Associated families of pluriharmonic maps and isotropy

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0. Introduction

Since the last century it is known that a minimal surface in 3-space (up to coverings) allows a one-parameter family of isometric deformations preserving the principal curvatures and rotating the principal curvature directions: the *associated family*. An example is the well-known deformation of the catenoid into the helicoid. The associated family deformation is constant only if the minimal surface is a plane. Associated families were also observed for minimal surfaces in spheres and complex projective spaces (e.g. cf. [EGT]), but in these target spaces there are interesting minimal surfaces with *constant* associated families which could be classified (cf. [C], [B], [DZ], [EW], [CW]). Eventually, the existence of an associated family was proven for harmonic maps of Riemann surfaces into any compact symmetric space (cf [U], [Hi], [BFPP], [DPW]), and it became a cornerstone for the loop group representation of these objects.

If we pass from surfaces to Kähler manifolds of higher dimension, we have to replace harmonic by *pluriharmonic* maps whose restrictions to all complex one-dimensional submanifolds are harmonic. A pluriharmonic isometric immersion is called *pluriminimal* or *(1,1)-geodesic*; its restriction to any complex one-dimensional submanifold is a minimal surface. Pluriharmonic maps also have an associated family. This was first shown by Ohnita and Valli [OV] if the target space is a compact Lie group, and was generalized by Burstall et al. [BFPP] to symmetric spaces of compact type, using the Cartan embedding of a symmetric space $S = G/K$ into its isometry group G. In fact, pluriharmonic maps are characterized by this property of having an associated family. One purpose of our paper is to give a simple direct proof of this fact for *any* symmetric space S using the geometry of S without passing to G. Further, we characterize the pluriharmonic maps with trivial associated family, the so called *isotropic* ones; it turns out that they all arise from holomorphic maps into a flag manifold or flag domain over S.

Besides minimal surfaces, also constant mean curvature surfaces in 3-space allow isometric deformations rotating the second fundamental form. These surfaces are generalized by Kähler submanifolds whose second fundamental form has parallel (1,1)-part. These $(1,1)$-*parallel* immersion allow also some kind of associated families (called "weak") which is the subject of our last chapter.

1. Associated families

Let $(M, \langle\ ,\ \rangle, J)$ be a Kähler manifold of complex dimension m. For any angle $\theta \in [0, 2\pi]$ let $\mathcal{R}_\theta : TM \to TM$,

$$\mathcal{R}_\theta(X) = \cos(\theta)X + \sin(\theta)JX.$$

This is a parallel endomorphism field on TM. As usual, the complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the parallel eigenbundles of J, called $T'M$ and $T''M$, corresponding to the eigenvalues i and $-i$, and the elements of $T'M$ and $T''M$ are called vectors of *type* (1,0) and (0,1). Clearly, $\mathcal{R}_\theta$ has eigenvalue $e^{i\theta}$ on $T'M$ and $e^{-i\theta}$ on $T''M$. Any linear map ω defined on TM will be complex linearly extended to $TM \otimes \mathbb{C}$; its restrictions to $T'M$ and $T''M$ will be denoted by ω' and ω''.

Further, let S be any Riemannian manifold with Riemannian metric $g = \langle\ ,\ \rangle$. Any naturally defined covariant derivative will be denoted by D.

A smooth map $f : M \longrightarrow S$ is called *pluriharmonic* if the (1,1)-part of its Hesseian vanishes, i.e.

$$D''d'f = D'd''f = 0$$

where $df : TM \longrightarrow f^*TS$ denotes the differential of f. Here, D is the covariant derivative in the bundle $\mathrm{Hom}(TM, f^*TS)$ which is induced by the Levi-Civita derivatives of M and S. A pluriharmonic isometric immersion $f : M \longrightarrow S$ is called *(1,1)-geodesic* since the (1,1)-part of its second fundamental form $\alpha = Ddf$ vanishes. Equivalently, $f|C$ is a minimal surface for any complex one-dimensional submanifold ("curve") $C \subset M$. Therefore, such an immersion is also called *pluriminimal.*

From now on, let S be a Riemannian symmetric space of compact, euclidean or noncompact type, and suppose that M is simply connected (but not necessarily complete). Let $f : M \longrightarrow S$ be a smooth map. An *associated family* for f is a smooth family of maps $f_\theta : M \longrightarrow S$ such that

$$(AF) \qquad\qquad \Phi_\theta \circ df_\theta = df \circ \mathcal{R}_\theta$$

for some parallel bundle isomorphism $\Phi_\theta : f_\theta^*TS \longrightarrow f^*TS$ which preserves the full curvature tensor R_S of S.

Theorem 1. *A smooth map $f : M \longrightarrow S$ is pluriharmonic if and only if there is an associated family for f.*

Recall the integrability condition for a differential (cf. [ET1]): If a smooth map $f : M \longrightarrow S$ is given, the differential $F = df : TM \longrightarrow E := f^*TS$ satisfies the following structural equations for all sections X, Y of TM and A of E:

$$DF(X,Y) = DF(Y,X) \tag{1}$$

$$R_E(X,Y)A = R_S(FX,FY)A \tag{2}$$

where R_E denotes the curvature tensor on E with its induced connection, R_S the curvature tensor (Lie triple product) on S and $DF(X,Y) = (D_XF).Y$. Vice versa, if a vector bundle E over M with connection D and a parallel Lie triple product R_S on each fibre, isomorphic to that of S, and a bundle map $F : TM \longrightarrow E$ satisfying (1) and (2) are given, then there exists a smooth map $f : M \longrightarrow S$ and a parallel bundle isomorphism $\Phi : f^*TM \longrightarrow E$ preserving R_S such that $\Phi \circ df = F$. Thus, to prove the theorem we only have to show that the pluriharmonicity for f is equivalent to (1) and (2) for $F_\theta = df \circ \mathcal{R}_\theta$.

