# Universität Augsburg

# Further Studies on Timed Testing of Concurrent Systems

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**Abstract**

The setting of [Vog95] is extended in three aspects: i) semantic equivalences are replaced by preorders, yielding implementation-specification-relations, ii) discrete time is generalized to continuous time and iii) the three variants of timed behaviour (liberal, mixed and strict) are completed by the 'dual' of mixed behaviour. As main results we derive: i) if a strictly timed system performs in both best- and worst-case as well as another strictly timed one, then both systems must be equivalent; ii) considering the basic-semantics, continuous time is in general more discriminating than discrete time, but never in tests; iii) the 'dual' of mixed behaviour cannot be related to some classical notion of concurrent behaviour in an equivalent way as it is possible for the original three variants.

## Contents



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# 1 Introduction

In [Vog95], the classical testing scenario of [DNH84] was modified to timed testing: not only the functional behaviour of concurrent systems was considered, but also the capability or necessity to perform some activity within a certain amount of time. Three different timing disciplines have been considered: liberal timing allows activity to be delayed and prolongated arbitrarily, whereas strict timing requires immediate start of possible actions and prohibits to exceed a fixed duration; this is also the case for mixed timing, where only the start but not the end of some action may be delayed arbitrarily. For liberal and mixed timing, only a best-case (may-) testing is appropriate, whereas for strict timing also worst-case (must-) testing is reasonable. In all variants, passage of time was modelled by discrete one-time-steps and the durations of actions were natural numbers. Finally, when comparing two systems w.r.t. their temporal and functional behaviour, semantic equivalences were considered. Here we address essentially three new topics:

Firstly, we consider semantic preorders rather than equivalences when comparing systems; this way, an implementation-relation is established: if $T(S)$ is the set of timed tests that are satisfied by some system $S$ and we have $T(I) \supseteq T(S)$ for some system $I$, then $I$ can be seen as a *faster implementation of specification* $S$, since $I$ not only performs successful in an environment whenever $S$ does, but also earlier in general. As a main result, it will turn out that a strictly timed system performs in both best- and worst-case as well as another strictly timed one, then both systems must be equivalent, i.e. satisfy exactly the same timed tests.

Secondly, passage of time is modelled by real-number steps. In principle, this gives much more liberty in temporal behaviour; we investigate in detail, whether and when this more detailed behaviour is actually observable. As a second main result, it will turn out that the discrete and continuous testing preorders coincide.

Thirdly, we complete the three timing variants by somewhat like the 'dual' of mixed behaviour: possible actions must start immediately, but may be prolongated arbitrarily. For the original three variants, a coincidence with classical time-free notions of concurrent behaviour could be established in [Vog95] (ST-sequences, step-sequences and maximal-step-sequences). As a third main result, it will turn out that this is not possible for the new variant.

The paper is structured as follows: Section 2 introduces labelled Petri nets as our system model and four variants of concurrent behaviour of such nets; the relationship between these variants is examined comprehensively. Basic knowledge of Petri nets and their behaviour is assumed; for further details see e.g. [Rei85]. Section 3 extends the setting by an explicit notion of time and introduces four variants of timed behaviour; the relationship between concurrent, discrete and continuous behaviour as well as the relationship between the four timed variants is studied. In Section 4 timed testing is explained, defined and applied as may-testing in all four timed variants and additionally as must-testing in the strict variant; the relationship between discrete and continuous testing is clarified, the discrete variants are characterized and – finally – the above mentioned result concerning the equivalence of strictly timed systems is derived.

## 2 Concurrent Behaviour of Labelled Petri Nets

We restrict attention to labelled safe Petri nets without isolated transitions and with arc-weights of at most 1; as a consequence, markings are sets of places. We also assume an (infinite) set $\Sigma$ of transition-labels or actions, which is understood to be common to all considered nets; later on, $\Sigma$ will be extended by special actions reserved for test nets.

**Definition 2.1**

A *labelled Petri net* $N = (S, T, F, M_N, l)$ (net for short) consists of disjoint sets of places $S$ and transitions $T$, the flow relation $F \subseteq (S \times T) \cup (T \times S)$, an initial marking $M_N \subseteq S$ and a labelling $l : T \mapsto \Sigma$, where $\Sigma = \{a, b, c, \dots\}$ is an infinite set of *actions*.

For a transition $t \in T$ let ${}^\bullet t = \{s \in S \mid (s,t) \in F\}$ and $t^\bullet = \{s \in S \mid (t,s) \in F\}$. We write $M[t\rangle$ if ${}^\bullet t \subseteq M$ for some $M \subseteq S$ and say that $t$ is activated (or enabled) under $M$. We define $M[T\rangle = \{t \in T \mid M[t\rangle\}$.

Let $T^\pm = \{t^+, t^- \mid t \in T\}$ be the *transition parts*, i.e. *transition-starts* $t^+$ and *transition-ends* $t^-$. Analogously, let $\Sigma^\pm = \{a^+, a^- \mid a \in \Sigma\}$ be the respective *action parts*. ■ 2.1

Two – not necessarily distinct – transitions $t_1$ and $t_2$ are *concurrently* enabled under some marking $M$, if $(M \setminus {}^\bullet t_1)[t_2\rangle$ and $(M \setminus {}^\bullet t_2)[t_1\rangle$; some transition $t$ is *self-concurrent* if $t$ is concurrently enabled with itself. Note that due to the above restrictions there are *no* self-concurrent transitions in the considered nets.

Since we are interested in the concurrent behaviour of nets, we consider transitions (and, hence, actions) to be non-atomic; rather, we distinguish between transition-starts and -ends, such that there is a chance to observe an overlapping of actions. Consequently, a state of a net will not only be described by the current marking $M$, but additionally by the set of currently firing transitions $C$. Note that in each state the set of possible transition-starts is completely determined by $M$, whereas the set of possible transition-ends is completely determined by $C$; however, the situation is not quite symmetric: starts may be in conflict (i.e. ${}^\bullet t_1 \cap {}^\bullet t_2 \neq \emptyset$ for some $t_1, t_2 \in M[T\rangle$), whereas ends never are in conflict; furthermore, since initially no transition is current, each transition-end must be preceded by the corresponding transition-start in a sequence, but not vice versa.

When comparing the behaviour of two nets, we will abstract from the identity of their transitions and will rather consider the actions they represent; since the labelling of transitions is not necessarily injective, the lack of self-concurrent transitions does not preclude *auto-concurrent* actions: an action $a$ is auto-concurrent under a marking $M$ if $M$ enables two (here: different) $a$-labelled transitions $t_1$ and $t_2$ concurrently. In order to connect each actions start with its end, we attach an 'event-tag' $e$ to both a transitions start and its end, and then lift this tag to the level of actions.

**Definition 2.2**

Let $N = (S, T, F, M_N, l)$ be a net and let $E$ be an infinite set of *events*, which is understood to be common to all considered nets.

