Parallelity and extrinsic homogeneity

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0. Introduction

It is a well known phenomenon in geometry that parallelity of certain structures implies homogeneity, at least locally: Sufficiently many parallel vector fields characterize homogeneous euclidean space forms. A Riemannian manifold with parallel curvature tensor is locally symmetric, and more generally, a smooth manifold equipped with a connection having parallel curvature and torsion tensors is locally homogeneous (cf. [K]). For submanifolds, parallelity often implies even *global* extrinsic homogeneity. E.g., a planar curve with constant curvature is a circle or a line, and a spatial curve with constant curvature and torsion is a helix. For submanifolds of higher dimension and codimension, a similar phenomenon was observed by Ferus [F1]: A closed submanifold of euclidean space with parallel second fundamental form is an extrinsic symmetric space, i.e. the reflections at all normal spaces leave the submanifold invariant. A simpler proof of this fact was given later by Strübing [S]. His idea was to show that on such a submanifold parallel frames along geodesics satisfy a linear differential equation with constant coefficients which is preserved by isometries of the ambient space. Olmos and Sanchez [OS] extended this idea to submanifolds which have parallel second fundamental form with respect to another connection D differing from the Levi-Civita connection by a D-parallel tensor, which resulted in a local characterization of the orbits of certain representations (so called s-representations). In the present paper we wish to derive the most general extension of Strübing's argument where the ambient space is no longer euclidean but a reductive homogeneous space (Ch. 2). The main difficulty arising in such an extension is that the group G acting transitively on the homogeneous space might be too small to contain the parallel displacements of D. Thus we need an extra condition ("G-connection") which is discussed in detail in Ch. 3. We want to point out that the arguments are in fact of purely affine character: Only connections are used, no metrics. A simple application is given in Ch. 4: a local characterization of extrinsic symmetric subspaces of Riemannian symmetric spaces. We hope that our results can be used to characterize isoparametric and equifocal submanifolds of Riemannian symmetric spaces.

1. Canonical connections

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. G acts on itself by left and right translations L_g and R_g; to save symbols we will use the notation of a matrix group and write gX and Xg for dL_gX and dR_gX where $X \in \mathfrak{g}$. In particular, $Ad(g)X = gXg^{-1}$.

Let $H \subset G$ be a closed subgroup. We consider the homogeneous space $P = G/H$ with its base point $p_0 = eH$. The left translations L_g induce an action of G on P; we will denote this actions and its differential simply by $g.p$ and $X.p$ (for $g \in G,\ X \in \mathfrak{g}$ and $p \in P$). Further we will assume that P is *reductive*, i.e. $\mathfrak{g}$ allows an $Ad(H)$-invariant vector space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}.$$

where $\mathfrak{h}$ denotes the Lie algebra of H. We may assume that G acts effectively and that $Ad(H)|_\mathfrak{p}$ is a closed subgroup of $Gl(\mathfrak{p})$ (otherwise, we can enlarge G). The complement $\mathfrak{p}$ defines a left invariant distribution on G which is right invariant under H and which defines a connection (the so called *canonical connection*) on the H-principal bundle G over $G/H = P$.

Now let E be a homogeneous vector bundle over P, i.e. the action of G on P is covered by an action on E via bundle isomorphisms. This action and its differential will be denoted also by $g.v$ and $X.v$ for $g \in G,\ X \in \mathfrak{g}$ and $v \in E$. Our standard example will be the tangent bundle $E = TP$. Any such vector bundle is associated to the H-principal bundle $G \to P$. Hence the canonical connection on G defined by $\mathfrak{p}$ induces a covariant derivative ∇ on E which can be computed as follows: Consider a section $W(t) = g(t).w$ along the curve $p(t) = g(t)p_0$ for some smooth curve $g(t)$ in G and some $w \in \mathfrak{p} = T_{p_0}P$. (Sections of this type form a basis for all sections along the curve $p(t)$.) Now the covariant derivative of W along this curve is

