

Embedding constraint relationships into C-semirings

Alexander Knapp, Alexander Schiendorfer

Angaben zur Veröffentlichung / Publication details:

Knapp, Alexander, and Alexander Schiendorfer. 2014. "Embedding constraint relationships into C-semirings." Augsburg: Fakultät für Angewandte Informatik, Universität Augsburg.

Nutzungsbedingungen / Terms of use:

licgercopyright

Dieses Dokument wird unter folgenden Bedingungen zur Verfügung gestellt: / This document is made available under these conditions:

Deutsches Urheberrecht

Weitere Informationen finden Sie unter: / For more information see:

<https://www.uni-augsburg.de/de/organisation/bibliothek/publizieren-zitieren-archivieren/publiz/>



UNIVERSITÄT AUGSBURG

**Embedding Constraint Relationships
into C-Semirings**

Alexander Knapp, Alexander Schiendorfer

Report 2014-03

April 2014

INSTITUT FÜR INFORMATIK
D-86135 AUGSBURG

INTRODUCTION

These notes provide technical details that are required to embed constraint relationships into the c-semiring framework presented in terms of category theory. It contains all steps required to map a dag to a partial order (Section 1), construct the *free* meet monoid from this partial order (Section 3) as well as the *free* c-semiring (Section 4). A constraint solving algorithm based on branch-and-bound search is presented in §34 for c-semirings and in §36 for meet monoids. A concrete instantiation for constraint relationships along with an example soft constraint problem concludes the report in Section 7.

1. PARTIAL ORDERS AND DIRECTED ACYCLIC GRAPHS

1. A *partial order* $(X, \leq)$ is given by a set X and a binary relation $\leq \subseteq X \times X$ such that $\leq$ is reflexive, transitive, and anti-symmetric on X . For $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$, and $x \geq y$ resp. $x > y$ if $y \leq x$ resp. $y < x$.

A *partial order homomorphism* $\varphi : P \rightarrow Q$ from a partial order $P = (|P|, \leq_P)$ to a partial order $Q = (|Q|, \leq_Q)$ is given by a map $\varphi : |P| \rightarrow |Q|$ such that $\varphi(p) \leq_Q \varphi(p')$ if $p \leq_P p'$ for all $p, p' \in |P|$.

The category PO of partial orders has the partial orders as objects and the partial order homomorphisms as morphisms.

2. A *directed acyclic graph*, or *dag*, $(X, \rightarrow)$ is given by a set X and a binary relation $\rightarrow \subseteq X \times X$ such that $\rightarrow^+$ is irreflexive. If $x \rightarrow y$, then x is a *predecessor* of y , and y is a *successor* of x .

A *dag homomorphism* $\varphi : G \rightarrow H$ from a dag $G = (|G|, \rightarrow_G)$ to a dag $H = (|H|, \rightarrow_H)$ is given by a map $\varphi : |G| \rightarrow |H|$ such that $\varphi(g) \rightarrow_H \varphi(g')$ if $g \rightarrow_G g'$ for all $g, g' \in |G|$.

The category DAG of dags has the dags as objects and the dag homomorphisms as morphisms.

3. Define the functor $PO\langle - \rangle : \text{DAG} \rightarrow \text{PO}$ by

$$PO\langle G \rangle = (|G|, \rightarrow_G^*),$$

$$PO\langle \varphi : G \rightarrow H \rangle = \varphi.$$

Define the functor $DAG : \text{PO} \rightarrow \text{DAG}$ by

$$DAG(P) = (|P|, <_P),$$

$$DAG(\varphi : P \rightarrow Q) = \varphi.$$

For each $G \in |\text{DAG}|$, define $\eta_G^{\text{PO}} : G \rightarrow DAG(PO\langle G \rangle)$ by $\eta_G^{\text{PO}}(g) = g$. Then $\eta^{\text{PO}} = (\eta_G^{\text{PO}})_{G \in |\text{DAG}|}$ is a natural transformation from 1_{DAG} to $DAG \circ PO\langle - \rangle$.

Let $G \in |\text{DAG}|$, $P \in |\text{PO}|$, and $\varphi : G \rightarrow DAG(P)$. Define $\varphi^{\sharp \text{PO}} : PO\langle G \rangle \rightarrow P$ by

$$\varphi^{\sharp \text{PO}}(g) = \varphi(g).$$

Then $DAG(\varphi^{\sharp \text{PO}})(\eta_G^{\text{PO}}(g)) = \varphi(g)$ and $\varphi^{\sharp \text{PO}}$ is unique with this property.

2. UPPER SEMI-LATTICES

4. A (*bounded*) *upper semi-lattice* $(X, \sqcup, \perp)$ is given by a set X , a binary operation $\sqcup : X \times X \rightarrow X$, and a constant $\perp \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:

$$(1) (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$$

$$(2) x \sqcup y = y \sqcup x$$

$$(3) x \sqcup x = x$$

$$(4) \ x \sqcup \perp = x$$

In words, $\sqcup$ is associative, commutative, and idempotent, and has $\perp$ as neutral element.

A (bounded) upper semi-lattice homomorphism $\varphi : U \rightarrow V$ from an upper semi-lattice $U = (|U|, \sqcup_U, \perp_U)$ to an upper semi-lattice $V = (|V|, \sqcup_V, \perp_V)$ is given by a map $\varphi : |U| \rightarrow |V|$ such that for all $u_1, u_2 \in |U|$:

- (1) $\varphi(u_1 \sqcup_U u_2) = \varphi(u_1) \sqcup_V \varphi(u_2)$
- (2) $\varphi(\perp_U) = \perp_V$

The category $\mathbf{uSL}$ of upper semi-lattices has the upper semi-lattices as objects and the upper semi-lattice homomorphisms as morphisms.

5. Let P be a partial order. Let $\mathcal{I}_{\text{fin}}(P)$ denote the set of finite subsets of $|P|$ which only contain pairwise incomparable elements w.r.t $\leq_P$. For a subset $S \subseteq |P|$, let $\text{Max}^{\leq_P}(S)$ denote the set of maximal elements of S w.r.t. $\leq_P$.

Define the binary operation $\cup_P : \mathcal{I}_{\text{fin}}(P) \times \mathcal{I}_{\text{fin}}(P) \rightarrow \mathcal{I}_{\text{fin}}(P)$ by

$$I \cup_P J = \text{Max}^{\leq_P}(I \cup J) .$$

LEMMA. $(\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset)$ is an upper semi-lattice.

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(P)$. For the associativity of $\cup_P$ we have

$$\begin{aligned} I \cup_P (J \cup_P K) &= \text{Max}^{\leq_P}(I \cup \text{Max}^{\leq_P}(J \cup K)) = \text{Max}^{\leq_P}(I \cup J \cup K) = \\ &= \text{Max}^{\leq_P}(\text{Max}^{\leq_P}(I \cup J) \cup K) = (I \cup_P J) \cup_P K , \end{aligned}$$

since $\text{Max}^{\leq_P}(I \cup \text{Max}^{\leq_P} X) = \text{Max}^{\leq_P}(I \cup X)$ for all $X \in \mathcal{P}_{\text{fin}} |P|$. $\cup_P$ inherits commutativity from $\cup$. For the idempotency of $\cup_P$ we have

$$I \cup_P I = \text{Max}^{\leq_P}(I \cup I) = \text{Max}^{\leq_P} I = I ,$$

since $I \in \mathcal{I}_{\text{fin}}(P)$. Finally, we have $I \cup_P \emptyset = I$. □

Define the functor $uSL\langle - \rangle : \mathbf{PO} \rightarrow \mathbf{uSL}$ by

$$\begin{aligned} uSL\langle P \rangle &= (\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset) , \\ uSL\langle \varphi : P \rightarrow Q \rangle &= \lambda\{p_1, \dots, p_n\} \in \mathcal{I}_{\text{fin}}(P) . \text{Max}^{\leq_Q}\{\varphi(p_1), \dots, \varphi(p_n)\} . \end{aligned}$$

6. Each upper semi-lattice U induces a partial ordering $\leq_U \subseteq |U| \times |U|$ on $|U|$ given by

$$u_1 \leq_U u_2 \iff u_1 \sqcup_U u_2 = u_2 .$$

Indeed, $\leq_U$ is reflexive on $|U|$ by the idempotency of $\sqcup_U$, $\leq_U$ is transitive by the associativity of $\sqcup_U$, and $\leq_U$ is anti-symmetric by the commutativity of $\sqcup_U$. Furthermore, $\perp_U$ is the smallest element w.r.t. $\leq_U$, i.e., $\perp_U \leq_U u$ for all $u \in |U|$, by the neutrality of $\perp_U$.

Define the functor $PO : \mathbf{uSL} \rightarrow \mathbf{PO}$ by

$$\begin{aligned} PO(U) &= (|U|, \leq_U) , \\ PO(\varphi : U \rightarrow V) &= \varphi , \end{aligned}$$

which is well-defined on objects by the remarks above and also morphisms since if $u_1 \leq_U u_2$, i.e., $u_1 \sqcup_U u_2 = u_2$, then $\varphi(u_1) \sqcup_V \varphi(u_2) = \varphi(u_1 \sqcup_U u_2) = \varphi(u_2)$, i.e., $\varphi(u_1) \sqcup_V \varphi(u_2) = \varphi(u_2)$.

For each $P \in |\mathbf{PO}|$, define $\eta_P^{\mathbf{uSL}} : P \rightarrow PO(uSL\langle P \rangle)$ by $\eta_P^{\mathbf{uSL}}(p) = \{p\}$. Then $\eta^{\mathbf{uSL}} = (\eta_P^{\mathbf{uSL}})_{P \in |\mathbf{PO}|}$ is a natural transformation from $1_{\mathbf{PO}}$ to $PO \circ uSL\langle - \rangle$.

Let $P \in |\mathbf{PO}|$, $U \in |\mathbf{uSL}|$, and $\varphi : P \rightarrow PO(U)$. Define $\varphi^{\#_{\mathbf{uSL}}} : uSL\langle P \rangle \rightarrow U$ by

$$\varphi^{\#_{\mathbf{uSL}}}(\{p_1, \dots, p_n\}) = \varphi(p_1) \sqcup_U \dots \sqcup_U \varphi(p_n)$$

for all $\{p_1, \dots, p_n\} \in \mathcal{I}_{\text{fin}}(P)$, where, if $n = 0$, the right hand side is to be understood as $\perp_U$; $\varphi^{\#_{\text{uSL}}}$ is indeed an upper semi-lattice homomorphism, since for each $\{p'_1, \dots, p'_n\} \in \mathcal{P}_{\text{fin}} |P|$ we have $\varphi^{\#_{\text{uSL}}}(\text{Max}^{\leq_P} \{p'_1, \dots, p'_n\}) = \varphi(p'_1) \sqcup_U \dots \sqcup_U \varphi(p'_n)$: if $p'_i \leq_P p'_j$, then $\varphi(p'_i) \leq_{PO(U)} \varphi(p'_j)$, i.e., $\varphi(p'_i) \sqcup_U \varphi(p'_j) = \varphi(p'_j)$.

Then $PO(\varphi^{\#_{\text{uSL}}})(\eta_P^{\text{uSL}}(p)) = \varphi(p)$ and $\varphi^{\#_{\text{uSL}}}$ is unique with this property.

7. The partial ordering of $PO(uSL\langle P \rangle)$ on $\mathcal{I}_{\text{fin}}(P)$ for a partial order P is called the *lower* or *Hoare* ordering on $\mathcal{I}_{\text{fin}}(P)$ which we denote by $\subseteq_P$; it is explicitly given by

$$\begin{aligned} I \subseteq_P J &\iff \text{Max}^{\leq_P}(I \cup J) = J \\ &\iff \forall p \in I. \exists q \in J. p \leq_P q \end{aligned}$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$. It is $\emptyset \subseteq_P I$ for all $I \in \mathcal{I}_{\text{fin}}(P)$.

The dual of the Hoare ordering is the *upper* or *Smyth ordering* $\subseteq^P$ on $\mathcal{I}_{\text{fin}}(P)$ defined by $I \subseteq^P J$ if, and only if, $J \subseteq_{P^{-1}} I$, where $P^{-1} = (|P|, \geq_P)$. Explicitly, the Smyth ordering is given by

$$\begin{aligned} I \subseteq^P J &\iff \text{Min}^{\leq_P}(I \cup J) = I \\ &\iff \forall q \in J. \exists p \in I. p \leq_P q \end{aligned}$$

where $\text{Min}^{\leq_P}(S)$ is the set of minimal elements of $S \subseteq |P|$. In particular, the Smyth ordering also induces a binary operation $\cup^P : \mathcal{I}_{\text{fin}}(P) \times \mathcal{I}_{\text{fin}}(P) \rightarrow \mathcal{I}_{\text{fin}}(P)$ given by

$$I \cup^P J = \text{Min}^{\leq_P}(I \cup J),$$

which is also associative, commutative, and idempotent. Here, $I \cup^P \emptyset = I$, i.e., $\emptyset$ is an absorptive element for $\cup^P$, and $I \subseteq^P \emptyset$ for all $I \in \mathcal{I}_{\text{fin}}(P)$, i.e., $\emptyset$ is the greatest element of $\mathcal{I}_{\text{fin}}(P)$ w.r.t. $\subseteq^P$.