An *instantaneous description* *ID* (*state* for short) of a net $N$ is a pair $(M, C)$, where $M \subseteq S$ is a marking and $C \subseteq T \times E$ is the set of *current* transitions. The *initial state* of a net $N$ is $ID_N = (M_N, \emptyset)$. We write $(M, C)[\varepsilon\rangle(M', C')$ if either

1. $\varepsilon = (t^+, e)$ where $(t, e) \in M[T\rangle \times E$ and $M' = M \setminus {}^\bullet t$ and $C' = C \cup \{(t, e)\}$, or
2. $\varepsilon = (t^-, e)$ where $(t, e) \in C$ and $M' = M \cup t^\bullet$ and $C' = C \setminus \{(t, e)\}$.

If $ID_0[\varepsilon_1\rangle ID_1 \ldots [\varepsilon_n\rangle ID_n$ for some $n \in \mathbb{N}_0$ and $v = \varepsilon_1 \ldots \varepsilon_n$, then we write $ID_0[v\rangle ID_n$ or $ID_0[v\rangle$. We define $l(v) \in (\Sigma^\pm \times E)^*$ inductively via $l(\lambda) = \lambda$ (the empty sequence), $l(v(t^+, e)) = l(v)(l(t)^+, e)$ and $l(v(t^-, e)) = l(v)(l(t)^-, e)$, and we write $ID[l(v)\rangle\rangle$ if $ID[v\rangle$. For $A \subseteq \Sigma$ we define $M[A\rangle\rangle = A \cap \{l(t) \mid t \in M[T\rangle\}$. ■ 2.2

Now the concurrent behaviour of a net can as usual be defined to be the set of all sequences of action (or transition) parts that are operationally derivable from the initial state; when doing so below, we will take into account two further aspects:

Firstly, we want do distinguish several kinds of concurrent behaviour; more precisely, we also want to consider restricted variants where the behaviour is maximal w.r.t. possible starts and/or ends of actions (or transitions) in each state. Without any such restriction, we gain the usual ST-sequences (cf. [Gla90]); if we require all possible ends to occur before the next start, then we gain the usual step sequences; if we require an end to occur only after no more start is possible and – additionally – all possible ends to occur before the next start, then we gain the usual maximum-step sequences (note that in this formulation the above mentioned asymmetry between starts and ends is reflected). If we give up the condition on ends in the definition of maximum-step sequences, then we gain a new variant, which does not seem to be expressible in conventional step-based terms.

Secondly, we want to ensure the uniqueness of each event-tag $e$ in a sequence; this reflects the presumed distinguishability of each occurrence of an action and will ease the deduction of the behaviour of a synchronized net from the behaviours of its components later on.

**Definition 2.3**

Let $N = (S, T, F, M_N, l)$ be a net. A sequence $v = \varepsilon_1 \ldots \varepsilon_n \in (T^\pm \times E)^*$ is *event-unique*, if each $e \in E$ occurs at most twice in $v$, and if $e$ occurs twice, then $v = v_1(t^+, e)v_2(t^-, e)v_3$ for some $t \in T$ and $v_1, v_2, v_3 \in (T^\pm \times E)^*$. For a sequence $w \in (\Sigma^\pm \times E)^*$, event-uniqueness is defined analogously, and obviously $w = l(v)$ is event-unique if and only if $v$ is event-unique. Now we define:

$\mathsf{IFS}(N) = \{v \mid ID_N[v\rangle$ and $v$ is event-unique$\}$,
containing the *ST-firing-sequences.*

$\mathsf{DFS}(N) = \{\varepsilon_1 \ldots \varepsilon_n \in \mathsf{IFS}(N) \mid$ if $\varepsilon_i = (t_1^-, e_1)$ and $\varepsilon_{i+1} = (t_2^+, e_2)$, then $C_i = \emptyset\}$,
containing the *(down-) step firing sequences* of $N$.

$\mathsf{UFS}(N) = \{\varepsilon_1 \ldots \varepsilon_n \in \mathsf{IFS}(N) \mid$ if $\varepsilon_i = (t_1^+, e_1)$ and $\varepsilon_{i+1} = (t_2^-, e_2)$, then $M_i[T\rangle = \emptyset\}$,
containing the *up-step firing sequences* of $N$.

$\mathsf{MFS}(N) = \mathsf{DFS}(N) \cap \mathsf{UFS}(N)$,
containing the *maximum-step firing sequences* of $N$.

For $\mathsf{X} \in \{\mathsf{I}, \mathsf{D}, \mathsf{U}, \mathsf{M}\}$, we let $\mathsf{XL}(N) = \{l(w) \mid w \in \mathsf{XFS}(N)\}$, be the *ST-language, (down-) step language, up-step language and maximum-step language* resp. of $N$. ■ 2.3

In [Vog95], event-uniqueness is partly already guaranteed by the operational behaviour: a $t^+$ may only be attached with an $e$ that is not yet used in $C$; however, the complete condition is established by the same (w.r.t. to the sequence) global predicate as here; in the end, both definitions coincide. It should be mentioned that event-uniqueness could as well established by purely operational (i.e. 'local') restrictions: extend the states of a net by a set $H$ ('history') of already used events and adjust rule 1. in Definition 2.2 accordingly. We have not chosen this way in order to keep the operational rules as simple as possible.

Note that – as for event-uniqueness – also the conditions which distinguish the four variants in Definition 2.3 could already have been realized by refining the operational rules of Definition 2.2: extend the states of a net by a boolean flag with values in e.g. $\{+, -\}$ indicating, whether the last event was a start or an end resp. and adjust rules 1. and 2. in Definition 2.2 according to the conditions on the four different variants. However, this would be technically considerably more involved and it would complicate the refinement to timed behaviour as carried out in the next section.

The four variants of concurrent behaviour have been introduced, since each of them will be closely related to a variant of timed behaviour for a special class of nets considered later on; hence, we are yet interested in the relationship between these timed-free concurrent variants. We have chosen a rather unconventional presentation of the usual step-based notions, since this will allow i) a facile comparison of the variants, ii) a clarification of the technical relation between concurrent and timed behaviour and iii) a neat integration of the new $\mathsf{UL}$-variant.