$$\nabla_t W = (g')_{g\mathfrak{h}}.w$$

where $(g')_{g\mathfrak{h}}$ denotes the projection of $g' \in T_gG = g\mathfrak{h} \oplus g\mathfrak{p}$ onto the vertical subspace $g\mathfrak{h}$. This covariant derivative is G-invariant, i.e. each $g \in G$ commutes with ∇. The ∇-geodesics through $p = g.p_0$ are $\gamma(t) = \exp(tX).p$

for $X \in Ad(g)\mathfrak{p}$ while the parallel sections along γ are $v(t) = \exp(tX).v$ for arbitrary $v \in E_p$. Consequently, the holonomy group at p_0 is contained in H (considered as a subgroup of $Gl(\mathfrak{p})$). For $E = TP$ this connection is in general not torsion free; in fact, its torsion tensor at p_0 is given by $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{p}}$, so it vanishes iff P is an affine symmetric space. However, the curvature and torsion tensors are ∇-parallel (cf. [K] for details in this section).

2. Homogeneity in terms of connections

Theorem 1. *Let $P = G/H$ be a reductive homogeneous space with canonical connection ∇ on TP and let $M \subset P$ a closed submanifold. Suppose that there is a covariant derivative D on the vector bundle $E := TP|_M$ with the following two properties:*

(a) $TM \subset E$ is a D-parallel subbundle,

(b) $\Gamma := \nabla - D : TM \to End(E)$ is D-parallel.

Suppose that $g \in G$ preserves Γ at some point $p \in M$, more precisely, $g.p \in M$ and $g.T_pM = T_{g.p}M$ for some $p \in M$, and $g.\Gamma_x v = \Gamma_{g.x} g.v$ for all $x \in T_pM$ and $v \in T_pP$. Then g leaves M invariant and preserves D.

Proof. Let γ be a D-geodesic in M with $\gamma(0) = p$ and $B(t) = (B_1(t), ..., B_n(t))$ a D-parallel basis along γ with $B_1(t) = \gamma'(t)$. We will show that $g.\gamma$ is again a D-geodesic in M and $g.B(t)$ a D-parallel basis along $g.\gamma$. In fact, from $D_t B_j = 0$ we get

$$\nabla_t B_j(t) = \Gamma_{\gamma'(t)} B_j(t).$$

Since Γ as well as B_j and γ' are parallel, we have

$$\Gamma_{\gamma'} B_j = \sum_i c_{ij} B_i$$

for some constant matrix $C = (c_{ij})$. Hence for $B = (B_1, ..., B_n)$ we get (in matrix notation):

$$\nabla_t B = B \cdot C. \qquad (*)$$

Now let $\tilde{\gamma}$ be the D-geodesic starting at $\tilde{\gamma}(0) = g.p$ with initial vector $\tilde{\gamma}'(0) = g.\gamma'(0)$. Let $\tilde{B}(t)$ be the D-parallel frame along $\tilde{\gamma}$ with $\tilde{B}(0) = g.B(0)$. Since g preserves Γ at p, the matrix of the linear map $\Gamma_{\tilde{\gamma}'(0)}$ with respect to the basis $\tilde{B}(0)$ is the same matrix $C = (c_{ij})$. So $\tilde{B}$ satisfies the same covariant differential equation $(*)$. But also $g.B(t) = (g.B_1(t), ..., g.B_n(t))$ satisfies $(*)$ since g commutes with ∇_t. Since the initial conditions for $\tilde{B}$ and $g.B$ agree, we obtain $\tilde{B} = g.B$. In particular we have $\tilde{\gamma}' = g.\gamma'$ since $\gamma' = B_1$ and $\tilde{\gamma}' = \tilde{B}_1$. Thus $\tilde{\gamma} = g.\gamma$.