The *convex* or *Plotkin ordering* on $\mathcal{I}_{\text{fin}}(P)$ is defined by the intersection of $\subseteq_P$ and $\subseteq^P$, which means

$$I (\subseteq_P \cap \subseteq^P) J \iff (\forall p \in I. \exists q \in J. p \leq_P q) \wedge (\forall q \in J. \exists p \in I. p \leq_P q)$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$.

Finally, $\cup_P$ is monotonic w.r.t. $\subseteq_P$, and $\cup^P$ is monotonic w.r.t. $\subseteq^P$, i.e., for all $I, J, K \in \mathcal{I}_{\text{fin}}(P)$,

$$\begin{aligned} I \subseteq_P J \text{ implies } I \cup_P K \subseteq_P J \cup_P K, \\ I \subseteq^P J \text{ implies } I \cup^P K \subseteq^P J \cup^P K. \end{aligned}$$

3. PARTIALLY ORDERED MONOIDS

8. A *partially ordered monoid* $(X, \cdot, \varepsilon, \leq)$ is given by a set X , a binary operation $\cdot : X \times X \rightarrow X$, a constant $\varepsilon \in X$, and a partial order relation $\leq \subseteq X \times X$ such that the following axioms are satisfied for $x, x', y, y', z \in X$:

- (1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- (2) $x \cdot y = y \cdot x$
- (3) $x \cdot \varepsilon = x$
- (4) if $x \leq x'$ and $y \leq y'$, then $x \cdot y \leq x' \cdot y'$

In words, $(X, \cdot, \varepsilon)$ is a commutative monoid with unity ε and $\leq$ is monotone w.r.t. $\cdot$.

A *partially ordered monoid homomorphism* $\varphi : M \rightarrow N$ from a partially ordered monoid $M = (|M|, \cdot_M, \varepsilon_M, \leq_M)$ to a partially ordered monoid $N = (|N|, \cdot_N, \varepsilon_N, \leq_N)$ is given by a map $\varphi : |M| \rightarrow |N|$ such that for all $m, n \in |M|$:

- (1) $\varphi(m \cdot_M n) = \varphi(m) \cdot_N \varphi(n)$
- (2) $\varphi(\varepsilon_M) = \varepsilon_N$
- (3) if $m \leq_M n$, then $\varphi(m) \leq_N \varphi(n)$

The category poMon of partially ordered monoids has the partially ordered monoids as objects and the partially ordered monoids homomorphisms as morphisms.

9. A partially ordered monoid M is a *join monoid* if for all $m, n \in |M|$

$$m \leq_M m \cdot_M n.$$

This requirement is equivalent to requiring that ε_M is the smallest element w.r.t. $\leq_M$. Indeed, if $m \leq_M m \cdot_M n$ holds for all $m, n \in |M|$, then $\varepsilon_M \leq_M \varepsilon_M \cdot_M n = n$ for all $n \in |M|$. Conversely, if $\varepsilon_M \leq_M n$ for all $n \in |M|$, then $m = m \cdot_M \varepsilon_M \leq_M m \cdot_M n$ for all $m, n \in |M|$ by the monotonicity of $\leq_M$.

Dually, a partially ordered monoid M is a *meet monoid* if for all $m, n \in |M|$

$$m \cdot_M n \leq_M m,$$

and this requirement is equivalent to requiring that ε_M is the greatest element w.r.t. $\leq_M$.

The full sub-categories of poMon having all join monoids respectively meet monoids as objects are denoted by jMon and mMon , respectively.

There are functors

$$\begin{aligned} jMon : \text{mMon} &\rightarrow \text{jMon} & mMon : \text{jMon} &\rightarrow \text{mMon} \\ jMon(M) &= (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) & mMon(M) &= (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) \\ jMon(\varphi : M \rightarrow N) &= \varphi & mMon(\varphi : M \rightarrow N) &= \varphi \end{aligned}$$

such that $jMon \circ mMon = 1_{\text{jMon}}$ and $mMon \circ jMon = 1_{\text{mMon}}$. Note that meet monoids have also been referred to as partial valuation structures [3] or ic-monoids [4].

10. For a set X let $\mathcal{M}_{\text{fin}}(X)$ be the set of finite multisets over X . We write $\{x_1, \dots, x_m\}$ with $x_i \in X$ for $1 \leq i \leq m$ or $\{l_1x_1, \dots, l_nx_n\}$ with $x_i \in X$ and $l_i \in \mathbb{N}$ for $1 \leq i \leq n$ for an element of $\mathcal{M}_{\text{fin}}(X)$, $T \uplus U$ for the multiset union of the multisets T and U , and $T \subseteq U$ for the sub-multiset relation, which is a partial ordering relation on $\mathcal{M}_{\text{fin}}(X)$.

For a partial order P , the *lower* or *Hoare ordering* on $\mathcal{M}_{\text{fin}}|P|$ is the binary relation $\subseteq_P \subseteq (\mathcal{M}_{\text{fin}}|P|) \times (\mathcal{M}_{\text{fin}}|P|)$ given by the transitive closure of

$$\begin{aligned} T \subseteq U \text{ implies } T \subseteq_P U, \\ p \leq_P q \text{ implies } T \uplus \{p\} \subseteq_P T \uplus \{q\}. \end{aligned}$$

If $T \subseteq_P U$, then $T \uplus \{r\} \subseteq_P U \uplus \{r\}$ for all $r \in X$, since this holds for both defining clauses of the ordering.

For an element $T = \{l_1x_1, \dots, l_nx_n\} \in \mathcal{M}_{\text{fin}}(X)$ with $l_1, \dots, l_n > 0$, $x_i \neq x_j$ for all $1 \leq i \neq j \leq n$, and $n \geq 0$ let $\mathcal{S}(T) = \bigcup_{1 \leq i \leq n} \{(j, x_i) \mid 1 \leq j \leq l_i\}$.

LEMMA. $T \subseteq_P U$ if, and only if, there is an injective mapping $f : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ with $p \leq_P q$ if $f(j, p) = (k, q)$ for all $(j, p) \in \mathcal{S}(T)$.

Proof. Let first $T \subseteq_P U$ hold. Then there are an $n > 1$ and $T_1, \dots, T_n \in \mathcal{M}_{\text{fin}}(X)$ such that $T_1 = T$, $T_n = U$, and either $T_i \subseteq T_{i+1}$ or $T_i = T'_i \uplus \{p\}$ and $T_{i+1} = T'_i \uplus \{q\}$ with $p \leq_P q$ for all $1 \leq i < n$. For each $1 \leq i < n$ there is a map $f_i : \mathcal{S}(T_i) \rightarrow \mathcal{S}(T_{i+1})$ as required in the claim as follows: If $T_{i-1} \subseteq T_i$, then we choose $f_i = 1_{\mathcal{S}(T_i)}$. If $T_i = T'_i \uplus \{p\}$ and $T_{i+1} = T'_i \uplus \{q\}$ with $p \leq_P q$, then we choose $f_i = 1_{\mathcal{S}(T'_i)} \cup \{(j, p) \mapsto (k, q)\}$ where $j = |\{l \mid (l, p) \in \mathcal{S}(T'_i)\}| + 1$ and $k = |\{l \mid (l, q) \in \mathcal{S}(T'_i)\}| + 1$. Then $f_n \circ \dots \circ f_1 : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ as required in the claim.

For the converse, we prove that if $f : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ is a mapping as required in the claim, then $T \subseteq_P U$ by induction on the cardinality of $\mathcal{S}(T)$. Let $f : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ be given. If $|\mathcal{S}(T)| = 0$, then $\emptyset = T \subseteq U$. Now let $|\mathcal{S}(T)| > 0$ and let $(j, p) \in \mathcal{S}(T)$ such that j is maximal. Then $f(j, p) = (k, q)$

with $p \leq_P q$. Define $g : \mathcal{S}(U) \rightarrow \mathcal{S}(U) \setminus \{(k, q)\}$ by $g(l, r) = (l, r)$ if $r \neq q$ or $l < k$, and $g(l, q) = (l - 1, q)$ if $l > k$. Let $T', U' \in \mathcal{M}_{\text{fin}}(X)$ be defined by $T = T' \uplus \{p\}$ and $U = U' \uplus \{q\}$. Then $\mathcal{S}(T') = \mathcal{S}(T) \setminus \{(j, p)\}$ and $f' : \mathcal{S}(T') \rightarrow \mathcal{S}(U')$ defined by $f'(l, r) = g(f(l, r))$ for all $(l, r) \in \mathcal{S}(T')$ is an injective mapping as required in the claim. By induction hypothesis $T' \subseteq_P U'$ and thus, by the remark above, $T = T' \uplus \{p\} \subseteq_P U' \uplus \{p\} \subseteq_P U' \uplus \{q\} = U$. $\square$

We call such a map a *witness* for $T \subseteq_P U$.

11. The relation $\subseteq_P$ is obviously transitive and reflexive on $\mathcal{M}_{\text{fin}}|P|$. It is also antisymmetric: Assume for a contradiction that there are T and U with $T \subseteq_P U$ and $U \subseteq_P T$, but $T \neq U$ and choose an T with minimal cardinality satisfying this property. Then $T \neq \emptyset$. Let $f : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ and $g : \mathcal{S}(U) \rightarrow \mathcal{S}(T)$ be witnessing maps for $T \subseteq_P U$ and $U \subseteq_P T$. Choose an element $(j, p) \in \mathcal{S}(T)$ such that p is maximal w.r.t. $\leq_P$ in T . Let $f(j, p) = (k, q) \in \mathcal{S}(U)$. Then $p \leq_P q$. If $p \neq q$, there would be a $(j', p') \in \mathcal{S}(T)$ with $p \leq_P q \leq_P p'$ but $p \neq p'$ contradicting the maximality of p in T ; thus $f(j, p) = (k, p)$. Assume, without loss of generality, that j and k are maximal. Remove the occurrence of p from T , obtaining T' , and from U , obtaining U' . Then $T' \subseteq_P U'$ and $U' \subseteq_P T'$, since $f' : \mathcal{S}(T') \rightarrow \mathcal{S}(U')$ with $f'(x) = f(x)$ if $x \neq (j, p)$ and $g' : \mathcal{S}(U') \rightarrow \mathcal{S}(T')$ with $g'(y) = g(y)$ if $y \neq (k, p)$ are witnessing maps, contradicting the minimality of T .

Furthermore, $\uplus$ is associative and commutative; it is also monotonic w.r.t. $\subseteq_P$, i.e., for all $T, U, V \in \mathcal{M}_{\text{fin}}|P|$,

$$T \subseteq_P U \text{ implies } T \uplus V \subseteq_P U \uplus V ;$$

and $\emptyset \subseteq_P T$ for all $T \in \mathcal{M}_{\text{fin}}|P|$. Thus,

LEMMA. $(\mathcal{M}_{\text{fin}}|P|, \uplus, \emptyset, \subseteq_P)$ is a join monoid. $\square$

Define the functor $jMon\langle - \rangle : \text{PO} \rightarrow \text{jMon}$ by

$$jMon\langle P \rangle = (\mathcal{M}_{\text{fin}}|P|, \uplus, \emptyset, \subseteq_P) ,$$

$$jMon\langle \varphi : P \rightarrow Q \rangle = \lambda \{p_1, \dots, p_n\} \in \mathcal{M}_{\text{fin}}|P| . \{ \varphi(p_1), \dots, \varphi(p_n) \} .$$

12. Dually, the *upper* or *Smyth ordering* on $\mathcal{M}_{\text{fin}}|P|$ is the binary relation $\subseteq^P \subseteq (\mathcal{M}_{\text{fin}}|P|) \times (\mathcal{M}_{\text{fin}}|P|)$, defined by $T \subseteq^P U$ if, and only if, $U \subseteq_{P^{-1}} T$; more explicitly, the Smyth ordering on $\mathcal{M}_{\text{fin}}|P|$ is given by the transitive closure of

$$T \supseteq U \text{ implies } T \subseteq^P U ,$$

$$p \leq_P q \text{ implies } T \uplus \{p\} \subseteq^P T \uplus \{q\} ,$$

i.e., $T \subseteq^P U$ if, and only if, there is an injective mapping $g : \mathcal{S}(U) \rightarrow \mathcal{S}(T)$ with $p \leq_P q$ if $g(k, q) = (j, p)$ for all $(k, q) \in \mathcal{S}(U)$; we call such a map a *witness* for $T \subseteq^P U$. The relation $\subseteq^P$ is also a partial ordering on $\mathcal{M}_{\text{fin}}|P|$, and, again, $\uplus$ is monotonic w.r.t. $\subseteq^P$, i.e.,

$$T \subseteq^P U \text{ implies } T \uplus V \subseteq^P U \uplus V ;$$

and $\emptyset \subseteq^P T$ for all $T \in \mathcal{M}_{\text{fin}}|P|$. Thus,

LEMMA. $(\mathcal{M}_{\text{fin}}|P|, \uplus, \emptyset, \subseteq^P)$ is a meet monoid. $\square$

