Topology Optimization of Conductive Media Described by Maxwell's Equations

Ronald H. W. Hoppe, Svetozara I. Petrova, Volker H. Schulz

Angaben zur Veröffentlichung / Publication details:

Hoppe, Ronald H. W., Svetozara I. Petrova, and Volker H. Schulz. 2001. "Topology Optimization of Conductive Media Described by Maxwell's Equations." *Lecture Notes in Computer Science* 1988: 414–22. https://doi.org/10.1007/3-540-45262-1_48.

Topology Optimization of Conductive Media Described by Maxwell's Equations*

Ronald H. W. Hoppe[1], Svetozara I. Petrova[1], and Volker H. Schulz[2]

[1] Institut für Mathematik, Universität Augsburg
Universitätsstraße 14, D-86135 Augsburg, Germany
[2] Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39, D-10117 Berlin, Germany

Abstract. The problem of an energy dissipation optimization in a conductive electromagnetic media is considered. The domain is known a priori and is fixed throughout the optimization process. We apply a perturbed and damped interior–point Newton method for the primal–dual formulation of the nonlinear programming problem. Nonnegative slack variables are added to the inequality constraints in the optimization problem. Computational results concerning a two–dimensional isotropic system are included.

1 Introduction

Computation of electromagnetic fields in various settings, analysis and different approaches for the spatial discretization of the Maxwell equations have been a subject of intense research in the last decade, see, e.g., [2,7,10]. In this paper we consider problems concerning topology optimization in electromagnetic media. For a general overview on the field of structural optimization and topology design, we refer to [3]. We are looking for an optimal distribution of conductivity in a fixed geometrical configuration.

Let $\Omega \subset R^3$ be a domain occupied by a conductor with a conductivity $\sigma > 0$. The rest of the space is vacuum. To simplify the presentation, we consider the stationary case, i.e., constant currents are available in the conductor ($\operatorname{div} \mathbf{J} = 0$). In this case the Maxwell equations read:

$$\mathbf{curl}\, \mathbf{E} = -\partial_t \mathbf{B}, \quad \mathbf{curl}\, \mathbf{H} = \mathbf{J}, \quad \operatorname{div} \mathbf{D} = \rho, \quad \operatorname{div} \mathbf{B} = 0, \tag{1}$$

* This work was supported in part by the Alexander von Humboldt Foundation. The second author has also been supported by the Bulgarian Ministry for Education, Science, and Technology under Grant MM–98#801.

supplemented by the following material laws:

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu\, \mathbf{H}, \quad \mathbf{J} = \sigma\, \mathbf{E}. \tag{2}$$

Here, the fundamental electromagnetic quantities are the *electric field* $\mathbf{E}$, the *magnetic induction* $\mathbf{B}$, the *magnetic field* $\mathbf{H}$, the *electric induction* $\mathbf{D}$, the *electric current density* $\mathbf{J}$, and the *space charge density* ρ. We consider only linear and isotropic materials, so that the *electric permeability* ε, the *magnetic permeability* μ, and the *electric conductivity* σ are supposed bounded scalar functions of the spatial variable $\mathbf{x}$ with $\varepsilon \geq \varepsilon_0 > 0,\ \mu \geq \mu_0 > 0$, and $\sigma > 0$. Steep jumps of these coefficients may occur at material interfaces. One can introduce a scalar electric potential φ and a magnetic vector potential $\mathbf{A}$, so that

$$\mathbf{E} = -\mathbf{grad}\, \varphi - \partial_t \mathbf{A} \quad \text{and} \quad \mathbf{B} = \mathbf{curl}\, \mathbf{A}. \tag{3}$$

To specify $\mathbf{A}$, which is not uniquely defined, we use the Coulomb *gauge*, namely, $\operatorname{div} \mathbf{A} = 0$. From (2) and (3) one gets $\mathbf{J} = \sigma\, \mathbf{E} = -\sigma \mathbf{grad}\, \varphi - \sigma \partial_t \mathbf{A}$, which yields

$$\operatorname{div} \mathbf{J} = \operatorname{div} (\mathbf{curl}\, \mathbf{H}) = 0 = -\operatorname{div} (\sigma\, \mathbf{grad}\, \varphi) - \operatorname{div} (\sigma\, \partial_{\mathrm{t}} \mathbf{A}). \tag{4}$$