Let M_1 be the open subset of all points $q \in M$ which can be connected to p by a D-geodesic. We have shown that g maps M_1 into M and preserves D. Repeating this argument for any $p_1 \in M_1$ in place of p, we get the same statement for the set M_2 containing the points which can be connected to M_1 by a D-geodesic, and by induction it holds for M_k (containing the points with a D-geodesic connection to M_{k-1}). Since M is connected and any curve can be approximated by a D-geodesic polygon, we have $M = \bigcup_k M_k$ and we are done. $\square$

Now we can characterize extrinsic homogeneous submanifolds of P in terms of connections. A closed submanifold $M \subset P$ will be called *extrinsic homogeneous* if it is a reductive orbit of a closed subgroup of G. A connection D on $E = TP|_M$ is called a *G-connection* if for any piecewise smooth curve $c : [a,b] \to M$ the D-parallel transport $\tau_c : T_{c(a)}P \to T_{c(b)}P$ is given by some group element, i.e. there exists some $g \in G$ such that for all $v \in T_{c(a)}P$ we have

$$\tau_c(v) = g.v \ .$$

If $P = \mathbb{R}^n$ and G is the full affine group (resp. the full isometry group of $\mathbb{R}^n$), then any connection (resp. any metric connection) is a G-connection. If P is an affine symmetric space and G the full group of affine diffeomorphisms, then a connection D on $TP|_M$ is a G-connection if and only if the parallel transports τ_c preserve the curvature tensor of P.

Theorem 2. *A closed submanifold $M \subset P$ is extrinsic homogeneous if and only if there exists a G-connection D on $E := TP|_M$ such that $TM \subset E$ is a parallel subbundle and the tensor $\Gamma = \nabla - D : TM \to End(E)$ is parallel with respect to D. This connection is a canonical connection on M.*

Proof. Suppose first that $M \subset P$ is a reductive orbit, i.e. $M = A.p$ for some $p \in P$ and a closed subgroup $A \subset G$ such that M is a reductive homogeneous space: If $B = A_p$ denotes the isotropy group of p under the action of A and $\mathfrak{b}$ its Lie algebra, we have an $Ad(B)$-invariant decomposition of the Lie algebra $\mathfrak{a}$ of A:

$$\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{m}.$$

Since $E = TP|_M$ is a homogeneous vector bundle over $M = A/B$ and $TM \subset E$ an A-invariant subbundle, the canonical connection on the principal bundle $A \to M$ defined by $\mathfrak{m}$ induces a covariant derivative D on E which leaves TM invariant. In particular, D is a G-connection since $A \subset G$.

Let $\Gamma := \nabla - D$ where ∇ is the canonical connection of TP. We put $p = g.p_0$ and $v = g.w$ for some $w \in T_{p_0}P$, and we let $g(t) = \exp(tX)g$. Then $v(t) = \exp(tX)v = g(t)w$ is D-parallel, $D_t v(t) = 0$, and we have for $x = X.p$

$$\Gamma_x v = \nabla_t v(t)|_{t=0} = g'(0)_{g\mathfrak{h}}.w = (Xg)_{g\mathfrak{h}}.w = X_{Ad(g)\mathfrak{h}}.v \ .$$

We have to show that Γ is D-parallel. Thus let $x(t)$ and $v(t)$ be parallel along a D-geodesic $p(t) = a_t.p_0$ with $a_t = \exp(tY)p$ for some $Y \in \mathfrak{m}$. Then we have $v(t) = a_t.v$ and $x(t) = a_t.x = a_t.X.p = (Ad(a_t)X).p(t)$. Thus

$$\Gamma_{x(t)}v(t) = (Ad(a_t)X)_{Ad(a_t)\mathfrak{h}}a_t.v = a_t.X_{\mathfrak{h}}.v$$

which is a D-parallel section of E along $p(t)$. This shows that Γ is D-parallel.

