ADAPTIVE FINITE ELEMENT METHODS FOR DOMAIN DECOMPOSITION ON NONMATCHING GRIDS

B. ENGELMANN*, R.H.W. HOPPE*, Y. ILIASH†, Y.A. KUZNETSOV‡, Y. VASSILEVSKI†, AND B. WOHLMUTH*

Abstract. We consider the parallel solution of elliptic boundary value problems discretized by domain decomposition methods on nonmatching grids involving mortar finite elements. We start from an initial nonoverlapping decomposition of the computational domain and independent triangulations of the subdomains realizing weak continuity conditions on the internal subdomain boundaries by means of appropriately chosen Lagrangian multipliers. The solution process features a preconditioned Lanczos iteration for the resulting saddle point problem using a block diagonal preconditioner and an adaptive local mesh refinement on the basis of efficient and reliable residual based or hierarchical a posteriori error estimators. In the parallel implementation of the code, the data related to the subdomains are appropriately distributed among the available processors. The efficiency of the parallel implementation and the benefits of the adaptive grid refinement process are illustrated by numerical simulation results obtained on a IBM SP2 for some selected test problems including fully potential flow around profiles.

1. Introduction. Domain decomposition methods on nonmatching grids based on the mortar finite element approach have attracted a lot of attention during the last couple of years. Beginning with the very first papers on this subject by Bernardi, Maday and Patera in the late eighties [13, 14] there have been many important contributions by various authors mainly focusing on such issues as a priori error estimates, the analysis of preconditioned iterative solvers and the efficient parallel implementation on parallel architectures (cf., e.g., [2–4, 10–12, 16, 18, 24–27]). On the other hand, as far as the realization of adaptivity concepts based on efficient and reliable a posteriori error estimators is concerned, much less work has been done so far. Indeed, to the authors' knowledge the papers [32] and [38] are the only ones that have addressed this equally important subject.

In this paper, we will consider mortar finite element approximations of elliptic boundary value problems both under the aspect of the multilevel preconditioned iterative solution of the discretized problems and with regard to a posteriori error estimators as a fundamental tool for local adaptive mesh refinement.

*Institute of Mathematics, University of Augsburg, Universitaetsstr. 14, D-86159 Augsburg, GERMANY.

†Institute of Numerical Mathematics, Russian Academy of Sciences, Gubkina 8, 117333 Moscow, RUSSIA.

‡Department of Mathematics, University of Houston, Houston, TX 77204-3476, U.S.A.

The paper is organized as follows:
In section 2, we will give a brief introduction into the idea of domain decomposition on nonmatching grids by mortar finite element techniques. Then, in section 3 we will focus on the iterative solution process featuring a multilevel preconditioned generalized Lanczos iteration with a blockdiagonal preconditioner involving appropriate subdomain preconditioners and a preconditioner for the Schur complement that arises from static condensation of the unknowns associated with the individual subdomains. In section 4, we will study a residual based error estimator that does provide local lower bounds and a global upper bound for the total error measured in the energy norm. We will further investigate a hierarchical type estimator that can be derived by a localization of the error equation on the subdomains' level leading to individual Neumann type problems that can be solved by the principle of defect correction in higher order ansatz spaces combined with a suitable localization by a hierarchical two-level splitting. Finally, in section 5 we will document the benefits of the adaptive approach by some illustrative numerical examples and also present numerical simulation results for the application of the techniques to fully potential flow problems with emphasis on the parallel performance of the algorithms.

2. The mortar finite element method. We consider a boundary value problem for a linear second order elliptic differential operator on a bounded polygonal domain $\Omega \subset \mathbf{R}$

$$Lu := -\nabla \cdot (a \nabla u) + b u = f \text{ in } \Omega , \tag{2.1}$$

$$\mathbf{n} \cdot a \nabla u = 0 \text{ on } \Gamma = \partial \Omega \tag{2.2}$$

where for simplicity we have chosen homogeneous Neumann boundary conditions. We assume $f \in L^2(\Omega)$ and a, b to be a symmetric, matrix-valued function $a = (a_{ij})_{i,j=1}^2$ with $a_{i,j} \in L^\infty(\Omega), 1 \le i, j \le 2$, and a scalar function $b \in L^\infty(\Omega)$ satisfying

$$\underline{\alpha} \mid \xi \mid^2 \le \sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \le \bar{\alpha} \mid \xi \mid^2 , \ \xi \in \mathbf{R}^2 , \ 0 < \underline{\alpha} \le \bar{\alpha} , \tag{2.3}$$

$$0 \le \underline{\beta} \le b(x) \le \bar{\beta} \tag{2.4}$$

for almost all $x \in \Omega$. We note that (2.1), (2.2) admits a unique weak solution $u \in H^1(\Omega)$ if $\underline{\beta} > 0$. Otherwise we have to impose another condition on the data of the problem as, for instance, $\int_\Omega f \, dx = 0$.

For discretization, we start from a nonoverlapping decomposition

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i , \ \Omega_i \cap \Omega_j = \emptyset , \ 1 \le i \ne j \le N , \tag{2.5}$$

of the computational domain Ω into N mutually disjoint, polygonal subdomains $\Omega_i, 1 \le i \le N$, and we assume that the partition is geometrically

conforming in the sense that any edge of $\partial\Omega_i$ either is part of the boundary Γ of the entire domain Ω or coincides with the edge of an adjacent subdomain (cf. Figures 1, 2).
We set $\mathcal{T} = \cup_{i=1}^N \mathcal{T}_i$ and denote by $\mathcal{N}^D$ and $\mathcal{E}^D$ the sets of vertices and edges of $\mathcal{T}$ in $D \subseteq \bar{\Omega}$.
On the subdomains Ω_i we consider individual, simplicial triangulations $\mathcal{T}_i$ regardless the situation on the interfaces between the subdomains so that typically nonconforming nodal points will occur on the boundary between two adjacent subdomains (cf. Figure 1).
On the subdomains we choose the standard conforming $P1$ approximation, i.e., we use continuous, piecewise linear finite elements with respect to the triangulation $\mathcal{T}_i$ and we denote by

$$S_1(\Omega_i; \mathcal{T}_i) := \{ v \in H^1(\Omega) \mid v|_T \in P_1(T) , T \in \mathcal{T}_i \}$$

the associated finite element space.

The individual $P1$ approximations on the subdomains give rise to the bilinear form $a : \prod_{i=1}^N S_1(\Omega_i; \mathcal{T}_i) \times \prod_{i=1}^N S_1(\Omega_i; \mathcal{T}_i) \to \mathbf{R}$

$$a(v,w) := \sum_{i=1}^N a_i(v,w) , \quad v,w \in \prod_{i=1}^N S_1(\Omega_i; \mathcal{T}_i) \tag{2.6}$$

where

$$a_i(v,w) := \int_{\Omega_i} [a \nabla u \cdot \nabla w + b\, v\, w]\, dx .$$

Obviously, the product space $\prod_{i=1}^N S_1(\Omega_i; \mathcal{T}_i)$ is not a subspace of $H^1(\Omega)$ due to the nonconformity on the internal subdomain boundaries and therefore, we have to impose proper matching conditions on the interfaces in order to achieve consistency. For that purpose we consider the so-called skeleton of the decomposition

$$S = \bigcup_{i=1}^N (\partial\Omega_i \setminus \partial\Omega) = \bigcup \{ \Gamma_{ij} \mid \bar{\Omega}_i \cap \bar{\Omega}_j \neq \emptyset , 1 \leq i \neq j \leq N \} \tag{2.7}$$

consisting of all subdomain boundaries that are not part of the boundary of the entire domain or, in other words, that is the union of all common boundaries Γ_{ij} of two adjacent subdomains.
We decompose the skeleton S according to

$$S = \bigcup_{\ell=1}^L \bar{\gamma}_\ell , \quad \gamma_\ell \cap \gamma_m = \emptyset , \quad 1 \leq \ell \neq m \leq L \tag{2.8}$$

into the so-called mortars where each mortar γ_ℓ, $1 \leq \ell \leq L$, is the entire edge of a subdomain $\Omega_{M(\ell)}$, $M(\ell) \in \{1, ...N\}$. We denote by

$\Omega_{\bar{M}(\ell)}$, $\bar{M}(\ell) \in \{1,...N\} \setminus \{M(\ell)\}$, the adjacent subdomain and refer to its common boundary Γ_ℓ with $\Omega_{M(\ell)}$ as the nonmortar. This formal distinction between mortars and nonmortars is essential, since due to the nonconformity on the interfaces the trace $W_{M(\ell)}(\gamma_\ell)$ of $S_1(\Omega_{M(\ell)}; \mathcal{T}_{M(\ell)})$ on γ_ℓ does not coincide with the trace $W_{\bar{M}(\ell)}(\Gamma_\ell)$ of $S_1(\Omega_{\bar{M}(\ell)}; \mathcal{T}_{\bar{M}(\ell)})$ on Γ_ℓ. We assume that γ_ℓ and Γ_ℓ are chosen in such a way that the nonmortar Γ_ℓ inherits its trace from the finer triangulation, i.e., card $\mathcal{N}_{\Gamma_\ell} \geq$ card $\mathcal{N}_{\gamma_\ell}$. We impose weak continuity constraints on the skeleton S by Lagrangian multipliers from the multiplier space

$$M_1(S) := \prod_{\ell=1}^{L} \tilde{W}_{\bar{M}(\ell)}(\Gamma_\ell) \tag{2.9}$$

where

$$\begin{aligned} \tilde{W}_{\bar{M}(\ell)}(\Gamma_\ell) := \{ \mu \in C(\Gamma_\ell) \mid \exists\, v \in S_1(\Omega_{\bar{M}(\ell)}; \mathcal{T}_{\bar{M}(\ell)}) \,,\ \mu\,|_E = v\,|_E \,, \\ E \in \mathcal{E}^{\Gamma_\ell^{int}} \,,\ \mu\,|_E \in P_0(E) \,,\ E \in \mathcal{E}^{\Gamma_\ell} \,,\ E \cap \partial\Gamma_\ell \neq \emptyset \} \,. \end{aligned} \tag{2.10}$$

Note that $\tilde{W}_{\bar{M}(\ell)}(\Gamma_\ell)$ is a subset of $W_{\bar{M}(\ell)}(\Gamma_\ell)$ of codimension 2.

