle\ulcorner(\mathsf{exh}\, a) \ .\end{aligned}$$

The first step uses (24), the second one star unfold, the third one distributivity and neutrality of 1, the fourth one again (24).

So $\ulcorner(\mathsf{exh}\, a)$ has to be a fixpoint of $f(p) = \neg\ulcorner a + |a\rangle p$ which by Noethericity of a and Corollary 5 is unique. Hence our claim is shown if 1 also is a fixpoint of f. This is easy, since $f(1) = \neg\ulcorner a + |a\rangle 1 = \neg\ulcorner a + \ulcorner a = 1$. □

This theorem shows again that modal Kleene algebra is more adequate for termination analysis than omega algebra. To see this, consider the algebra LAN of formal languages which is both an omega algebra and a $\ulcorner$-KA with complete test algebra $\mathsf{test}(\text{LAN}) = \{0, 1\}$. In LAN we have $|a\rangle 1 = \ulcorner a = 1 \neq 0$ when $a \neq 0$ and hence $\mathcal{N}(a) \Leftrightarrow a = 0$. Moreover, distinguishing the cases $a = 0$ and $a \neq 0$, easy calculations show that in LAN we have $\mathsf{exh}\, a = \neg\ulcorner a$. This mirrors the fact that by totality of concatenation a nonempty language can be iterated indefinitely without reaching a terminal element. But we also have $a^\omega = 0$ whenever $1 \sqcap a = 0$. Therefore, unlike in the relational model, $a^\omega = 0 \ \not\Rightarrow\ \ulcorner(\mathsf{exh}\, a) = 1$, while still $\nu|a\rangle = 0 \Rightarrow \ulcorner(\mathsf{exh}\, a) = 1$. Hence, for termination analysis in KAs more general than the relational model the element $\nu|a\rangle$ seems more adequate than a^ω.

9 Additivity of Termination

It has been shown that many statements of abstract rewriting that depend on termination assumptions can be proved in ω-algebra [19], among them an abstract variant of Bachmair's and Dershowitz's well-founded union theorem [2]. As we have seen in the previous section, there is a gap between termination in ω-algebra and in $\ulcorner$-KA. Here, we provide a proof of Bachmair's and Dershowitz's theorem in $\ulcorner$-KA.

Consider a KA K and $a, b \in K$. We say that a *semi-commutes* over b, if $ba \leq a^+ b^*$. a *quasi-commutes* over b, if $ba \leq a(a + b)^*$. We write $sc(a, b)$ if a semi-commutes over b and $qc(a, b)$, iff a quasi-commutes over b. Semi-commutation and quasi-commutation state conditions for permuting certain steps to the left of others. In general, sequences with a-steps and b-steps can be split into a "good" part with all a-steps occurring to the left of b-steps and into a "bad" part where both kinds of steps are mixed.

Lemma 9. *For a KA K and all $a, b \in K$,*

$$(a + b)^* = a^*b^* + a^*b^+a(a + b)^*. \tag{25}$$

The following lemma lifts semi-commutation and quasi-commutation to sequences of b-steps.

Lemma 10. *For a KA K and all $a, b \in K$,*
(i) $sc(a, b) \Leftrightarrow b^*a \le a^+b^*$,
(ii) $qc(a, b) \Leftrightarrow b^+a \le a(a + b)^*$,

Proofs for these two lemmas can be found in [19]. The following lemma compares quasi-commutation and semi-commutation.

Lemma 11. *Consider a KA K and $a, b \in K$.*
(i) $sc(a, b) \Rightarrow qc(a, b)$.
(ii) If K is an extensional $\ulcorner$-KA and $a \in \mathcal{N}(K)$ then $qc(a, b) \Rightarrow sc(a, b)$.

Proof. (i) Let a semi-commute over b. By Kleene algebra, $a^+b^* = a(a^*b^*) \le a(a + b)^*$.

(ii) Let a quasi-commute over b and let a be Noetherian. First, we calculate

$$\begin{aligned} a(a + b)^* &= a(a^*b^* + a^*b^+a(a + b)^*) = a^+b^* + a^+b^+a(a + b)^* \\ &\le a^+b^* + a^+a(a + b)^*(a + b)^* = a^+b^* + a^+a(a + b)^*. \end{aligned}$$

The first step uses Lemma 9, the second one distributivity and the definition of a^+, the third one Lemma 10 (ii), the fourth one $x^*x^* = x^*$.