Vice versa, assume that we have a connected submanifold $M \subset P = G/H$ having such a connection D on $E = TP|_M$. Let $c : [a,b] \to M$ be any piecewise smooth curve. Then there exists some (unique) $g_c \in G$ with $g_c(c(a)) = c(b)$ and $g_c.v = \tau_c v$ for all $v \in T_pP$. Since Γ is D-parallel, $\Gamma|_p$ is preserved under the parallel transport τ_c and hence under g_c. Thus g_c leaves M invariant and preserves D (cf. Theorem 1). It follows that the closed subgroup $A \subset G$ generated by all g_c acts transitively on M as a group of affine transformation with respect to D. We have to show only that M is reductive. Let T be the torsion tensor of D. Fix some $p \in M$. Let

$$\mathfrak{m} = \{X \in \mathfrak{a};\ D_yX = T(y, X.p)\ \forall y \in T_pM\}$$

where $X \in \mathfrak{a}$ is considered as the tangent vector field $q \mapsto X.q$ on M (affine Killing field). Let $B = A_p$ be the isotropy group at p and $\mathfrak{b}$ its Lie algebra; we have

$$\mathfrak{b} = \{X \in \mathfrak{a};\ X.p = 0\}.$$

Since $B \subset A$ preserves D and hence T (cf. Theorem 1), $\mathfrak{m}$ is an $Ad(B)$-invariant subspace. Further we have $\mathfrak{m} \cap \mathfrak{b} = 0$ since an affine Killing field is determined by its value and derivative at one point, and for $X \in \mathfrak{m} \cap \mathfrak{b}$ we have $X.p = 0$ and $DX|_p = 0$, hence $X = 0$. It remains to show that for any $x \in T_pM$ there exists $X \in \mathfrak{m}$ with $X.p = x$; then $\mathfrak{m}$ is a complement of $\mathfrak{b}$ by reasons of dimension.

In fact, let γ be the D-geodesic starting from $\gamma(0) = p$ with $\gamma'(0) = x$. For any $t, u \in \mathbb{R}$ sufficiently close to 0 let $\tau_{t,u} : T_{\gamma(u)}M \to T_{\gamma(t+u)}M$ be the parallel displacement along γ. There exists $g_t \in A$ with $g_t(\gamma(0)) = \gamma(t)$ and $g_t.y = \tau_{t,0}y$ for all $y \in T_pM$. Since g_t is affine, it is a transvection along γ, i.e. it translates γ and the parallel vector fields along γ. From $\tau_{s,t} \circ \tau_{t,0} = \tau_{s+t,0}$ we obtain $g_sg_t = g_{s+t}$, hence the family (g_t) forms a one-parameter subgroup: $g_t = \exp(tX)$ for some $X \in \mathfrak{a}$ with $X.p = \gamma'(0) = x$. If we extend any $y \in T_pM$ to the parallel vector field $Y(t) = g_t.y$ along γ, we have $D_yX = D_xY + T(y,x)$ (no Lie bracket term since Y is invariant under the flow of X), but $D_xY = D_tY(t)|_{t=0} = 0$. Thus we have $X \in \mathfrak{m}$ which finishes the proof that M is reductive. $\square$

3. G-connections

The flaw of our last theorem is that it might be hard to decide whether a connection is a G-connection. The next theorem gives a more simple characterization.

Theorem 3. *Let $M \subset P = G/H$ be a closed submanifold. A connection D on $E = TP|_M$ preserving $TM \subset E$ is a G-connection if and only if at any point $p = g.p_0 \in M$, the tensor $\Gamma = \nabla - D : TM \to End(E)$ takes values in $Ad(g)\mathfrak{h} \subset End(T_pP)$. If Γ is D-parallel, this property need to be satisfied only at some $p \in M$.*

Proof. Suppose that D is a G-connection on $TP|_M$. Let $B(t)$ be a D-parallel frame along some curve $p(t)$. Then $B(t) = g(t).b$ and $p(t) = g(t).p_0$ for some smooth curve $g(t)$ in G and some basis b of $T_{p_0}P$. Since B is D-parallel, we have $\nabla_t B = \Gamma_{p'} B$, and on the other hand, $\nabla_t B = (g')_{g\mathfrak{h}}.b$, thus

$$\Gamma_{p'} B = (g')_{g\mathfrak{h}}.g^{-1}.B = (g'g^{-1})_{Ad(g)\mathfrak{h}}.B$$

which shows that $\Gamma_{p'} \in Ad(g)\mathfrak{h}$.