We need another piece of preparation. Recall the substitute for Gauss and Codazzi equations for a smooth map $f : M \longrightarrow S$: For any $X, Y, Z \in TM$,

$$R_E(X,Y)df.Z = df.(R(X,Y)Z) + D_X(Ddf)(Y,Z) - D_Y(Ddf)(X,Z) \tag{3}$$

where $E = f^*TS$. As a consequence, we get

Lemma. *(cf. [OU], p. 374) Let $f: M \to S$ be a pluriharmonic map. Then $R_S(df.X, df.Y)$ and $R_E(X,Y)$ vanish for all $X, Y \in T'M$.*

Proof. Since $Ddf^{(1,1)}$ vanishes by pluriharmonicity and since $T'M$ and $T''M$ are parallel, we get $(D_X(Ddf))^{(1,1)} = 0$ for any X. Thus the right hand side of (3) vanishes for $X, Y \in T'M$ and $Z \in T''M$; recall that $R(X,Y)Z = 0$ by the Kähler property of M. Thus $R_E(X,Y)df.Z = 0$ by (3) and hence $R_S(df.X, df.Y)df.Z = 0$ by (2). In particular,

$$\langle R_S(df.X, df.Y)df.\bar{X}, df.\bar{Y}\rangle = 0$$

for all $X, Y \in T'M$. Since the curvature operator $R_S : \Lambda^2 TS \to \Lambda^2 TS$ is semi-definite, we obtain $R_S(df.X, df.Y) = 0$. The result for R_E follows from (2). □

After these preparations, we can prove Theorem 1. Assume first that $f: M \to S$ is pluriharmonic. Put $E = f^*TS$ and $F = df : TM \to E$. Fix $\theta \in (0, 2\pi)$. Let $F_\theta = F \circ \mathcal{R}_\theta$. We have to show that F_θ satisfies (1) and (2). In fact, by parallelity of $\mathcal{R}_\theta$ we have for all X, Y

$$DF_\theta(X,Y) = (DF)(X, \mathcal{R}_\theta.Y). \tag{4}$$

Thus, if X and Y have both the same type ((1,0) or (0,1) vectors), then $DF_\theta(X,Y) = e^{\pm i\theta}DF(X,Y)$ which is symmetric in X and Y, while $DF(X, \mathcal{R}_\theta Y)$ vanishes by pluriharmonicity if X and Y have different type. This shows (1). Equation (2) holds since by the Lemma above, both sides vanish if X, Y have the same type, and if they have different type, the two factors $e^{i\theta}$ and $e^{-i\theta}$ from $\mathcal{R}_\theta$ on the right hand side cancel each other. This proves the existence of a map $f_\theta : M \to S$ with $df_\theta = F_\theta$ up to a parallel isomorphism between f_θ^*TS and f^*TS. Clearly f_θ is again pluriharmonic since $\mathcal{R}_\theta$ preserves type.

Vice versa, suppose only that $f : M \to S$ is a smooth map with differential $F = df : TM \to f^*TS$ and that $F_\theta := F \circ \mathcal{R}_\theta$ satisfies (1) for all θ. Then we have for all $X \in T'M$, $Y \in T''M$ that $DF_\theta(X,Y) = e^{-i\theta}F(X,Y)$ while $DF_\theta(Y,X) = e^{i\theta}F(Y,X)$ by (4). Therefore, (1) implies that $DF(X,Y) = 0$ for all $X \in T'M$, $Y \in T''M$; in other words, f is pluriharmonic. □

Remark. If $f : M \to S$ is a pluriharmonic isometric immersion then so is f_θ since $\mathcal{R}_\theta$ is an isometry on TM. Note that the parallel isomorphism $\Phi_\theta : f_\theta^*TS \to f^*TS$ with $\Phi_\theta \circ df_\theta = df \circ \mathcal{R}_\theta$ maps $df_\theta(TM)$, the tangent bundle of f_θ, onto $df(TM)$. Thus Φ_θ restricts to a parallel map between the normal bundles of f_θ and f. So the geometries of the tangent and the normal bundle of f and f_θ agree, but by (4), the second fundamental forms are different. In fact, since $\alpha(X,Y)=0$ if X and Y have different type, we can express (4) also in the following way:

$$\alpha_\theta(X,Y)=\alpha(\mathcal{R}_{\theta/2}X,\mathcal{R}_{\theta/2}Y) \tag{5}$$

for all X and Y.

2. Isotropic pluriharmonic maps

Now let us consider the special case of those pluriharmonic maps $f: M\to S$ where the associated family is trivial, i.e. $f_\theta=f$ for all θ. Let $E=f^*TS$. Adopting a notion from [EW] for surfaces in $\mathbb{C}P^n$, we will call these pluriharmonic maps *isotropic*. By Ch.1, Equation (AF), a smooth map $f: M\to S$ is isotropic pluriharmonic if and only if there is a family of parallel automorphism Φ_θ of (E,R_S) (called *associated rotations*) such that

$$(AR)\qquad \Phi_\theta\circ df=df\circ\mathcal{R}_\theta.$$

Examples.