The four variants can be related w.r.t. their 'degree of concurrency' as follows:

**Proposition 2.4**

Let $N$ be a net and let $\longrightarrow$ denote (set-) inclusion; then

$$
\begin{array}{ccccc}
 & & \mathsf{IFS}(N) & & \\
 & \nearrow & & \nwarrow & \\
\mathsf{DFS}(N) & & & & \mathsf{UFS}(N) \\
 & \nwarrow & & \nearrow & \\
 & & \mathsf{MFS}(N) & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
 & & \mathsf{IL}(N) & & \\
 & \nearrow & & \nwarrow & \\
\mathsf{DL}(N) & & & & \mathsf{UL}(N) \\
 & \nwarrow & & \nearrow & \\
 & & \mathsf{ML}(N) & &
\end{array}
$$

*Proof:* Directly from Definition 2.3. ■ 2.4

Consider the net $N$ depicted in Figure 1. From the $\mathsf{IL}$-view, it is observable that $c$ can be performed without a preceding $a$ (choosing the lower $b$ and the lower $c$); this is not possible from the $\mathsf{UL}$-view, where $a$ and the lower $b$ have to start immediately, and $c$ can thus only occur after the end of $a$. Furthermore, from the $\mathsf{IL}$-view, it is observable that $b$ can overlap both $a$ and $c$ (choosing the lower $b$ and the upper $c$); this is not possible from the $\mathsf{DL}$-view, where $b$ can only occur either with $a$ or $c$. Finally, from the $\mathsf{ML}$-view it is not observable, whether $c$ must be preceded by both $a$ and $b$, or not. Intuitively, information on concurrency is decreased when moving in opposite direction of the arrows in Proposition 2.4.

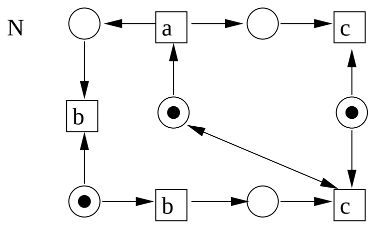



Figure 1: Different views yield different degrees of concurrency.

A further outcome of the above mentioned asymmetry between possible starts and possible ends of actions is the following result: $\mathsf{DL}(N)$ can be constructed from $\mathsf{IL}(N)$, and $\mathsf{ML}(N)$ can be constructed from $\mathsf{UL}(N)$, but neither can $\mathsf{UL}(N)$ be constructed from $\mathsf{IL}(N)$, nor can $\mathsf{ML}(N)$ be constructed from $\mathsf{DL}(N)$ in general. Informally, the reason is the following: action-ends must be preceded by the corresponding starts and, thus, maximality w.r.t. (never conflicting) possible ends is checkable syntactically for a sequence in $\mathsf{IL}(N)$ or $\mathsf{UL}(N)$, deciding whether this sequence is also in $\mathsf{DL}(N)$ or $\mathsf{ML}(N)$ resp. In contrast, maximality w.r.t. possible action-starts is not checkable for a given sequence. In order to state this result formally for sequences, we first define syntactic criteria which have a close connection to the semantic condition on step-sequences:

**Definition 2.5**

Let $N$ be a net and let $v \in \mathsf{IFS}(N)$. We say that $v$ is *terminated*, if $v = v_1(t^+, e)v_2$ implies $v_2 = v_2'(t^-, e)v_2''$. We say that $v$ is *step-partitioned*, if $v = v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ implies that $v_1(t_1^-, e_1)$ is terminated.

Termination and step-partition are defined analogously for $w \in (\Sigma^\pm \times E)^*$, and $w = l(v)$ is terminated (step-partitioned) if and only if $v$ is terminated (step-partitioned). ■ 2.5

Here, termination is an auxiliary notion in the definition of step-partition, which distinguishes the sequences in $\mathsf{DL}(N)$ and $\mathsf{ML}(N)$ from sequences in $\mathsf{IL}(N) \setminus \mathsf{DL}(N)$ and $\mathsf{UL}(N) \setminus \mathsf{ML}(N)$ resp. in the following proposition. Later on, termination will also be helpful in other contexts.

**Proposition 2.6**

Let $N$ be a net and $w \in (\Sigma^\pm \times E)^*$; then

1. $w \in \mathsf{DL}(N)$ if and only if $w$ is step-partitioned and $w \in \mathsf{IL}(N)$.
2. $w \in \mathsf{ML}(N)$ if and only if $w$ is step-partitioned and $w \in \mathsf{UL}(N)$.

*Proof:*

Let $ID_N[v\rangle(M, C)$ for some $v \in \mathsf{IFS}(N)$. We first see that $C = \emptyset$ if and only if $v$ is terminated: $e \notin proj_2(C)$ iff $e$ does not occur in $v$ or – by the event-uniqueness of $v$

– it occurs twice in $v$ and we have $v = v_1(t^+, e)v_2(t^-, e)v_3$, hence iff $v$ is terminated. Now:

1. $w \in \mathsf{DL}(N)$ iff $w = l(v)$ for some $v \in \mathsf{IFS}(N)$, such that $v = v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ and $ID_N[v_1(t_1^-, e_1)\rangle(M, C)$ implies $C = \emptyset$, hence iff $w = l(v)$ for some $v \in \mathsf{IFS}(N)$, such that $v = v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ implies $v_1(t_1^-, e_1)$ terminated, thus iff $w = l(v)$ for some step-partitioned $v \in \mathsf{IFS}(N)$ iff $w$ is step-partitioned and $w \in \mathsf{IL}(N)$.

2. $w \in \mathsf{ML}(N)$ iff $w = l(v)$ for some $v \in \mathsf{DFS}(N) \cap \mathsf{UFS}(N)$ iff $w = l(v)$ for some $v \in \mathsf{IFS}(N) \cap \mathsf{UFS}(N)$, such that $v = v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ and $ID_N[v_1(t_1^-, e_1)\rangle(M, C)$ implies $C = \emptyset$, hence iff $w = l(v)$ for some $v \in \mathsf{UFS}(N)$, such that $v = v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ implies $v_1(t_1^-, e_1)$ terminated, thus iff $w = l(v)$ for some step-partitioned $v \in \mathsf{UFS}(N)$ iff $w$ is step-partitioned and $w \in \mathsf{UL}(N)$. ■ 2.6

In the following section, we will extend our setting by explicit introduction of time in states and behaviour of nets. We will study several different timing disciplines which are technically related to the four variants of concurrent behaviour considered in this section. Since we are going to compare these timing variants and to characterize the corresponding testing preorders via inclusion of some language later on, as a preparation we finish this section by checking for implications between XL-inclusion and YL-inclusion of two nets for $\mathsf{X}, \mathsf{Y} \in \{\mathsf{I}, \mathsf{D}, \mathsf{U}, \mathsf{M}\}$; the results will significantly back up the above mentioned comparisons in the timed setting.

**Proposition 2.7**

Let $N_1$ and $N_2$ be nets and $\mathsf{X}, \mathsf{Y} \in \{\mathsf{I}, \mathsf{D}, \mathsf{U}, \mathsf{M}\}$ with $\mathsf{X} \neq \mathsf{Y}$.