We introduce the bilinear form $b : M_1(S) \times \prod_{i=1}^{N} S_1(\Omega_i; \mathcal{T}_i) \to \mathbf{R}$ according to

$$b(\mu, v) := -\sum_{\ell=1}^{L} \int_{\Gamma_\ell} \mu\, [v]_J \, d\sigma \quad , \quad \mu \in M_1(S) \,,\ v \in \prod_{i=1}^{N} S_1(\Omega_i; \mathcal{T}_i) \tag{2.11}$$

where $[v]_J$ denotes the jump of v across the common boundary Γ_ℓ of the two adjacent subdomains $\Omega_{M(\ell)}$ and $\Omega_{\bar{M}(\ell)}$ as given by $[v]_J := v\,|_{\Omega_{\bar{M}(\ell)}} - v\,|_{\Omega_{M(\ell)}}$.

Setting

$$X_1(\Omega; \mathcal{T}) := \{ v \in \prod_{i=1}^{N} S_1(\Omega_i; \mathcal{T}_i) \mid b(\mu, v) = 0 \,,\ \mu \in M_1(S) \} \,, \tag{2.12}$$

the mortar finite element approximation of (2.1), (2.2) amounts to the computation of $u_{X_1} \in X_1(\Omega; \mathcal{T})$ such that

$$a(u_{X_1}, v) = (f, v)_{0;\Omega} \quad , \quad v \in X_1(\Omega; \mathcal{T}) \,. \tag{2.13}$$

Here, $(.,.)_{0;\Omega}$ stands for the standard L^2-inner product. In general, for $D \subseteq \Omega$ and $k \in \mathbf{N}_0$ we refer to $|\,.\,|_{k;D}$ and $\|.\|_{k;D}$ as the Sobolev seminorm and norm on $H^k(D)$, respectively. We further denote by $|||\,.\,|||$ the broken energy norm associated with the bilinear form $a(.,.)$ as given by $|||\,.\,||| = a(.,.)^{1/2}$.

Note that $a(.,.)$ is $X_1(\Omega;\mathcal{T})$-elliptic and hence, (2.13) admits a unique solution.

The variational equation (2.13) can be equivalently stated as a saddle point problem:
Find $(u_M, \lambda_M) \in \prod_{i=1}^N S_1(\Omega_i;\mathcal{T}_i) \times M_1(S)$ such that

$$a(u_M, v) + b(\lambda_M, v) = (f, v)_{0;\Omega} \ , \quad v \in \prod_{i=1}^N S_1(\Omega_i;\mathcal{T}_i) \ , \tag{2.14}$$

$$b(\mu, u_M) = 0 \ , \qquad \mu \in M_1(S) \ . \tag{2.15}$$

We remark that the Babuska-Brezzi condition is satisfied (cf., e.g., [13, 14]).

As has been shown, for instance in [13, 14], if $u \in \prod_{i=1}^N H^2(\Omega_i)$, then the following a priori error estimate holds true

$$||| u - u_M ||| \ \leq \ C\, h\, (\sum_{i=1}^N \|u\|^2_{2;\Omega_i})^{1/2} \tag{2.16}$$

where $C > 0$ is independent of $h := \max\{h_i \mid 1 \leq i \leq N\}$, $h_i := \max\{\text{diam}\, T \mid T \in \mathcal{T}_i\}$.
As far as the Lagrangian multiplier $\lambda_M \in M_1(S)$ is concerned, it can be interpreted as an approximation of the flux $\mathbf{n} \cdot a\, \nabla u$ on the skeleton S. Denoting by $\|.\|_{0;S}$ the weighted L^2-norm

$$\|v\|_{0;S} \ := \ (\sum_{E \subset S} h_E\, \|v\|^2_{0;E})^{1/2} \tag{2.17}$$

where h_E denotes the length of the edge E, under the same regularity assumption as above we have (cf., e.g., [10])

$$\begin{aligned} \|\lambda_M - \mathbf{n} \cdot a\, \nabla u\|_{0;S} \ &\leq \ C \inf_{\mu \in M_1(S)} \|\mu - \mathbf{n} \cdot a\, \nabla u\|_{0;S} \\ &\leq \ C\, h^{3/2} \left(\sum_{i=1}^N \|u\|^2_{2;\Omega_i} \right)^{1/2} \end{aligned} \tag{2.18}$$

again with $C > 0$ being independent of h.

3. Multilevel preconditioned iterative solvers. In its algebraic form the saddle point problem (2.14), (2.15) takes the form

$$\mathcal{A}\, x_M \ = \ \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_M \\ \lambda_M \end{pmatrix} \ = \ \begin{pmatrix} f \\ 0 \end{pmatrix} \ = \ b \tag{3.1}$$

Here, the first diagonal block A of the stiffness matrix $\mathcal{A}$ is a blockdiagonal matrix

$$A \ = \ \text{diag}\,(A_1, ..., A_N)$$

where A_i are the $n_i \times n_i$ subdomain stiffness matrices of dimension $n_i := \text{diam } S_1(\Omega_i; \mathcal{T}_i)$, $1 \leq i \leq N$. The off-diagonal blocks B and B^T represent the continuity constraints on the skeleton S.

Throughout the rest of this section we assume the coefficient functions a and b to be constant on the subdomains Ω_i, i.e.

$$(3.2) \quad a_i := a\,|_{\Omega_i} = \text{ const. } , \quad b_i := b\,|_{\Omega_i} = \text{ const. } , \quad 1 \leq i \leq N \ .$$

Setting $d := \max \{ \text{ diam } \Omega_i \mid 1 \leq i \leq N \}$, we further assume

$$(3.3) \qquad 0 < b_i \leq C\, d^{-2}\, a_i \ , \quad 1 \leq i \leq N$$

where the constant $C > 0$ is independent of d, h and a_i, b_i , $1 \leq i \leq N$. Moreover, the spectrum of a matrix A will be denoted by $\sigma(A)$. In particular, for two stiffness matrices A and B we will use the notation $A \sim B$ if A and B are spectrally equivalent, i.e., if there exist constants $\lambda \leq \Lambda$ independent of the grid size such that $\sigma(B^{-1}\, A) \subset [\lambda, \Lambda]$.

For the iterative solution of (3.1) we will use a blockdiagonal preconditioner

$$(3.4) \qquad \mathcal{R} \ = \ \begin{pmatrix} R_u & 0 \\ 0 & R_\lambda \end{pmatrix}$$

featuring a preconditioner

$$(3.5) \qquad R_u \ := \ \text{diag } (R_1, ..., R_N) \ , \quad R_i \ \sim \ A_i \ , \quad 1 \leq i \leq N$$

for the subdomain stiffness matrices and a preconditioner

$$(3.6) \qquad R_\lambda \ \sim \ S_\lambda \ := \ B\, A^{-1}\, B^T$$

for the Schur complement S_λ arising from static condensation of the unknowns associated with the individual subdomains.

Using a preconditioner $\mathcal{R}$ as in (3.4), the linear system (3.1) will be solved by the preconditioned generalized Lanczos method of minimal iterations:

Given some startiterate $x^0 \ = \ (u_M^0, \lambda_M^0)^T$, we compute x^ν , $\nu \geq 1$, according to

$$(3.7) \qquad x^\nu \ = \ x^{\nu-1} \ - \ \gamma_\nu\, p_\nu \ , \quad \nu \geq 1$$

with p_ν , $\nu \geq 1$, obtained by the recurrence formulae

$$(3.8) \qquad p_\nu \ = \ \begin{cases} \mathcal{R}^{-1} r^0 & \text{if } \ \nu = 1 \\ \mathcal{R}^{-1} \mathcal{A}\, p_1 - \alpha_2\, p_1 & \text{if } \ \nu = 2 \\ \mathcal{R}^{-1} \mathcal{A}\, p_{\nu-1} - \alpha_\nu\, p_{\nu-1} - \beta_\nu\, p_{\nu-2} & \text{if } \ \nu \geq 3 \end{cases}$$

where $r^\nu := \mathcal{A}\, x^\nu - b$, $\nu \geq 0$, and the coefficients α_ν, β_ν, and γ_ν are given by

$$
\begin{aligned}
\alpha_\nu &:= [(\mathcal{A}p_{\nu-1})^T(\mathcal{R}^{-1}\mathcal{A}p_{\nu-1})]^{-1}[(\mathcal{R}^{-1}\mathcal{A}p_{\nu-1})^T(\mathcal{A}\mathcal{R}^{-1}\mathcal{A}p_{\nu-1})], \nu \geq 2, \\
\beta_\nu &:= [(\mathcal{A}p_{\nu-2})^T(\mathcal{R}^{-1}\mathcal{A}p_{\nu-2})]^{-1}[(\mathcal{A}p_{\nu-1})^T(\mathcal{R}^{-1}\mathcal{A}p_{\nu-1})], \nu \geq 3, \\
\gamma_\nu &:= [(\mathcal{A}p_{\nu})^T(\mathcal{R}^{-1}\mathcal{A}p_{\nu})]^{-1}[(\mathcal{A}p_{\nu})^T(\mathcal{R}^{-1}r^{\nu-1})], \nu \geq 1.
\end{aligned} \tag{3.9}
$$

The choice of the iterative solver and the preconditioner is motivated by the following result:

LEMMA 3.1. *Let $\mathcal{A}$ and $\mathcal{R}$ be given by (3.1) and (3.4) with $R_u = A$ and $R_\lambda = S_\lambda$. Then there exist constants $\tau_1 \leq \tau_2 < 0 < \tau_3 \leq \tau_4$ independent of the grid size such that*

$$
\sigma(\mathcal{R}^{-1}\,\mathcal{A}) \subset [\tau_1, \tau_2] \cap [\tau_3, \tau_4] \ . \tag{3.10}
$$

Moreover, denoting by κ the quantity $\kappa := max\{|\tau_1|, \tau_4\}/min\{|\tau_2|, \tau_3\}$, for the preconditioned generalized Lanczos method (3.7)–(3.9) there holds

$$
\|r^{2\nu}\|_{\mathcal{R}^{-1}} < 2\, q^\nu \, \|r^0\|_{\mathcal{R}^{-1}} \ , \ q := (\kappa - 1)/(\kappa + 1) \ , \ \nu \geq 1 \tag{3.11}
$$

where $\|v\|_{\mathcal{R}^{-1}} := v^T\mathcal{R}^{-1}v$.