1. If S is hermitian symmetric with complex structure j and $f: M\to S$ is holomorphic, i.e. $df\circ J=j\circ df$, then f is isotropic pluriharmonic where Φ_θ is the rotation $r_\theta=\cos(\theta)I+\sin(\theta)j$ on S.
2. Consider a Kähler manifold Z with complex structure j, a symmetric space S and a Riemannian submersion $\pi: Z\to S$ whose fibres are complex submanifolds. Let $\hat{f}: M\to Z$ be a horizontal holomorphic map, i.e. $\hat{f}$ is holomorphic with $df(TM)\subset f^*\mathcal{H}$ where $\mathcal{H}\subset TZ$ is the horizontal subbundle. Then $f=\pi\circ\hat{f}$ is isotropic pluriharmonic. In fact, since the rotation $r_\theta=\cos(\theta)I+\sin(\theta)j$ on Z is parallel and preserves the vertical and horizontal components, it leaves invariant the curvature tensor of Z and also the O'Neill tensor $A_XY=(D_XY)_{vert}$ of the Riemannian submersion: $A_Xr_\theta Y=r_\theta A_XY$ for any two horizontal vector fields X,Y. Hence, by O'Neill's formula (cf. [CE], p.67f, (3.25), (3.30)), r_θ preserves also the curvature tensor R_S of S. (Using $d\pi$, we identify π^*TS with $\mathcal{H}$.) Thus the pullback of $r_\theta|_{\pi^*TS}$ by $\hat{f}$ defines a parallel automorphism Φ_θ on $E=f^*TS$. Many examples are of this type (e.g. [EW], [ErW], [OU], [ET2], [K]).
3. In certain cases, the submersion $\pi: Z\to S$ need not be Riemannian, i.e. $d\pi|_\mathcal{H}$ need not be isometric, but the values of $\hat{f}: M\to Z$ lie in a parallel subbundle $\mathcal{H}_1$ of $\mathcal{H}$ such that $d\pi|_{\mathcal{H}_1}$ is isometric, see the Remark following Theorem 2 below.

Proposition. *The associated rotations Φ_θ of a full isotropic pluriharonic map $f: M\to S$ have the following properties:*

(a) *They form a one-parameter group, i.e.* $\Phi_{\theta+\theta'} = \Phi_\theta \circ \Phi_{\theta'}$, *with* $\Phi_{2\pi} = I$.
(b) *There is a* Φ_θ*-invariant parallel subbundle* E_1 *containing the values of* df *where* Φ_θ *has precisely the eigenvalues* $e^{\pm i\theta}$.
(c) $\Phi_\pi = -I$ *and hence* $j := \Phi_{\pi/2}$ *is a parallel complex structure on* E.

(A map $f : M \to S$ will be called *full* if the values of f do not lie in a totally geodesic proper subspace of S.)

Proof. Let E_1 be the smallest parallel subbundle of $E = f^*TS$ containing the values of df. From (AR) and the parallelity of Φ_θ we see that Φ_θ preserves E_1 with eigenvalues $e^{\pm i\theta}$. Moreover from the group law $\mathcal{R}_\theta \circ \mathcal{R}_{\theta'} = \mathcal{R}_{\theta+\theta'}$ we get the corresponding group law for $\Phi_\theta|_{E_1}$. Since all Φ_θ are automorphisms for the curvature tensor R_S, we obtain the same group law on the smallest R_S-stable subbundle E_0 containing E_1. (A subbundle $E_0 \subset E$ is called *R_S-stable* if $R_S(A,B)C \in E_0$ for any $A, B, C \in E_0$.) Since R_S is parallel, also E_0 is a parallel subbundle, and moreover, E_0 is R_S-stable and contains $df(TM)$. Since f is full, we conclude $E_0 = E$ (cf. [ET1], Thm. 2). Further, $\Phi_{2\pi} = I$ on E_1 and hence on $E_0 = E$.

Now we consider the case $\theta = \pi$. Since $\Phi_\pi^2 = \Phi_{2\pi} = I$, the only eigenvalues of Φ_π are ± 1. Let $E_- \subset E$ be the (-1)-eigenbundle. This is parallel and contains $df(TM)$, and it is also R_S-stable since for any $A, B, C \in E_-$ we have

$$\Phi_\pi(R_S(A,B)C) = R_S(\Phi_\pi A, \Phi_\pi B)\Phi_\pi C = -R_S(A,B)C.$$

As before we conclude $E_- = E$ which finishes the proof. □

Corollary 1. *If there exists a full pluriharmonic isotropic map* $f : M \to S$, *then the symmetric space* S *is inner (in particular,* S *is even dimensional).*

Proof. A Riemannian symmetric space $S = G/K$ is called *inner* if the geodesic symmetry τ at the base point o lies in the connected component of K which is the connected automorphism group of the Lie triple (T_oS, R_S). (In particular, τ has a square root in K which is a complex structure on T_oS.) Assuming that $f(x_0) = o$ for some $x_0 \in M$, we have a one-parameter group $\Phi_\theta(x_0)$ of such automorphisms with $\Phi_\pi(x_0) = -I$, hence S is inner. □

Corollary 2. *Any isotropic pluriharmonic map* $f : M \to \mathbb{R}^{2n} = \mathbb{C}^n$ *is holomorphic up to isometries of* $\mathbb{R}^{2n}$.

Proof. If $S = \mathbb{R}^{2n}$, then the parallel complex structure $j = \Phi_{\pi/2}$ has a parallel extension to all of $\mathbb{R}^{2n}$. □

Remark. An isotropic pluriharmonic map $f : M \to S$ is also *pluriconformal*, i.e. $df(T'M)$ is isotropic (the complexified metric g vanishes there), since by (AR), $df(T'M)$ is contained in the isotropic subbundle

$E' = \{A \in E \otimes \mathbb{C};\ j(A) = iA\}$ where $j = \Phi_{\pi/2}$. Hence, if f is an immersion, f^*g is a compatible Kähler metric on M (cf. [ET2], and f is pluriminimal with respect to this metric.