Suppose now that σ is piecewise constant, i.e., independent of the spatial variable $\mathbf{x}$. Then $\operatorname{div} \mathbf{A} = 0$ results in $\operatorname{div} (\sigma\, \partial_{\mathrm{t}} \mathbf{A}) = 0$. From (4) we get the following coupled system of equations for φ and $\mathbf{A}$:

$$\operatorname{div}(\sigma\, \mathbf{grad}\, \varphi) = 0 \text{ in } \Omega, \quad \mathbf{n} \cdot \sigma\, \mathbf{grad}\, \varphi = \begin{cases} \mathrm{I}_\nu \text{ on } \Gamma_\nu \subset \partial\Omega \\ 0 \quad \text{otherwise} \end{cases} \tag{5}$$

$$\sigma\, \partial_t \mathbf{A} + \mathbf{curl}\, (\mu^{-1}\, \mathbf{curl} \mathbf{A}) = \begin{cases} -\sigma \mathbf{grad}\, \varphi & \text{in } \Omega \\ 0 & \text{in } R^3 \backslash \bar{\Omega} \end{cases}. \tag{6}$$

Here, the unit normal vector is denoted by $\mathbf{n}$. For the given electric current densities $\{I_\nu\}$ on the boundary $\Gamma_\nu \subset \partial\Omega$ we impose the compatibility condition $\sum_\nu I_\nu = 0$. The energy dissipation given by the Joule–Lenz law reads as follows:

$$f(\varphi, \sigma) := \int_\Omega \mathbf{J}\, \mathbf{E}\, dx = -\int_\Omega \mathbf{J} \cdot \mathbf{grad}\, \varphi\, dx = -\int_\Omega \operatorname{div}(\varphi\, \mathbf{J})\, dx. \tag{7}$$

Using the Gauss–Ostrogradski formula and the Neumann boundary conditions from (5) we get the following expression:

$$f(\varphi, \sigma) = -\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{J}\, \varphi\, ds = \sum_\nu \int_{\Gamma_\nu} I_\nu\, \varphi\, ds. \tag{8}$$

The remainder of this paper is organized as follows. In Section 2 we introduce the primal–dual formulation of our nonlinear nonconvex programming problem. Slack variables are added directly to the optimization problem. In Section 3 we discuss the steplength strategy and give the interior–point algorithm. In the last section, we include some numerical experiments concerning the conductivity distribution for a two–dimensional isotropic system.

2 Primal–Dual Approach

In this section, we formulate the nonlinear nonconvex optimization problem for a minimization of the energy dissipation given by (8).

$$\min_{\varphi,\sigma} f(\varphi,\sigma) \;=\; \min_{\varphi,\sigma} \sum_{\nu} \int_{\Gamma_\nu} I_\nu \varphi \; ds, \tag{9}$$

subject to the following constraints:

$$\begin{aligned} &\varphi \text{ satisfies (5)}, \\ &\int_\Omega \sigma \, dx = C \qquad\qquad \text{(mass constraint)}, \\ &\sigma_{\min} \le \sigma \le \sigma_{\max} \qquad \text{(conductivity box constraint)}. \end{aligned} \tag{10}$$

Here, $\sigma_{\min}$ and $\sigma_{\max}$ are a priori given positive limits for the conductivity and C is a fixed given value. Note that we formulate a constrained optimization problem, where the differential equation for φ (5) is part of the constraints. This is in contrast to many standard optimization approaches, which would consider φ as a function of the independent variable σ via the differential equation. However, this simultaneous optimization approach reduces the overall computational complexity of the resulting optimization algorithm.

We apply the primal–dual interior–point method, originally proposed for linear programs by [8]. This method has been recently extended to nonlinear programming in [1] and started to prove its impressive computational performance for nonlinear programming, see, e.g., [5,6,12]. We deal with the corresponding inequality constraints introducing nonnegative slack variables. This variant of the primal–dual approach has been used, e.g., in [1,9]. After a finite element discretization of the domain we get the following finite dimensional nonlinear programming problem:

$$\min_{\boldsymbol{\varphi},\boldsymbol{\sigma}} f(\boldsymbol{\varphi},\boldsymbol{\sigma}), \tag{11}$$