This meet monoid over a partial order P does not show suprema of finite sets, in general: Consider the partial order $P = (\{a, b, c\}, \{a < c, b < c\})$ (which does show suprema). In the meet monoid $(\mathcal{M}_{\text{fin}}|P|, \uplus, \emptyset, \subseteq^P)$, we have

$$\{c\} \subseteq^P \{a\}, \{b\} ,$$

$$\{a, b\} \subseteq^P \{a\}, \{b\}$$

and no $T \in \mathcal{M}_{\text{fin}} |P|$ exists with $\lambda a, b\rfloor, \lambda c\rfloor \subseteq^P T \subseteq^P \lambda a\rfloor, \lambda b\rfloor$ since, e.g., for $\lambda c\rfloor \subseteq^P T$, T can only be $\lambda\rfloor$ by the first rule (with $\lambda a, b\rfloor, \lambda c\rfloor \subseteq^P \lambda\rfloor$), or $\lambda a\rfloor$ or $\lambda b\rfloor$ by the second rule; but $\lambda a\rfloor$ and $\lambda b\rfloor$ are incomparable w.r.t. $\subseteq^P$

Define the functor $mMon\langle - \rangle : \text{PO} \rightarrow \text{mMon}$ by

$$\begin{aligned} mMon\langle P \rangle &= (\mathcal{M}_{\text{fin}} |P|, \cup, \lambda\rfloor, \subseteq^P), \\ mMon\langle \varphi : P \rightarrow Q \rangle &= \lambda \lambda p_1, \dots, p_n\rfloor \in \mathcal{M}_{\text{fin}} |P| . \lambda \varphi(p_1), \dots, \varphi(p_n)\rfloor . \end{aligned}$$

In particular, $jMon\langle P \rangle = jMon(mMon\langle P^{-1} \rangle)$ and $mMon\langle P \rangle = mMon(jMon\langle P^{-1} \rangle)$.

13. Finally, the *convex* or *Plotkin ordering* on $\mathcal{M}_{\text{fin}} |P|$ is the intersection of $\subseteq_P$ and $\subseteq^P$. Then $T (\subseteq_P \cap \subseteq^P) U$ if, and only if, there is a bijective mapping $h : \mathcal{S}(T) \rightarrow \mathcal{S}(U)$ with $p \leq_P q$ if $h(j, p) = (k, q)$ for all $(j, p) \in \mathcal{S}(T)$; we again call such a map a *witness* for $T (\subseteq_P \cap \subseteq^P) U$. The relation $\subseteq_P \cap \subseteq^P$ is also a partial ordering on $\mathcal{M}_{\text{fin}} |P|$, and again $\cup$ is monotonic w.r.t. this ordering.

LEMMA. $(\mathcal{M}_{\text{fin}} |P|, \cup, \lambda\rfloor, \subseteq_P \cap \subseteq^P)$ is a partially ordered monoid. $\square$

Define the functor $poMon\langle - \rangle : \text{PO} \rightarrow \text{poMon}$ by

$$\begin{aligned} poMon\langle P \rangle &= (\mathcal{M}_{\text{fin}} |P|, \cup, \lambda\rfloor, \subseteq_P \cap \subseteq^P), \\ poMon\langle \varphi : P \rightarrow Q \rangle &= \lambda \lambda p_1, \dots, p_n\rfloor \in \mathcal{M}_{\text{fin}} |P| . \lambda \varphi(p_1), \dots, \varphi(p_n)\rfloor . \end{aligned}$$

14. Define the functor $PO : \text{poMon} \rightarrow \text{PO}$ by

$$\begin{aligned} PO(M) &= (|M|, \leq_M), \\ PO(\varphi : M \rightarrow N) &= \varphi . \end{aligned}$$

For each $P \in |\text{PO}|$ and each $x \in \{po, j, m\}$ define the partial order homomorphisms $\eta_P^{x\text{Mon}} : P \rightarrow PO(xMon\langle P \rangle)$ by $\eta_P^{x\text{Mon}}(p) = \lambda p\rfloor$. Then each $\eta^{x\text{Mon}} = (\eta_P^{x\text{Mon}})_{P \in |\text{PO}|}$ is a natural transformation from 1_{PO} to $PO \circ xMon\langle - \rangle$.

Let $x \in \{po, j, m\}$, $P \in |\text{PO}|$, $M \in |xMon|$, and $\varphi : P \rightarrow PO(M)$. Define $\varphi^{\sharp_{x\text{Mon}}} : xMon\langle P \rangle \rightarrow M$ by

$$\varphi^{\sharp_{x\text{Mon}}}(\lambda p_1, \dots, p_n\rfloor) = \varphi(p_1) \cdot_M \dots \cdot_M \varphi(p_n)$$

for all $\lambda p_1, \dots, p_n\rfloor \in \mathcal{M}_{\text{fin}} |P|$, where, if $n = 0$, the right hand side is to be understood as ε_M .

Then for all $x \in \{po, j, m\}$, $PO(\varphi^{\sharp_{x\text{Mon}}})(\eta_P^{x\text{Mon}}(p)) = \varphi(p)$ and all $\varphi^{\sharp_{x\text{Mon}}}$ are unique with this property.

15. Each upper semi-lattice can also be viewed as a join monoid. Indeed, define the functor $jMon : \text{uSL} \rightarrow \text{jMon}$ by

$$\begin{aligned} jMon(U) &= (|U|, \sqcup_U, \perp_U, \leq_U), \\ jMon(\varphi : U \rightarrow V) &= \varphi . \end{aligned}$$

In particular, by §5 and §7, for each partial order P , $jMon(uSL\langle P \rangle) = (\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset, \subseteq_P)$ is a join monoid.

3.1. Constructing Meet Monoids from Dags

16. Let G be a dag. Consider the *single-predecessor lifting* $\rightsquigarrow_G^{\text{SPD}} \subseteq (\mathcal{M}_{\text{fin}} |G|) \times (\mathcal{M}_{\text{fin}} |G|)$ of G to the finite multisets $\mathcal{M}_{\text{fin}} |G|$ over the elements of G , given by

$$T \rightsquigarrow_G^{\text{SPD}} T \cup \lambda g\rfloor ,$$

$g \rightarrow_G h$ implies $T \uplus \downarrow g \rightsquigarrow_G^{\text{SPD}} T \uplus \downarrow h$.

Then $G^{\text{SPD}} = (\mathcal{M}_{\text{fin}} |G|, \rightsquigarrow_G^{\text{SPD}})$ is a dag, where $T \rightsquigarrow_G^{\text{SPD}} U$ expresses that U is *worse* than T . We write $\leq_G^{\text{SPD}}$ for $\geq_{PO\langle G^{\text{SPD}} \rangle} = ((\rightsquigarrow_G^{\text{SPD}})^*)^{-1}$. Furthermore, $\leq_G^{\text{SPD}}$ is monotonic w.r.t. multiset union and thus $(\mathcal{M}_{\text{fin}} |G|, \uplus, \downarrow, \leq_G^{\text{SPD}})$ is a meet monoid. We have

LEMMA. For each dag G , $mMon\langle PO\langle G \rangle^{-1} \rangle \cong (\mathcal{M}_{\text{fin}} |G|, \uplus, \downarrow, \leq_G^{\text{SPD}})$. $\square$

17. Let G be a dag. Consider the *transitive-predecessors lifting* $\rightsquigarrow_G^{\text{TPD}} \subseteq (\mathcal{M}_{\text{fin}} |G|) \times (\mathcal{M}_{\text{fin}} |G|)$ of G to the finite multisets $\mathcal{M}_{\text{fin}} |G|$ over the elements of G , given by

$$\begin{aligned} T &\rightsquigarrow_G^{\text{TPD}} T \uplus \downarrow g, \\ g_1, \dots, g_n &\rightarrow_G^* h \text{ implies } T \uplus \downarrow g_1, \dots, g_n \rightsquigarrow_G^{\text{TPD}} T \uplus \downarrow h. \end{aligned}$$

Then $G^{\text{TPD}} = (\mathcal{M}_{\text{fin}} |G|, \rightsquigarrow_G^{\text{TPD}})$ is a dag; we write $\leq_G^{\text{TPD}}$ for $\geq_{PO\langle G^{\text{TPD}} \rangle} = ((\rightsquigarrow_G^{\text{TPD}})^*)^{-1}$. Furthermore, $\leq_G^{\text{TPD}}$ is monotonic w.r.t. multiset union and thus $(\mathcal{M}_{\text{fin}} |G|, \uplus, \downarrow, \leq_G^{\text{TPD}})$ is a meet monoid. We have

LEMMA. Let G be a dag, let $S_1, S_2 \in \mathcal{P}_{\text{fin}} |G|$, and let $\bar{S}_1, \bar{S}_2 \in \mathcal{M}_{\text{fin}} |G|$ be the finite multisets corresponding to S_1 and S_2 , respectively. Then

$$\bar{S}_1 \leq_G^{\text{TPD}} \bar{S}_2 \text{ implies } \text{Max}^{PO\langle G \rangle}(S_2) \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(S_1).$$

Proof. Let $\bar{S}_1 \leq_G^{\text{TPD}} \bar{S}_2$ hold. It suffices to prove the claim for the two defining clauses for $\rightsquigarrow_G^{\text{TPD}}$. For the case that there is a $g \in S_1$ such that $S_2 = S_1 \setminus \{g\}$, we have

$$\text{Max}^{PO\langle G \rangle}(S_2) = \text{Max}^{PO\langle G \rangle}(S_1 \setminus \{g\}) \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(S_1) \setminus \{g\} \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(S_1).$$

For the case that there is an $h \in S_1$ and $g_1, \dots, g_n \in S_2$ with $g_1, \dots, g_n \rightarrow_G h$ such that $S_2 = (S_1 \setminus \{h\}) \cup \{g_1, \dots, g_n\}$, we have

$$\begin{aligned} \text{Max}^{PO\langle G \rangle}(S_2) &= \text{Max}^{PO\langle G \rangle}(S_1 \cup \{g_1, \dots, g_n\}) \setminus \{h\} \subseteq_{PO\langle G \rangle} \\ &\text{Max}^{PO\langle G \rangle}(S_1) \setminus \{h\} \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(S_1), \end{aligned}$$

since $h \in S_1$. $\square$

The converse is in general wrong: Let $G = (\{g, h\}, \{(g, h)\})$, i.e., $g \rightarrow_G h$; then indeed we have $\text{Max}^{PO\langle G \rangle}(\{g, h\}) \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(\{h\})$, but $\downarrow h \not\rightsquigarrow_G^{\text{TPD}} \downarrow g, h$. However, if $S_2 \in \mathcal{I}_{\text{fin}}(PO\langle G \rangle)$, then the converse follows from the map $f : \text{Max}^{PO\langle G \rangle}(S_2) \rightarrow \text{Max}^{PO\langle G \rangle}(S_1)$ witnessing $S_2 = \text{Max}^{PO\langle G \rangle}(S_2) \subseteq_{PO\langle G \rangle} \text{Max}^{PO\langle G \rangle}(S_1)$.

3.2. Lexicographic Product of Meet Monoids

18. For a meet monoid M , define its set of *collapsing elements* by [3]

$$\mathcal{C}(M) = \{m \in |M| \mid \exists m_1, m_2 \in |M|. m_1 <_M m_2 \wedge m_1 \cdot_M m = m_2 \cdot_M m\}.$$

EXAMPLE. Let P be a partial order.

(1) The set of collapsing elements of the meet monoid $mMon\langle P \rangle$ is empty: If $T \subsetneq^P U$, then $T \uplus \downarrow p = U \uplus \downarrow p$ for some $p \in |P|$ would imply that $T = U$.

(2) The set of collapsing elements of the meet monoid $mMon(jMon(uSL\langle P \rangle))$ is $\mathcal{I}_{\text{fin}}(P) \setminus \{\emptyset\}$: If $\emptyset \neq I \in \mathcal{I}_{\text{fin}}(P)$, then $I \supset_P \emptyset$, but $\emptyset \cup_P I = I = I \cup_P I$. $\square$

Generalising the first example, if M is a *strict* meet monoid, i.e., $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n, o \in |M|$, then $\mathcal{C}(M) = \emptyset$. Generalising the second example, all idempotent elements of a meet monoid M which are different from ε_M are collapsing: If $m \in |M|$ such that $m \neq \varepsilon_M$ and $m \cdot_M m = m$, then $m <_M \varepsilon_M$ but $m \cdot_M m = m = \varepsilon_M \cdot_M m$.

LEMMA. $|M| \setminus \mathcal{C}(M)$ is closed under $\cdot_M$.