Proof. The spectral equivalence (3.10) of $\mathcal{A}$ and $\mathcal{R}$ can be easily established by considering the generalized eigenvalue problem $\mathcal{A}\, x = \mu\, \mathcal{R}\, x$ (cf., e.g., [24]) whereas (3.11) is a well known result from the convergence analysis of the Lanczos method (cf., e.g., [28]). □

We will now focus on appropriate subdomain preconditioners R_i, $1 \leq i \leq N$, and a suitable preconditioner R_λ for the Schur complement S_λ. In particular, we will construct such preconditioners that are independent of the refinement level, the quantities d, h, and the values a_i, b_i of the coefficient functions. We begin with the subdomain preconditioners and denote by L_i and M_i the stiffness matrix for the Laplacian $-\Delta$ and the mass matrix with respect to the $P1$ approximation on the subdomains Ω_i, i.e., we take $a_i \equiv 1$, $b_i \equiv 0$ in case of L_i and $a_i \equiv 0$, $c_i \equiv 1$ for M_i, $1 \leq i \leq N$. Moreover, we refer to I_i as the $n_i \times n_i$ identity matrix and to $P_i := n_i^{-1}\,[p_1^{(i)} \mid ... \mid p_{n_i}^{(i)}]$, $p_j^{(i)} := (1, ..., 1)^T$, $1 \leq i, j \leq n_i$ as the orthogonal projection onto Ker L_i.

We have the following result (cf., e.g., [24]):

THEOREM 3.1. *Under the assumptions (3.2), (3.3) let $H_i = R_i^{-1}$ be given by*

$$
H_i \ = \ a_i^{-1}\,(I_i - P_i)\,H_i^L\,(I_i - P_i) \ + \ b_i^{-1}\,h^{-2}\,P_i \tag{3.12}
$$

where $H_i^L = (R_i^L)^{-1}$ with

$$
R_i^L \ \sim \ L_i \ + \ d^{-2}\,M_i \ , \ 1 \leq i \leq N \ . \tag{3.13}
$$

Then there exist constants $0 < \gamma_i \leq \Gamma_i$ *independent of* d, h *and* a_i, b_i *such that*

$$\sigma(R_i^{-1} A_i) \subset [\gamma_i, \Gamma_i] \ , \ 1 \leq i \leq N \ . \tag{3.14}$$

Proof. In view of assumption (3.3) we have $\sigma(L_i) \setminus \{0\} \subset [c_i^{(1)} d^{-2} h^2, c_i^{(2)}]$ with constants $c_i^{(\nu)} > 0$, $1 \leq \nu \leq 2$, independent of d and h. Since $M_i \sim h^2 I_i$, we infer

$$L_i + d^{-2} h^2 P_i \sim L_i + d^{-2} M_i \tag{3.15}$$

and thus

$$A_i = a_i L_i + b_i M_i \sim \hat{R}_i$$

where

$$\hat{R}_i = a_i L_i + b_i h^{-2} P_i = a_i (I_i - P_i) (L_i + d^{-2} h^2 P_i) (I_i - P_i) + b_i h^2 P_i \ .$$

It remains to be shown that

$$\hat{R}_i^{-1} = a_i^{-1} (I_i - P_i) (L_i + d^{-2} h^2 P_i)^{-1} (I_i - P_i) + b_i^{-1} h^{-2} P_i$$

is spectrally equivalent to H_i as given by (3.12). This follows by considering the Rayleigh quotient

$$r(v) = (v^T H_i v)^{-1} v^T \hat{R}_i^{-1} v$$

associated with the generalized eigenvalue problem $\mu H_i v = \hat{R}_i^{-1} v$ and taking into account the spectral equivalences (3.13) and (3.15). □

REMARK 3.1. There are several possible approaches for the construction of a preconditioner R_i^L satisfying (3.13) (cf., e.g., [24]). In the present paper, we may consider $\mathcal{T}_i$ as the finest grid $\mathcal{T}_i = \mathcal{T}_{i\ell}$ of an adaptively generated nested hierarchy $(\mathcal{T}_{ij})_{j=0}^{\ell}$ of triangulations of Ω_i. Then $H_i^L = (R_i^L)^{-1}$ can be easily realized by means of the BPX-preconditioner (cf., e.g., [15, 39]). In terms of $H_i = R_i^{-1}$ as given by (3.12) we thus obtain a subdomain preconditioner that is independent of d, h, ℓ, a_i, b_i and of optimal arithmetical complexity $O(n_i)$.

We will now deal with the construction of an appropriate preconditioner R_λ for the Schur complement $S_\lambda = B A^{-1} B^T$. For this purpose, we decompose the matrices A_i and B_i^T according to

$$A_i = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \ , \quad B_i^T = \begin{pmatrix} 0 \\ (B_{\Gamma\Gamma}^{(i)})^T \end{pmatrix} \tag{3.16}$$

where the subindices I and Γ refer to the unknowns associated with nodal points in the interior of Ω_i and on the boundary $\partial\,\Omega_i$, respectively. Then, the Schur complement S_λ admits the corresponding partition

$$S_\lambda \;=\; \sum_{i=1}^{N} B_{\Gamma\Gamma}^{(i)}\,(S_{\Gamma\Gamma}^{(i)})^{-1}\,(B_{\Gamma\Gamma}^{(i)})^T \tag{3.17}$$

where

$$S_{\Gamma\Gamma}^{(i)} \;:=\; A_{\Gamma\Gamma}^{(i)} \;-\; A_{\Gamma I}^{(i)}\,(A_{II}^{(i)})^{-1}\,A_{I\Gamma}^{(i)}\;,\quad 1\le i\le N \tag{3.18}$$

denote the subdomain Schur complements arising from block elimination of the unknowns associated with the nodal points in the interior of Ω_i. Consequently, if we construct appropriate preconditioners $R_{\Gamma\Gamma}^{(i)}$ for the subdomain Schur complements $S_{\Gamma\Gamma}^{(i)}$, $1\le i\le N$, in view of (3.17) we obtain a preconditioner R_λ for S_λ.

As in the construction of the subdomain preconditioners, we refer to $S_{\Gamma\Gamma}^{L,(i)}$ as the subdomain Schur complements for the Laplacian $-\Delta$ and to $P_{\Gamma\Gamma}^{(i)}$ as the orthogonal projections onto $\mathrm{Ker}\;S_{\Gamma\Gamma}^{L,(i)}$, $1\le i\le N$. We further denote by $I_\Gamma^{(i)}$ the $n_{\Gamma_i}\times n_{\Gamma_i}$ identity matrix, where n_{Γ_i} is the number of nodes on $\partial\,\Omega_i$, and by $M_{\Gamma\Gamma}^{(i)}$, $1\le i\le N$ the second diagonal block in the decomposition of the mass matrix M_i as in (3.16).
Then, we have the following result (cf., e.g., [24]).

THEOREM 3.2. *Under the assumptions (3.2), (3.3) let* $H_\lambda=\hat R_\lambda^{-1}$ *be given by*

$$\begin{aligned} H_\lambda \;=\; \sum_{i=1}^{N} B_{\Gamma\Gamma}^{(i)} \Big[& a_i^{-1}\,(I_\Gamma^{(i)}-P_{\Gamma\Gamma}^{(i)})\,H_{\Gamma\Gamma}^{L,(i)}\,(I_\Gamma^{(i)}-P_{\Gamma\Gamma}^{(i)}) \\ & + b_i^{-1}d^{-1}h^{-1}\,P_{\Gamma\Gamma}^{(i)}\Big]\,(B_{\Gamma\Gamma}^{(i)})^T \end{aligned} \tag{3.19}$$

where $H_{\Gamma\Gamma}^{L,(i)}=(R_{\Gamma\Gamma}^{L,(i)})^{-1}$ *with*

$$R_{\Gamma\Gamma}^{L,(i)} \;\sim\; S_{\Gamma\Gamma}^{L,(i)} \;+\; d\,M_{\Gamma\Gamma}^{(i)}\;,\quad 1\le i\le N\;. \tag{3.20}$$

Then there exist constants $0<\gamma_\lambda\le\Gamma_\lambda$ *independent of* d,h, *and* a_i,b_i *such that*

$$\sigma(\hat R_\lambda^{-1}\,S_\lambda)\;\subset\;[\gamma_\lambda,\Gamma_\lambda]\;. \tag{3.21}$$

Proof. The assertions can be shown by following the lines of proof of Theorem 3.1:

Due to assumption (3.3) we find $\sigma(S_{\Gamma\Gamma}^{L,(i)}) \setminus \{0\} \subset [c_{\Gamma_i}^{(1)} d^{-1} h, c_{\Gamma_i}]$ with constants $c_{\Gamma_i}^{(\nu)} > 0$, $1 \le \nu \le 2$, independent of d and h. Moreover, we have $M_{\Gamma\Gamma}^{(i)} \sim h\, I_{\Gamma}^{(i)}$ which gives

$$S_{\Gamma\Gamma}^{L,(i)} + d^{-1}\, h\, P_{\Gamma\Gamma}^{(i)} \sim S_{\Gamma\Gamma}^{L,(i)} + d\, M_{\Gamma\Gamma}^{(i)} . \tag{3.22}$$

It follows that

$$S_{\Gamma\Gamma}^{(i)} = a_i\, S_{\Gamma\Gamma}^{L,(i)} + b_i\, M_{\Gamma\Gamma}^{(i)} \sim \hat{R}_{\Gamma\Gamma}^{(i)}$$

where

$$\begin{aligned} \hat{R}_{\Gamma\Gamma}^{(i)} &= a_i\, S_{\Gamma\Gamma}^{(i)}\, b_i\, d\, h\, P_{\Gamma\Gamma}^{(i)} \\ &= a_i\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)})\, (S_{\Gamma\Gamma}^{(i)} + d^{-1} h P_{\Gamma\Gamma}^{(i)})\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)}) + b_i d h P_{\Gamma\Gamma}^{(i)} . \end{aligned}$$