$$\begin{aligned} \text{subject to} \qquad & A(\boldsymbol{\sigma})\,\boldsymbol{\varphi} - \mathbf{b} = 0, \qquad \sigma_{\min}\mathbf{e} - \boldsymbol{\sigma} + \mathbf{s} = 0, \qquad \mathbf{s} \ge 0, \\ & g(\boldsymbol{\sigma}) - \mathrm{C} = 0, \qquad \boldsymbol{\sigma} - \sigma_{\max}\mathbf{e} + \mathbf{t} = 0, \qquad \mathbf{t} \ge 0, \end{aligned} \tag{12}$$

where $A(\boldsymbol{\sigma})$ is the finite element stiffness matrix, $\mathbf{b}$ is the discrete load vector and $g(\boldsymbol{\sigma})$ is a discrete approximation of $\int_\Omega \sigma \, dx$. Here, $\mathbf{e} \in \mathcal{R}^N$, $\mathbf{e} = (e_1,\ldots,e_N)^T$, $e_i = 1$, $1 \le i \le N$, and $\boldsymbol{\sigma},\mathbf{s},\mathbf{t} \in \mathcal{R}^N$, where N is the number of finite elements. Note that the lower bound $\sigma_{\min}$ plays a crucial role keeping the ellipticity of the discrete problem.

The *Lagrangian function* associated with problem (11)-(12) is:

$$\begin{aligned} \mathcal{L}(\boldsymbol{\varphi},\boldsymbol{\sigma},\boldsymbol{\lambda},\eta,\mathbf{z},\mathbf{w},\mathbf{s},\mathbf{t},\boldsymbol{\alpha},\boldsymbol{\beta}) := {} & f(\boldsymbol{\varphi},\boldsymbol{\sigma}) + \boldsymbol{\lambda}^T(\mathrm{A}(\boldsymbol{\sigma})\boldsymbol{\varphi} - \mathbf{b}) + \eta\,(g(\boldsymbol{\sigma}) - \mathrm{C}) \\ & + \mathbf{z}^T(\sigma_{min}\mathbf{e} - \boldsymbol{\sigma} + \mathbf{s}) + \mathbf{w}^T(\boldsymbol{\sigma} - \sigma_{max}\mathbf{e} + \mathbf{t}) \\ & - \boldsymbol{\alpha}^T\mathbf{s} - \boldsymbol{\beta}^T\mathbf{t}. \end{aligned} \tag{13}$$

Here, $\boldsymbol{\lambda}$, η, $\mathbf{z} \ge 0$, $\mathbf{w} \ge 0$ and $\boldsymbol{\alpha} \ge 0$, $\boldsymbol{\beta} \ge 0$ are the Lagrange multipliers for the equality and inequality constraints in (12), respectively. Our purpose is to

find an isolated (locally unique) local minimum of the problem (11)-(12) under the assumption that at least one such point exists. We suppose that the standard conditions for the application of Newton's method, see, e.g., [4], are satisfied. Denote by $\boldsymbol{\Phi} := (\boldsymbol{\varphi}, \boldsymbol{\sigma}, \boldsymbol{\lambda}, \eta, \mathbf{z}, \mathbf{w}, \mathbf{s}, \mathbf{t})$ the vector of the unknown variables. The complementarity conditions $Z\,\mathbf{s} = 0$ and $W\,\mathbf{t} = 0$ are replaced by the *perturbed complementarity conditions* $Z\,\mathbf{s} = p\,\mathbf{e}$ and $W\,\mathbf{t} = p\,\mathbf{e}$. At each iteration, the positive parameter p is decreased by a certain amount.

The necessary first–order Karush–Kuhn–Tucker (KKT) optimality conditions lead to the following nonlinear equation:

$$F_p(\boldsymbol{\Phi}) := \begin{pmatrix} \nabla_{\boldsymbol{\varphi}}\mathcal{L} \\ \nabla_{\boldsymbol{\sigma}}\mathcal{L} \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} \\ \nabla_{\eta}\mathcal{L} \\ \nabla_{\mathbf{z}}\mathcal{L} \\ \nabla_{\mathbf{w}}\mathcal{L} \\ \nabla_{\mathbf{s}}\mathcal{L} \\ \nabla_{\mathbf{t}}\mathcal{L} \end{pmatrix} = \begin{pmatrix} \nabla_{\boldsymbol{\varphi}} f + \mathrm{A}(\boldsymbol{\sigma})^T\boldsymbol{\lambda} \\ \partial_{\boldsymbol{\sigma}}(\boldsymbol{\lambda}^T A(\boldsymbol{\sigma})\boldsymbol{\varphi}) + \eta\nabla g(\boldsymbol{\sigma}) - \mathbf{z} + \mathbf{w} \\ A(\boldsymbol{\sigma})\boldsymbol{\varphi} - \mathbf{b} \\ g(\boldsymbol{\sigma}) - \mathrm{C} \\ \sigma_{min}\mathbf{e} - \boldsymbol{\sigma} + \mathbf{s} \\ \boldsymbol{\sigma} - \sigma_{max}\mathbf{e} + \mathbf{t} \\ Z\,\mathbf{s} - p\,\mathbf{e} \\ W\,\mathbf{t} - p\,\mathbf{e} \end{pmatrix} = 0, \tag{14}$$

where $\nabla_{\mathbf{s}}\mathcal{L} = Z\,\mathbf{s} - p\,\mathbf{e}$, $\nabla_{\mathbf{t}}\mathcal{L} = W\,\mathbf{t} - p\,\mathbf{e}$. The *search direction* is given by $\triangle\boldsymbol{\Phi} := (\triangle\boldsymbol{\varphi}, \triangle\boldsymbol{\sigma}, \triangle\boldsymbol{\lambda}, \triangle\eta, \triangle\mathbf{z}, \triangle\mathbf{w}, \triangle\mathbf{s}, \triangle\mathbf{t})$. The update $\boldsymbol{\Phi} \leftarrow \boldsymbol{\Phi} + \triangle\boldsymbol{\Phi}$ is determined by the increment $\triangle\boldsymbol{\Phi}$ computed by using the Newton method for the following $p-$dependent system of equations.

$$F_p'(\boldsymbol{\Phi})\,\triangle\boldsymbol{\Phi} = -F_p(\boldsymbol{\Phi}), \tag{15}$$

where (15) is often referred to as the *primal–dual* system and solved at each iteration with a decreasing parameter p. More precisely, (15) is equivalent to:

$$\begin{pmatrix} 0 & \mathcal{L}_{\boldsymbol{\varphi}\boldsymbol{\sigma}} & \mathcal{L}_{\boldsymbol{\varphi}\boldsymbol{\lambda}} & 0 & 0 & 0 & 0 & 0 \\ \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\varphi}} & \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} & \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\lambda}} & \mathcal{L}_{\boldsymbol{\sigma}\eta} & -\mathrm{I} & \mathrm{I} & 0 & 0 \\ \mathcal{L}_{\boldsymbol{\lambda}\boldsymbol{\varphi}} & \mathcal{L}_{\boldsymbol{\lambda}\boldsymbol{\sigma}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_{\eta\boldsymbol{\sigma}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{I} & 0 & 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \\ 0 & 0 & 0 & 0 & S & 0 & Z & 0 \\ 0 & 0 & 0 & 0 & 0 & T & 0 & W \end{pmatrix} \begin{pmatrix} \triangle\boldsymbol{\varphi} \\ \triangle\boldsymbol{\sigma} \\ \triangle\boldsymbol{\lambda} \\ \triangle\eta \\ \triangle\mathbf{z} \\ \triangle\mathbf{w} \\ \triangle\mathbf{s} \\ \triangle\mathbf{t} \end{pmatrix} = - \begin{pmatrix} \nabla_{\boldsymbol{\varphi}}\mathcal{L} \\ \nabla_{\boldsymbol{\sigma}}\mathcal{L} \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} \\ \nabla_{\eta}\mathcal{L} \\ \nabla_{\mathbf{z}}\mathcal{L} \\ \nabla_{\mathbf{w}}\mathcal{L} \\ \nabla_{\mathbf{s}}\mathcal{L} \\ \nabla_{\mathbf{t}}\mathcal{L} \end{pmatrix}, \tag{16}$$

where I stands for the identity matrix, $S = \mathrm{diag}(s_i)$, $Z = \mathrm{diag}(z_i)$, $T = \mathrm{diag}(t_i)$, and $W = \mathrm{diag}(w_i)$ are diagonal matrices. Note that $\mathcal{L}_{\boldsymbol{\lambda}\boldsymbol{\varphi}} = A(\boldsymbol{\sigma})$ is the stiffness matrix of the electric potential equation, $\mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}$ is a diagonal matrix, and $\mathcal{L}_{\eta\boldsymbol{\sigma}} = \nabla^T g(\boldsymbol{\sigma})$ is just one row vector.