In order to apply Noethericity, we now pass to the modal operator semiring. To enhance readability, we write α for $|a\rangle$ and β for $|b\rangle$ and ζ for $|0\rangle$.

$$\begin{aligned} \alpha(\alpha + \beta)^* - \alpha^+\beta^* &\le (\alpha^+\beta^* + \alpha^+\alpha(\alpha + \beta)^*) - \alpha^+\beta^* \\ &= (\alpha^+\beta^* - \alpha^+\beta^*) + (\alpha\alpha^+(\alpha + \beta)^* - \alpha^+\beta^*) \\ &\le \alpha\alpha^+(\alpha + \beta)^* - \alpha^+\alpha^+\beta^* \\ &= \alpha^+(\alpha(\alpha + \beta)^* - \alpha^+\beta^*). \end{aligned}$$

The first step uses isotonicity of minus in its first argument. The second step uses $(p+q)-r = (p - r) + (q - r)$. The third step uses $p - p = 0$, $a^+a^+ \le a^+$ and antitonicity of subtraction in its second argument. The fourth step uses $aa^+ = a^+a$ and distributivity.

By Lemma 1(vi) we know that a is Noetherian iff a^+ is. Therefore $\alpha^+(\alpha+\beta)^*-\alpha^+\beta^* \le \zeta$, whence $\alpha^+(\alpha + \beta)^* \le \alpha^+\beta^*$. The claim then follows from $\alpha \le \alpha^+$ and extensionality. □

Lemma 12. *Let K be a $\ulcorner$-KA.*
(i) For all $a \in \mathcal{N}(K)$ and $b \in K$, $qc(a, b) \Rightarrow b^*a \le a^+b^*$.
(ii) For all $a, b \in K$, $qc(a, b)$ and $a \in \mathcal{N}(K)$ imply $b^*a \in \mathcal{N}(K)$.
(iii) For all $b, b^*a \in \mathcal{N}(K)$, $(a + b) \in \mathcal{N}(K)$.

Proof. We use the same abbreviations as in the previous proof.

(i) Immediate from Lemma 11 and Lemma 10 (i).

(ii) Let $a \in \mathcal{N}(K)$ and $\alpha\beta \le (\alpha + \beta)^*\alpha$. Then by (i), $\alpha\beta^* \le \beta^*\alpha^+$. Now let $p \le \beta^*\alpha p$, whence $p \le \alpha^+\beta^*p$ and in particular $\beta^*p \le \alpha^+\beta^*p$. Since by Lemma1 (vi) a is Noetherian iff a^+ is, we have that $\beta^*p \le 0$ by assumption. This can only be the case if $p \le 0$.

(iii) We calculate $(a + b)^+ = (b^*a)^*b^*(a + b) \le (b^*a)^+ + b^+$.
Now $a+b$ is Noetherian if $(a+b)^+$ is. Let $p \le (\alpha + \beta)^+p$. Then $p \le (\beta^*\alpha)^+p+\beta^+p$ and $p \le 0$ follows from the assumptions. □

Lemma 12 (ii) and (iii) immediately imply the main theorem of this section. It generalizes the Bachmair-Dershowitz well-founded union theorem from relations to modal Kleene algebra.

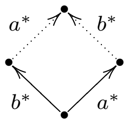
Theorem 13. *Let K be an extensional $^{\ulcorner}$-KA. For all $a, b \in K$, if $qc(a, b)$, then $(a + b) \in \mathcal{N}(K)$, iff $a, b \in \mathcal{N}(K)$.*

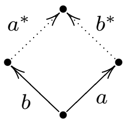
These results show that modal Kleene algebra provides proofs for abstract rewriting which are as simple as those in omega algebra. Note that the proofs in [2] are rather informal, while also previous diagrammatic proofs (e.g. [9]) suppress many elementary steps. In contrast, the algebraic proofs are complete, formal and still simple. An extensive discussion of the relation between the proofs in omega algebra and their diagrammatic counterparts can be found in [19]. In particular, the algebraic proofs mirror precisely the diagrammatic ones. This also holds for the modal proofs we present here.

10 Newman's Lemma and Normal Forms

We now turn from semi-commutation to commutation and confluence. For their direct algebraic characterization one either has to use converse at the element level or a combination of forward and backward modalities at the operator level. Since we do not have converse available, we have to choose the second alternative.

We say that $a, b \in K$ *commute* if $\langle b^*||a^*\rangle \le |a^*\rangle\langle b^*|$, and *commute locally* if $\langle b||a\rangle \le |a^*\rangle\langle b^*|$. As a special case, we call $a \in K$ *(locally) confluent* if it (locally) commutes with itself. These definitions can be visualized as





If one wants to avoid combinations of forward and backward modalities, one can express commutation and confluence by the Geach formula [4] that replaces the expression $a^{\smile}b \le cd^{\smile}$ of a modal KA with converse by $|b\rangle|d] \le |a]|c\rangle$ (see [15] for a proof). Equivalence to the above formulation is shown by

$$|b\rangle|d] \le |a]|c\rangle \Leftrightarrow \langle a||b\rangle|d] \le |c\rangle \Leftrightarrow \langle a||b\rangle \le |c\rangle\langle d|.$$

The first step uses (10), the second one (12). Consequently, commutation and local commutation are equivalent to the following formulas:

$$|a^*\rangle|b^*] \le |b^*]|a^*\rangle, \qquad |a\rangle|b^*] \le |b]|a^*\rangle.$$

However, these formulas are much less intuitive than our original ones. Still, the proof we present below can be carried out in this unidirectional form as well.