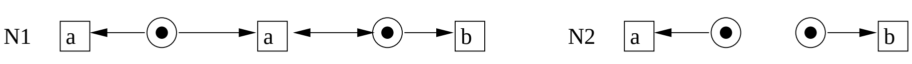
1. $\mathsf{IL}(N_1) \subseteq \mathsf{IL}(N_2)$ implies $\mathsf{DL}(N_1) \subseteq \mathsf{DL}(N_2)$.
2. $\mathsf{UL}(N_1) \subseteq \mathsf{UL}(N_2)$ implies $\mathsf{ML}(N_1) \subseteq \mathsf{ML}(N_2)$.
3. If $(\mathsf{X}, \mathsf{Y}) \notin \{(\mathsf{I}, \mathsf{D}), (\mathsf{U}, \mathsf{M})\}$, then there are nets $N_1$ and $N_2$, such that $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$, but $\mathsf{YL}(N_1) \not\subseteq \mathsf{YL}(N_2)$.

*Proof:*

1. Follows from Proposition 2.6.1.
2. Follows from Proposition 2.6.2.
3. We distinguish several cases:

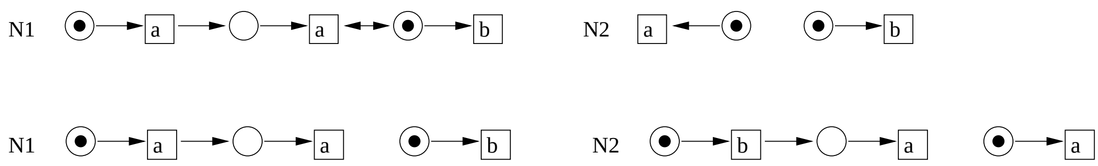
**$\mathsf{X} \in \{\mathsf{I}, \mathsf{D}\}$, $\mathsf{Y} \in \{\mathsf{U}, \mathsf{M}\}$:**

We have $(a^+, e)(a^-, e) \in (\mathsf{UL}(N_1) \cap \mathsf{ML}(N_1)) \setminus (\mathsf{UL}(N_2) \cup \mathsf{ML}(N_2))$:




**$\mathsf{X} \in \{\mathsf{U}, \mathsf{M}\}$, $\mathsf{Y} \in \{\mathsf{I}, \mathsf{D}\}$:**

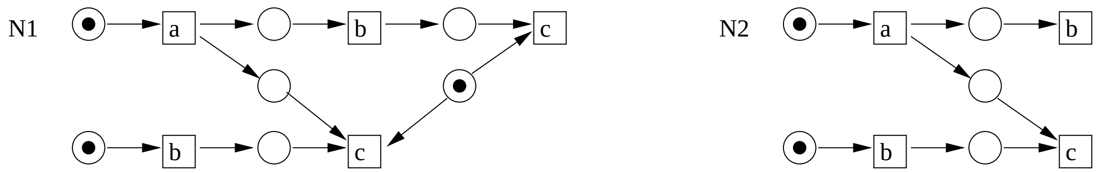
We have $(a^+, e)(a^-, e)(a^+, e') \in (\mathsf{IL}(N_1) \cap \mathsf{DL}(N_1)) \setminus (\mathsf{IL}(N_2) \cup \mathsf{DL}(N_2))$:




X = M, Y = U:

We have We have $(a^+,e)(b^+,e')(a^-,e)(a^+,e'') \in \mathsf{UL}(N_1) \setminus \mathsf{UL}(N_2)$:

X = D, Y = I: consider




We first argue that $\mathsf{DL}(N_1) \subseteq \mathsf{DL}(N_2)$: it suffices to consider a behaviour of $N_1$ in which the additional $c$ is enabled; this is possible if and only if $a$ and the upper $b$ have fired in $N_1$, and additionally the lower $b$ either i) has not occurred yet or ii) occurred with $a$ or iii) occurred with the upper $b$. This can be simulated in $N_2$ by firing $a$ either i) followed by the lower $b$ or ii) together with the lower $b$ and followed by the upper $b$ or iii) followed by both the upper and lower $b$ simultaneously.

But $(a^+,e_a)(b^+,e_b)(a^-,e_a)(b^+,e_b')(b^-,e_b')(c^+,e_c) \in \mathsf{IL}(N_1) \setminus \mathsf{IL}(N_2)$. ■ 2.7

# 3 Timed Nets and their Behaviour in Time

In the previous section, we have studied the concurrent behaviour of nets without using an explicit notion of time. We rather made qualitative distinctions essentially by checking whether actions can overlap each other, i.e. can occur independently. We now add a quantitative notion of time to the operational behaviour and, therefore, first extend the nets defined in the previous section by introducing durations for transitions.

**Definition 3.1**

A *timed labelled Petri net* $N = (S, T, F, M_N, l, \delta)$ (timed net for short) consists of a labelled net $(S, T, F, M_N, l)$ and a transition *duration* $\delta : T \mapsto \mathbb{N}$. $N$ is called *untimed*, if $\delta(t) = 1$ for all $t \in T$. ■ 3.1

Note that this definition allows equally labelled transitions to have different durations; this is not only in order to enhance flexibility or generality, but actually allows to model systems, where the duration of the same action varies with the systems internal situation; in particular, when testing systems via synchronization on equal actions with a test net in section 4, we can keep the tested net untimed and, hence, let all transition durations be determined by the test net.

We have chosen durations to be natural numbers although we will allow real-valued passage of time. This is a generalization (towards reality) of the setting in [Vog95], where time is modelled to pass in discrete unit-time-steps; consequently, a fair amount of this section will be devoted to answer the question, whether and when the refinement of discrete to continuous time allows to distinguish timed nets that where formerly considered to be equal.

The operational behaviour given in Definition 2.2 is now extended to timed nets and passage of (real) time, where we also distinguish four different disciplines: L-behaviour (called 'liberal' in [Vog95]) allows arbitrary passage of time in any state and only requires that a transition fires at least for its duration. E-behaviour (called 'mixed' in [Vog95]) allows an enabled transition to delay its start for an arbitrary amount of time, but its firing time must be exactly its duration. A-behaviour requires an enabled transition to be started or deactivated immediately, but allows its firing time to exceed its duration; this is a new variant not yet considered in [Vog95]. Finally, S-behaviour (called 'strict' in [Vog95]) requires an enabled transition to be started or deactivated immediately, and its firing time must be exactly its duration. In order to keep track of the firing time of a transition, the states of a net are extended by function $\rho$, which yields the residual (firing) time of all current transitions.

**Definition 3.2**

A *timed instantaneous description* $TD = (M, C, \rho)$ (*timed state* for short) of a timed net $N$ consists of a state $(M, C)$ of $N$ and the *residual time* of the current transitions $\rho : C \mapsto \mathbb{R}_0^+$. The *initial timed state* of a timed net $N$ is $TD_N = (M_N, \emptyset, \emptyset)$.