Now we shall give another geometric interpretation of the associated rotations. Consider again a full isotropic pluriharmonic map $f : M \to S$ where $S = G/K$ is an inner symmetric space of compact or noncompact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra of G. Let Φ_θ be the associated rotations on $E = f^*TS$. By the Proposition above, $\Phi_\theta = \exp(\theta \cdot \xi)$ for some parallel derivation ξ of (E, R_S). Moreover, since $\Phi_\pi = -I$, all eigenvalues of ξ are of the form ik where $(i = \sqrt{-1}$ and) k is an odd integer, and on the subbundle E_1, the eigenvalues are $\pm i$. For any $x \in M$, we consider $\Phi_\theta(x)$ as a one-parameter subgroup of $G_{f(x)}$ (the isotropy group of S at the point $f(x)$) and $\xi(x)$ as an element in its Lie algebra $\mathfrak{g}_{f(x)} \subset \mathfrak{g}$. In particular, $\Phi_\pi(x) = \exp \pi\xi(x)$ is the geodesic symmetry of S at $f(x)$. Since ξ is parallel, all $\xi(x) \in \mathfrak{g}$ are conjugate to $\xi_0 := \xi(x_0) \in \mathfrak{k}$ by a parallel translation along some curve from $o = f(x_0)$ to $s = f(x)$ in S, hence by some $g \in G$ with $g(o) = s$ (recall that G is generated by transvections). Thus ξ may be considered as a smooth map $\xi : M \to Ad(G)\xi_0$.

This adjoint orbit has been extensively studied (cf. [BR]). Its isotropy group is the centralizer H of ξ_0,

$$H = \{h \in G;\ Ad(h)\xi_0 = \xi_0\}$$

We claim that $H \subset K$. In fact, recall that the Cartan involution τ of G corresponding to S is the conjugation with $\exp(\pi \cdot \xi_0)$. If $h \in H$, then $Ad(h)$ fixes ξ_0 and hence h commutes with $\exp(t\xi_0)$ for all $t \in \mathbb{R}$, and in particular, h lies in the fixed point set of τ which is K.

Therefore we have a fibration

$$\pi : Ad(G)\xi_0 \to S,\quad \pi(Ad(g)\xi_0) = g(o)$$

which is (abstractly) just the canonical map $\pi : G/H \to G/K$. We may consider $Ad(G)\xi_0$ as a subbundle of $\mathrm{End}(TS)$ which is invariant under parallel displacement. So the Levi-Civita connection on S (given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ on the principal bundle $G \to S$) induces a horizontal distribution on $Ad(G)\xi_0$: By definition, horizontal curves in $Ad(G)\xi_0$ are given by parallel displacements of the endomorphism $ad(\xi_0)$ on $\mathfrak{p} = T_oS$. In other terms, the bundle $Ad(G)\xi_0 \to S$ is associated to the principal K-bundle $G \to S$ with associated fibre $Ad(K)\xi_0$, and it inherits a horizontal structure $\mathcal{H}$ from the Levi-Civita connection on the principal bundle $G \to S$. If we identify $Ad(G)\xi_0$ with $Z := G/H$, this is the horizontal structure given by $\mathfrak{p}$, more precisely, if $\mathfrak{k} = \mathfrak{h} + \mathfrak{q}$ is a reductive decomposition, the horizontal subbundle is

$$\mathcal{H} = G \times_{Ad(H)} \mathfrak{p} \subset G \times_{Ad(H)} (\mathfrak{p} + \mathfrak{q}) = TZ.$$

In [BR], $Z = G/H$ is called *flag manifold* over S if S is of compact type and *flag domain* if S is of noncompact type, and the embedding

$$\hat{\xi} : G/H \overset{\cong}{\rightarrow} Ad(G)\xi_0 \subset \mathfrak{g}, \;\; \hat{\xi}(gH) = Ad(g)\xi_0$$

is called *canonical section*. It is well known that Z is a complex manifold (in fact, a Kähler manifold, but if S is of noncompact type, this Kähler metric will be indefinite, cf. [BR], p.48), and $\mathcal{H} \subset TZ$ is a complex subbundle; see Remark below for the definition of the complex structure j.

Returning to our isotropic pluriharmonic map $f : M \longrightarrow S$, we consider our map $\xi : M \longrightarrow Ad(G)\xi_0$ as a smooth mapping $\hat{f} : M \longrightarrow Z$ with $\pi \circ \hat{f} = f$ (a lift of f) by putting

$$\hat{\xi}(\hat{f}(x)) = \xi(x).$$

Since ξ is a parallel section of $\mathrm{End}(f^*TS)$, this map $\hat{f}$ is horizontal, i.e. $d\hat{f}$ takes values in $\mathcal{H}$. From (AR) we have $df \circ J = Ad(\exp \frac{\pi}{2}\xi) \circ df$. On the other hand, $d\hat{f}$ takes values in the so called *superhorizontal* (cf. [BR]) subbundle $\mathcal{H}_1 \subset \mathcal{H}$ where the eigenvalues of $ad(\hat{\xi})$ are only $\pm i$ (this is equivalent to the fact that df takes values in E_1). But on $\mathcal{H}_1$ we have in fact $Ad(\exp \frac{\pi}{2}\xi) = j$ and therefore $\hat{f}$ is holomorphic.