In the relational setting, the generalization from confluence to commutation has been used in [17] for a theory of term-rewriting with pre-congruences that extends the traditional equational case. This also leads to generalizations of the Church-Rosser theorem and of Newman's lemma. While the Church-Rosser case has already been proved in Kleene algebra in [18], it has been argued in [19] that a proof of Newman's lemma does not work in pure Kleene or omega algebra.

For the equational case, [16] gives a calculational proof of Newman's lemma in relation algebra. But it cannot be adapted to our case, since it uses a notion of unique normal form that does not exist in the commutation-based setting. Moreover, conceptually it is nicer to completely uncouple confluence from normal forms.

We will faithfully reconstruct the diagrammatic proof using Noetherian induction [17]; it will turn out that modal Kleene algebra is very well suited for this. A calculational proof that is close in spirit occurs in [8]. However, it is more complex in that it uses full residuation, whereas we can make do with the much weaker concept of modal operators. (The modal box operator corresponds to the monotype factor that is also used in [8].) Also, the theorem there is more restricted in application, since it only covers the relational case, whereas our result also applies to e.g. the path algebra.

Now we are ready for our generalization of Newman's lemma.

Theorem 14. *Let K be a modal KA with complete test algebra. If $a+b \in \mathcal{N}(K)$ and a and b commute locally then a and b commute.*

Proof. The central idea of our proof is to use a generalized predicate that characterizes the set of all points on which a and b commute and to retrieve C as a special case. If we can show that this predicate is contracted by $|a+b]$ then, by the second form (15) of Noethericity, we are done.

So let us define (rc stands for "restricted commutation")

$$rc(p,a,b) \Leftrightarrow \langle b^*|\langle p\rangle|a^*\rangle \leq |a^*\rangle\langle b^*| \ .$$

$rc(p,a,b)$ states that a and b commute on all points in p. We have used the notation $\langle p\rangle$ to enhance the symmetry of the formulation; this is justified, since $|p\rangle = \langle p|$ for all tests p. Clearly, a and b commute iff $rc(1,a,b)$. Moreover, by isotonicity rc is downward closed, i.e., $rc(p,a,b) \wedge q \leq p \Rightarrow rc(q,a,b)$. We now define (the supremum exists by completeness of $\mathsf{test}(K)$)

$$r = \sup\{p \mid rc(p,a,b)\} \ .$$

This represents the set of all points on which a and b commute. Now, completeness of $\mathsf{test}(K)$ implies that $\cdot$ distributes over all suprema in $\mathsf{test}(K)$, so that $|r\rangle = \sup\{|p\rangle \mid rc(p,a,b)\}$. Moreover, composition with diamonds is universally disjunctive in both arguments, so that we may infer $rc(r,a,b)$. Together with downward closure of rc we therefore obtain the characterization

$$p \leq r \Leftrightarrow rc(p,a,b) \ . \tag{26}$$

We now show that r is contracted by $|a+b]$, so that $a+b \in \mathcal{N}(K)$ implies $r=1$. For this we first calculate

$$\begin{aligned}
(|a+b]r \leq r) &\Leftrightarrow (\forall\, p \,.\, p \leq |a+b]r \Rightarrow p \leq r)\\
&\Leftrightarrow (\forall\, p \,.\, \langle a+b|p \leq r \Rightarrow p \leq r)\\
&\Leftrightarrow (\forall\, p \,.\, \langle a|p \leq r \wedge \langle b|p \leq r \Rightarrow p \leq r)\\
&\Leftrightarrow (\forall\, p \,.\, rc(p_a,a,b) \wedge rc(p_b,a,b) \Rightarrow rc(p,a,b)).
\end{aligned}$$

The first step uses indirect inequality, the second one the Galois connection (5), the third one distributivity and Boolean algebra, the fourth one (26) and the definition $p_x = \langle x|p$.