The primal–dual matrix $F_p'(\boldsymbol{\Phi})$ in (15) is sparse, nonsymmetric, indefinite, and usually well–conditioned. Our approach is to transform $F_p'(\boldsymbol{\Phi})$ to a smaller (so called *condensed*) matrix, which is inherently ill–conditioned, but the ill–conditioning should not necessarily be avoided and has no negative consequences. For detailed discussion, see, e.g., [12]. We eliminate the increments for $\mathbf{s}$ and $\mathbf{t}$

from the 5th and 6th rows of (16), namely, $\triangle\mathbf{s} = \triangle\boldsymbol{\sigma} - \nabla_{\mathbf{z}}\mathcal{L}$, $\triangle\mathbf{t} = -\triangle\boldsymbol{\sigma} - \nabla_{\mathbf{w}}\mathcal{L}$. From the last two rows of (16) we obtain the increments for $\mathbf{z}$ and $\mathbf{w}$:

$$\begin{aligned} \triangle\mathbf{z} &= S^{-1}(-\nabla_{\mathbf{s}}\mathcal{L} - Z\,(\triangle\boldsymbol{\sigma} - \nabla_{\mathbf{z}}\mathcal{L})) \\ \triangle\mathbf{w} &= T^{-1}(-\nabla_{\mathbf{t}}\mathcal{L} - W\,(-\triangle\boldsymbol{\sigma} - \nabla_{\mathbf{w}}\mathcal{L})). \end{aligned} \tag{17}$$

Substituting (17) in the second row of (16), we get the following linear system:

$$\begin{pmatrix} 0 & \mathcal{L}_{\boldsymbol{\varphi}\boldsymbol{\sigma}} & \mathcal{L}_{\boldsymbol{\varphi}\boldsymbol{\lambda}} & 0 \\ \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\varphi}} & \tilde{\mathcal{L}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} & \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\lambda}} & \mathcal{L}_{\boldsymbol{\sigma}\eta} \\ \mathcal{L}_{\boldsymbol{\lambda}\boldsymbol{\varphi}} & \mathcal{L}_{\boldsymbol{\lambda}\boldsymbol{\sigma}} & 0 & 0 \\ 0 & \mathcal{L}_{\eta\boldsymbol{\sigma}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \triangle\boldsymbol{\varphi} \\ \triangle\boldsymbol{\sigma} \\ \triangle\boldsymbol{\lambda} \\ \triangle\eta \end{pmatrix} = - \begin{pmatrix} \nabla_{\boldsymbol{\varphi}}\mathcal{L} \\ \tilde{\nabla}_{\boldsymbol{\sigma}}\mathcal{L} \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L} \\ \nabla_{\eta}\mathcal{L} \end{pmatrix}, \tag{18}$$

where $\tilde{\mathcal{L}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \mathcal{L}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} + S^{-1}\,Z + T^{-1}\,W$ and the modified entry for the right–hand side is

$$\tilde{\nabla}_{\boldsymbol{\sigma}}\mathcal{L} = \nabla_{\boldsymbol{\sigma}}\mathcal{L} + S^{-1}(\nabla_{\mathbf{s}}\mathcal{L} - Z\,\nabla_{\mathbf{z}}\mathcal{L}) - T^{-1}(\nabla_{\mathbf{t}}\mathcal{L} - W\,\nabla_{\mathbf{w}}\mathcal{L}).$$

Transforming iterations, proposed in [11], for the null space decomposition of the condensed matrix, are applied to compute the search direction, see, [9].

3 Interior–Point Method

We apply the *line–search* version of the Newton method. After computation of the search direction $\triangle\boldsymbol{\Phi}$, a common steplength α ($\alpha > 0$) is employed to update the solution $\boldsymbol{\Phi} \leftarrow \boldsymbol{\Phi} + \alpha\triangle\boldsymbol{\Phi}$. In all Newton–type methods, $\alpha = 1$ is almost always the "ideal" value. The method for choosing α at each iteration becomes more complex, as it is well known that for general nonlinear problems with a poor initial estimate, Newton's method may diverge. Complete convergence analysis of the Newton interior–point method for nonlinear programming is given by [1] provided the Jacobian $F_p'(\boldsymbol{\Phi})$ of the system (14) remains nonsingular.