Observing

$$\begin{aligned} (\hat{R}_{\Gamma\Gamma}^{(i)})^{-1} &= a_i^{-1}\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)})\, (S_{\Gamma\Gamma}^{L,(i)} + d^{-1} h P_{\Gamma\Gamma}^{(i)})^{-1}\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)}) \\ &\quad + b_i^{-1} d^{-1} h^{-1} P_{\Gamma\Gamma}^{(i)} , \end{aligned}$$

we conclude in view of the spectral equivalences (3.20) and (3.22). □

In practice, we use a Schur complement preconditioner R_λ by means of inner Chebyshev iterations

$$R_\lambda^{-1} = [\, I_\lambda - \prod_{\ell=1}^{L_C} (\, I_\lambda - \beta_\ell\, \hat{R}_\lambda^{-1}\, \hat{S}_\lambda \,) \,]\, \hat{S}_\lambda^{-1} \tag{3.23}$$

where

$$\hat{S}_\lambda := \sum_{i=1}^{N} B_{\Gamma\Gamma}^{(i)} \Big[a_i^{-1}\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)})\, H_{\Gamma\Gamma}^{L,(i)}\, (I_{\Gamma}^{(i)} - P_{\Gamma\Gamma}^{(i)}) + b_i d^{-1} h^{-1} P_{\Gamma\Gamma}^{(i)} \Big]\, (B_{\Gamma\Gamma}^{(i)})^T ,$$

I_λ is the $n_\lambda \times n_\lambda$ identity matrix with n_λ denoting the number of nodal points on S, and β_ℓ, $1 \le \ell \le L_C$, are the Chebyshev parameters with L_C being of order $O((h^{-1}d)^{1/2})$ (cf., e.g., [35]).

Since $S_\lambda \sim \hat{S}_\lambda$, we obtain:

COROLLARY 3.1. *Under the assumptions of Theorem 3.2 the spectral equivalence (3.21) remains valid, if we replace $\hat{R}_\lambda^{-1}$ by R_λ^{-1} as given by (3.23).*

REMARK 3.2. Again, if we consider $\mathcal{T}_i$ as the finest grid $\mathcal{T}_i = \mathcal{T}_{i\ell}$ of a nested hierarchy $(\mathcal{T}_{ij})_{j=0}^{\ell}$, a suitable realization of $H_{\Gamma\Gamma}^{L,(i)} = (R_{\Gamma\Gamma}^{L,(i)})^{-1}$ is given by means of the corresponding boundary blocks of the BPX-preconditioner with respect to that hierarchy. This results in an overall preconditioner R_λ of optimal arithmetical complexity.

An alternative construction can be achieved in view of a result due to Nepomnyashikh [29]: If $A_i^{(1)}$ and $A_i^{(2)}$ are two stiffness matrices with respect to triangulations $\mathcal{T}_i^{(1)} \neq \mathcal{T}_i^{(2)}$ having the same trace on the boundary $\partial\,\Omega_i$, then the associated Schur complements $S_{\Gamma\Gamma}^{1,(i)}$ and $S_{\Gamma\Gamma}^{2,(i)}$ are spectrally equivalent. Hence, for the construction of $R_{\Gamma\Gamma}^{(i)}$ we may use a grid $\tilde{\mathcal{T}}_i$ that has the same trace on $\partial\,\Omega_i$ as $\mathcal{T}_i$ but is much coarser in the interior of Ω_i.

REMARK 3.3. The practical implementation of the inner Chebyshev iterations (3.23) requires the solution of a linear algebraic system of the form $\hat{R}_\lambda\, w = z$ that can be interpreted as a coarse grid system of dimension n_λ (for details see, e.g., [24]).

4. A posteriori error estimators. In this section, we consider two efficient and reliable a posteriori error estimators that are cheaply computable by means of their elementwise contributions and do provide local lower bounds and a global upper bound for the discretization error measured in the $|||\,.\,|||$-norm.

The first estimator relies on a proper evaluation of the residual with respect to the mortar finite element approximation. We note that the concept of residual based error estimation can be traced back to the early work by Babuska and Rheinboldt [5, 6] and has been subsequently further developed and analyzed by various authors [9, 21, 36]. In case of nonconforming finite element methods, such estimators have been recently considered in [17]. For a comprehensive treatment and additional references we refer to [37].

We assume that $(\mathcal{T}_i^{(k)})_{k\in\mathbf{N_0}}$ are regular, locally quasiuniform, nested sequences of simplicial triangulations of Ω_i , $1 \le i \le N$.
Throughout the following we will refer to $c, C > 0$ as generic constants that only depend on the shape regularity of $(\mathcal{T}_i^{(0)})$, $i \le i \le N$, and possibly on the constants $\underline{\alpha}, \bar{\alpha}, \beta, \bar{\beta}$ from (2.3), (2,4).
We remind that $\mathcal{N}_k^D$ and $\mathcal{E}_k^D$ stand for the sets of vertices and edges of $\mathcal{T}_k = \cup_{i=1}^N \mathcal{T}_i^{(k)}$ in $D \subseteq \bar{\Omega}$ and we further denote by h_T, h_E the diameter of $T \in \mathcal{T}_k$ and the length of an edge $E \in \mathcal{E}_k(\bar{\Omega})$. The regularity of the sequence implies

$$c\,h_E \;\le\; h_T \;\le\; C\,h_E \;\;,\;\; E \in \mathcal{E}_k^T \;,\; T \in \mathcal{T}_i^{(k)} \;,\; 1 \le i \le N \tag{4.1}$$

whereas the local quasiuniformity infers that for $T \in \mathcal{T}_i^{(k)}$, $1 \le i \le N$

$$\begin{aligned} &\text{card}\,\{E \in \mathcal{E}_k^{\bar{\Omega}_i} \mid E \cap \partial T \neq \emptyset\} \le C \;, \\ &\text{card}\,\{T' \in \mathcal{T}_i^{(k)} \mid \partial T' \cap \partial T \neq \emptyset\} \le C \;. \end{aligned} \tag{4.2}$$

We note that any adaptively generated nested sequence $(\mathcal{T}_i^{(k)})_{k\in\mathbf{N_0}}$, $1 \le i \le N$, obtained by bisection following the refinement rules of [7] and [33] satisfies the properties of regularity and local quasiuniformity.

We remind that $u \in H^1(\Omega)$ and $(u_M, \lambda_M) \in \prod_{i=1}^N S_1(\Omega_i; \mathcal{T}_i^{(k)}) \times M_1(S)$ denote the weak solution of (2.1), (2.2) and the mortar finite element approximation (2.14), (2.15), respectively. Then, if $u \in \prod_{i=1}^N H^2(\Omega_i)$ and $[\mathbf{n} \cdot a \nabla u]_J = 0$ on S, it is easy to see that the discretization error $e := u - u_M$ satisfies the variational equation

$$a(e, v) = r(v) \ , \ v \in \prod_{i=1}^N H^1(\Omega_i) \tag{4.3}$$

where the residual $r(\cdot)$ is given by

$$r(v) := (f, v)_{0;\Omega} - b(\mathbf{n} \cdot a \nabla u, v) - a(u_M, v) \ . \tag{4.4}$$

We note that in case of the standard conforming $P1$ approximation a global upper bound for the discretization error can be derived by evaluating the residual in the dual norm of $H^1(\Omega)$ using Clément's interpolation operator and taking into account Galerkin orthogonality (cf., e.g., [37]).

Here, the nonconformity of the approach requires the use of a particular interpolation operator

$$P_S := \sum_{i=1}^N P_{S;\Omega_i} \ , \ P_{S;\Omega_i} : H^1(\Omega_i) \to S_1(\Omega_i; \mathcal{T}_i^{(k)}) \ , \ 1 \le i \le N \tag{4.5}$$

where the operators $P_{S;\Omega_i}$, $1 \le i \le N$ are given as follows:
For $p \in \mathcal{N}_k^{\Omega_i}$ we set

$$D_p := \bigcup \{T \in \mathcal{T}_i^{(k)} \mid p \in \mathcal{N}_k(T)\} \ , \ n_p := \text{card } D_p \tag{4.6}$$

and refer to ψ_p^T , $p \in \Omega_i$, $T \in D_p$, and ψ_p^E , $p \in \partial\Omega_i$, $E \in \mathcal{E}_k^{\partial D_p}$, as the functions being dual to the nodal basis functions φ_q^T , $q \in \mathcal{N}_k^T$ in the sense that $(\varphi_q^T, \psi_p^T)_{0;T} = \delta_{pq}$ and $(\varphi_q^T, \psi_p^E)_{0;E} = \delta_{pq}$.

We define $P_{S;\Omega_i} \in S_1(\Omega_i; \mathcal{T}_i)$ by means of

$$(P_{S;\Omega_i} v)(p) := \begin{cases} n_p^{-1} \sum_{T \in D_p} (v, \psi_p^T)_{0;T} \ , \ p \in \Omega_i \ , \\ \dfrac{1}{2} \displaystyle\sum_{E \in \mathcal{E}_k^{\partial D_p}} (v, \psi_p^E)_{0;E} \ , \ p \in \partial\Omega_i \ . \end{cases} \tag{4.7}$$

The operators $P_{S;\Omega_i}$ that have been used in [38] and represent a modification of an operator from [34] (cf. also [30]) admit the following local properties:

LEMMA 4.1. Let $P_{S;\Omega_i} : H^1(\Omega_i) \to S_1(\Omega_i; \mathcal{T}_i^{(k)})$, $1 \le i \le N$, be as in (4.7) and consider the sets D_T and D_E given by

$$D_T := \bigcup \{D_p \mid p \in \mathcal{N}_k^T\} \ , \ T \in \mathcal{T}_i^{(k)} \ , \tag{4.8}$$

$$D_E := \bigcup \{D_p \mid p \in \mathcal{N}_k^E\} \ , \ E \in \mathcal{E}_k(\bar{\Omega}_i) \ . \tag{4.9}$$

Then, for $v \in H^1(\Omega_i)$, $1 \le i \le N$, there holds

$$\|P_{S;\Omega_i} v\|_{1;T} \le C \, \|v\|_{1;D_T} \ , \qquad T \in \mathcal{T}_i^{(k)} \ , \tag{4.10}$$

$$\|v - P_{S;\Omega_i} v\|_{0;T} \le C \, h_T \, \|v\|_{1;D_T} \ , \qquad T \in \mathcal{T}_i^{(k)} \ , \tag{4.11}$$

$$\|v - P_{S;\Omega_i} v\|_{0;E} \le C \, h_E^{1/2} \, \|v\|_{1;D_E} \ , \quad E \in \mathcal{E}_k^{\bar{\Omega}_i} \ . \tag{4.12}$$