Vice versa, if a superhorizontal holomorphic map $\hat{f} : M \longrightarrow Z$ is given, then $f = \pi \circ \hat{f} : M \longrightarrow S$ is a full isotropic pluriharmonic map. In fact, since $\hat{f}$ is horizontal, $\xi := \hat{\xi} \circ \hat{f}$ defines a parallel derivation of (E, R_S) where $E = \hat{f}^*\mathcal{H} = f^*TS$. From holomorphicity and the definition of the complex structure on $\mathcal{H}_1$ we get on $(\mathcal{H}_1)_{\hat{f}(x)}$ for any $x \in M$:

$$d\hat{f}_x \circ J = Ad(\exp \frac{\pi}{2}\xi(x)) \circ d\hat{f}_x = ad(\xi(x)) \circ d\hat{f}_x$$

Thus putting $\Phi(\theta) = \exp(\theta \cdot \xi) = \cos(\theta)I + \sin(\theta)ad(\xi)$, we obtain (AR). So we have proved:

Theorem 2. *Let S be an inner symmetric space of compact (resp. noncompact) type and $f : M \longrightarrow S$ a full smooth map. Then f is isotropic pluriharmonic if and only if there is a flag manifold (resp. flag domain) Z over S with canonical projection $\pi : Z \longrightarrow S$ and a holomorphic superhorizontal map $\hat{f} : M \longrightarrow Z$ such that $f = \pi \circ \hat{f}$.*

Remark. If S is of compact type, the complex structure j on $Z = G/H$ is defined as follows. Fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ with $\xi_0 \in \mathfrak{t}$ (thus $\mathfrak{t} \subset \mathfrak{h}$). For each positive root α of $(\mathfrak{g}, \mathfrak{t})$, the (real) root space $\mathfrak{g}_\alpha$ is spanned by two nonzero vectors X_α, Y_α such that for all $\xi \in \mathfrak{t}$

$$[\xi, X_\alpha] = \alpha(\xi) Y_\alpha, \quad [\xi, Y_\alpha] = -\alpha(\xi) X_\alpha.$$

Corresponding to Z we have the reductive decomposition $\mathfrak{g} = \mathfrak{z} + \mathfrak{h}$ where $\mathfrak{z}$ is the sum of those root spaces $\mathfrak{g}_\alpha$ with $\alpha(\xi_0) \neq 0$. Now j is defined on $\mathfrak{z}$ by $j(X_\alpha) = Y_\alpha$, $j(Y_\alpha) = -X_\alpha$. The Kähler metric on $\mathfrak{z}$ is defined by $\langle X, Y \rangle = -B(\xi_0, [X, jY])$ where B denotes the Killing form of $\mathfrak{g}$. Since $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{k}$, the roots of $\mathfrak{k}$ form a subset of the root set of $\mathfrak{g}$ (cf. also [H], p.424), and we have $\mathfrak{z} = \mathfrak{p} + \mathfrak{q}$ where $\mathfrak{q}$ contains the root spaces in $\mathfrak{k}$ and $\mathfrak{p}$ contains those which are not in $\mathfrak{k}$. Let $\mathfrak{p}_1 \subset \mathfrak{p}$ be the sum of the root spaces $\mathfrak{g}_\alpha$ with $\alpha(\xi_0) = 1$. On $\mathfrak{p}_1$ we have $j = ad(\xi_0)$ and $\langle\ ,\ \rangle = -B$. The subspaces $\mathfrak{p}$ and $\mathfrak{p}_1$ extend to the horizontal and the superhorizontal bundles $\mathcal{H}$ and $\mathcal{H}_1$ on Z. Thus the submersion $\pi : Z \to S$ is "partial Riemannian", namely $d\pi|_{\mathcal{H}_1}$ is isometric. Example 2 is precisely the case where $\mathfrak{p} = \mathfrak{p}_1$.

3. (1,1)-parallel immersions

Isometric deformations are known not only for minimal surfaces, but also for constant mean curvature surfaces in 3-space. The generalizations of these surfaces to higher-dimensional Kähler manifolds are the *(1,1)-parallel* immersions.

Throughout this chapter, let $f : M \to S$ be a full isometric immersion of a Kähler manifold M into a symmetric space S. Let NM denote the normal bundle of f and $\alpha = Ddf : TM \otimes TM \to NM$ the second fundamental form. Using df, we consider TM as a subbundle of $E = f^*TS$; hence $E = TM \oplus NM$, and df becomes simply the inclusion $TM \subset E$. We shall consider only immersions $f : M \to S$ which are *adapted* to the structure of S (which is no condition if S is a space of constant sectional curvature): We assume that $T_pM \subset T_{f(p)}M$ is R_S-invariant (a Lie subtriple) for all $p \in M$, and the rotations $\mathcal{R}_\theta$ are automorphisms of $R_S|_{T_pM}$. Consequently, $R_S(X, Y)$ preserves the splitting or E into tangent and normal bundle, for any two tangent vector fields X, Y of M.

We complexify α to a $\mathbb{C}$-linear map $\alpha : T^cM \otimes T^cM \to N^cM$ (complexified bundles) and consider its decomposition

$$\alpha = \alpha^{(2,0)} + \alpha^{(1,1)} + \alpha^{(0,2)}$$

where $\alpha^{(1,1)}$ (resp. $\alpha^{(2,0)}$, $\alpha^{(0,2)}$) is the restriction of α to $T'M \otimes T''M + T''M \otimes T'M$ (resp. $T'M \otimes T'M$, $T''M \otimes T''M$). The immersion f is called *(1,1)-parallel* if $\alpha^{(1,1)}$ is parallel with respect to the normal connection $D^\perp$

on N^cM. If M is a surface ($m = 1$), then $\alpha^{(1,1)} = \langle .,. \rangle \cdot \eta$ where η is the mean curvature vector of the immersion. Hence a surface immersion is $(1,1)$-parallel iff it has parallel mean curvature vector.