Proof. We show that $m \cdot_M n \in \mathcal{C}(M)$ if, and only if, $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$: If $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$, then obviously $m \cdot_M n \in \mathcal{C}(M)$. Conversely, if $m \cdot_M n \in \mathcal{C}(M)$, then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$, and $m_1 \cdot_M m \cdot_M n = m_2 \cdot_M m \cdot_M n$; but if $m \notin \mathcal{C}(M)$, then $m_1 \cdot_M m <_M m_2 \cdot_M m$, and if also $n \notin \mathcal{C}(M)$, then $m_1 \cdot_M m \cdot_M n <_M m_2 \cdot_M m \cdot_M n$. $\square$

Note that in particular $\varepsilon_M \notin \mathcal{C}(M)$ since $m_1 <_M m_2$ implies $m_1 \cdot_M \varepsilon_M <_M m_2 \cdot_M \varepsilon_M$. Thus, $(|M| \setminus \mathcal{C}(M), \cdot_M, \varepsilon_M, \leq_M)$ forms a meet monoid.

19. A meet monoid M is *bounded* if $|M|$ has a smallest element w.r.t. $\leq_M$; we denote this element by $\perp_M$ if it exists.

In a bounded meet monoid M it holds that $m \cdot_M \perp_M = \perp_M$ for all $m \in |M|$, i.e., $\perp_M$ is an absorbing element. Furthermore, $\perp_M \in \mathcal{C}(M)$. We call a bounded meet monoid *weakly strict* if $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n \in |M|$ and $\perp_M \neq o \in |M|$.

Each meet monoid M which is not bounded can be *lifted* into a bounded meet monoid $M_\perp = (|M| \uplus \{\perp\}, \cdot_{M_\perp}, \varepsilon_{M_\perp}, \leq_{M_\perp})$ setting $m \cdot_{M_\perp} \perp = \perp$, $\varepsilon_{M_\perp} = \varepsilon_M$, and $\perp \leq_{M_\perp} m$ for all $m \in |M| \uplus \{\perp\}$.

20. Let M be a meet monoid and let N be a bounded meet monoid. Let

$$L = ((|M| \setminus \mathcal{C}(M)) \times |N|) \cup (\mathcal{C}(M) \times \{\perp_N\}).$$

Define the binary operation $\cdot_L : L \times L \rightarrow L$ by

$$(m_1, n_1) \cdot_L (m_2, n_2) = (m_1 \cdot_M m_2, n_1 \cdot_N n_2).$$

This is well-defined: If $(m_1, n_1) \in (|M| \setminus \mathcal{C}(M)) \times |N|$ and $(m_2, n_2) \in \mathcal{C}(M) \times \{\perp_N\}$, then $m_1 \cdot_M m_2 \in \mathcal{C}(M)$, but also $n_2 = \perp_N$ and thus $n_1 \cdot_N n_2 = n_1 \cdot_N \perp_N = \perp_N$. $\cdot_L$ inherits associativity and commutativity from M and N .

Further define the element $\varepsilon_L \in L$ by

$$\varepsilon_L = (\varepsilon_M, \varepsilon_N),$$

which is also well-defined, since $\varepsilon_M \notin \mathcal{C}(M)$. Also $(m, n) \cdot_L \varepsilon_L = (m, n)$.

Finally, define the *lexicographic ordering* $\leq_L \subseteq L \times L$ on L by

$$(m_1, n_1) \leq_L (m_2, n_2) \iff (m_1 \neq m_2 \text{ and } m_1 \leq_M m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 \leq_N n_2).$$

LEMMA. $(L, \cdot_L, \varepsilon_L, \leq_L)$ is a meet monoid.

Proof. Since $(L, \cdot_L, \varepsilon_L)$ is a commutative monoid, it only remains to show the monotonicity of $\cdot_L$ w.r.t. $\leq_L$. Let $(m_1, n_1) \leq_L (m_2, n_2)$ and an $(m, n) \in L$ be given.

Case $m_1 <_M m_2$: If not $m_1 \cdot_M m <_M m_2 \cdot_M m$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$, then $m \in \mathcal{C}(M)$ and thus $n = \perp_N$. Hence

$$\begin{aligned} (m_1, n_1) \cdot_L (m, n) &= (m_1 \cdot_M m, n_1 \cdot_N n) = (m_1 \cdot_M m, \perp_N) = \\ &= (m_2 \cdot_M m, \perp_N) = (m_2 \cdot_M m, n_2 \cdot_N n) = (m_2, n_2) \cdot_L (m, n). \end{aligned}$$

Case $m_1 = m_2$ and $n_1 \leq_N n_2$: Then $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n \leq_N n_2 \cdot_N n$. $\square$

Let us write $M \ltimes N$ for $(L, \cdot_L, \varepsilon_L, \leq_L)$.

21. Let M and N be meet monoids such that N is bounded. Then $\mathcal{C}(M \ltimes N) = (\mathcal{C}(M) \times \{\perp_N\}) \cup ((|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N))$. Indeed, let $(m, n) \in \mathcal{C}(M \ltimes N)$. Then there are $(m_1, n_1), (m_2, n_2) \in |M \ltimes N|$ with $(m_1, n_1) <_{M \ltimes N} (m_2, n_2)$, i.e., $m_1 <_M m_2$ or $m_1 = m_2$ and $n_1 <_N n_2$, and $(m_1, n_1) \cdot_{M \ltimes N} (m, n) = (m_2, n_2) \cdot_{M \ltimes N} (m, n)$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n = n_2 \cdot_N n$. If $m_1 <_M m_2$, then $m \in \mathcal{C}(M)$ and therefore $n = \perp_N$; if $m_1 = m_2$, then $n \in \mathcal{C}(N)$. Conversely, let first $(m, n) \in \mathcal{C}(M) \times \{\perp_N\}$; then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$ and $m_1 \cdot_M m = m_2 \cdot_M m$, hence $(m_1, \perp_N) <_{M \ltimes N} (m_2, \perp_N)$ with $(m_1, \perp_N) \cdot_{M \ltimes N} (m, n) = (m_2, \perp_N) \cdot_{M \ltimes N} (m, n)$. Now let $(m, n) \in (|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N)$; then there are $n_1, n_2 \in |N|$ with $n_1 <_N n_2$ and $n_1 \cdot_N n = n_2 \cdot_N n$, hence $(m, n_1) <_{M \ltimes N} (m, n_2)$ with $(m, n_1) \cdot_{M \ltimes N} (m, n) = (m, n_2) \cdot_{M \ltimes N} (m, n)$. Abbreviate $|M| \setminus \mathcal{C}(M)$ by $R(M)$. Then

$$\begin{aligned} |M \ltimes N| \setminus \mathcal{C}(M \ltimes N) &= R(M \ltimes N) = \\ &= ((R(M) \times |N|) \cup (\mathcal{C}(M) \times \{\perp_N\})) \setminus ((R(M) \times \mathcal{C}(N)) \cup (\mathcal{C}(M) \times \{\perp_N\})) = \\ &= R(M) \times R(N) = (|M| \setminus \mathcal{C}(M)) \times (|N| \setminus \mathcal{C}(N)). \end{aligned}$$

Thus

$$\begin{aligned} |(M \ltimes N) \ltimes O| &= \\ &= (R(M \ltimes N) \times |O|) \cup (\mathcal{C}(M \ltimes N) \times \{\perp_O\}) = \\ &= (R(M) \times R(N) \times |O|) \cup (R(M) \times \mathcal{C}(N) \times \{\perp_O\}) \cup (\mathcal{C}(M) \times \{(\perp_N, \perp_O)\}) = \\ &= (R(M) \times ((R(N) \times |O|) \cup (\mathcal{C}(N) \times \{\perp_O\}))) \cup (\mathcal{C}(M) \times \{\perp_{N \ltimes O}\}) = \\ &= (R(M) \times |N \ltimes O|) \cup (\mathcal{C}(M) \times \{\perp_{N \ltimes O}\}) = \\ &= |M \ltimes (N \ltimes O)|, \end{aligned}$$

from which it follows that $\ltimes$ is associative.

4. C-SEMIRINGS

22. A *c-semiring* [1] $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is given by a set X , two binary operations $\oplus, \otimes : X \times X \rightarrow X$, and two constants $\mathbf{0}, \mathbf{1} \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:

- (1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
- (2) $x \oplus y = y \oplus x$
- (3) $x \oplus \mathbf{1} = \mathbf{1}$
- (4) $x \oplus \mathbf{0} = x$
- (5) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
- (6) $x \otimes y = y \otimes x$
- (7) $x \otimes \mathbf{0} = \mathbf{0}$
- (8) $x \otimes \mathbf{1} = x$
- (9) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

In words, $\oplus$ is associative and commutative, has $\mathbf{1}$ as annihilator and $\mathbf{0}$ as neutral element; $\otimes$ is associative and commutative, has $\mathbf{0}$ as annihilator and $\mathbf{1}$ as neutral element; and $\otimes$ distributes over $\oplus$. A *c-semiring homomorphism* $\varphi : A \rightarrow B$ from a c-semiring $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ to a c-semiring $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ is given by a map $\varphi : |A| \rightarrow |B|$ such that for all $a_1, a_2 \in |A|$:

- (1) $\varphi(a_1 \oplus_A a_2) = \varphi(a_1) \oplus_B \varphi(a_2)$
- (2) $\varphi(a_1 \otimes_A a_2) = \varphi(a_1) \otimes_B \varphi(a_2)$
- (3) $\varphi(\mathbf{0}_A) = \mathbf{0}_B$
- (4) $\varphi(\mathbf{1}_A) = \mathbf{1}_B$

The category $\mathbf{cSRng}$ of c-semirings has the c-semirings as objects and the c-semiring homomorphisms as morphisms.

23. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\oplus$ is idempotent:

$$x \oplus x = (x \otimes \mathbf{1}) \oplus (x \otimes \mathbf{1}) = x \otimes (\mathbf{1} \oplus \mathbf{1}) = x \otimes \mathbf{1} = x .$$

Thus, there is a functor $uSL : \mathbf{cSRng} \rightarrow \mathbf{uSL}$, defined by

$$uSL(A) = (|A|, \oplus_A, \mathbf{0}) ,$$

$$uSL(\varphi : A \rightarrow B) = \varphi .$$

For a c-semiring A , the thereby induced ordering $\leq_{uSL(A)}$, explicitly given by $a \leq_{uSL(A)} b$ if, and only if, $a \oplus_A b = b$, will be written as $\preceq_A$.

With this definition, for all $a, b, c \in |A|$ it holds that

- (1) $\mathbf{0} \preceq_A a \preceq_A \mathbf{1}$;
- (2) $a \preceq_A a \oplus_A b$ and $b \preceq_A a \oplus_A b$;
- (3) if $a \preceq_A c$ and $b \preceq_A c$, then $a \oplus_A b \preceq_A c$.

Also $\oplus_A$ is monotone w.r.t. $\preceq_A$ in both arguments, i.e.,

$$a \preceq_A a' \text{ and } b \preceq_A b' \text{ implies } a \oplus_A b \preceq_A a' \oplus_A b' .$$

24. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\otimes$ is monotone w.r.t. the induced ordering $\preceq$, since if $x \preceq x'$, i.e., $x \oplus x' = x'$, then

$$(x \otimes y) \oplus (x' \otimes y) = (x \oplus x') \otimes y = x' \otimes y ,$$

i.e., $x \otimes y \preceq x' \otimes y$, from which it follows that

$$x \preceq x' \text{ and } y \preceq y' \text{ implies } x \otimes y \preceq x' \otimes y' .$$

Furthermore, for all $x, y \in X$

$$x \otimes y \preceq x \text{ and } x \otimes y \preceq y ,$$

since

$$(x \otimes y) \oplus x = (x \otimes y) \oplus (x \otimes \mathbf{1}) = x \otimes (y \oplus \mathbf{1}) = x \otimes \mathbf{1} = x .$$

Thus, there is a functor $mMon : \mathbf{cSRng} \rightarrow \mathbf{mMon}$, given by

$$mMon(A) = (|A|, \otimes_A, \mathbf{1}_A, \preceq_A) ,$$

$$mMon(\varphi : A \rightarrow B) = \varphi .$$

Note that $mMon(A)$ is a bounded meet monoid with $\perp_{mMon(A)} = \mathbf{0}_A$.

25. Consider the c-semiring of *boolean values* $B = (\{\perp, \top\}, \vee, \wedge, \perp, \top)$ where $\vee$ and $\wedge$ have their usual meaning.

LEMMA. B is initial in $\mathbf{cSRng}$. □

26. Consider the c-semiring $T = (\{*\}, \cdot, \cdot, *, *)$ with $* \cdot * = *$.

LEMMA. T is terminal in $\mathbf{cSRng}$. □

27. Let $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ and $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ be c-semirings. Define $\oplus_{A \times B}, \otimes_{A \times B} : (|A| \times |B|) \times (|A| \times |B|) \rightarrow |A| \times |B|$ by

$$\begin{aligned} (a_1, b_1) \oplus_{A \times B} (a_2, b_2) &= (a_1 \oplus_A a_2, b_1 \oplus_B b_2) \\ (a_1, b_1) \otimes_{A \times B} (a_2, b_2) &= (a_1 \otimes_A a_2, b_1 \otimes_B b_2) \end{aligned}$$

Then $A \times B = (|A| \times |B|, \oplus_{A \times B}, \otimes_{A \times B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring.