So assume $rc(p_a,a,b) \wedge rc(p_b,a,b)$. By the star fixpoint law (9) and distributivities

$$\langle b^*|\langle p\rangle|a^*\rangle \leq \langle b^*|\langle p\rangle + \langle b^*|\langle b|\langle p\rangle|a\rangle|a^*\rangle + \langle p\rangle|a^*\rangle \ .$$

The outer two of these summands are below $|a^*\rangle\langle b^*|$ by isotonicity, $p \leq 1 \leq x^*$ and neutrality of $|1\rangle$. For the middle summand we first state

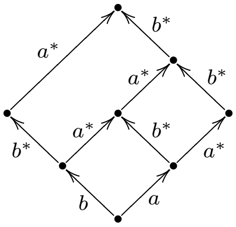
$$\langle p\rangle|x\rangle = \langle p\rangle|x\rangle\langle p_x\rangle \leq |x\rangle\langle p_x\rangle \ , \qquad \langle x|\langle p\rangle = \langle p_x\rangle\langle x|\langle p\rangle \leq \langle p_x\rangle\langle x| \ . \tag{27}$$

This follows by isotonicity, since the definition of p_a and right neutrality of codomain imply $px = px(p_x) \leq x(p_x)$. Now we calculate

$$\begin{aligned}
\langle b^*|\langle b|\langle p\rangle|a\rangle|a^*\rangle &\leq \langle b^*|\langle p_b\rangle\langle b||a\rangle\langle p_a\rangle|a^*\rangle \\
&\leq \langle b^*|\langle p_b\rangle|a^*\rangle\langle b^*|\langle p_a\rangle|a^*\rangle \\
&\leq |a^*\rangle\langle b^*|\langle b^*|\langle p_a\rangle|a^*\rangle \\
&\leq |a^*\rangle\langle b^*|\langle p_a\rangle|a^*\rangle \\
&\leq |a^*\rangle|a^*\rangle\langle b^*| \\
&\leq |a^*\rangle\langle b^*|.
\end{aligned}$$

The first step uses idempotence of $\langle p\rangle$, codomain propagation (27) twice and compositionality, the second one $\mathrm{LC}(a,b)$, the third one the assumption $rc(p_a,a,b)$, the fourth one idempotence of star and compositionality, the fifth one the assumption $rc(p_b,a,b)$, the sixth one idempotence of star and compositionality.

This finishes the proof. We reflect this last calculation in a diagram in which the bottom point is in p and the two points in the next higher layer are in p_b and p_a, respectively.



□

We conclude this section by showing that confluence implies uniqueness of normal forms. As in Section 8, for $a \in K$ the element $\mathsf{exh}\, a = a^*\neg\ulcorner a$ describes the exhaustive iteration of a, the points in $(\mathsf{exh}\, a)\urcorner$ being the *normal forms*. Now, a Kleene element b assigns to each point in its domain at most one point in its codomain iff b is *deterministic*, i.e., iff $\langle b||b\rangle \leq \langle 1\rangle$. This formula corresponds to the relational characterization $\breve{b}b \leq 1$ of determinacy of b. Now we can show

Lemma 15. *If a is confluent then $\mathsf{exh}\, a$ is deterministic.*

Proof. Plugging in the definition of $\mathsf{exh}\, a$ we calculate

$$\begin{aligned}
\langle a^*\neg\ulcorner a||a^*\neg\ulcorner a\rangle &= \langle\neg\ulcorner a\rangle\langle a^*||a^*\rangle\langle\neg\ulcorner a\rangle \leq \langle\neg\ulcorner a\rangle|a^*\rangle\langle a^*|\langle\neg\ulcorner a\rangle \\
&= |\neg\ulcorner a a^*\rangle\langle\neg\ulcorner a a^*| = |\neg\ulcorner a\rangle\langle\neg\ulcorner a| \leq \langle 1\rangle.
\end{aligned}$$

The first step uses compositionality, the second one confluence of a, the third one compositionality again, the fourth one the star fixpoint law, distributivity and (gla), the fifth one isotonicity and idempotence of $\langle 1\rangle$. □

11 Conclusion

We have used modal KA for termination analysis, introducing and comparing different notions of termination that arise in this context and applying our techniques to two examples from abstract rewriting. All proofs are abstract, concise and entirely calculational. Together with previous work [18,19] our case study in abstract rewriting shows that large parts of this theory can be reconstructed in modal Kleene algebra. Due to its simplicity, our approach has considerable potential for mechanization. There are strong connections with automata-theoretic decision procedures.

From the proof of Newman's lemma and the associated diagram it becomes clear that modal Kleene algebra allows one to perform induction in the middle of an expression. This is not possible in pure Kleene or omega algebra due to the shape of the star and omega induction rules. This shows that modal Kleene algebra allows "context-free" induction, whereas pure Kleene or omega algebra only admits "regular" induction. Therefore, in [8] residuals are used to move the point of induction from the middle of an expression to its ends and back.

The results of this paper contribute to an attempt to establish modal Kleene algebra as a formalism that enhances safe cross-theory reasoning and therefore interoperability between different calculi for program analysis. We envision three main lines of further work. First, the integration of our results into Hoare-style reasoning and into Kleene algebras for the weakest precondition semantics. Second, a further exploitation of the mentioned connection with the modal μ-calculus. Third, further applications of our technique to the analysis of programs and protocols.

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