Vice versa, suppose that $\Gamma|_{g.p_0}$ takes values in $Ad(g)\mathfrak{h}$. Let $B(t)$ be a D-parallel basis along some curve $p(t)$ in M, i.e. we have

$$\nabla_t B = \Gamma_{p'} B. \qquad (*)$$

We claim that this equation can be solved by the ansatz $B(t) = g(t).b$ for some fixed basis b of $T_{p_0}P$ and some smooth curve $g(t)$ in G with $p(t) = g(t).p_0$. In fact, then $\nabla_t B = (g')_{g\mathfrak{h}} b = (g')_{g\mathfrak{h}} g^{-1} B$, thus by $(*)$, we get for the vertical component of g':

$$(g')_{g\mathfrak{h}} = \Gamma_{p'} g$$

(note that $\Gamma_{p'} g$ lies in fact in $g\mathfrak{h}$). On the other hand, the horizontal component of g' is determined by p':

$$(g')_{g\mathfrak{p}} = \hat{p'}$$

where $\hat{v} \in T_g G$ denotes the horizontal lift of $v \in T_{g.p_0}P$. Thus we obtain an ODE for the curve $g(t)$ in G, and $B(t) := g(t).b$ solves $(*)$. This shows that D is a G-connection.

Now suppose that Γ is D-parallel and that $\Gamma|_{p_0}$ takes values in $\mathfrak{h} \subset End(T_{p_0}P)$ (we may assume $p = p_0$). We claim that D is a G-connection. In fact, let $B(t)$ be a D-parallel basis along a D-geodesic γ in M starting at $\gamma(0) = p_0$, and let $B(0) = b$. By parallelity we have $\Gamma_{\gamma'} B = B \cdot C$ for some constant matrix $C = (c_{ij})$, and since $\Gamma|_{p_0}$ takes values in $\mathfrak{h}$, there exists $X \in \mathfrak{h}$ such that

$$b \cdot C = \Gamma_{\gamma'(0)} b = X.b\ .$$

Now the parallelity of $B(t)$ is equivalent to

$$\nabla_t B = B \cdot C \tag{$**$}$$

which will again be solved by the ansatz $B(t) = g(t)b$ for some curve $g(t)$ in G with $g(t).p_0 = \gamma(t)$. In fact, then we have $\nabla_t B = (g')_{g\mathfrak{h}} b$, thus $(**)$ is equivalent to $(g')_{g\mathfrak{h}} b = g.b \cdot C = g.X.b$. Hence $B(t) = g(t)b$ solves $(**)$ if $g(t)$ solves the ODE

$$(g')_{g\mathfrak{h}} = gX, \;\; (g')_{g\mathfrak{p}} = \hat{\gamma}'.$$

This shows that the parallel transport along D-geodesics is given by elements of G. Since any curve can be approximated by geodesic polygons, D is a G-connection. □

4. Extrinsic symmetric subspaces

As an application, we now consider a Riemannian symmetric space $P = G/H$ where G is the isometry group of P. A submanifold $M \subset P$ is called *extrinsic symmetric* if for any $p \in M$ there exists an isometry $\rho \in G$ fixing p and leaving M invariant and acting as id on the normal space N_pM and as $-id$ on the tangent space T_pM. Extrinsic symmetric spaces in euclidean space $P = \mathbb{R}^n$ are classified (cf. [KN], [F2], [EH]), and they are characterized by the property that the second fundamental form $\alpha : TM \otimes TM \to NM$ is parallel with respect to the natural connections on TM and NM. It is easy to see that also for an arbitrary ambient space P this property is necessary: $D\alpha : TM \otimes TM \otimes TM \to NM$ is invariant under ρ, but while the three arguments in T_pM change sign under ρ, the value under $D\alpha$ in N_pM stays the same which shows $D\alpha = 0$. However, this property is no longer sufficient: If $M \subset P$ is extrinsic symmetric, then N_pM and T_pM must be *totally geodesic*, i.e. invariant under the curvature tensor R of P since they are the fixed spaces of the isometries ρ and $\rho \circ \sigma$ where σ denotes the symmetry of P at p. But there are totally geodesic submanifolds ($\alpha = 0$), e.g. geodesics, whose normal spaces are not totally geodesic. Instead, we have the following characterization of extrinsic symmetric spaces (the equivalence of the first two statements has been proved already in [NT]):