Define $\pi_1 : A \times B \rightarrow A$ by $\pi_1(a, b) = a$ and $\pi_2 : A \times B \rightarrow B$ by $\pi_2(a, b) = b$. Then π_1 and π_2 are c-semiring homomorphisms. Furthermore, for any c-semiring $C = (|C|, \oplus_C, \otimes_C, \mathbf{0}_C, \mathbf{1}_C)$ and two c-semiring homomorphisms $\varphi_1 : C \rightarrow A$ and $\varphi_2 : C \rightarrow B$, the c-semiring homomorphism $\langle \varphi_1, \varphi_2 \rangle : C \rightarrow A \times B$ defined by $\langle \varphi_1, \varphi_2 \rangle(c) = (\varphi_1(c), \varphi_2(c))$ is unique for the property $\varphi_1 = \langle \varphi_1, \varphi_2 \rangle ; \pi_1$ and $\varphi_2 = \langle \varphi_1, \varphi_2 \rangle ; \pi_2$.

LEMMA. cSRng has finite products. □

28. The collapsing elements $\mathcal{C}(A)$ of a c-semiring A are the collapsing elements of $mMon(A)$. A c-semiring A is total if for all $a, a_1, a_2 \in |A|$:

$$a_1 \prec_A a_2 \text{ or } a_1 = a_2 \text{ or } a_2 \prec_A a_1 .$$

Let A and B be c-semirings where A is total. Let $L = ((|A| \setminus \mathcal{C}(A)) \times |B|) \cup (\mathcal{C}(A) \times \{\mathbf{0}_B\})$. Define $\oplus_{A \times B}, \otimes_{A \times B} : L \times L \rightarrow L$ by

$$\begin{aligned} (a_1, b_1) \oplus_{A \times B} (a_2, b_2) &= \begin{cases} (a_1, b_1) & \text{if } a_2 \prec_A a_1 \\ (a_2, b_2) & \text{if } a_1 \prec_A a_2 , \\ (a_1, b_1 \oplus_B b_2) & \text{if } a_1 = a_2 \end{cases} \\ (a_1, b_1) \otimes_{A \times B} (a_2, b_2) &= (a_1 \otimes_A a_2, b_1 \otimes_B b_2) . \end{aligned}$$

Then $(a, b) \otimes_{A \times B} ((a_1, b_1) \oplus_{A \times B} (a_2, b_2)) = ((a, b) \otimes_{A \times B} (a_1, b_1)) \oplus_{A \times B} ((a, b) \otimes_{A \times B} (a_2, b_2))$ for all $(a, b), (a_1, b_1), (a_2, b_2) \in L$, where the right hand side is $(a \otimes_A a_1, b \otimes_B b_1) \oplus_{A \times B} (a \otimes_A a_2, b \otimes_B b_2)$. If $a_1 = a_2$, then $(a_1, b_1) \oplus_{A \times B} (a_2, b_2) = (a_1, b_1 \oplus_B b_2)$ and $a \otimes_A a_1 = a \otimes_A a_2$, from which the claim follows by the distributivity of $\otimes_B$ over $\oplus_B$. If $a_2 \prec_A a_1$ (the case $a_1 \prec_A a_2$ is symmetric), then $(a_1, b_1) \oplus_{A \times B} (a_2, b_2) = (a_1, b_1)$. If additionally $a \notin \mathcal{C}(A)$, then $a \otimes_A a_2 \prec_A a \otimes_A a_1$; if $a \in \mathcal{C}(A)$, then $b = \mathbf{0}_B$, and in both cases the claim follows.

Thus $A \times B = (L, \oplus_{A \times B}, \otimes_{A \times B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring, the *lexicographic product* of A and B .

29. Let M be meet monoid. We write $\mathcal{I}_{\text{fin}}(M)$ for $\mathcal{I}_{\text{fin}}(PO(M))$. Define the operations $\tilde{\cdot}_M, \tilde{\cup}_M : \mathcal{I}_{\text{fin}}(M) \times \mathcal{I}_{\text{fin}}(M) \rightarrow \mathcal{I}_{\text{fin}}(M)$ by

$$\begin{aligned} I \tilde{\cdot}_M J &= \text{Max}^{\leq M} \{m \cdot_M n \mid m \in I, n \in J\} , \\ I \tilde{\cup}_M J &= \text{Max}^{\leq M} (I \cup J) . \end{aligned}$$

LEMMA. $(\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\})$ is a c-semiring.

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(M)$.

The operation $\tilde{\cup}_M$ is associative and commutative and has $\emptyset$ as neutral element by §5. Furthermore, $I \tilde{\cup}_M \{\varepsilon_M\} = \{\varepsilon_M\}$, since ε_M is the greatest element of $|M|$ w.r.t. $\leq_M$.

For the associativity of $\tilde{\cdot}_M$ we have

$$\begin{aligned} I \tilde{\cdot}_M (J \tilde{\cdot}_M K) &= \\ \text{Max}^{\leq M} \{m_I \cdot_M m_{JK} \mid m_I \in I, m_{JK} \in \text{Max}^{\leq M} \{m_J \cdot_M m_K \mid m_J \in J, m_K \in K\}\} &= \end{aligned}$$

$$\begin{aligned} & \text{Max}^{\leq M} \{m_I \cdot_M m_J \cdot_M m_K \mid m_I \in I, m_J \in J, m_K \in K\} = \\ & \text{Max}^{\leq M} \{m_{IJ} \cdot_M m_K \mid m_{IJ} \in \text{Max}^{\leq M} \{m_I \cdot_M m_J \mid m_I \in I, m_J \in J\}, m_K \in K\} = \\ & (I \tilde{\cdot}_M J) \tilde{\cdot}_M K, \end{aligned}$$

since

$$\text{Max}^{\leq M} \{m \cdot_M n \mid m \in I, n \in \text{Max}^{\leq M}(X)\} = \text{Max}^{\leq M} \{m \cdot_M n \mid m \in I, n \in X\}$$

for all $X \in \mathcal{P}_{\text{fin}} |M|$. Assume $n \in X$ but $n \notin \text{Max}^{\leq M}(X)$; then there exists some (maximal) $n' \in X$ such that $n <_M n'$; Since $n \leq n'$ implies $m \cdot_M n \leq m \cdot_M n'$ by the monotonicity of $\cdot_M$, $m \cdot_M n'$ Also $\tilde{\cdot}_M$ inherits commutativity from $\cdot_M$; $I \tilde{\cdot}_M \emptyset = \emptyset$ by definition; and $I \tilde{\cdot}_M \{\varepsilon_M\} = I$, since ε_M is neutral in M .

Finally, $\tilde{\cdot}_M$ distributes over $\tilde{\cup}_M$:

$$\begin{aligned} & I \tilde{\cdot}_M (J \tilde{\cup}_M K) = \\ & \text{Max}^{\leq M} \{m_I \cdot_M m_{JK} \mid m_I \in I, m_{JK} \in \text{Max}^{\leq M}(J \cup K)\} = \\ & \text{Max}^{\leq M} \{m_I \cdot_M m_{JK} \mid m_I \in I, m_{JK} \in J \cup K\} = \\ & \text{Max}^{\leq M} (\{m_I \cdot_M m_J \mid m_I \in I, m_J \in J\} \cup \{m_I \cdot_M m_K \mid m_I \in I, m_K \in K\}) = \\ & \text{Max}^{\leq M} (\text{Max}^{\leq M} \{m_I \cdot_M m_J \mid m_I \in I, m_J \in J\} \cup \\ & \quad \text{Max}^{\leq M} \{m_I \cdot_M m_K \mid m_I \in I, m_K \in K\}) = \\ & (I \tilde{\cdot}_M J) \tilde{\cup}_M (I \tilde{\cdot}_M K), \end{aligned}$$

since

$$\text{Max}^{\leq M} (I \cup \text{Max}^{\leq M}(X)) = \text{Max}^{\leq M} (I \cup X)$$

for all $X \in \mathcal{P}_{\text{fin}} |M|$. □

Let $\varphi : M \rightarrow N$ be a meet monoid homomorphism. For $X \in \mathcal{P}_{\text{fin}} |M|$, we have

$$\text{Max}^{\leq N} (\varphi(\text{Max}^{\leq M}(X))) = \text{Max}^{\leq N} (\varphi(X)).$$

Indeed, on the one hand, $\text{Max}^{\leq N} (\varphi(\text{Max}^{\leq M}(X))) \subseteq \text{Max}^{\leq N} (\varphi(X))$, since $\text{Max}^{\leq M}(X) \subseteq X$. For $\text{Max}^{\leq N} (\varphi(X)) \subseteq \text{Max}^{\leq N} (\varphi(\text{Max}^{\leq M}(X)))$ it suffices to show that for each $n \in \varphi(X)$ there is an $n' \in \varphi(\text{Max}^{\leq M}(X))$ such that $n \leq_M n'$. Thus, let $n \in \varphi(X)$, i.e., $n = \varphi(m)$ for some $m \in X$. Then there is an $m' \in \text{Max}^{\leq M}(X)$ with $m \leq_M m'$, hence $n = \varphi(m) \leq_N \varphi(m')$, and $\varphi(m') \in \varphi(\text{Max}^{\leq M}(X))$.

Define the functor $cSRng\langle - \rangle : \text{mMon} \rightarrow cSRng$ by

$$\begin{aligned} cSRng\langle M \rangle &= (\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\}), \\ cSRng\langle \varphi : M \rightarrow N \rangle &= \lambda\{m_1, \dots, m_k\} \in \mathcal{I}_{\text{fin}}(M). \text{Max}^{\leq N} \{\varphi(m_1), \dots, \varphi(m_k)\}. \end{aligned}$$

Indeed, $cSRng\langle \varphi : M \rightarrow N \rangle$ is a c-semiring homomorphism from $cSRng\langle M \rangle$ to $cSRng\langle N \rangle$:

$$\begin{aligned} cSRng\langle \varphi \rangle(\emptyset) &= \emptyset, \\ cSRng\langle \varphi \rangle(\{\varepsilon_M\}) &= \{\varphi(\varepsilon_M)\} = \{\varepsilon_N\}, \\ cSRng\langle \varphi \rangle(I_1 \tilde{\cup}_M I_2) &= cSRng\langle \varphi \rangle(\text{Max}^{\leq M}(I_1 \cup I_2)) = \\ & \text{Max}^{\leq N} (\varphi(\text{Max}^{\leq M}(I_1 \cup I_2))) = \text{Max}^{\leq N} (\varphi(I_1 \cup I_2)) = \text{Max}^{\leq N} (\varphi(I_1) \cup \varphi(I_2)) = \\ & cSRng\langle \varphi \rangle(I_1) \tilde{\cup}_N cSRng\langle \varphi \rangle(I_2), \\ cSRng\langle \varphi \rangle(I_1 \tilde{\cdot}_M I_2) &= cSRng\langle \varphi \rangle(\text{Max}^{\leq M} \{m_1 \cdot_M m_2 \mid m_1 \in I_1, m_2 \in I_2\}) = \\ & \text{Max}^{\leq N} (\varphi(\text{Max}^{\leq M} \{m_1 \cdot_M m_2 \mid m_1 \in I_1, m_2 \in I_2\})) = \end{aligned}$$

$$\begin{aligned}
& \text{Max}^{\leq_N} \{ \varphi(m_1 \cdot_M m_2) \mid m_1 \in I_1, m_2 \in I_2 \} = \\
& \text{Max}^{\leq_N} \{ \varphi(m_1) \cdot_N \varphi(m_2) \mid m_1 \in I_1, m_2 \in I_2 \} = \\
& \text{Max}^{\leq_N} \{ n_1 \cdot_N n_2 \mid n_1 \in \varphi(I_1), n_2 \in \varphi(I_2) \} = \\
& cSRng\langle \varphi \rangle(I_1) \cdot_N cSRng\langle \varphi \rangle(I_2) .
\end{aligned}$$

30. For each $M \in |\mathbf{mMon}|$, define $\eta_M^{\text{cSRng}} : M \rightarrow mMon(cSRng\langle M \rangle)$ by $\eta_M^{\text{cSRng}}(m) = \{m\}$. Then $\eta^{\text{cSRng}} = (\eta_M^{\text{cSRng}})_{M \in |\mathbf{mMon}|}$ is a natural transformation from $1_{\mathbf{mMon}}$ to $mMon \circ cSRng\langle - \rangle$. Let $M \in |\mathbf{mMon}|$, $A \in |\mathbf{cSRng}|$, and $\varphi : M \rightarrow mMon(A)$. Define $\varphi^{\#_{\text{cSRng}}} : cSRng\langle M \rangle \rightarrow A$ by

$$\varphi^{\#_{\text{cSRng}}}(\{m_1, \dots, m_n\}) = \varphi(m_1) \oplus_A \dots \oplus_A \varphi(m_n)$$

for all $\{m_1, \dots, m_n\} \in \mathcal{I}_{\text{fin}}(M)$, where, if $n = 0$, the right hand side is to be understood as 0_A ; $\varphi^{\#_{\text{cSRng}}}$ is indeed a c-semiring homomorphism, since for each $\{m'_1, \dots, m'_n\} \in \mathcal{P}_{\text{fin}} |M|$ we have $\varphi^{\#_{\text{cSRng}}}(\text{Max}^{\leq_M} \{m'_1, \dots, m'_n\}) = \varphi(m'_1) \oplus_A \dots \oplus_A \varphi(m'_n)$: if $m'_i \leq_M m'_j$ then $\varphi(m'_i) \leq_{mMon(A)} \varphi(m'_j)$, i.e., $\varphi(m'_i) \preceq_A \varphi(m'_j)$, and thus $\varphi(m'_i) \oplus_A \varphi(m'_j) = \varphi(m'_j)$.