For $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$ we write $(M, C, \rho)[\varepsilon\rangle_{\mathsf{X}}(M', C', \rho')$ if one of the following cases applies:

1. $\varepsilon = (t^+, e)$ and $(M, C)[(t^+, e)\rangle(M', C')$ and $\rho'(t, e) = \delta(t)$ and $\rho'|_C = \rho$.
2. $\varepsilon = (t^-, e)$ and $(M, C)[(t^-, e)\rangle(M', C')$ and $\rho(t, e) = 0$ and $\rho' = \rho|_{C'}$.
3. $\varepsilon = (r)$ for $r \in (0; 1]$, such that $(M', C') = (M, C)$ and $\rho' = \rho \dot{-} r$, and
   - if $\mathsf{X} = \mathsf{E}$, then $r \leq \rho(t, e)$ for all $(t, e) \in C$.
   - if $\mathsf{X} = \mathsf{A}$, then $M[T\rangle = \emptyset$.
   - if $\mathsf{X} = \mathsf{S}$, then $M[T\rangle = \emptyset$ and $r \leq \rho(t, e)$ for all $(t, e) \in C$.

If $TD_0[\varepsilon_1\rangle_{\mathsf{X}} TD_1 \dots [\varepsilon_n\rangle_{\mathsf{X}} TD_n$ for $n \in \mathbb{N}_0$ and $v = \varepsilon_1 \dots \varepsilon_n$, then we write $TD_0[v\rangle_{\mathsf{X}} TD_n$ or $TD_0[v\rangle_{\mathsf{X}}$. We define $l(v) \in ((\Sigma^{\pm} \times E) \cup \{(r) \mid r \in (0; 1]\})^*$ analogously as in Definition 2.2, where we additionally let $l(v(r)) = l(v)(r)$, and we write $TD[l(v)\rangle\rangle_{\mathsf{X}}$ if $TD[v\rangle_{\mathsf{X}}$.

For a sequence $w \in ((T^{\pm} \times E) \cup \{(r) \mid r \in (0; 1]\})^*$ or $w \in ((\Sigma^{\pm} \times E) \cup \{(r) \mid r \in (0; 1]\})^*$ let $seq(w)$ denote the sequence of transition or action parts in $w$ and let $dur(w)$ be the sum of time steps in $w$. If $w = \varepsilon_1 \dots \varepsilon_n$ and $1 \leq i < j \leq n$, then $\varepsilon_j$ occurs *after* $\varepsilon_i$ in $w$; if additionally $dur(\varepsilon_{i+1} \dots \varepsilon_{j-1}) > 0$, then $\varepsilon_j$ occurs *later than* $\varepsilon_i$ in $w$. ■ 3.2

In Definition 3.2, rules 1. and 2. are refinements of the corresponding rules from Definition 2.2: by rule 1., a started transition $t$ has residual time $\delta(t)$ (its duration) and it must fire at least for its duration by rule 2 and rule 3. The latter allows passage of time, where marking and current transitions do not change, but the residual time of the current transitions is updated according to the time step; here, in the liberal case ($\mathsf{X} = \mathsf{L}$), passage of time is always possible in any timed state, and transition durations may be exceeded due to $\rho' = \rho \dot{-} r := \min(\rho - r, 0)$. The additional conditions restrict this behaviour for the other three variants: in the mixed case ($\mathsf{X} = \mathsf{E}$), time may only pass if the duration of any current transition will not be exceeded; if $\mathsf{X} = \mathsf{A}$, then time may pass only if no more transition can start in the current state, i.e. an activated transition starts or is deactivated as soon as possible; in the strict case ($\mathsf{X} = \mathsf{S}$), both restrictions apply together.

Note that Definition 3.2 allows to distinguish the four variants already by *local operational* restrictions, whereas Definition 2.3 applies *global* restrictions to operationally derivable sequences for this purpose.

Quite obviously, if a transition $t$ is current during performance of a sequence $v$, leading from timed state $TD$ to $TD'$, then the residual time of $t$ in $TD'$ coincides with the difference between its residual time in $TD$ and the duration $dur(v)$ of the sequence $v$ – unless $\delta(t)$ has been exceeded already before $TD'$ is reached. Furthermore, if a transition start $t_1^+$ is followed immediately (in particular: not later) by a transition end $t_2^-$, then $t_1$ and $t_2$ must be different (since $\delta(t_1) > 0$), and $t_1$ may as well start after (but not necessarily later than!) $t_2$ ends (since $t_2^-$ can only increase the marking which already enables $t_1$); additionally, this permutation leads to the same timed state as before. These two properties are of rather technical nature but important in many future developments and are therefore stated formally:

**Lemma 3.3**

Let $N$ be a timed net with timed states $TD$, $TD'$ and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$.

1. If $TD[v\rangle_{\mathsf{X}} TD'$ and $(t, e) \in C \cap C'$, then $\rho'(t, e) = \rho(t, e) \dot{-} dur(v)$.
2. If $TD[(t_1^+, e_1)(t_2^-, e_2)\rangle_{\mathsf{X}} TD'$, then also $TD[(t_2^-, e_2)(t_1^+, e_1)\rangle_{\mathsf{X}} TD'$.

*Proof:*

1. We perform induction on $|v|$, where for $v = \lambda$ we have $\rho'(t, e) = \rho(t, e)$ and $dur(v) = 0$; hence assume the claim to hold for some $v$ and consider $v' = v\varepsilon$ where $TD[v\rangle_{\mathsf{X}} TD''[\varepsilon\rangle TD'$.

If $\varepsilon = (t_1^+, e_1)$, then $dur(v') = dur(v)$ and $\rho'(t, e) = \rho''(t, e) = \rho(t, e) \dot{-} dur(v) = \rho(t, e) \dot{-} dur(v')$ by induction. If $\varepsilon = (t_1^-, e_1)$, then we have $e_1 \neq e$ (otherwise $(t, e) \notin C'$) and as above $dur(v') = dur(v)$ and $\rho'(t, e) = \rho''(t, e) = \rho(t, e) \dot{-} dur(v) = \rho(t, e) \dot{-} dur(v')$ by induction. If $\varepsilon = (r)$, then $dur(v') = dur(v) + r$ and $\rho'(t, e) = \rho''(t, e) \dot{-} r = (\rho(t, e) \dot{-} dur(v)) \dot{-} r = \rho(t, e) \dot{-} (dur(v) + r) = \rho(t, e) \dot{-} dur(v')$ by induction.