A standard approach for choosing the steplength α is to define a suitable merit function, that measures the progress towards the solution. The squared l_2-norm of the residual as a merit function was introduced in [1] as

$$M(\boldsymbol{\Phi}) = \|F(\boldsymbol{\Phi})\|_2^2, \tag{19}$$

where $F(\boldsymbol{\Phi}) := F_p(\boldsymbol{\Phi}) + p\,\hat{\mathbf{e}}$, see (14), and $\hat{\mathbf{e}} = (0, \ldots, 0, 1, \ldots, 1)$ is a vector with $2N$ ones. We accept the following notations: $M_k = M_k(0) = M(\boldsymbol{\Phi}_k)$ and $M_k(\alpha) = M(\boldsymbol{\Phi}_k + \alpha\triangle\boldsymbol{\Phi}_k)$, where $\boldsymbol{\Phi}_k$ is the computed solution at a given iteration.

To specify the selection of α, we apply the algorithm proposed by [1]. Let $\boldsymbol{\Phi}_0 = (\boldsymbol{\varphi}_0, \boldsymbol{\sigma}_0, \boldsymbol{\lambda}_0, \eta_0, \mathbf{z}_0, \mathbf{w}_0, \mathbf{s}_0, \mathbf{t}_0)$ be a given starting point with $(\mathbf{z}_0, \mathbf{w}_0, \mathbf{s}_0, \mathbf{t}_0) > 0$. Let

$$\tau = \min(Z_0\,\mathbf{s}_0, W_0\,\mathbf{t}_0)/[(\mathbf{z}_0^T\mathbf{s}_0 + \mathbf{w}_0^T\mathbf{t}_0)/(2\,N)].$$

We denote by

$$\boldsymbol{\Phi}(\alpha) := (\boldsymbol{\varphi}(\alpha), \boldsymbol{\sigma}(\alpha), \boldsymbol{\lambda}(\alpha), \eta(\alpha), \mathbf{z}(\alpha), \mathbf{w}(\alpha), \mathbf{s}(\alpha), \mathbf{t}(\alpha)) = \boldsymbol{\Phi} + \alpha \triangle \boldsymbol{\Phi}.$$

For a given iteration k, we define

$$q_k(\alpha) = \min(Z(\alpha)\mathbf{s}(\alpha), W(\alpha)\mathbf{t}(\alpha)) - \gamma_k \tau \, (\mathbf{z}(\alpha)^T \mathbf{s}(\alpha) + \mathbf{w}(\alpha)^T \mathbf{t}(\alpha))/(2\,N),$$

where $\gamma_k \in (0,1)$ is a constant. The steplength α_k is determined as

$$\alpha_k = \max_{\alpha \in (0,1]} \{\alpha : q_k(\alpha') \geq 0, \text{ for all } \alpha' \leq \alpha\}. \tag{20}$$

Note that the function $q_k(\alpha)$ is piecewise quadratic and, hence, α_k is either one or the smallest positive root of $q_k(\alpha)$ in $(0,1]$.

We describe now the primal–dual Newton interior–point algorithm.

Interior–point algorithm:

1. Choose $\boldsymbol{\Phi}_0 = (\boldsymbol{\varphi}_0, \boldsymbol{\sigma}_0, \boldsymbol{\lambda}_0, \eta_0, \mathbf{z}_0, \mathbf{w}_0, \mathbf{s}_0, \mathbf{t}_0)$ such that $(\mathbf{z}_0, \mathbf{w}_0, \mathbf{s}_0, \mathbf{t}_0) > 0$ and $\beta \in (0, 1/2]$. Set $k = 0$, $\gamma_{k-1} = 1$, and compute $M_0 = M(\boldsymbol{\Phi}_0)$. For $k = 0, 1, 2, \ldots$, do the following steps:
2. Test for convergence: if $M_k \leq \epsilon_{\text{exit}}$, stop.
3. Choose $\xi_k \in (0,1)$; for $\boldsymbol{\Phi} = \boldsymbol{\Phi}_k$, compute the perturbed Newton direction $\triangle \boldsymbol{\Phi}_k$ from (15) with a perturbation parameter

$$p_k = \xi_k (\mathbf{z}_k^T \, \mathbf{s}_k + \mathbf{w}_k^T \, \mathbf{t}_k)/(2\,N). \tag{21}$$