Proof. The estimates can be established using the affine equivalence of the elements and Bramble-Hilbert type arguments. □

Now, using the interpolation operator P_S, we split the residual according to

$$r(v) \ = \ r(P_S v) \ + \ r(v - P_S v) \ , \ v \in \prod_{i=1}^{N} H^1(\Omega_i) \ . \tag{4.13}$$

Note that we do not have Galerkin orthogonality. Instead, in view of (2.15) we obtain

$$r(P_S v) \ = \ a(e, P_S v) \ = \ b(\lambda_M - \mathbf{n} \cdot a \nabla u, P_S v) \ . \tag{4.14}$$

On the other hand, from (4.4) it follows that

$$\begin{aligned} r(v - P_S v) \ = \ & (f - L u_M, v - P_S v)_{0;\Omega} - \ b(\mathbf{n} \cdot a \nabla u, v - P_S v) \\ & - \sum_{T \in \mathcal{T}_k} (\mathbf{n} \cdot a \nabla u_M, v - P_S v)_{0;\partial T} \ . \end{aligned} \tag{4.15}$$

Using (4.14) and (4.15) in (4.13), we get

$$\begin{aligned} r(v) \ = \ & (f - L u_M, v - P_S v)_{0;\Omega} \ - \sum_{T \in \mathcal{T}_k} (\mathbf{n} \cdot a \nabla u_M, v - P_S v)_{0;\partial T} \\ & + b(\lambda_M, P_S v) \ - \ b(\mathbf{n} \cdot a \nabla u, v) \ . \end{aligned} \tag{4.16}$$

We note that

$$\begin{aligned} & \sum_{T \in \mathcal{T}_k} (\mathbf{n} \cdot a \nabla u_M, v - P_S v)_{0;\partial T} \ = \\ & \sum_{E \in \mathcal{E}_k^{\bar{\Omega}}} \int_E \{ \, [\mathbf{n} \cdot a \nabla u_M]_J \, [v - P_S v]_A + \ [\mathbf{n} \cdot a \nabla u_M]_A \, [v - P_S v]_J \, \} \, d\sigma \end{aligned} \tag{4.17}$$

where $[.]_J$ stands for the jump and $[.]_A$ for the average across the edges of $\mathcal{T}_k$.

We further refer to $\Pi_0 f$ and $\Pi_{\gamma_\ell} \lambda_M$ as the L^2-projections

$$(\Pi_0 f, v)_{0;\Omega} = (f, v)_{0;\Omega} \ , \qquad v \in W_0(\Omega; \mathcal{T}_k) \ , \tag{4.18}$$

$$(\Pi_{\gamma_\ell} \lambda_M, v)_{0;\gamma_\ell} = (\lambda_M, v)_{0;\gamma_\ell} \ , \quad v \in \tilde{W}_{M(\ell)}(\gamma_\ell) \tag{4.19}$$

where $W_0(\Omega;\mathcal{T}_k) := \{v \in L^2(\Omega) \mid v\,|_T \in P_0(T)\ ,\ T \in \mathcal{T}_k\}$ and $\tilde{W}_{M(\ell)}(\gamma_\ell)$ is defined as in (2.10) with Γ_ℓ and $M(\ell)$ replaced by γ_ℓ and $M(\ell)$, respectively. Then, setting $v = e$ in (4.16) and observing (4.17) as well as $[e - P_S e]_J = 0$ on $E \in \mathcal{E}_k^{\Omega\setminus S}$ and $[e]_J = [u_M]_J$ on $E \in \mathcal{E}_k^{\Gamma_\ell}$, for arbitrarily chosen $\mu \in M_1(S)$ we obtain

$$
\begin{aligned}
|||\, e \,|||^2 \;&=\; r(e) \\
&=\; (\Pi_0 f - Lu_M, e - P_S e)_{0;\Omega} \;+\; (f - \Pi_0 f, e - P_S e)_{0;\Omega} \\
&\quad + \sum_{E\in\mathcal{E}_k^{\Omega\setminus S}} ([\mathbf{n}\cdot a\nabla u_M]_J, [e - P_S e]_A)_{0;E} \\
&\quad + \sum_{E\in\mathcal{E}_k^{\partial\Omega}} (\mathbf{n}\cdot a\nabla u_M, e - P_S e)_{0;E} \\
&\quad + \sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\gamma_\ell}} \Big\{ (\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M, e - P_S e)_{0;E} \\
&\quad + (\lambda_M - \Pi_{\gamma_\ell}\lambda_M, e - P_S e)_{0;E} \Big\} \\
&\quad + \sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\Gamma_\ell}} (\lambda_M - \mathbf{n}\cdot a\nabla u_M, e - P_S e)_{0;E} \\
&\quad + \sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\Gamma_\ell}} (\mu - \mathbf{n}\cdot a\nabla u, [u_M]_J)_{0;E}\ .
\end{aligned}
\tag{4.20}
$$

The last term on the right-hand side in (4.20) involves the approximation of the flux $\mathbf{n}\cdot a\nabla u$ on the skeleton S by the multipliers from $M_1(S)$ and can be handled under the saturation assumption

$$
\inf_{\mu\in M_1(S)} \|\mu - \mathbf{n}\cdot a\nabla u\|_{0;S} \;\le\; C\ |||\, u - u_M \,|||\ . \tag{4.21}
$$

We remark that (4.21) is supported by the a priori estimates (2.16) and (2.18).

Taking into account the properties (4.10), (4.11), and (4.12) of the interpolation operator P_S and the saturation assumption (4.21), (4.20) gives rise to the following element-oriented a posteriori error estimator

$$
\eta_{R_1} := \Big(\sum_{T\in\mathcal{T}_k} \eta_{R_1;T}^2 \Big)^{1/2}, \quad \eta_{R_1;T} := \Big(\sum_{\nu=1}^{6} \big(\eta_{R_1;T}^{(\nu)}\big)^2 \Big)^{1/2}\ . \tag{4.22}
$$

The elementwise error terms $\eta_{R_1;T}^{(\nu)}$, $1 \le \nu \le 6$, are given by

$$
\eta_{R_1;T}^{(1)} := h_T\, \|\Pi_0 f - Lu_M\|_{0;T}\ ,
$$

$$\eta^{(2)}_{R_1;T} := \Big(\sum_{E\in\mathcal{E}_k^I\cap\mathcal{E}_k^T} h_E \, \|[\mathbf{n}\cdot a\nabla u_M]_J\|^2_{0;E} \Big)^{1/2} ,$$

$$\eta^{(3)}_{R_1;T} := \Big(\sum_{E\in\mathcal{E}_k^\Gamma\cap\mathcal{E}_k^T} h_E \, \|\mathbf{n}\cdot a\nabla u_M\|^2_{0;E} \Big)^{1/2} ,$$

$$\eta^{(4)}_{R_1;T} := \Big(\sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\gamma_\ell}\cap\mathcal{E}_k^T} h_E \, \|\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M\|^2_{0;E} \Big)^{1/2} ,$$

$$\eta^{(5)}_{R_1;T} := \Big(\sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\Gamma_L}\cap\mathcal{E}_k^T} h_E \, \|\lambda_M - \mathbf{n}\cdot a\nabla u_M\|^2_{0;E} \Big)^{1/2} ,$$

$$\eta^{(6)}_{R_1;T} := \Big(\sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\Gamma_\ell}\cap\mathcal{E}_k^T} h_E^{-1} \, \|[u_M]_J\|^2_{0;E} \Big)^{1/2}$$

We further define the "higher-order" terms

$$\eta_{R_2} := \Big(\sum_{T\in\mathcal{T}_k} \eta^2_{R_2;T} \Big)^{1/2} , \quad \eta_{R_2;T} := \Big(\sum_{\nu=1}^{2} \big(\eta^{(\nu)}_{R_2;T}\big)^2 \Big)^{1/2} \tag{4.23}$$

where

$$\eta^{(1)}_{R_2;T} := h_T \, \|f - \Pi_0 f\|_{0;T} ,$$

$$\eta^{(2)}_{R_2;T} := \Big(\textstyle\sum_{\ell=1}^{L} \sum_{E\in\mathcal{E}_k^{\gamma_\ell}\cap\mathcal{E}_k^T} h_E \, \|\lambda_M - \Pi_{\gamma_\ell}\lambda_M\|^2_{0;E} \Big)^{1/2}$$

PROPOSITION 4.1. *Let η_{R_1} and η_{R_2} be given by (4.22), (4.23) and assume that the saturation assumption (4.21) is satisfied. Then there exist constants $\Gamma_{R_1}, \Gamma_{R_2} > 0$, depending only on the shape regularity of $\mathcal{T}_0$ and on $\underline{\alpha}, \bar{\alpha}, \underline{\beta}, \bar{\beta}$ in (2.3), (2.4) such that*

$$|||\, e \,||| \;\le\; \Gamma_{R_1}\, \eta_{R_1} \;+\; \Gamma_{R_2}\, \eta_{R_2} \; . \tag{4.24}$$

Proof. The assertion is an immediate consequence of the representation (4.20) of the discretization error e and the local estimates (4.10)–(4.12) from Lemma 3.1 as well as the saturation assumption (4.21) taking further into account the implications (4.1), (4.2) of the regularity and local quasi-uniformity of the sequences $(\mathcal{T}_i^{(k)})_{k\in\mathbf{N}_0}$. □

We will now show that the local contributions of the error estimator η_{R_1} do provide lower bounds for the associated local parts of the discretization errors in u and the flux $\mathbf{n}\cdot a\nabla u$.