2. Let $(M, C, \rho)[(t_1^+, e_1)\rangle_{\mathsf{X}}(M_1, C_1, \rho_1)[(t_2^-, e_2)\rangle_{\mathsf{X}}(M', C', \rho')$; then $t_1 \neq t_2$ by $\rho_1(t_1, e_1) = \delta(t_1) > 0$ and $M' = (M \setminus {}^\bullet t_1) \cup t_2^\bullet$, $C' = (C \cup \{(t_1, e_1)\}) \setminus \{(t_2, e_2)\}$ and $\rho'(t, e) = \rho(t, e)$ for all $(t, e) \in C \setminus \{(t_2, e_2)\}$ and $\rho'(t_1, e_1) = \delta(t)$. Now $t_1 \neq t_2$ implies also $(t_2, e_2) \in C$ and $\rho(t_2, e_2) = \rho'(t_2, e_2) = 0$, hence $(M, C, \rho)[(t_2^-, e_2)\rangle_{\mathsf{X}}(M_1', C_1', \rho_1')$ with

$M_1' = M \cup t_2^\bullet$, $C_1' = C \setminus \{(t_2, e_2)\}$ and $\rho_1' = \rho|_{C_1'}$. Now $M_1'[t_1\rangle$ by $M_1' \supseteq M$, hence $(M_1', C_1', \rho_1')[(t_1^+, e_1)\rangle_{\mathsf{X}}(M'', C'', \rho'')$ with $M'' = M_1' \setminus {}^\bullet t_1 = (M \cup t_2^\bullet) \setminus {}^\bullet t_1 = M'$ since $M[t_1\rangle$ and $C'' = C_1' \cup \{(t_1, e_1)\} = (C \setminus \{(t_2, e_2)\}) \cup \{(t_1, e_1)\} = C'$ since $(t_2, e_2) \in C$, and finally $\rho''(t, e) = \rho(t, e) = \rho'(t, e)$ for all $(t, e) \in C \setminus \{(t_2, e_2)\}$ and $\rho''(t_1, e_1) = \rho'(t_1, e_1) = \delta(t)$, thus $(M'', C'', \rho'') = (M', C', \rho')$. ■ 3.3

## 3.1 Behaviour in Continuous and Discrete Time

Up to now, we have defined four timed operational variants, and – analogously to Definition 2.3 – we could define the according variants of timed behaviour to be the sets of all event-unique operationally derivable sequences. But by anticipating phenomena that play an important role when testing nets in Section 4, we will impose further conditions on the considered sequences.

In [Vog95], timed behaviour of a timed net is essentially characterized by considering only operationally derivable sequences that are *event-unique*, *max-caused* and *terminated*:

**Definition 3.4**

Let $N$ be a timed net and let $v \in ((T^{\pm} \times E) \cup \{(r) \mid r \in (0;1]\})^*$. We say that

- $v$ is *max-caused* if whenever $v = v_1(t_1^+, e_1)v_2(t_2^-, e_2)v_3$, then $dur(v_2) > 0$.
- $v$ is *wellformed* if $v$ is event-unique and max-caused.
- $v$ is *time-complete* if $v = v'(r)$ for some $r \in (0;1]$.
- $v$ is *terminated* if whenever $v = v_1(t^+, e)v_2$, then $v_2 = v_2'(t^-, e)v_2''$.

For a sequence $w \in ((\Sigma^{\pm} \times E) \cup \{(r) \mid r \in (0;1]\})^*$, max-causedness, wellformedness, time-completeness and termination are defined analogously, and obviously $w = l(v)$ shares exactly the properties of $v$. ■ 3.4

The notion of max-causedness is motivated as follows: assume $TD_N[v_1(t_1^-, e_1)(t_2^+, e_2)v_2\rangle_{\mathsf{X}}$ for some timed net $N$ and $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$); then $v_1(t_2^+, e_2)(t_1^-, e_1)v_2$ might or might not be an operationally derivable sequence, too; in the latter case, we can conclude that the start of $t_2$ requires the tokens provided by the end of $t_1$; however, this information could not be gained from a purely 'observational' point of view (which abstracts causal dependencies), since both events – start of $t_2$ and end of $t_1$ – happen at the same time. Hence, we will restrict attention to max-caused timed sequences: exactly those sequences *without* subsequences of the form $(t_2^+, e_2)(t_1^-, e_1)$; this way, a (potential) causal *in*dependence of the start of $t_2$ from the end of $t_1$ is left invisible. The important point is that by Lemma 3.3.2 the restriction to max-caused sequences does not change the observable behaviour: if $v_1(t_2^+, e_2)(t_1^-, e_1)v_2$ is operationally derivable, then also $v_1(t_1^-, e_1)(t_2^+, e_2)v_2$ is (but not necessarily vice versa).

It must be pointed out that max-causedness has been enforced operationally for mixed ($\mathsf{E}$-) and strict ($\mathsf{S}$-) behaviour in [Vog95], where rule 1. in the definition according to our Definition 3.2 allows to start a transition only after all current transitions with elapsed residual time have finished ('if $\mathsf{X} \in \{\mathsf{E},\mathsf{S}\}$, then $\rho(t,e) > 0$ for all $(t,e) \in C$'). Actually, this requirement is even stricter than max-causedness: by the additional operational restriction, any max-caused sequence can be extended to a time-complete sequence. Somewhat astonishingly, this does not work in general with our Definition 3.2 and Definition 3.4; as an example consider the timed net $N$ depicted in Figure 2 below (all transition durations are 1 and omitted). We can derive max-caused $w = (a^+, e_a)(b^+, e_b)(1)(a^-, e_a)(c^+, e_c)$ operationally in both our E- and S-variant, but $w$ cannot be extended to a time-complete sequence: $c$ has started although $b$ with elapsed residual time has not finished yet; now $b^-$ immediately after $c^+$ would violate max-causedness, and a time step immediately after $c^+$ is impossible by Definition 3.2.3, since $b$ has no residual time left. With the additional restriction in [Vog95], $c$ could not have started before $b$ has finished.

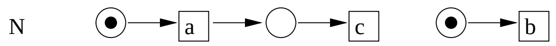



Figure 2: Some max-caused sequences cannot be time-completed.

Since we strongly intend to reuse the existing results from [Vog95] and, hence, aim at the coincidence of both definitions of behaviour in discrete time, we could simply adopt the operational restriction for rule 1. in Definition 3.2 in case $\mathsf{X} \in \{\mathsf{E},\mathsf{S}\}$. We desist from this solution for the following reason: when comparing continuous and discrete behaviour later on, we will sometimes construct discrete traces from continuous ones by induction on their length, where in intermediate states we cannot guarantee wellformedness; however, such intermediate sequences can be transformed to wellformed ones by iterated application of Lemma 3.3.2, but the operational restriction would already inhibit to construct even such intermediate sequences. In order to reconcile our operational Definition 3.2 with the corresponding one in [Vog95], we consider as timed behaviour all prefixes of some wellformed and time-complete operationally derivable sequence:

**Definition 3.5**

Let $N$ be a timed net. For $\mathsf{X} \in \{\mathsf{L},\mathsf{E},\mathsf{A},\mathsf{S}\}$ we define

$$\begin{aligned}
\mathsf{XFS}^c(N) &= \{v \mid v \text{ is a prefix of some wellformed time-complete } v' \text{ with } TD_N[v'\rangle_{\mathsf{X}}\} \\
\mathsf{XFS}(N) &= \{v \in \mathsf{XFS}^c(N) \mid \text{if } (r) \text{ is a time step in } v\text{, then } r = 1\} \\
\mathsf{XL}^c(N) &= \{w = l(v) \mid v \in \mathsf{XFS}^c(N)\} \\
\mathsf{XL}(N) &= \{w = l(v) \mid v \in \mathsf{XFS}(N)\}
\end{aligned}$$