4. Steplength selection.
   (4a) Choose $1/2 \leq \gamma_k \leq \gamma_{k-1}$; compute α_k from (20).
   (4b) Let $\alpha_k = \alpha_k/(2^n)$, where $n > 0$ is the smallest integer such that

$$M_k(\alpha_k) \leq M_k(0) + \alpha_k \, \beta \, \nabla M_k^T \triangle \boldsymbol{\Phi}_k.$$

5. Let $\boldsymbol{\Phi}_{k+1} = \boldsymbol{\Phi}_k + \alpha_k \triangle \boldsymbol{\Phi}_k$ and $k \leftarrow k+1$. Go to 2.

It was shown in [1] that for the proposed choice of p_k in (21), the search direction $\triangle \boldsymbol{\Phi}_k$, generated by the interior–point algorithm, gives descent for the merit function $M(\boldsymbol{\Phi}_k)$, i.e., $\nabla M_k^T \triangle \boldsymbol{\Phi}_k < 0$, where ∇M_k is the derivative of $M_k(\alpha)$ at $\alpha = 0$.

4 Numerical Experiments

In this section, we give some details concerning our computations. We solve the optimization problem (11)-(12) with an objective function defined in (8). The first equality constraint is related to solving elliptic differential equation for the electric potential $\boldsymbol{\varphi}$, see (5). We allow here some modification in the conductivity, namely, we consider

$$\operatorname{div}(h(\sigma)\,\mathbf{grad}\,\varphi) = 0 \text{ in } \Omega, \quad \mathbf{n} \cdot h(\sigma)\,\mathbf{grad}\,\varphi = \begin{cases} \mathrm{I}_\nu \text{ on } \Gamma_\nu \subset \partial\Omega \\ 0 \quad \text{otherwise} \end{cases}, \tag{22}$$

where

$$h(\sigma) = \left(\frac{\sigma - \sigma_{\min} + 0.01}{\sigma_{\max} - \sigma_{\min}} \right)^2 \tag{23}$$

is treated as a conductivity. Neumann boundary conditions were imposed, assuming that the compatibility condition from Section 1 is satisfied. The computations have been carried through a rectangular domain Ω decomposed into N uniform quadrilateral finite elements. We suppose that the domain is an isotropic conductor. The conductivity is computed at the center points of the finite elements and the electric potential is approximated at the midpoints of the edges. Due to the definition (23), the diagonal matrix $\mathcal{L}\boldsymbol{\sigma}\boldsymbol{\sigma}$ does not vanish.

Our primal–dual code was written in C++ using double precision binary arithmetic. All numerical tests were run on Alpha PC164LX machine. We choose lower and upper limits for the conductivity $\sigma_{\min} = 0.01$ and $\sigma_{\max} = 1$, respectively. In all runs, an initial homogeneous distribution was proposed with $\sigma = 0.45$. The constant C in (11) is computed in accordance with this initialization. The following parameters for the interior–point algorithm in Section 3 are used: $\xi_k = \min(0.2, 100\,(\mathbf{z}_k^T\,\mathbf{s}_k + \mathbf{w}_k^T\,\mathbf{t}_k))$, $\beta = 0.0001$, and $\epsilon_{exit} = 10^{-6}$.

The most expensive (in terms of CPU–time) part of the algorithm during a given iteration is to solve the condensed primal–dual system finding the increments. Two transforming iterations have been used with a zero initial guess. The preconditioned conjugate gradient (PCG) method is applied with the symmetric successive overrelaxation (SSOR) iteration as a preconditioner for the stiffness matrix. We choose a relaxation parameter $\omega = 1.5$ and a stopping criterion for both iterative procedures $\mathbf{r}^T A(h(\boldsymbol{\sigma}))\mathbf{r} < 10^{-10}$, where $\mathbf{r}$ is the current residual.