The possible deformations of such an immersion must be more general than the asssociated family (which by Theorem 1 exists only if $\alpha^{(1,1)} = 0$); in fact, one changes Ddf instead of df. Namely, for any $\theta \in [0, \pi]$ let $\alpha_\theta : TM \otimes TM \to NM$,

$$\alpha_\theta(X,Y) = \alpha(\mathcal{R}_\theta X, \mathcal{R}_\theta Y). \tag{6}$$

A *weak associated family* for f is a smooth family of adapted isometric immersions $f_\theta : M \to S$, such that f_θ and f have the "same" normal bundle (up to a parallel isometric isomorphism of the normal bundles which interchanges with $R_S(X,Y)$ for all tangent vectors X, Y of M) and the same second fundamental form $Ddf_\theta = \alpha_\theta$. By equation (5) of Ch.1, an associated family is also a weak associated family (with f_θ replaced with $f_{\theta/2}$). Under some additional assumption, $(1,1)$-parallel immersions are characterized by the existence of a weak associated family:

Theorem 3. *Let $f : M \to S$ be an adapted isometric immersion such that*

$$R^\perp(X,Y)\xi = R_S(X,Y)\xi \tag{7}$$

for any $X, Y \in T'M$ and $\xi \in NM$ where $R^\perp$ is the curvature tensor of $(NM, D^\perp)$. Then f has a weak associated family if and only if f is (1,1)-parallel.

Proof. (cf. [FT] for the case $S = \mathbb{R}^n$.) Let D^T and $D^\perp$ denote the connections in the tangent and normal bundles of f. Let $\alpha = Ddf$ be the second fundamental form and $A_\xi(X) = (D_X\xi)^T$ the Weingarten map (for $X \in TM$, $\xi \in NM$). We define a new connection D^θ on the bundle $E = TM \oplus NM$ as follows: for $X, Y \in TM$, $\xi \in NM$ put

$$D^\theta_X Y = D^T_X Y + \alpha_\theta(X,Y)$$
$$D^\theta_X \xi = D^\perp_X \xi + A^\theta_\xi X$$

where

$$\alpha_\theta(X,Y) = \alpha(\mathcal{R}_\theta X, \mathcal{R}_\theta Y), \quad A^\theta_\xi = \mathcal{R}_\theta^{-1} A_\xi \mathcal{R}_\theta.$$

We have to show the structure equations (1) and (2), Ch.1, for $E = TM \oplus NM$ with the connection D^θ and $F : TM \to E$ the inclusion. (1) is trivial since $(D^\theta_X F)Y = \alpha_\theta(X,Y)$ is symmetric by definition. (2) is equivalent to Gauss, Codazzi and Ricci equations:

$$\begin{aligned}
&\langle R_S(X,Y)Z,W\rangle - \langle R(X,Y)Z,W\rangle \\
&\qquad = -\ \langle \alpha(\mathcal{R}_\theta Y,\mathcal{R}_\theta Z),\alpha(\mathcal{R}_\theta X,\mathcal{R}_\theta W)\rangle \\
&\qquad\quad + \langle \alpha(\mathcal{R}_\theta X,\mathcal{R}_\theta Z),\alpha(\mathcal{R}_\theta Y,\mathcal{R}_\theta W)\rangle \qquad \text{(2a)}\\
&\quad 0 = (R_S(X,Y)Z)^\perp \\
&\qquad =(D^\perp_X\alpha)(\mathcal{R}_\theta Y,\mathcal{R}_\theta Z) - (D^\perp_Y\alpha)(\mathcal{R}_\theta X,\mathcal{R}_\theta Z) \qquad \text{(2b)}\\
&(R_S(X,Y)\xi)^\perp - R^\perp(X,Y)\xi \\
&\qquad =\alpha(\mathcal{R}_\theta X, A_\xi\mathcal{R}_\theta Y) - \alpha(\mathcal{R}_\theta Y, A_\xi\mathcal{R}_\theta X) \qquad \text{(2c)}
\end{aligned}$$

The verification is straight-forward: We compare the desired equations (2a), (2b), (2c) for arbitrary θ with the given ones for $\theta = 0$. Since f is adapted we know that $R_S(X,Y)Z = 0$ if X, Y, Z have the same type because $e^{\pm 3i\theta}$ is not an eigenvalue of $\mathcal{R}_\theta$. Thus, if $X, Y, Z, W \in T'M \cup T''M$, the right hand side of (2a) picks up a common factor $e^{ik\theta}$ while the left hand side vanishes unless two of the four vector are of type (1,0) and the other two (0,1) in which case the common factor is 1. This shows (2a). A similar argument holds for (2c): If X, Y have the same type, the left hand side vanishes by assumption (7), and the right hand side picks up a common factor, and if X and Y have different type, the factors at the right hand side cancel each other. In (2b), the left hand side is always zero since f is adapted. If all three types are equal, the right hand side picks up a common factor; otherwise, if f is (1,1)-parallel (this is the only point where we use this assumption), one of the terms vanishes while the other term picks up a factor. Thus the structure equations hold, and we get adapted immersions f_θ with the same tangent and normal connections and second fundamental form α_θ as desired.

Vice versa, if such immersions f_θ are given, we use (2b) in the case where Y, Z have different type and X, Z equal type. Then only the second term at the right hand side picks up a factor $e^{\pm 2i\theta}$ which shows that both terms vanish, hence $D^\perp_X \alpha^{(1,1)} = 0$ for all X. □

Remark. If f is a $(1,1)$-geodesic (or pluriminimal) immersion, the assumption (7) in Theorem 3 is automatically satisfied. To see this observe that the terms like $\alpha(X, A_\xi Y)$ arising in the Ricci equation (cf. (2c) for $\theta = 0$) must vanish for $X, Y \in T'M$ since the Weingarten maps A_ξ interchange $T'M$ and $T''M$: For any $Y \in T'M$ and $\bar{Z} \in T''M$ we have $\langle A_\xi Y, \bar{Z}\rangle = 0$ and hence $A_\xi Y \in (T''M)^\perp = T''M$ (remember that $T''M$ is maximal isotropic).