Then $mMon(\varphi^{\#_{\text{cSRng}}})(\eta_M^{\text{cSRng}}(m)) = \varphi(m)$ and $\varphi^{\#_{\text{cSRng}}}$ is unique with this property.

5. SOFT CONSTRAINTS

31. We redefine essential notions introduced in [5]. A *constraint domain* (X, D) is given by a set X of *variables* and a family $D = (D_x)_{x \in X}$ of *variable domains* where each D_x is a set. A constraint domain (X, D) is *finite* if X and $\bigcup_{x \in X} D_x$ are finite.

A *valuation* for a constraint domain (X, D) is a dependent map $v \in \Pi x \in X . D_x$, i.e., $v(x) \in D_x$; we abbreviate $\Pi x \in X . D_x$ by $[X \rightarrow D]$.

A *constraint* c over a constraint domain (X, D) , or (X, D) -*constraint*, is given by a map $c : [X \rightarrow D] \rightarrow \mathbb{B}$. We also write $v \models c$ for $c(v) = tt$.

32. Given a constraint domain (X, D) and a c-semiring G , a G -*soft constraint* γ over (X, D) , or (X, D) - G -*soft constraint*, is given by a map $\gamma : [X \rightarrow D] \rightarrow |G|$. In particular, a constraint over (X, D) can be considered a B-soft constraint over (X, D) .

Let Γ be a finite set of (X, D) - G -soft constraints. For a $v \in [X \rightarrow D]$ let the *solution degree* for Γ of v be

$$\Gamma(v) = \bigotimes_G \{ \gamma(v) \mid \gamma \in \Gamma \} .$$

Define a binary relation $\leq_\Gamma \subseteq [X \rightarrow D] \times [X \rightarrow D]$ by

$$v \leq_\Gamma w \iff \Gamma(v) \preceq_G \Gamma(w) ,$$

meaning that valuation w is a *better solution* for Γ than the valuation v .

The *maximum solution degrees* of Γ are given by

$$\Gamma^* = \text{Max}^{\preceq_G} \{ \Gamma(v) \mid v \in [X \rightarrow D] \} ,$$

and the *maximum solutions* by

$$\text{Max}^{\leq_\Gamma} [X \rightarrow D] = \{ v \in [X \rightarrow D] \mid \Gamma(v) \in \Gamma^* \} .$$

33. For a constraint domain (X, D) we fix an *extended* constraint domain $(X, D^?)$ setting $D^? = (D_x^?)_{x \in X}$ with $D_x^? = D_x \uplus \{?\}$, where $?$ is fresh.

A valuation $p \in \Pi x \in X . D_x^? = [X \rightarrow D^?]$ is called a *partial valuation* for (X, D) .

The *domain of definition* $\text{def}(p)$ of a partial valuation p for (X, D) is the set $\{x \in X \mid p(x) \neq ?\}$. For $p, q \in [X \rightarrow D^?]$, we write $p \sqsubseteq q$ if $x \in \text{def}(p)$ implies $x \in \text{def}(q)$ and $q(x) = p(x)$ for each $x \in X$; by $p \uparrow$ we denote the set $\{v \in [X \rightarrow D] \mid p \sqsubseteq v\}$ of (X, D) -valuations.

34. An $(X, D^?)$ - G -soft constraint $\omega : [X \rightarrow D^?] \rightarrow |G|$ is *bounding* if $\omega(v) \preceq_G \omega(p)$ for all $p \in [X \rightarrow D^?]$ and $v \in p \uparrow$. A bounding $(X, D^?)$ - G -soft constraint ω is *tight* for a finite set of (X, D) - G -soft constraints Γ if $\omega(v) = \Gamma(v)$ for all $v \in [X \rightarrow D]$.

A pair (π, ω) of $(X, D^?)$ - G -soft constraints forms a *bounding pair* if ω is bounding and for each $p\{x \mapsto d\} \in [X \rightarrow D^?]$ there is a $v \in p \uparrow$ with $\pi(p\{x \mapsto d\}) \preceq_G \omega(v)$; a bounding pair (π, ω) is *tight* for a finite set of (X, D) - G -soft constraints Γ if ω is tight for Γ .

For a bounding pair (π, ω) of $(X, D^?)$ - G -soft constraints the following “branch & bound” algorithm $\text{maxSolDeps}_{(\pi, \omega)}$ computes, given a partial valuation $p \in [X \rightarrow D^?]$ and a finite set of lower bounds $L \subseteq |G|$ (which we assume to contain only elements which are pairwise incomparable w.r.t. $\preceq_G$), the maximum degrees in $L \cup \{\omega(v) \mid v \in p \uparrow\}$ w.r.t. $\preceq_G$, i.e., in particular, if $p = (\lambda x \in X . ?)$ and $L = \emptyset$, the maximum degrees in $\{\omega(v) \mid v \in [X \rightarrow D]\}$:

Assume: – (X, D) finite constraint domain
– G c-semiring
– (π, ω) bounding pair of $(X, D^?)$ - G -soft constraints
In: – $p \in [X \rightarrow D^?]$ partial valuation for (X, D)
– $L \subseteq |G|$ finite and pairwise incomparable w.r.t. $\preceq_G$
Return: $\text{Max}^{\preceq_G}(L \cup \omega(p \uparrow))$

$\text{maxSolDeps}_{(\pi, \omega)}(p, L) \equiv$
if $\forall x \in X . p(x) \neq ?$
then return $\text{Max}^{\preceq_G}(L \cup \{\omega(p)\})$
 $x \leftarrow$ **choose** $\{x \in X \mid p(x) = ?\}$
 $L \leftarrow \text{Max}^{\preceq_G}(L \cup \{\pi(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg \exists l \in L . \omega(p\{x \mapsto d\}) \preceq_G l$
then $L \leftarrow \text{maxSolDeps}_{(\pi, \omega)}(p\{x \mapsto d\}, L)$ **fi od**
return L

We prove the claim that

$$\text{maxSolDeps}_{(\pi, \omega)}(p, L) = \text{Max}^{\preceq_G}(L \cup \omega(p \uparrow))$$

by a first induction on the cardinality n of $\{x \in X \mid p(x) = ?\}$. If $n = 0$, i.e., $p \in [X \rightarrow D]$, then $\text{maxSolDeps}_{(\pi, \omega)}(p, L) = \text{Max}^{\preceq_G}(L \cup \{\omega(p)\})$ and $\{\omega(p)\} = \omega(p \uparrow)$. If $n > 0$, then let $x \in X$ with $p(x) = ?$, and let $d_1, \dots, d_r$ be an enumeration of D_x . Let $P = \{p\{x \mapsto d_i\} \mid 1 \leq i \leq r\}$ and define

$$L_0 = \text{Max}^{\preceq_G}(L \cup P),$$

and inductively

$$L_k = \begin{cases} L_{k-1} & \text{if } \exists l \in L_{k-1} . \omega(p\{x \mapsto d_k\}) \preceq_G l \\ \text{maxSolDeps}_{\omega}(p\{x \mapsto d_k\}, L_{k-1}) & \text{otherwise} \end{cases}$$

for $1 \leq k \leq r$. We prove the sub-claim that

$$L_k = \text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq j \leq k} \omega(p\{x \mapsto d_j\}\uparrow))$$

for all $0 \leq k \leq r$ by a second induction on k : For $k = 0$, $L_0 = \text{Max}^{\preceq_G}(L \cup P)$ by definition. For $k > 0$, let there first be an $l \in L_{k-1}$ with $\omega(p\{x \mapsto d_k\}) \preceq_G l$. Since $\omega(v) \preceq_G \omega(p\{x \mapsto d_k\}) \preceq_G l$ for all $v \in p\{x \mapsto d_k\}\uparrow$ and $l \in L_{k-1} = \text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$ by the second induction hypothesis, the sub-claim follows. Otherwise, if no such $l \in L_{k-1}$ exists, $\text{maxSolDegr}_\omega(p\{x \mapsto d_k\}, L_{k-1}) = \text{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow))$ by the first induction hypothesis, which is applicable since, by the second induction hypothesis, $L_{k-1} = \text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$, and hence L_{k-1} is pairwise incomparable w.r.t. $\preceq_G$; therefore,

$$\begin{aligned} L_k &= \text{maxSolDegr}_\omega(p\{x \mapsto d_k\}, L_{k-1}) = \\ &\text{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ &\text{Max}^{\preceq_G}((\text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))) \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ &\text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k} \omega(p\{x \mapsto d_i\}\uparrow)), \end{aligned}$$

which establishes the sub-claim. Thus, $L_r = \text{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq r} \omega(p\{x \mapsto d_i\}\uparrow)) = \text{Max}^{\preceq_G}(L \cup \omega(p\uparrow)) = \text{maxSolDegr}_\omega(p, L)$, since $d_1, \dots, d_r$ is an enumeration of D_x and for each $1 \leq i \leq r$ there is a $v \in p\uparrow$ with $\pi(p\{x \mapsto d_i\}) \preceq_G \omega(v)$, which yields the claim.

In particular, if $(\Gamma?, \Gamma?)$ is a tight bounding pair of $(X, D^?)$ - G -soft constraints for a finite set of (X, D) - G -soft constraints Γ , then

$$\begin{aligned} \text{maxSolDegr}_{(\Gamma?, \Gamma?)}(\lambda x \in X.?, \emptyset) &= \text{Max}^{\preceq_G}(\emptyset \cup \Gamma^?((\lambda x \in X.?)\uparrow)) = \\ &\text{Max}^{\preceq_G}\{\Gamma^?(v) \mid v \in [X \rightarrow D]\} = \text{Max}^{\preceq_G}\{\Gamma(v) \mid v \in [X \rightarrow D]\} = \Gamma^*. \end{aligned}$$

35. Given a meet monoid M and a constraint domain (X, D) , an M -soft constraint presentation μ over (X, D) , or (X, D) - M -soft constraint presentation, is given by a map $\mu : [X \rightarrow D] \rightarrow |M|$. Each (X, D) - M -soft constraint presentation induces an (X, D) - $cSRng\langle M \rangle$ -soft constraint $\eta_M^{\text{cSRng}} \circ \mu$ (viz., $(\eta_M^{\text{cSRng}} \circ \mu)(v) = \{\mu(v)\}$).

Let M be a finite set of (X, D) - M -soft constraint presentations. Then

$$\bigotimes_{cSRng\langle M \rangle} \{\eta_M^{\text{cSRng}}(\mu(v)) \mid \mu \in M\} = \{\prod_M \{\mu(v) \mid \mu \in M\}\},$$

and thus

$$\begin{aligned} v \leq_{\eta_M^{\text{cSRng}} \circ M} w &\iff \{\prod_M \{\mu(v) \mid \mu \in M\}\} \preceq_{cSRng\langle M \rangle} \{\prod_M \{\mu(w) \mid \mu \in M\}\} \iff \\ &\prod_M \{\mu(v) \mid \mu \in M\} \leq_M \prod_M \{\mu(w) \mid \mu \in M\}. \end{aligned}$$

We write $cSRng\langle M \rangle$ for the set of (X, D) - $cSRng\langle M \rangle$ -soft constraints $\{\eta_M^{\text{cSRng}} \circ \mu \mid \mu \in M\}$.

In analogy to the notions for soft constraints, we define the *solution degree* for a $v \in [X \rightarrow D]$ by

$$M(v) = \prod_M \{\mu(v) \mid \mu \in M\},$$

and the *maximum solution degrees* by

$$M^* = \text{Max}^{\leq_M}\{M(v) \mid v \in [X \rightarrow D]\}.$$

36. The algorithm $\text{maxSolDegr}_{(\pi, \omega)}$ for a bounding pair (π, ω) of $(X, D^?)$ - G -soft constraints from §34 in fact also works under the assumptions that (X, D) is a constraint domain, M is a meet monoid, and α and ζ are $(X, D^?)$ - M -soft constraint representations, such that $(\eta_M^{\text{cSRng}} \circ \alpha, \eta_M^{\text{cSRng}} \circ \zeta)$ is a bounding pair. We call (α, ζ) itself a *bounding pair* if $\zeta(v) \leq_M \zeta(p)$ for all $p \in [X \rightarrow D^?]$ and $v \in p\uparrow$, and if for each $p\{x \mapsto d\} \in [X \rightarrow D^?]$ there is a $v \in p\uparrow$ such that $\alpha(p\{x \mapsto d\}) \leq_M \zeta(v)$.