LEMMA 4.2. *For $E \in \mathcal{E}_k^{\bar\Omega}$ let $\bar{D}_E := \{T \in \mathcal{T}_k \mid E \in \mathcal{E}_k^T\}$. Then there exist constants $C_\nu > 0$, $1 \le \nu \le 6$, such that*

$$h_T\|\Pi_0 f - Lu_M\|_{0;T} \le C_1(|||\, e \,|||_T + h_T\|\Pi_0 f - f\|_{0;T}),\ T \in \mathcal{T}_k \ , \tag{4.25}$$

$$h_E^{1/2}\|[\mathbf{n}\cdot a\nabla u_M]_J\|_{0;E} \le C_2(|||\, e\,|||_{\tilde{D}_E} + h_E\|\Pi_0 f - Lu_M\|_{0;\tilde{D}_E} + h_E\|\Pi_0 f - f\|_{0;\tilde{D}_E}), \quad E \in \mathcal{E}_k^I, \tag{4.26}$$

$$h_E^{1/2}\|\mathbf{n}\cdot a\nabla u_M\|_{0;E} \le C_3(|||\, e\,|||_{\tilde{D}_E} + h_E\|\Pi_0 f - Lu_M\|_{0;\tilde{D}_E} + h_E\|\Pi_0 f - f\|_{0;\tilde{D}_E}), \quad E \in \mathcal{E}_k^\Gamma, \tag{4.27}$$

$$h_E^{1/2}\|\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;E} \le C_4(|||\, e\,|||_{\tilde{D}_E} + h_E\|\Pi_0 f - Lu_M\|_{0;\tilde{D}_E} + h_E\|\Pi_0 f - f\|_{0;\tilde{D}_E} + h_E^{1/2}\|\lambda_M - \Pi_{\gamma_\ell}\lambda_M\|_{0;E}), \quad E \in \mathcal{E}_k^{\gamma_\ell}, \tag{4.28}$$

$$h_E^{1/2}\|\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;E} \le C_5(|||\, e\,|||_{\tilde{D}_E} + h_E\|\Pi_0 f - Lu_M\|_{0;\tilde{D}_E} + h_E\|\Pi_0 f - f\|_{0;\tilde{D}_E}), \quad E \in \mathcal{E}_k^{\Gamma_\ell}, \tag{4.29}$$

$$h_E^{-1/2}\,\|[u_M]_J\|_{0;E} \;\le\; C_6\;\; |||\, e\,|||_{\tilde{D}_E}\;\;, \quad E \in \mathcal{E}_k^{\Gamma_\ell}\,. \tag{4.30}$$

Proof. The local estimates (4.25)–(4.29) can be proved by using the local element bubble functions $\varphi_T := 27\,\lambda_1^T\lambda_2^T\lambda_3^T$, $T \in \mathcal{T}_k$, and edge bubble functions $\varphi_E := 4\,\lambda_1^T\lambda_2^T$, $p_1, p_2 \in \mathcal{N}_k^E$, $E \in \mathcal{E}_k^{\bar{\Omega}}$ where λ_ν^T, $1 \le \nu \le 3$, are the barycentric coordinates of T.
In particular, the proofs of (4.25), (4.26), and (4.27) follow along the lines of the standard conforming case (cf., e.g., [37]).
For the proof of (4.28) we set $q := \Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M$ and denote by $q^{\tilde{D}_E}$ its extension to $\tilde{D}_E$ as, e.g., in [37]. Then, by Green's formula we obtain

$$\begin{aligned}
&\|\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;E}^2 \\
&\quad\le C\int_E (\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M) q\varphi_E\, d\sigma \\
&\quad= C\Big[a(e, q^{\tilde{D}_E}\varphi_E) - (f - Lu_M, q^{\tilde{D}_E}\varphi_E)_{0;E} \\
&\qquad + b(\mathbf{n}\cdot a\nabla u_M - \lambda_M, q\varphi_E) + (\Pi_{\gamma_\ell}\lambda_M - \lambda_M, q\varphi_E)_{0;E}\Big] \\
&\quad\le C\Big[h_E^{-1/2}\;|||\, e\,|||_{\tilde{D}_E} + h_E^{1/2}\|f - Lu_M\|_{0;\tilde{D}_E} + \|\Pi_{\gamma_\ell}\lambda_M - \lambda_M\|_{0;E} \\
&\qquad + \|\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;E}\Big]\,\|\Pi_{\gamma_\ell}\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;E}.
\end{aligned} \tag{4.31}$$

The estimate (4.29) can be verified similarly. Finally, for the proof of (4.30) we refer to [38]. □

The preceding local lower bounds result in the following global lower bound:

PROPOSITION 4.2. *Let η_{R_1} and η_{R_2} be given by (4.22) and (4.23). Then, under the saturation assumption (4.21) there exist constants γ_{R_1}, $\gamma_{R_2} > 0$ depending only on the shape regularity of $\mathcal{T}_0$ and on $\underline{\alpha}, \bar{\alpha}, \underline{\beta}, \bar{\beta}$ in (2.3), (2.4) such that*

$$\gamma_{R_1}\,\eta_{R_1} \;-\; \gamma_{R_2}\,\eta_{R_2} \;\le\; |||\, e\,|||\;\;. \tag{4.32}$$

Proof. The saturation assumption (4.21) implies $\|\lambda_M - \mathbf{n}\cdot a\nabla u_M\|_{0;S} \leq C \;|||\, e \,|||$ and hence, (4.32) follows from (4.25)–(4.30) observing again (4.1), (4.2). □

Finally, combining the bounds given by (4.24) and (4.32) results in the following two-sided estimate:

THEOREM 4.1. *Under the saturation assumption (4.21) and with the notations of Propositions 4.1 and 4.2 there holds*

$$\gamma_{R_1}\,\eta_{R_1} - \gamma_{R_2}\,\eta_{R_2} \;\leq\; |||\, e \,||| \;\leq\; \Gamma_{R_1}\,\eta_{R_1} + \Gamma_{R_2}\,\eta_{R_2} \tag{4.33}$$

REMARK 4.1. The quantities $h_E^{1/2}\,\|\Pi_{\gamma_\ell}\lambda_M - \lambda_M\|_{0;E}$, $E \in \mathcal{E}_k^{\gamma_\ell}$, in η_{R_2} can be considered as "higher-order" terms provided the ratios $h_{\gamma_\ell}/h_{\Gamma_\ell}$, $1 \leq \ell \leq L$, remain bounded where $h_{\gamma_\ell} := \min\{h_E \mid E \in \mathcal{E}_k^{\gamma_\ell}\}$, $h_{\Gamma_\ell} := \min\{h_E \mid E \in \mathcal{E}_k^{\Gamma_\ell}\}$. Otherwise, these quantities will obscure the lower bound in (4.32) and thus effect the efficiency of the estimator. Therefore, in practice it is advisable to monitor these quantities carefully in order to guarantee the boundedness of $h_{\gamma_\ell}/h_{\Gamma_\ell}$, $1 \leq \ell \leq L$.

The concept of hierarchical type a posteriori error estimation relies on a discretization of the defect problem satisfied by the error with respect to a higher order finite element approximation combined with a suitable localization of that problem by a hierarchical two-level splitting.

In case of the standard conforming $P1$ approximation there are two different approaches:
The first one due to Bank and Weiser [8] starts from a localization of the defect problem on the elements' level resulting in local Neumann problems which are then solved by using continuous, piecewise quadratic elements and their standard hierarchical decomposition.
The second technique which can be attributed to Deuflhard, Leinen, and Yserentant [19] does it the other way around. It uses a conforming P2 approximation of the global defect problem followed by a localization of the discretized problem by means of the hierarchical two-level splitting. Note that hierarchical type error estimators for nonconforming finite element discretizations have been considered in [22, 23].

Here, in case of the mortar finite element approximation we suggest a somewhat hybrid approach in so far as we begin with a localization of the defect problem (4.3) on the subdomains' level and then consider a conforming P2 approximation of the resulting Neumann problems on the individual subdomains.
In particular, setting $e_i := (u - u_M)\,|_{\Omega_i}$, $1 \leq i \leq N$, from (4.3) we get the Neumann problems

$$a_i(e_i, v) = (f, v)_{0;\Omega_i} - a_i(u_M, v) + \int\limits_{\partial\Omega_i \setminus \Gamma} \mathbf{n}\cdot a\nabla u v d\sigma\,, \quad v \in H^1(\Omega_i) \tag{4.34}$$

involving the unknown Neumann boundary data $\mathbf{n} \cdot a\nabla u$ on $\partial\Omega_i \setminus \Gamma$. On the other hand, we know that the multiplier $\lambda_M \in M_1(S)$ does provide an approximation of the flux and hence, the idea is to replace $\mathbf{n} \cdot a\,\nabla u$ on $\partial\Omega_i \setminus \Gamma$ by the available multiplier λ_M and to approximate the resulting local Neumann problems in terms of the conforming P2 approximation. Denoting by $S_2(\Omega_i; \mathcal{T}_i^{(k)})$, $1 \le i \le N$, the space of continuous, piecewise quadratic finite element with respect to the triangulations $\mathcal{T}_i^{(k)}$, we thus end up with the following discrete defect problems on the subdomains: Find $\tilde{e}_i \in S_2(\Omega_i; \mathcal{T}_i^{(k)})$ such that

$$a_i(\tilde{e}_i, v) \;=\; \tilde{r}_i(v) \;\;,\;\; v \in S_2(\Omega_i; \mathcal{T}_i^{(k)}) \;\;, \tag{4.35}$$

where

$$\tilde{r}_i(v) \;:=\; (f, v)_{0;\Omega_i} \;-\; a_i(u_M, v) \;+\; \int\limits_{\partial\Omega_i \setminus \Gamma} \lambda_M \, v \, d\sigma \;\;.$$

We localize (4.35) by means of the hierarchical two-level splitting

$$S_2(\Omega_i; \mathcal{T}_i^{k)}) \;=\; S_1(\Omega_i; \mathcal{T}_i^{(k)}) \;\oplus\; \tilde{S}_2(\Omega_i; \mathcal{T}_i^{(k)}) \tag{4.36}$$

where $\tilde{S}_2(\Omega_i; \mathcal{T}_i^{(k)})$ stands for the hierarchical surplus $\tilde{S}_2(\Omega_i; \mathcal{T}_i^{(k)}) = \text{span}\,\{\varphi_E \mid E \in \mathcal{E}_k^{\bar{\Omega}_i \setminus \Gamma}\}$ spanned by the quadratic bubble functions φ_E associated with the midpoints of the edges. It is well-known that the stiffness matrices associated with (4.35) are 2×2-block matrices with respect to the decomposition (4.36) which are spectrally equivalent to their block-diagonals (cf., e.g., [19]). Moreover, the second diagonal block is spectrally equivalent to its diagonal so that

$$c \sum_{E \in \mathcal{E}_k^{\bar{\Omega}_i}} \alpha_E^2 \;|||\, \varphi_E \,|||^2 \;\le\; |||\, \tilde{e}_i \,|||^2 \;\le\; C \sum_{E \in \mathcal{E}_k^{\bar{\Omega}_i}} \alpha_E^2 \;|||\, \varphi_E \,|||^2 \tag{4.37}$$

where $\alpha_E := \tilde{r}_i(\varphi_E)/a_i(\varphi_E, \varphi_E)$.
This means that we only have to solve scalar equations associated with the edges $E \in \mathcal{E}_k^{\bar{\Omega}_i}$.