The set $\mathsf{XFS}^c(N)$ ($\mathsf{XL}^c(N)$) contains the *continuous* X-firing-sequences (X-traces); the set $\mathsf{XFS}(N)$ ($\mathsf{XL}(N)$) contains the *discrete* X-firing-sequences (X-traces). ■ 3.5

In order to show the coincidence of Definition 3.5 of $\mathsf{XFS}(N)$ and $\mathsf{XL}(N)$ with the corresponding one in [Vog95] for $\mathsf{X} \in \{\mathsf{L},\mathsf{E},\mathsf{S}\}$, we first develop some tools of general usability:

**Definition 3.6**

For a timed state $TD = (M, C, \rho)$ of a timed net $N$ define inductively the sets

$$\begin{array}{rcll} end(C) & = & \bigcup_{(t,e)\in C \wedge \rho(t,e)=0} & (\{(t^-,e)\} \circ end(C \setminus \{(t,e)\})) \\ start(M) & = & \bigcup_{(t,e)\in M[T\rangle \times E} & (\{(t^+,e)\} \circ start(M \setminus {}^\bullet t)) \end{array}$$

where $\circ$ denotes language concatenation and $\bigcup_{(t,e)\in\emptyset} := \{\lambda\}$. ■ 3.6

Informally, a sequence from $end(C)$ finishes all current transitions with elapsed residual time. Analogously, performing a sequence from $start(M)$ yields a timed state, where no more transition is activated. Formally:

### Lemma 3.7

Let $N$ be a timed net with timed state $TD = (M, C, \rho)$ and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$.

1. $v \in end(C)$ implies $TD[v\rangle_\mathsf{X}(M', C', \rho')$ for some $(M', C', \rho')$, such that $C' \subseteq C$ and $\rho'(t,e) > 0$ for all $(t,e) \in C'$.
2. $v \in start(M)$ implies $TD[v\rangle_\mathsf{X}(M', C', \rho')$ for some $(M', C', \rho')$, such that $M'[T\rangle = \emptyset$, $C' \supseteq C$ and $\rho'(t,e) \geq 1$ for all $(t,e) \in C' \setminus C$.

*Proof:* Straightforward induction on $|v|$, where in the base cases $v = \lambda$ by Definition 3.6 we must have $\rho(t,e) > 0$ for all $(t,e) \in C$ considering $end(C)$ and $M[T\rangle = \emptyset$ considering $start(M)$. ■ 3.7

We now compare the original definition of discrete behaviour in [Vog95] described by the more restricted operational rule $[\,\rangle'_\mathsf{X}$ with the one given in Definition 3.5:

### Proposition 3.8

For a timed net $N$ with timed states $TD$, $TD'$, $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{S}\}$ and $\varepsilon \in ((T^\pm \times E) \cup \{(1)\})^*$ write $TD[\varepsilon\rangle'_\mathsf{X} TD'$ if

- $TD[\varepsilon\rangle_\mathsf{X} TD'$ and
- $\varepsilon = (t^+, e)$ and $\mathsf{X} \in \{\mathsf{E}, \mathsf{S}\}$ implies $\rho(t', e') > 0$ for all $(t', e') \in C$.

Extend $[\,\rangle'_\mathsf{X}$ to sequences $v$ as usual. Then $\mathsf{XFS}(N) = \{\text{wellformed } v \mid TD_N[v\rangle'_\mathsf{X}\}$.

*Proof:*

'$\subseteq$': Take some $v \in \mathsf{XFS}(N)$; then $v$ is wellformed and $TD_N[v\rangle_\mathsf{X} TD$ for some $TD$; hence it suffices to show $TD_N[v\rangle'_\mathsf{X} TD$ by induction on $|v|$, where the base case $v = \lambda$ is clear. Thus, assume the claim to hold for some $v$ and consider $v' = v\varepsilon$. If $\varepsilon \neq (t^+, e)$ or $\mathsf{X} = \mathsf{L}$, then we immediately have $TD_N[v\rangle'_\mathsf{X} TD[\varepsilon\rangle'_\mathsf{X} TD'$ by induction and the definition of $[\,\rangle'_\mathsf{X}$. If $\varepsilon = (t^+, e)$ and $\mathsf{X} \in \{\mathsf{E}, \mathsf{S}\}$, then by $v\varepsilon \in \mathsf{XFS}(N)$ we have $TD_N[v\rangle_\mathsf{X} TD[(t^+, e)\rangle_\mathsf{X} TD'[v^+\rangle_\mathsf{X} TD^+[(1)\rangle_\mathsf{X}$ for some $v^+ \in (T^+ \times E)^*$, since $v\varepsilon$ must be a prefix of some wellformed time-complete sequence from $\mathsf{XFS}(N)$. Now $C \subseteq C^+$, hence $TD^+[(1)\rangle_\mathsf{X}$ implies $\rho(t', e') \geq 1 > 0$ for all $(t', e') \in C \subseteq C^+$ by Definition 3.2.3.

'$\supseteq$': Take some wellformed $v \in ((T^\pm \times E) \cup \{(1)\})^*$ with $TD_N[v\rangle'_\mathsf{X} TD$; then also $TD_N[v\rangle_\mathsf{X} TD$ by the first condition on $[\,\rangle'_\mathsf{X}$, and if $v$ is time-complete, we are done. Otherwise, we have to show $TD[u\rangle_\mathsf{X} TD'[(1)\rangle_\mathsf{X}$ for some $u(1) \in ((T^\pm \times E) \cup \{(1)\})^*$ such that $vu(1)$ is wellformed. If $\mathsf{X} = \mathsf{L}$, we can choose $u(1) = (1)$. Now let $\mathsf{X} \in \{\mathsf{E}, \mathsf{S}\}$; if $v$ ends $(t^-, e)$, then we choose $u = u^- u^+$, where $u^- \in end(C)$ (yielding $TD[u^-\rangle_\mathsf{X}(M^-, C^-, \rho^-)$

with $\rho^-(t',e') > 0$, hence $\rho^-(t',e') \geq 1$ by $\rho(t',e') \in \mathbb{N}_0$ for all $(t',e') \in C^-$ by Lemma 3.7.1) and $u^+ \in \mathit{start}(M^-)$, such that $(M^-, C^-, \rho^-)[u^+\rangle_{\mathsf{X}}(M', C', \rho') = TD'$ with $\rho'(t',e') \geq 1$ for all $(t',e') \in C'$ and $M'[T\rangle = \emptyset$ by Lemma 3.7.2 (since $\rho'(t',e') \geq 1$ for $(t',e') \in C^-$ and $\rho'(t',e') \geq 1$ for $(t',e') \in C' \setminus C^-$); thus, $TD'[(1)\rangle_{\mathsf{X}}$ and we can assume $vu(1)$ to be wellformed, since $u^+$ can be chosen event-unique w.r.t. $vu^-$. If $v$ ends $(t^+, e)$, then already $\rho(t',e') \geq 1$ for all $(t',e') \in C$ by the second condition on $[\,\rangle'_{\mathsf{X}}$ and the above, hence it suffices to let $u \in \mathit{start}(M)$, such that again $\rho'(t',e') \geq 1$ for all $(t',e') \in C'$ and $M'[T\rangle = \emptyset$, thus $TD'[(1)\rangle_{\mathsf{X}}$ and $vu(1)$ can be assumed wellformed. ■ 3.8