Theorem 4. *Let P be a Riemannian symmetric space and $M \subset P$ a closed submanifold. Then the following statements are equivalent:*

(1) M is extrinsic symmetric,
(2) $D\alpha = 0$ and T_pM and N_pM are totally geodesic for all $p \in M$,
(3) $D\alpha = 0$ and for some $p_0 \in M$ there is a linear map $X : T_{p_0}M \to \mathfrak{h}$ with $\alpha(v, w) = X(v).w$ for all $v, w \in T_{p_0}M$.

Proof. For a Riemannian symmetric space P, the canonical connection ∇ is the Levi-Civita connection. Let D be the Levi-Civita connection on $TP|_M = TM \oplus NM$, i.e.

$$D_V W = (\nabla_V W)_{TM}, \ \ D_V \xi = (\nabla_V \xi)_{NM}$$

for all tangent fields V, W and normal fields ξ on M. Then $\Gamma = \nabla - D$ maps TM into NM and vice versa, and

$$\Gamma_V W = \alpha(V, W), \ \ \langle \Gamma_V \xi, W\rangle = -\langle \xi, \alpha(V, W)\rangle.$$

Consequently, Γ is D-parallel iff so is α.

We have already seen that (1) implies (2). To show the converse we recall that any linear isometry of T_pP preserving the curvature tensor R at p extends to an isometry of P fixing p. If T_pP splits orthogonally into totally geodesic subspaces $T_pP = T_pM \oplus N_pM$, then $\rho := -id_{T_pM} \oplus id_{N_pM}$ leaves R invariant: We only have to consider expressions R_{abcd} with $a, b, c, d \in T_pM \cup N_pM$. But if precisely one of the four arguments lies in T_pM or in N_pM, the expression is zero since these subspaces are totally geodesic. In all remaining cases, the number of arguments of each type is even and so the expression is invariant under ρ. Clearly ρ commutes with α and hence with Γ. Thus we see from Theorem 1 that $\rho \in G$ preserves M and thus M is extrinsic symmetric.

The equivalence of (1) and (3) is a consequence of the Theorems 2 and 3: The asumption in (3) says precisely that $\Gamma|_{p_0}$ takes values in $\mathfrak{h}$, thus $(3) \Rightarrow (1)$ follows. For the converse statement note that for an extrinsic symmetric space $M \subset P$, the group generated by all the reflections ρ contains the transvections, hence the canonical connection on $TP|_M$ must be the Levi-Civita connection D and the additional property in (3) follows from Theorem 3. □

References

[EH] J.-H. Eschenburg, E. Heintze: Extrinsic symmetric spaces and orbits of s-representations, manuscripta math. **88** (1995), 517–524

[F1] D. Ferus: Immersions with parallel second fundamental form, Math. Z. **140** (1974), 87–93

[F2] D. Ferus: Symmetric submanifolds of euclidean space, Math. Ann. **247** (1980), 81–93

[KN] S. Kobayashi, T. Nagano: On filtered Lie algebras and geometric structures I, J. Math. Mech. **13** (1964), 875–907

[K] O. Kowalski: Generalized Symmetric Spaces, Springer Lecture Notes in Mathematics **805**, 1980

[NT] H. Naitoh, M. Takeuchi: Symmetric submanifolds of symmetric spaces, Sugaku Expositions, 2(2), 157–188 (1989)

[OS] C. Olmos, C. Sánchez: A geometric characterization of the orbits of s-representations, J. Reine u. Angew. Math. **420** (1991), 195–202

[St] W. Strübing: Symmetric submanifolds of Riemannian manifolds, Math. Ann. **245** (1979), 37–44