On the other hand, in view of

$$|||\, e \,|||^2 = \sum_{i=1}^{N} \tilde{r}_i(e\,|_{\Omega_i}) \;+\; \sum_{\ell=1}^{L} \int\limits_{\Gamma_\ell} (\mathbf{n} \cdot a\nabla u - \lambda_M)\,[u_M]_J \, d\sigma \tag{4.38}$$

we additionally have to take into account the jumps $[u_M]_J$ across the interfaces.
Altogether, this gives rise to the following hierarchical a posteriori error estimator

$$\eta_H := \Big(\sum_{T\in\mathcal{T}_k} \eta_{H;T}^2\Big)^{1/2} ,$$

$$\eta_{H;T} := \Big(\sum_{E\in\mathcal{E}_k^\Omega\cap\mathcal{E}_k^T} \alpha_E^2 \,|||\, \varphi_E \,|||^2 + \sum_{E\in\mathcal{E}_k^S\cap\mathcal{E}_k^T} h_E^{-1}\|[u_M]_J\|_{0;E}^2\Big)^{1/2}, T\in\mathcal{T}_k. \tag{4.39}$$

In addition to (4.21) we need a further saturation assumption that is known from the hierarchical approach in a posteriori error estimation and adopted here to the macro-hybrid finite element method under consideration. Denoting by $u_{2,i}\in S_2(\Omega_i;\mathcal{T}_i^{(k)})$, $1\le i\le N$, the solution of

$$a_i(u_{2,i},v) = (f,v)_{0;\Omega_i} + \int_{\partial\Omega_i\setminus\Gamma} \mathbf{n}\cdot a\nabla u\, v\, d\sigma \ , \ v\in S_2(\Omega_i;\mathcal{T}_i^{(k)} \ ,$$

we require the existence of constants $0\le\beta_i\le\beta<1$, $1\le i\le N$, independent of the refinement level such that

$$|||\, u_{2,i}-u\,|||_{\Omega_i} \le \beta_i \,|||\, e\,|_{\Omega_i}|||_{\Omega_i} \ , \ 1\le i\le N \ . \tag{4.40}$$

THEOREM 4.2. *Let η_H be the hierarchical error estimator as given by (4.39). Then, under the saturation assumptions (4.21) and (4.40) there exist constants $0<\gamma_H\le\Gamma_H$ depending only on the shape regularity of $\mathcal{T}_0$ and on $\underline{\alpha},\bar{\alpha},\underline{\beta},\bar{\beta}$ in (2.3), (2.4) such that*

$$\gamma_H\,\eta_H \le |||\, e\,||| \le \Gamma_H\,\eta_H \tag{4.41}$$

Proof. The first inequality follows from (4.30) and (4.37). On the other hand, the second one can be deduced from (4.37), (4.38) and $\|\lambda_M-\mathbf{n}\cdot a\nabla u\|_{0;S}\le C\,|||\, e\,|||$ which is a consequence of the saturation assumption (4.21). □

REMARK 4.2. In practice, we do not compute $(u_M,\lambda_M)\in\prod_{i=1}^N S_1(\Omega_i;\mathcal{T}_i^{(k)})\times M_1(S)$ exactly but only some approximation $(\tilde{u}_M,\tilde{\lambda}_M)$ by means of an iterative solution process as, e.g., that one described in the previous section. In this case, the iteration errors $|||\, u_M-\tilde{u}_M\,|||$ and $\|\lambda_M-\tilde{\lambda}_M\|_{0;S}$ additionally enter the bounds in (4.33) and (4.41). However, if the chosen iterative scheme is optimal, the iteration errors can be controlled during the iterative solution, for instance, by monitoring the residuals with respect to the computed iterates.

REMARK 4.3. If we are dealing with differential operators of type (2.1) featuring rapidly varying or even discontinuous coefficients, it is advisable to use error estimators η_R and η_H whose local contributions $\eta_{R;T}$ and $\eta_{H;T}$, $T\in\mathcal{T}_k$, incorporate local bounds for the coefficient functions a and b (cf., e.g., [38]).

5. Numerical results. In this section, we will give numerical results illustrating both the benefits of the adaptive approach and the parallel efficiency of the iterative solution process.

First, we consider an example for testing the effectivity of the residual based and the hierarchical type a posteriori error estimators η_{R_1} and η_H as given by (4.22) and (4.39), respectively. In particular, we have chosen the diffusion equation $-\text{div}\,(a\,\text{grad}\,u) + b\,u = f$ in $\Omega = (0,1)^2$ with a discontinuous diffusion coefficient a varying from 1 to 100 as indicated in Figure 1a. The right-hand side f and the (inhomogeneous) Neumann boundary data have been selected such that $u(x,y) = a^{-1}\,(y-x)(1-x-y)$, $(x,y) \in \Omega$, is the solution of the problem. The subdomain partitioning corresponds to the four regions of the varying diffusion coefficient and the initial triangulation $\mathcal{T}_0$ is given as shown in Figure 1b.

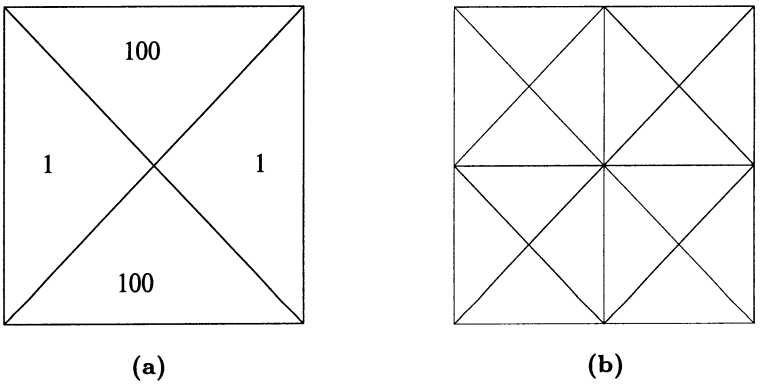


FIG. 1. *(a) Diffusion coefficient a. (b) Initial triangulation.*

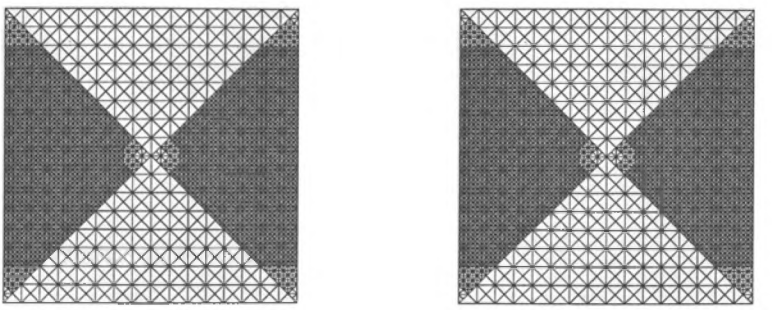
Figures 2a and 2b display the adaptively generated final triangulation in case of the residual based error estimator (Fig. 2a) and the hierarchical type error estimator (Fig. 2b). In both cases, the refinement process has been terminated when the estimated error was less than 10^{-2}.

Tables 1a and 1b contain the history of the refinement process displaying the number of unknowns, the estimated and the true error, as well as the effectivity index per level of refinement. Note that the effectivity index is the ratio of the estimated and the true error.

We see that in both cases the adaptively generated triangulations are similar with a slightly larger number of unknowns generated by the hierarchical type estimator. The effectivity index approaches its optimal value 1 rapidly.

If we have a closer look at Figures 2a and 2b, we see that we have a pronounced refinement in the two subdomains with the smaller diffusion coefficient along with a sharp resolution of the interfaces. This is obviously

in contrast to the use of standard conforming meshes where, due to the continuity requirements, we also encounter strong refinement in the other regions close to the interfaces.



(a) (b)

FIG. 2. *(a) Residual based error estimator. (b) Hierarchical error estimator.*

TABLE 1
(a) *Residual based error estimator.*

Level	# Nodes	Est. Error	Error	Eff. Index
0	24	$0.983 \cdot 10^{-1}$	0.408	0.240
1	60	0.140	0.147	0.954
2	116	$0.734 \cdot 10^{-1}$	$0.759 \cdot 10^{-1}$	0.966
3	312	$0.389 \cdot 10^{-1}$	$0.401 \cdot 10^{-1}$	0.974
4	1122	$0.201 \cdot 10^{-1}$	$0.203 \cdot 10^{-1}$	0.991
5	4132	$0.102 \cdot 10^{-1}$	$0.103 \cdot 10^{-1}$	0.989
6	15804	$0.514 \cdot 10^{-2}$	$0.520 \cdot 10^{-2}$	0.989

(b) *Hierarchical error estimator.*

Level	# Nodes	Est. Error	Error	Eff. Index
0	24	0.241	0.408	0.592
1	60	0.150	0.147	1.02
2	116	$0.764 \cdot 10^{-1}$	$0.759 \cdot 10^{-1}$	1.01
3	320	$0.392 \cdot 10^{-1}$	$0.393 \cdot 10^{-1}$	0.997
4	1150	$0.196 \cdot 10^{-1}$	$0.197 \cdot 10^{-1}$	0.998
5	4372	$0.100 \cdot 10^{-1}$	$0.991 \cdot 10^{-2}$	1.01
6	17044	$0.497 \cdot 10^{-2}$	$0.495 \cdot 10^{-2}$	1.00

We think that mortar finite element methods are especially well suited for problems with highly varying or even discontinuous coefficients and can result in a considerable savings of nodal points and thus computational time, an effect that is expected to be even more pronounced for 3D problems.