Assume: – (X, D) finite constraint domain
– M meet monoid
– (α, ζ) bounding pair of $(X, D^?)$ - M -soft constraint presentations
In: – $p \in [X \rightarrow D^?]$ partial valuation for (X, D)
– $L \subseteq |M|$ finite and pairwise incomparable w.r.t. $\leq_M$
Return: $\text{Max}^{\leq_M}(L \cup \zeta(p\uparrow))$

```

maxSolDegr(α,ζ)(p, L) ≡
  if ∀x ∈ X . p(x) ≠ ?
  then return Max≤M(L ∪ {ζ(p)}) fi
  x ← choose {x ∈ X | p(x) = ?}
  L ← Max≤M(L ∪ {α(p{x ↦ d}) | d ∈ Dx})
  for d ∈ Dx
  do if ¬∃l ∈ L . ζ(p{x ↦ d}) ≤M l
    then L ← maxSolDegr(α,ζ)(p{x ↦ d}, L) fi od
  return L

```

Note that $\text{maxSolDegr}_{(\alpha, \zeta)}(p, L) = L \oplus_{\text{cSRng}\langle M \rangle} \text{Max}^{\leq_M} \zeta(p\uparrow)$.

A bounding pair (α, ζ) of $(X, D^?)$ - M -soft constraint presentations is *tight* for a finite set of (X, D) - M -soft constraint presentations M if $\zeta(v) = M(v)$ for all $v \in [X \rightarrow D^?]$. For a tight bounding pair $(M_?, M^?)$ for a finite set of (X, D) - M -soft constraint presentations M we again obtain

$$\text{maxSolDegr}_{(M_?, M^?)}(\lambda x \in X . ?, \emptyset) = \text{Max}^{\leq_M} \{M(v) \mid v \in [X \rightarrow D]\} = M^*.$$

6. CONSTRAINT HIERARCHIES

37. A *constraint hierarchy* [2] $H = (C_k)_{1 \leq k \leq n}$ over a constraint domain (X, D) , or (X, D) -*constraint hierarchy*, is given by a family of sets of (X, D) -constraints. The constraints in level $1 \leq k \leq n$ are considered as *strictly more important* than the constraints in level $k + 1$. An (X, D) -constraint hierarchy is *finite* if $\bigcup_{1 \leq k \leq n} C_k$ is finite.

Let $H = (C_k)_{1 \leq k \leq n}$ be a finite (X, D) -constraint hierarchy, let $W = (M_i)_{1 \leq i \leq n}$ be a corresponding family of meet monoids, and let for each $1 \leq k \leq n$ and each $c \in C_k$, $\mu(c)$ be an (X, D) - M_k -soft constraint presentation. We call $H = (M_k)_{1 \leq k \leq n}$ with $M_k = \{\mu(c) \mid c \in C_k\}$ for $1 \leq k \leq n$ a (X, D) - W -*soft constraint hierarchy presentation*. For a $v \in [X \rightarrow D]$ the *solution degree* for $(M_k)_{1 \leq k \leq n}$ of v is defined to be $(M_k(v))_{1 \leq k \leq n}$. Define a binary relation $<_H \subseteq [X \rightarrow D] \times [X \rightarrow D]$ by

$$v <_H w \iff \exists 1 \leq k \leq n . (\forall 1 \leq i \leq k-1 . M_i(v) = M_i(w)) \wedge M_k(v) <_{M_k} M_k(w),$$

saying that the valuation w is *strictly better* than the valuation v , and denote its reflexive closure on $[X \rightarrow D]$ by $\leq_H$, which is the lexicographic order on the set $\{(M_k(v))_{1 \leq k \leq n} \mid v \in [X \rightarrow D]\}$. In particular,

$$v \leq_H w \iff (M_k(v))_{1 \leq k \leq n} \leq_{M_1 \times \dots \times M_n} (M_k(w))_{1 \leq k \leq n}$$

if, on the one hand, every M_k is a bounded meet monoid for all $2 \leq k \leq n$, and, on the other hand, $M_k(v), M_k(w) \notin \mathcal{C}(M_k)$ for all $1 \leq k \leq n$, or, equivalently, if $\mu(c)(v), \mu(c)(w) \notin \mathcal{C}(M_k)$ for each $c \in C_k, 1 \leq k \leq n$. The first requirement, that each M_k is bounded, can be achieved by moving from M_k to its lifted variant $(M_k)_\perp$.

38. Consider a single level k of a finite (X, D) -constraint hierarchy $H = (C_k)_{1 \leq k \leq n}$, and let $C = C_k$. The *locally-predicate-better (LPB) comparator* for C corresponds to requiring

$$v <_C^{\text{LPB}} w \iff \{c \in C \mid w \not\models c\} \subseteq \{c \in C \mid v \not\models c\}.$$

This can be expressed by choosing the meet monoid $M = (\mathcal{P}_{\text{fin}}(C), \cup, \emptyset, \supseteq)$ and the set of (X, D) - M -soft constraint representations $M = \{\mu(c) \mid c \in C\}$ with $\mu(c)(v) = \{c\}$ if $v \not\models c$ and $\mu(c)(v) = \emptyset$ otherwise, for each $c \in C$. However, all elements of M are idempotent, and thus the collapsing elements of M are $\mathcal{P}_{\text{fin}}(C) \setminus \{\emptyset\}$. Hence, M is not suitable for a lexicographic product.

Choosing instead the meet monoid $N = (\mathcal{M}_{\text{fin}}(C), \sqcup, \sqcap, \supseteq)$ which has no collapsing elements and the set of (X, D) - N -soft constraint representations $N = \{\nu(c) \mid c \in C\}$ with $\nu(c)(v) = \{c\}$ if $v \not\models c$ and $\nu(c)(v) = \sqcap$ otherwise, for each $c \in C$, deviates this situation, since we have

$$M(v) \leq_M M(w) \iff N(v) \leq_N N(w)$$

for all $v, w \in [X \rightarrow D]$.

39. A *weighting* for a set C of (X, D) -constraints is given by a function $g : C \times [X \rightarrow D] \rightarrow \mathbb{R}_{\geq 0}$ with $g(c, v) = 0$ if $v \models c$ for $v \in [X \rightarrow D]$ and $c \in C$.

We consider the following *level weightings* for a level k of a finite (X, D) -constraint hierarchy $H = (C_k)_{1 \leq k \leq n}$, letting $C = C_k$:

- *Weighted sum*: $W(v) = \sum_{c \in C} g(c, v)$.
- *Least squares*: $W(v) = \sqrt{\sum_{c \in C} g(c, v)^2}$.
- *Worst case*: $W(v) = \max\{g(c, v) \mid c \in C\}$.

Each of these level weightings W induces a relation $\leq_C^W \subseteq [X \rightarrow D] \times [X \rightarrow D]$ on valuations defined by

$$v \leq_C^W w \iff W(w) \leq W(v),$$

where $\leq$ is the usual order over the real numbers.

Let $R = (\mathbb{R}_{\geq 0}, \otimes, 0, \geq)$ be a meet monoid (where $\leq$ is the usual order over the real numbers) and let $g : C \times [X \rightarrow D] \rightarrow \mathbb{R}_{\geq 0}$ be a weighting for C . Consider $W_R(v) = \bigotimes_{c \in C} g(c, v)$. Then we have the following correspondences:

- *Weighted sum*: $r \otimes s = r + s$.
- *Least squares*: $r \otimes s = \sqrt{r^2 + s^2}$.
- *Worst case*: $r \otimes s = \max\{r, s\}$.

For any $p > 0$, $r \otimes s = (r^p + s^p)^{1/p}$ is strictly monotonic in both arguments and thus the corresponding R has no collapsing elements, which covers the cases of weighted sum and least squares.

In the case of worst case, $\mathcal{C}(R) = \mathbb{R}_{\geq 0} \setminus \{0\}$, since $\otimes$ is idempotent. Assume that C has three different constraints c_1, c_2 , and c_3 ; that there are valuations v_1 violating only c_1 , v_2 violating only c_2 , v_{13} violating exactly c_1 and c_3 , and v_{23} violating exactly c_2 and c_3 ; that $g(c, v) = 0$ if, and only if, $c \in \{c_1, c_2, c_3\}$ is satisfied by $v \in \{v_1, v_2, v_{13}, v_{23}\}$; and that the weightings are independent of the valuation, i.e., $g(c_1, v_1) = g(c_1, v_{13})$ and $g(c_2, v_2) = g(c_2, v_{23})$ and $g(c_3, v_{13}) = g(c_3, v_{23})$. Also assume that the level weightings for the valuations v_1, v_2, v_{13} , and v_{23} for the worst case are related by

$$\begin{aligned} W_R(v_1) &= g(c_1, v_1) > g(c_2, v_2) = W_R(v_2), \\ W_R(v_{13}) &= \max\{g(c_1, v_{13}), g(c_3, v_{13})\} = \max\{g(c_2, v_{23}), g(c_3, v_{23})\} = W_R(v_{23}). \end{aligned}$$

Any set of (X, D) - M -soft constraint representations $M = \{\mu(c) \mid c \in C\}$ reflecting the ordering induced by W_R on valuations, i.e., $M(v) \leq_M M(w) \iff W_R(w) \leq W_R(v)$, would thus have $\mu(c_3)$ as collapsing element in M .

7. CONSTRAINT RELATIONSHIPS

40. A *constraint relationship* over a constraint domain (X, D) , or (X, D) -*constraint relationship*, is given by a dag C , where $|C|$ is a set of (X, D) -constraints. We think of a constraint $c' \in |C|$ as *more important* than another constraint $c \in |C|$ if $c \rightarrow_C c'$. An (X, D) -constraint relationship C is *finite* if $|C|$ is finite.

41. Let C be an (X, D) -constraint relationship and let M be a meet monoid with a partial order homomorphism $\varphi : PO\langle C \rangle \rightarrow PO(M)$. For each $c \in |C|$, define the (X, D) - M -soft constraint representation $c_{M,\varphi} : [X \rightarrow D] \rightarrow |M|$ by

$$c_{M,\varphi}(v) = \begin{cases} \varphi(c) & \text{if } v \not\models c \\ \varepsilon_M & \text{otherwise} \end{cases}.$$

We write $C_{M,\varphi}$ for the set of (X, D) - M -soft constraint presentations $\{c_{M,\varphi} \mid c \in |C|\}$.

EXAMPLE. Let C be a finite constraint relationship over (X, D) .

(1) We first consider the single-predecessor lifting introduced in §16.

Let $M_C = mMon(jMon\langle PO\langle C \rangle \rangle) = mMon\langle PO\langle C \rangle^{-1} \rangle$ and define $m_C : PO\langle C \rangle \rightarrow PO(M_C)$ by $m_C(c) = \eta_{PO\langle C \rangle^{-1}}^{mMon}(c) = \downarrow c$; in particular $\varepsilon_{M_C} = \downarrow$. Then for $v, w \in [X \rightarrow D]$

$$\begin{aligned} v \leq_{cSRng\langle C_{M_C, m_C} \rangle} w &\iff \\ \prod_{M_C} \{c_{M_C, m_C}(v) \mid c \in |C|\} &\leq_{M_C} \prod_{M_C} \{c_{M_C, m_C}(w) \mid c \in |C|\} \iff \\ \downarrow c \mid v \not\models c, c \in |C| &\subseteq^{PO\langle C \rangle^{-1}} \downarrow c \mid w \not\models c, c \in |C| \iff \\ \downarrow c \mid w \not\models c, c \in |C| &\subseteq_{PO\langle C \rangle} \downarrow c \mid v \not\models c, c \in |C|. \end{aligned}$$

Thus, w is considered a better solution if each constraint that is not satisfied by w can be paired off with a constraint that is not satisfied by v and which is more important.

(2) We now consider the transitive-predecessors lifting introduced in §17.

Let $U_C = mMon(jMon(uSL\langle PO\langle C \rangle \rangle))$ and define $u_C : PO\langle C \rangle \rightarrow PO(U_C)$ by $u_C(c) = \{c\}$; in particular, $\varepsilon_{U_C} = \emptyset$. Then for $v, w \in [X \rightarrow D]$

$$\begin{aligned} v \leq_{cSRng\langle C_{U_C, u_C} \rangle} w &\iff \\ \prod_{U_C} \{c_{U_C, u_C}(v) \mid c \in |C|\} &\leq_{U_C} \prod_{U_C} \{c_{U_C, u_C}(w) \mid c \in |C|\} \iff \\ \text{Max}^{\leq_C} \{c \mid v \not\models c, c \in |C|\} &\supseteq_{PO\langle C \rangle} \text{Max}^{\leq_C} \{c \mid w \not\models c, c \in |C|\} \iff \\ \text{Max}^{\leq_C} \{c \mid w \not\models c, c \in |C|\} &\subseteq_{PO\langle C \rangle} \text{Max}^{\leq_C} \{c \mid v \not\models c, c \in |C|\} \iff \\ \forall c \in \{c_w \in |C| \mid w \not\models c_w\}. \exists c' \in \{c_v \in |C| \mid v \not\models c_v\}. c &\leq_{PO\langle C \rangle} c'. \end{aligned}$$

Thus, w is considered a better solution if each constraint that is not satisfied by w can be covered by a constraint that is not satisfied by v and which is more important. $\square$

42. The *scope* of a constraint c over a constraint domain (X, D) is given by the set of variables it depends on, i.e.,

$$\text{sc}(c) = \{x \in X \mid \exists v \in [X \rightarrow D], d_1 \neq d_2 \in D_x. c(v\{x \mapsto d_1\}) \neq c(v\{x \mapsto d_2\})\}.$$

For a partial valuation $p \in [X \rightarrow D^?]$, we write $p \not\models c$ if $\text{sc}(c) \subseteq \text{def}(p)$ and $v \not\models c$ for some $v \in p\uparrow$ (which is well-defined, since then c only depends on variables that are in the domain of definition of p).