Now we consider the case of an adapted (1,1)-parallel immersion $f : M \to S$ which satisfies assumption (7) of Theorem 3 and whose weak associated family is *constant*: $f_\theta = f$ for all θ; such an immersion will be called *isotropic (1,1)-parallel*. By the previous theorem, this holds if and only if there is a parallel endomorphism family $\Psi_\theta : NM \to NM$ which commutes with $R_S(X,Y)|_{NM}$ for any $X, Y \in TM$ such that

$$\alpha_\theta = \Psi_\theta \circ \alpha, \tag{8}$$

where α_θ is defined by (6). Equivalently, the extension of Ψ_θ to $E = TM \oplus NM$ by the identity on TM is a parallel R_S-automorphism from (E, D) to (E, D^θ).

Theorem 4. *An adapted immersion $f : M \to S$ with (7) is isotropic $(1,1)$-parallel if and only if its complexified normal bundle N^cM splits orthogonally as*

$$N^cM = N^2M \oplus N^0M \oplus N^{-2}M$$

where the factors N^kM are subbundles of N^cM which are parallel with respect to $D^\perp$ and invariant under $R_S(X,Y)$ for all $X, Y \in TM$ such that the values of $\alpha^{(2,0)}$ ($\alpha^{(1,1)}$, $\alpha^{(0,2)}$) are contained in N^2M (N^0M, $N^{-2}M$).

Proof. Any eigenbundle of Ψ_θ in NM is parallel and stable under $R_S(X,Y)$ for all $X, Y \in TM$. In particular, let N^2M, N^0M and $N^{-2}M$ be the eigenbundles corresponding to the eigenvalues $e^{2i\theta}$, 1 and $e^{-2i\theta}$. Then by (8), the components $\alpha^{(2,0)}$, $\alpha^{(1,1)}$ and $\alpha^{(0,2)}$ of α take values in these three bundles. Since f is full, they must form a complete decomposition of N^cM (cf. [ET1]).

Vice versa, if such a splitting of N^cM is given, we define a linear bundle map Ψ_θ on N^cM with $\Psi_\theta = \lambda_k \cdot id$ on N^kM where $\lambda_k = e^{ik\theta}$ for $k \in \{2, 0, -2\}$. Then Ψ_θ is parallel and commutes with $R_S(X,Y)$ for all $X, Y \in TM$. Putting $\alpha_\theta = \mathcal{R}_\theta^*\alpha$ as in (6) we obtain that $\alpha_\theta = \Psi_\theta \circ \alpha$. Thus $f_\theta := f$ is a weak associated family for f (with isomorphism Ψ_θ between the normal bundles of f and $f_\theta = f$). Hence f is $(1,1)$-parallel by Theorem 3, and from $f_\theta = f$ we see that f is isotropic $(1,1)$-parallel. □

Examples. 1. Any isotropic minimal surface in a sphere S^{n-1} is isotropic $(1,1)$-parallel in $\mathbb{R}^n$. Higher dimensional examples of this type (other than surfaces) do not exist since by the Lemma in Ch.1, the differential of a pluriharmonic map into the sphere (having positive curvature operator) must have rank ≤ 2.

2. Let M be complete Kähler with no euclidean factor in its universal cover and $f : M \to \mathbb{R}^n$ an isometric immersion with $D^\perp\alpha = 0$. Then $f(M) \subset \mathbb{R}^n$ is extrinsic symmetric and $f : M \to f(M)$ is a covering map (cf. [F]), but $f(M)$ need not to be Kähler (e.g. $f(M)$ can be the real projective plane). Clearly, f is (1,1)-parallel. Moreover, (7) holds: For all $X, Y \in T'M$ and $V, W \in TM$ we have by the parallelity of α and the Kähler property of M:

$$R^\perp(X,Y)(\alpha(V,W)) = \alpha(R(X,Y)V, W) + \alpha(V, R(X,Y)W) = 0. \tag{$*$}$$

This shows (7) since for a full immersion with parallel α, the normal space is spanned by vectors of the type $\alpha(V, W)$. Now we want to show that f is isotropic. Fix $p \in M$. Since $R(X, Y) = 0$ if X and Y have the same type, we have $R(\mathcal{R}_\theta X, \mathcal{R}_\theta Y) = R(X, Y)$ for all $X, Y \in T_pM$, and therefore, $\mathcal{R}_\theta$ is an automorphism of the Lie triple (T_pM, R). Hence, $\mathcal{R}_\theta$ is the differential of an isometry ϕ_θ of M fixing p. Since M is a symmetric space without local euclidean factor the connected component of its isometry group is generated by compositions of geodesic symmetries, and these have an extension to the ambient space $\mathbb{R}^n$. Hence ϕ_θ extends to an isometry Φ_θ of $\mathbb{R}^n$ fixing $f(p)$, more precisely, $f \circ \phi_\theta = \Phi_\theta \circ f$. Let $\Psi_{\theta,p} = d(\Phi_\theta)_{f(p)}|_{N_pM}$. Now letting $p \in M$ be variable, we get a map $\Psi_\theta : p \mapsto \Psi_{\theta,p}$ which is a section of $\mathrm{End}(NM)$ with

$$\Psi_\theta(\alpha(X, Y)) = \alpha(\mathcal{R}_\theta X, \mathcal{R}_\theta Y)$$

for all $X, Y \in TM$. Since $\mathcal{R}_\theta$ and α are parallel and the values of α span NM, Ψ_θ must be parallel, too.

The classification of extrinsic symmetric spaces (cf. [KN]) shows that the only examples are the standard embedded hermitean symmetric spaces (see below) and the standard embedded Grassmannians $G_2(\mathbb{R}^p)$ of 2-planes in $\mathbb{R}^p$. These are not Kähler manifolds but doubly covered by the space of *oriented* 2-planes in $\mathbb{R}^p$ which is Kähler hermitean symmetric (it is isometric to the hyperquadric in complex projective $(p-1)$-space).