The performance of the multilevel preconditioned iterative solver and its parallel efficiency have been tested for incompressible and compressible fully potential flows around airfoils. In particular, the benchmark problems NACA0012 and BiNACA0012 have been chosen as test examples (cf. Figures 3a,b). All numerical results reported below have been obtained by the implementation of the algorithm on the IBM SP 2 of the Leibniz Computing Laboratory, Munich.

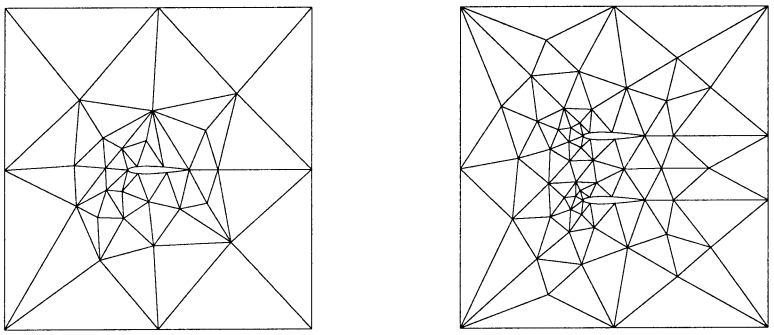


FIG. 3. *(a) NACA0012. (b) BiNACA0012.*

The governing equation for incompressible flow is the Laplace equation

$$\Delta \varphi = 0 \text{ in } \Omega \tag{5.1}$$

whereas that one for compressible flow is given by

$$\nabla \cdot (\rho(|\nabla\varphi|^2)\nabla\varphi) = 0 \text{ in } \Omega \ . \tag{5.2}$$

Here, φ stands for the potential of the velocity field $\mathbf{u} = (u_1, u_2)^T$ and $\rho(\cdot)$ in (4.2) is the density function

$$\rho(|\nabla\varphi|^2) = \rho_0 \left(1 - \frac{\gamma - 1}{2c_0^2} |\nabla\varphi|^2\right)^{1/(\gamma-1)}$$

where c_0, ρ_0, and γ refer to the speed of sound and the density in motionless gas and to the gas constant, respectively.
In case of the NACA0012, the computational domain Ω is simply connected with boundary $\Gamma = \Gamma_\infty \cup S \cup \Sigma$ where Γ_∞ is the outer boundary, S is the

TABLE 2
(a) *Dependence of n_{it} and t_{ex} on the level p.*

p	N	$L_A = 1$					$L_A = 5$				
		L	incompress.		compress.		L	incompress.		compress.	
			n_{it}	t_{ex}	n_{it}	t_{ex}		n_{it}	t_{ex}	n_{it}	t_{ex}
5	8568	4	247	9.2	1034	37.5	16	77	4.7	323	19.6
6	31416	5	258	15.2	1186	70.1	24	71	9.6	314	41.7
7	120120	7	254	31.2	1232	163.3	34	70	29.4	336	142.5
8	469560	10	259	102.9	1316	521.5	48	66	104.6	385	635.7

(b) Dependence of n_{it} and t_{ex} on n_{pr} and p (incompressible flow).

p	N	$L_A = 1$					$L_A = 5$				
		n_{pr}	7	14	28	56	n_{pr}	7	14	28	56
		n_{it}					n_{it}				
6	31416	258	41.3	24.3	15.8	15.3	71	46.1	26.5	18.3	9.6
7	120120	254	148.9	80.7	46.0	31.2	70	182.4	95.3	52.5	29.4
8	469560	259	619.9	332.1	185.6	102.9	66	747.3	379.4	212.6	104.6

(c) Dependence of n_{it} and t_{ex} on n_{pr} and p (compressible flow).

p	N	$L_A = 1$					$L_A = 5$				
		n_{pr}	7	14	28	56	n_{pr}	7	14	28	56
		n_{it}					n_{it}				
6	31416	1188	211.9	121.0	71.3	70.1	314	230.1	118.0	67.5	41.7
7	120120	1232	741.5	397.9	230.9	163.3	336	852.9	474.0	247.4	142.5
8	469560	1316	3335	1720	905.7	521.5	385	4361	2214	1170	635.7

boundary of the airfoil and Σ the slit connecting the trailing edge with the outer boundary. For the BiNACA0012 we have $\Gamma = \Gamma_\infty \cup S_1 \cup S_2 \cup \Sigma_1 \cup \Sigma_2$ with the corresponding meanings of S_i and Σ_i , $1 \leq i \leq 2$. Note that the introduction of the slits is done to ensure the existence of a unique global potential φ on Ω. The equations (5.1) and (5.2) have to be completed by appropriate boundary conditions. We refer to [20] for a detailed discussion. For compressible flow problems, the nonlinear equation (5.2) is solved by means of the iterative scheme $\varphi^{\nu+1} = \varphi^\nu + \omega^{\nu+1}$, $\nu \geq 0$, where the correction $\omega^{\nu+1}$ is computed as the solution of the linear problem

$$\nabla \cdot (\rho(|\nabla\varphi^\nu|^2)\nabla\omega^{\nu+1}) = -\nabla \cdot (\rho(|\varphi^\nu|^2)\nabla\varphi^\nu) .$$

(cf., e.g., [31]).

In particular, we have chosen a symmetric, nonlifting flow around the NACA0012 with $|\mathbf{u}_\infty| = 160$ m/sec where $\mathbf{u}_\infty = \mathbf{u}|_{\Gamma_\infty}$. The computa-

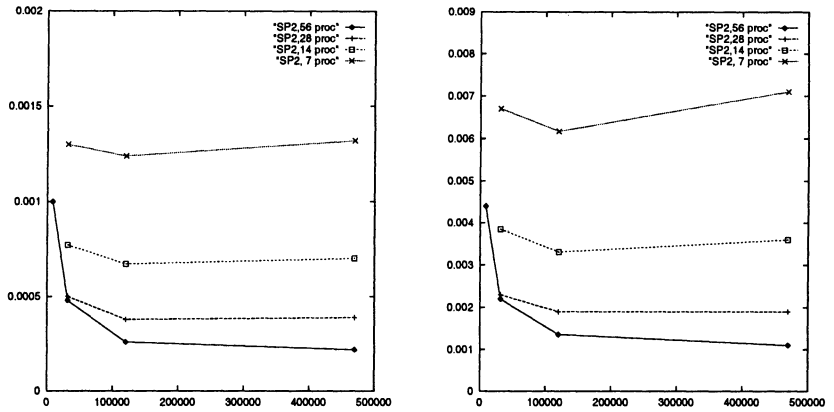


FIG. 4. *(a) Asymptotical optimality (incompressible flow). (b) Asymptotical optimality (compressible flow).*

tional domain Ω is decomposed into 56 geometrical conforming, nonoverlapping subdomains as shown in Figure 3a. The subdomains are distributed to a certain number of processors where the distribution is done in a preprocessing step taking into account the information provided by the Chebyshev iterative parameters to guarantee a proper load balancing (see [20] for details).

Table 2a contains the number N of unknowns, the number L of inner Chebyshev iterations, the total number n_{it} of outer Lanczos iterations, and the execution times t_{ex} for different levels p of the refinement process. Note that in this case 56 processors have been used. The results underpin the optimal order of arithmetical complexity of the method.

The same data are given in Tables 2b and 2c for incompressible and compressible flow in case of a different number n_{pr} of processors. These data are visualized in Figures 4a,b which reflect the asymptotic optimality of the algorithm by displaying the execution times per unknown in dependence of the total number of unknowns.

Finally, Figures 5a,b show the dependence of the parallel efficiencies on the number of processors for different levels of the refinement process (p = 6,7, and 8). We see that both for incompressible flow (Figure 5a) and compressible flow (Figure 5b), on level $p = 8$ (469560 unknowns) the parallel efficiencies slightly decrease to 0.8 in case of 56 processors indicating a reasonably well parallel performance of the algorithm.

For the BiNACA0012 we have tested the dependence of the total number n_{it} of Lanczos iterations and of the execution times t_{ex} on the number m of subdomains used in the decomposition of the computational domain. The corresponding results are listed in Table 3.

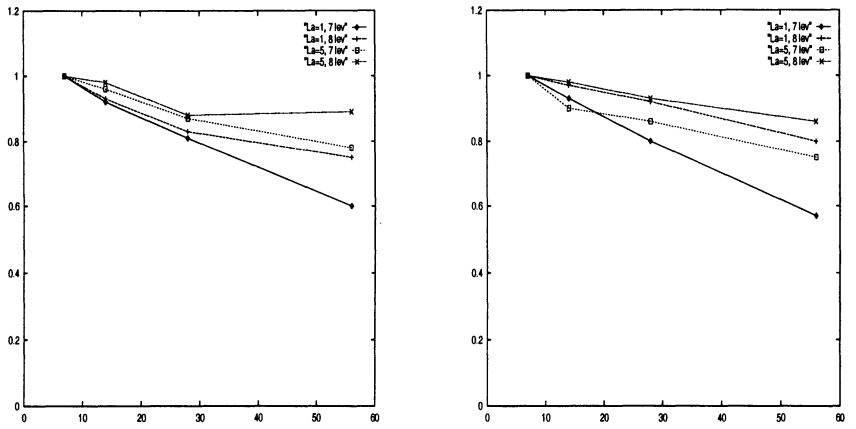


FIG. 5. *(a) Parallel efficiencies (incompressible flow). (b) Parallel efficiencies (compressible flow).*

TABLE 3
Dependence of n_{it} and t_{ex} on the number of subdomains.

$L_A = 1$				$L_A = 5$			
$m = 111$ $p = 7$		$m = 168$ $p = 6$		$m = 111$ $p = 7$		$m = 168$ $p = 6$	
n_{it}	t_{ex}	n_{it}	t_{ex}	n_{it}	t_{ex}	n_{it}	t_{ex}
295	59.2	247	24.4	74	58.8	69	23.1

We have further run various experiments concerning lifting flows around the NACA0012 and the BiNACA0012 for different angles of attack which gave similar results as in case of the test problem reported above. For a comprehensive documentation we refer to [20].

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