Let C be a finite constraint relationship over (X, D) , let M be a meet monoid, and let $\varphi : PO\langle C \rangle \rightarrow PO(M)$. Define $\alpha_{M,\varphi}, \zeta_{M,\varphi} : [X \rightarrow D^?] \rightarrow |M|$ by

$$\begin{aligned}\alpha_{M,\varphi}(p) &= \prod_M \{\varphi(c) \mid c \in |C|, \text{sc}(c) \subseteq \text{def}(p), p \not\models c\} \cdot_M \prod_M \{\varphi(c) \mid \text{sc}(c) \not\subseteq \text{def}(p)\}, \\ \zeta_{M,\varphi}(p) &= \prod_M \{\varphi(c) \mid c \in |C|, \text{sc}(c) \subseteq \text{def}(p), p \not\models c\}.\end{aligned}$$

LEMMA. $(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is a tight bounding pair of $(X, D^?)$ - M -soft constraint presentations for $C_{M,\varphi}$.

Proof. For a $p \in [X \rightarrow D^?]$ let $V(p) = \{c \in |C| \mid \text{sc}(c) \subseteq \text{def}(p), p \not\models c\}$ and $W(p) = \{c \in |C| \mid \text{sc}(c) \not\subseteq \text{def}(p)\}$. Then $\alpha_{M,\varphi}(p) = \prod_M \varphi(V(p)) \cdot_M \prod_M \varphi(W(p))$ and $\zeta_{M,\varphi}(p) = \prod_M \varphi(V(p))$ for all $p \in [X \rightarrow D^?]$.

Let $p \in [X \rightarrow D^?]$ and $v \in p\uparrow$ be given. Then $V(p) \subseteq V(v)$, and thus $\zeta_{M,\varphi}(v) \leq_M \zeta_{M,\varphi}(p)$: For a $c \in V(p)$, i.e., $\text{sc}(c) \subseteq \text{def}(p)$ and $p \not\models c$, also $c \in V(v)$, since $v \in p\uparrow$ and thus $v \not\models c$.

Now let $p' = p\{x \mapsto d\} \in [X \rightarrow D^?]$ and let $v \in p\uparrow$ be arbitrary. Then $V(v) \subseteq V(p') \cup W(p')$, and thus $\alpha_{M,\varphi}(p') \leq_M \zeta_{M,\varphi}(v)$: Let $c \in V(v)$, i.e., $v \not\models c$. If $\text{sc}(c) \subseteq \text{def}(p')$, then $p' \not\models c$, and hence $c \in V(p')$; otherwise $c \in W(p')$.

Finally, $\zeta_{M,\varphi}(v) = C_{M,\varphi}(v)$ and thus $(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is tight for $C_{M,\varphi}$. $\square$

EXAMPLE. Consider the constraint domain (X, D) given by

$$\begin{aligned}X &= \{x, y, z\}, \\ D_x &= D_y = D_z = \{1, 2, 3\}\end{aligned}$$

as well as the constraint relationship $C = (\{c_1, c_2, c_3\}, \{(c_2, c_1), (c_3, c_1)\})$, i.e., $c_2 \rightarrow_C c_1$ and $c_3 \rightarrow_C c_1$, with

$$\begin{aligned}c_1 &: x + 1 = y, \\ c_2 &: z = y + 2, \\ c_3 &: x + y \leq 3.\end{aligned}$$

Let $M_C = mMon\langle PO\langle C \rangle^{-1} \rangle$ and $m_C = \eta_{PO\langle C \rangle^{-1}}^{\text{mMon}}$. Then

$$\begin{aligned}\alpha_{M_C, m_C}(p) &= \{c \in |C| \mid \text{sc}(c) \subseteq \text{def}(p), p \not\models c\} \cup \{c \in |C| \mid \text{sc}(c) \not\subseteq \text{def}(p)\}, \\ \zeta_{M_C, m_C}(p) &= \{c \in |C| \mid \text{sc}(c) \subseteq \text{def}(p), p \not\models c\},\end{aligned}$$

such that, for example,

$$\begin{aligned}\alpha_{M_C, m_C}(\{x \mapsto 1, y \mapsto 1, z \mapsto ?\}) &= \{c_1, c_2\}, & \zeta_{M_C, m_C}(\{x \mapsto 1, y \mapsto 1, z \mapsto ?\}) &= \{c_1\}, \\ \alpha_{M_C, m_C}(\{x \mapsto 1, y \mapsto 2, z \mapsto ?\}) &= \{c_2\}, & \zeta_{M_C, m_C}(\{x \mapsto 1, y \mapsto 2, z \mapsto ?\}) &= \{\}, \\ \alpha_{M_C, m_C}(\{x \mapsto 2, y \mapsto 3, z \mapsto ?\}) &= \{c_2, c_3\}, & \zeta_{M_C, m_C}(\{x \mapsto 2, y \mapsto 3, z \mapsto ?\}) &= \{c_3\};\end{aligned}$$

in particular

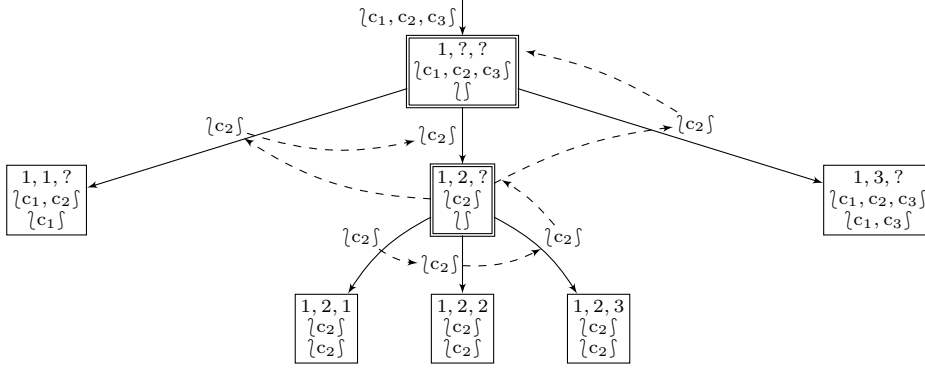
$$\zeta_{M_C, m_C}(\{x \mapsto 1, y \mapsto 1, z \mapsto ?\}) = \{c_1\} <_{M_C} \{c_2\} = \alpha_{M_C, m_C}(\{x \mapsto 1, y \mapsto 2, z \mapsto ?\}).$$

We abbreviate α_{M_C, m_C} by α and ζ_{M_C, m_C} by ζ . We follow an execution of $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto ?, y \mapsto ?, z \mapsto ?\}, \emptyset)$, choosing the variables x, y , and z in this order and running through $\{1, 2, 3\}$ in the natural order; we select x and y first, since $\text{sc}(c_1) = \{x, y\}$ and c_1 is the top element in C .

The first step in evaluating $\text{maxSolDeps}_{(\alpha, \zeta)}(\lambda x \in X. ?, \emptyset)$ is to evaluate $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 1, y \mapsto ?, z \mapsto ?\}, \{c_1, c_2, c_3\})$, since

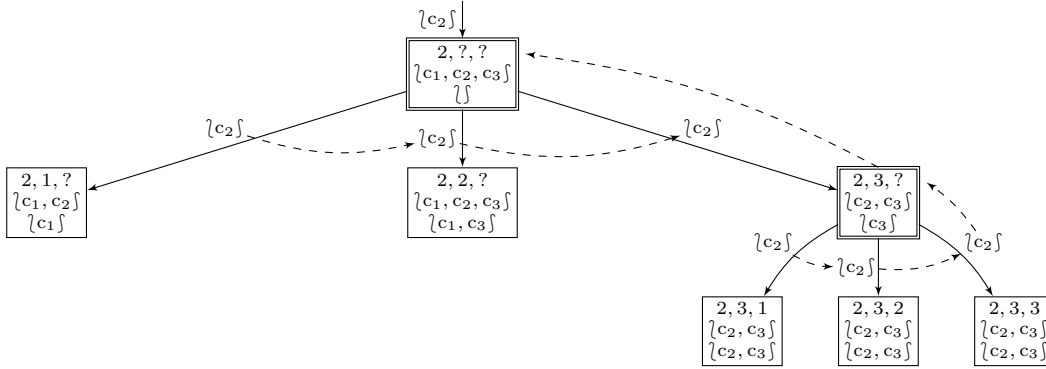
$$\alpha((\lambda x \in X. ?)\{x \mapsto d\}) = \{c_1, c_2, c_3\},$$

for all $d \in \{1, 2, 3\}$ and $\zeta((\lambda x. X)\{x \mapsto 1\}) = \lambda\}$; which leads to the following graph:

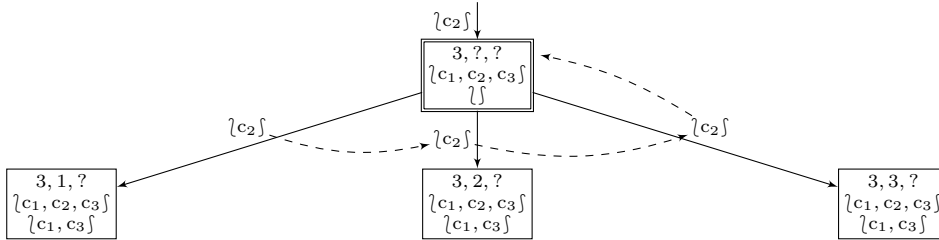


The annotations on the solid edges represent the current values of the lower bound set L , where we have omitted the set braces. Each node gives the respective partial valuation p for x , y , and z at the top, $\alpha(p)$ in the middle, and $\zeta(p)$ at the bottom. Doubly outlined nodes represent calls to $\text{maxSolDeps}_{(\alpha, \zeta)}(p, L)$; singly outlined nodes represent the successful test whether $\zeta(p)$ already is dominated by a lower bound in L . Finally, the dashed edges show the flow of the lower bounds.

Thus, $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 1, y \mapsto ?, z \mapsto ?\}, \emptyset) = \{\lambda c_2\}$, and $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 2, y \mapsto ?, z \mapsto ?\}, \{\lambda c_2\})$ is executed:



Hence, $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 2, y \mapsto ?, z \mapsto ?\}, \{\lambda c_2\}) = \{\lambda c_2\}$ and $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 3, y \mapsto ?, z \mapsto ?\}, \{\lambda c_2\})$ is executed:



Therefore, $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto 3, y \mapsto ?, z \mapsto ?\}, \{\lambda c_2\}) = \{\lambda c_2\}$, and we have as the final result that $\text{maxSolDeps}_{(\alpha, \zeta)}(\{x \mapsto ?, y \mapsto ?, z \mapsto ?\}, \emptyset) = \{\lambda c_2\}$. $\square$

REFERENCES

- [1] Stefano Bistarelli, Ugo Montanari, Francesca Rossi, Thomas Schiex, Gérard Verfaillie, and Hélène Fargier. Semiring-Based CSPs and Valued CSPs: Frameworks, Properties, and Comparison. *Constraints*, 4(3):199–240, 1999.
- [2] Alan Borning, Bjorn Freeman-Benson, and Molly Wilson. Constraint Hierarchies. *LISP Symb. Comp.*, 5:223–270, 1992.
- [3] Fabio Gadducci, Matthias M. Hözl, Giacomina V. Monreale, and Martin Wirsing. Soft Constraints for Lexicographic Orders. In Félix Castro, Alexander Gelbukh, and Miguel González, editors, *Proc. 12th Mexican Int. Conf. Artificial Intelligence (MICAI'13)*, Lect Notes Comp. Sci. 8265, pages 68–79. Springer, 2013.
- [4] Matthias Hözl, Max Meier, and Martin Wirsing. Which Soft Constraints do you Prefer? In *Proc. 7th Int. Wsh. Rewriting Logic and its Applications (WRLA'08)*, volume 238 of *Electronic Notes in Theoretical Computer Science*, pages 189–205, 2009.
- [5] P. Meseguer, Francesca Rossi, and T. Schiex. Soft Constraints. In Francesca Rossi, P. van Beek, and Toby Walsh, editors, *Handbook of Constraint Programming*, chapter 9. Elsevier, 2006.