3. A subcase of the previous example leads also to isotropic (1,1)-parallel immersions in other symmetric spaces: Let $f : M \to \mathbb{R}^n$ be the *standard embedding* of an hermitean symmetric space M. These embeddings are characterized by the assumption

$$\alpha(JX, JY) = \alpha(X, Y) \qquad (**)$$

for any $X, Y \in TM$ (cf. [F]), in other words $\alpha^{(2,0)} = 0$; in particular, they are extrinsic symmetric (cf. Remark 3 below). Recall from [F] or [EH] that an extrinsic symmetric space is a certain orbit of the isotropy representation of a symmetric space. Hence in the above example 2 we may assume that the receiving space $\mathbb{R}^n$ is the tangent space $\mathfrak{p} = T_oS$ of a symmetric space $S = G/K$ (where $o = eK$) and that $f(M)$ is a K-orbit in $\mathfrak{p}$. Let $e_t : T_oS \to S$, $x \mapsto \exp_o(tx)$. We consider the immersions (in fact embeddings)

$$f_t = e_t \circ f : M \to S$$

for every sufficiently small $t > 0$. Since e_t is K-equivariant and K contains the geodesic symmetries of $f(M)$, we see that $f_t(M) \subset S$ is again *extrinsic symmetric*, i.e. the geodesic symmetry at each point $p \in M$ extends to an isometry τ_p of the ambient space S fixing $f_t(p)$ and the normal space at $f_t(p)$. As in the euclidean case this implies that the second fundamental form α of f_t

is parallel: If we apply $d\tau_p$ to both sides of the equation $\xi := (D_x\alpha)(y, z)$ where $x, y, z \in T_pM$, $\xi \in N_pM$, then the right hand side changes sign (three $-$-signs) while the left hand side stays the same; thus $\xi = 0$. In particular, f_t is $(1, 1)$-parallel.

Further, recall that $df_p(T_pM) \subset \mathfrak{p} = T_oS$ is the (-1)-eigenspace of $d(\tau_p)_o$ which is an automorphism of R_S. Thus $df_p(T_pM)$ is a Lie subtriple of T_oS. The same holds for $d(f_t)_p(T_pM) \subset T_{f_t(p)}S$, being the (-1)-eigenspace of $d(\tau_p)_{f_t(p)}$. Since $\mathcal{R}_\theta(p)$ extends to an intrinsic isometry of M and hence to an extrinsic isometry of $f_t(M)$, it is an automorphism of $R_S|_{df_t(T_pM)}$. This shows that f_t is adapted to S.

It remains to show (7). As in the previous example (cf. $(*)$) $D\alpha = 0$ implies that $R^\perp(X, Y)\xi = 0$ for all $X, Y \in T'M$ and $\xi \in NM$. Hence we have to show $R_S(X, Y)\xi = 0$. For this we need the extra assumption $(**)$: It implies that the isometry $j \in K$ whose differential at p is the complex structure J on T_pM extends as identity on N_pM. Now

$$j(R_S(X, Y)\xi) = R_S(jX, jY)j\xi = -R_S(X, Y)\xi$$

for any $X, Y \in T'_pM$, but -1 is not an eigenvalue of j. Hence $R_S(X, Y)\xi = 0$ which completes the proof of (7).

As above, the weak associated family is trivial since $\mathcal{R}_\theta$ extends to an isometry. So f_t is isotropic (1,1)-parallel.

Remarks. 1.) We do not know other isotropic (1,1)-parallel immersions.

2.) The exclusion of local euclidean factors in example 2 is necessary: The cylinder $\mathbb{R} \times S^1 \subset \mathbb{R}^3$ and the torus $S^1 \times S^1 \subset \mathbb{R}^4$ are extrinsic symmetric and Kähler but not isotropic.

3.) One might ask also for the isometric immersions $f : M \to S$ where the *(2,0)-part* of α is parallel ($(2, 0)$-*parallel immersions*). If $S = \mathbb{R}^n$, the Codazzi equations imply immediately that α is parallel since we can always assume that two of the three argument of $D^\perp\alpha$ have the same type. Ferus [F] has already noticed that these spaces are precisely the standard embedded hermitean symmetric spaces.

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Note added in proof. F. Burstall has pointed out to us that our arguments yield in fact the following improvement of Theorem 3:

Theorem 3'. *Let $f : M \to S$ be any isometric immersion of a Kähler manifold M with second fundamental form α such that $T_pM \subset T_{\{f(p)\}}S$ is R_S-invariant for any $p \in M$. Then f has a weak associated family if and only if the following three conditions are satisfied:*

(a) $\{\mathcal{R}\}_\theta$ *is an automorphism of* $R_S|_{\{}T_pM\}$ *for any* $p \in M$.
(b) $D^\perp \alpha^{\{}(1,1)\} = 0$,
(c) $R^\perp(X,Y)\xi = R_S(X,Y)\xi$ *for all* $X, Y \in T'M$ *and* $\xi \in NM$.

In fact, the "if" statement has already been proved in the paper. For the "only if" statement we have to conclude (a)–(c) *from equations* $(2a)$–$(2c)$ *in the proof of Theorem 3. Clearly,* (c) *follows from* $(2c)$ *for* $X, Y \in T'M$ *since the right hand side picks up a common factor* $e^{\{}2i\theta\}$ *and thus must vanish, and the tangent part of* $R_S(X,Y)\xi$ *vanishes anyway since* TM *is* R_S*-invariant. Further,* (b) *follows from* $(2b)$ *by choosing* $X, Z \in T'M$ *and* $Y \in T''M$*; moreover we get* $D^\perp_{\{}T''M\}\alpha^{\{}(2,0)\} = 0$. *Finally,* (a) *follows from* $(2a)$ *since the right hand side and the second term on the left hand side are unchanged if* X, Y, Z, W *are replaced by their images under* $\{\mathcal{R}\}_\theta$.

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