The results from our numerical experiments are reported in Table 1 for various number of contacts NC and various number of finite elements N. The dimension of the stiffness matrix is denoted by NP. We report as well the global number of iterations in the main optimization loop denoted by iter, the perturbation parameter p and the merit function $M(\boldsymbol{\Phi})$ at the last iteration.

Table 1. Results from applications of the interior–point algorithm

NC	N	NP	iter	p	$M(\boldsymbol{\Phi})$
2	30	71	20	1.13e-4	5.11e-7
2	40	93	14	2.17e-5	3.42e-8
2	80	178	18	7.03e-5	5.76e-8
2	80	178	22	5.08e-4	2.05e-7
2	120	262	34	3.93e-5	1.27e-8
3	30	71	25	6.03e-4	4.18e-7
3	64	144	41	5.17e-5	8.03e-8
4	96	212	45	3.12e-4	4.32e-7
5	180	388	42	1.18e-4	2.84e-7

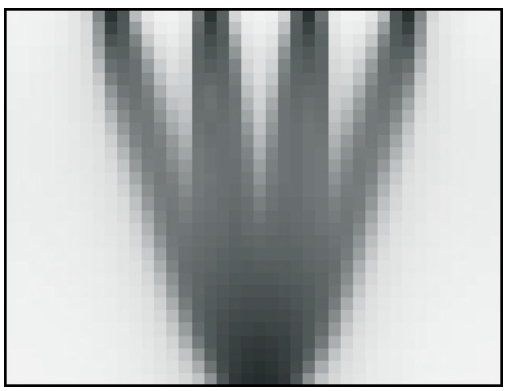

Fig. 1. Conductivity distribution for a mesh 30×40 with 5 contacts

Figure 1 shows the conductivity distribution for a mesh 30×40 with five contacts. The black color indicates elements where the conductivity is very close to $\sigma_{\max}$ and the white color indicates those elements with a conductivity close to $\sigma_{\min}$.

References

1. El–Bakry, A. S., Tapia, R. A., Tsuchiya, T., Zhang, Y.: On the formulation and theory of the Newton interior–point method for nonlinear programming. Journal of Optimization Theory and Applications, **89** (1996) 507-541. 416, 418, 419
2. Beck, R., Deuflhard, P., Hiptmair, R., Hoppe, R. H. W., Wohlmuth, B.: Adaptive miltilevel methods for edge element discretizations of Maxwell's equations. Surveys of Math. in Industry, **8** (1999) 271-312. 414
3. Bendsøe, M. P.: Optimization of Structural Topology, Shape, and Material. Springer, (1995). 414
4. Fiacco, A. V., McCormick, G. P.: Nonlinear Programming: Sequential Unconstrained Minimization Techniques. John Wiley and Sons, New York, (1968). Republished by SIAM, Philadelphia, (1990). 417
5. Forsgen, A., Gill, P.: Primal–dual interior methods for nonconvex nonlinear programming. SIAM Journal on Optimization, **8** (1998) 1132-1152. 416
6. Gay, D. M., Overton, M. L., Wright, M. H.: A primal–dual interior method for nonconvex nonlinear programming. Technical Report 97-4-08, Computing Science Research Center, Bell Laboratories, Murray Hill, New Jersey, (1997). 416
7. Hoppe, R. H. W.: Mortar edge elements in R^3. East–West J. Numer. Anal., **7** (1999) 159-173. 414
8. Kojima, M., Mizuno, S., Yoshise, A.: A primal–dual method for linear programming. Progress in Mathematical Programming Interior Point and Related Methods, Edited by N. Megiddo, Springer Verlag, New York, (1989). 416
9. Maar, B., Schulz, V.: Interior point multigrid methods for topology optimization. Preprint 98-57, IWR, University of Heidelberg, Germany, (1998), Structural Optimization (to appear). 416, 418

10. Monk, P.: Analysis of finite element method for Maxwell's equations. SIAM J. Numer. Anal., **29** (1992) 714-729. 414
11. Wittum, G.: On the convergence of multigrid methods with transforming smoothers. Theory with applications to the Navier–Stokes equations. Numer. Math., **57** (1989) 15-38. 418
12. Wright, M. H.: Ill–conditioning and computational error in interior methods for nonlinear programming. Technical Report 97-4-04, Computing Science Research Center, Bell Laboratories, Murray Hill, New Jersey, (1997). 416, 417