By this, we can actually carry over all results concerning discrete $\mathsf{L}$-, $\mathsf{E}$- or $\mathsf{S}$-behaviour from [Vog95], in particular the characterization techniques for test-equivalences. In particular, for discrete $\mathsf{L}$-behaviour we can as well restrict attention to all those sequences, which are terminated and both begin and end with a time step:

**Proposition 3.9**

For a timed net $N$ and $w \in ((\Sigma^\pm \times E) \cup \{(1)\})^*$ the following items are equivalent:

1. $w \in \mathsf{LL}(N)$.
2. $(1)w \in \mathsf{LL}(N)$.
3. $w$ is the prefix of some terminated and time-complete $w' \in \mathsf{LL}(N)$.

*Proof:*

'1. $\Leftrightarrow$ 2.': Since $TD_N[(1)\rangle\!\rangle_{\mathsf{L}} TD$ if and only if $TD = TD_N$.

'1. $\Rightarrow$ 3.': Let $w = l(v)$ for some $v \in \mathsf{LFS}(N)$, let $TD_N[v\rangle_{\mathsf{L}}(M, C, \rho)$ and let $n = \max_{(t,e) \in C} \rho(t,e)$; then $n \in \mathbb{N}_0$ and $(M, C, \rho)[(1)^n\rangle_{\mathsf{L}}(M', C', \rho')$, such that $\rho'(t,e) = 0$ for all $(t,e) \in C' = C$ by Lemma 3.3.1. Hence, we have $(M', C', \rho')[v'\rangle_{\mathsf{L}}(M'', \emptyset, \rho'')$ for any $v' \in \mathit{end}(C')$ by Lemma 3.7.1, thus $v(1)^n v'$ is terminated. Finally, $(M'', \emptyset, \rho'')[(1)\rangle_{\mathsf{L}}$ by Definition 3.2.3, hence we can choose wellformed, terminated and time-complete $w' = l(v(1)^n v'(1)) \in \mathsf{LL}(N)$.

'3. $\Rightarrow$ 1.': Directly from Definition 3.5. ■ 3.9

**Proposition 3.10**

Let $N$ be a timed net and let $\longrightarrow$ denote (set-) inclusion; then

$$\begin{array}{ccccccc} & & & \mathsf{LFS}^c(N) & & & \\ & & & \uparrow & & & \\ & & & \mathsf{LFS}(N) & & & \\ & & \nearrow & & \nwarrow & & \\ \mathsf{EFS}^c(N) & \leftarrow & \mathsf{EFS}(N) & & \mathsf{AFS}(N) & \rightarrow & \mathsf{AFS}^c(N) \\ & & \nwarrow & & \nearrow & & \\ & & & \mathsf{SFS}(N) & & & \\ & & & \downarrow & & & \\ & & & \mathsf{SFS}^c(N) & & & \end{array}$$

and

$$
\begin{array}{ccccccc}
 & & & \mathsf{LL}^c(N) & & & \\
 & & & \uparrow & & & \\
 & & & \mathsf{LL}(N) & & & \\
 & & \nearrow & & \nwarrow & & \\
\mathsf{EL}^c(N) & \leftarrow & \mathsf{EL}(N) & & \mathsf{AL}(N) & \rightarrow & \mathsf{AL}^c(N) \\
 & & \nwarrow & & \nearrow & & \\
 & & & \mathsf{SL}(N) & & & \\
 & & & \downarrow & & & \\
 & & & \mathsf{SL}^c(N) & & &
\end{array}
$$

*Proof:* Directly from Definition 3.5. ■ 3.10

## 3.2 Comparing Discrete and Concurrent Behaviour

The variants of concurrent behaviour introduced in Section 2 are closely related to the variants of discrete behaviour of untimed nets. The following results have been established in [Vog95] and – due to Proposition 3.8 – apply in our setting as well:

**Proposition 3.11**

Let $N_1$ and $N_2$ be untimed nets and let $(\mathsf{X}, \mathsf{Y}) \in \{(\mathsf{L}, \mathsf{I}), (\mathsf{E}, \mathsf{D}), (\mathsf{S}, \mathsf{M})\}$.
Then $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$ if and only if $\mathsf{YL}(N_1) \subseteq \mathsf{YL}(N_2)$.

*Proof:*

By Proposition 3.8, LL-, EL- and SL-semantics coincide with liberal-, mixed- and strict-behaviour resp. defined in [Vog95]. From the developments there, we can directly conclude that liberal-, mixed- and strict-behaviour inclusion of untimed nets coincide with ST-language-, step-language and maximal-step-language inclusion resp., which in turn are IL-, DL- and ML-inclusion resp. by Definition 2.3. ■ 3.11

The discrete A-variant was not yet treated in [Vog95] and is related to the concurrent U-variant. Somewhat unexpectedly, this relationship is not an equivalence (as in all three other cases) but only an implication from A- to U-behaviour inclusion:

**Proposition 3.12**

1. For untimed nets $N_1, N_2$, if $\mathsf{AL}(N_1) \subseteq \mathsf{AL}(N_2)$ then $\mathsf{UL}(N_1) \subseteq \mathsf{UL}(N_2)$.
2. There are untimed nets $N_1, N_2$ with $\mathsf{UL}(N_1) \subseteq \mathsf{UL}(N_2)$, but $\mathsf{AL}(N_1) \not\subseteq \mathsf{AL}(N_2)$.

*Proof:*

1. Assume $\mathsf{AL}(N_1) \subseteq \mathsf{AL}(N_2)$ and take some $w \in \mathsf{UL}(N_1)$; then $w = l(v)$ for some $v \in \mathsf{UFS}(N_1)$ of the form $v = v_0^+ v_1^- v_1^+ \dots v_n^- v_n^+$ for some $n \in \mathbb{N}_0$, such that $v_i^+ \in (T_1^+ \times E)^+$ for $i = 0, \dots, n-1$ and $v_i^- \in (T_1^- \times E)^+$ for $i = 1, \dots, n$ and $v_n^+ \in (T_1^+ \times E)^*$. Furthermore, if $(M_i, C_i)$ is reached after $v_i^+$, then $M_i[T\rangle = \emptyset$ for $i = 0, \dots, n-1$; hence we have $TD_N[v_0^+\rangle_{\mathsf{A}}\, TD_0[(1)v_1^- v_1^+\rangle_{\mathsf{A}}\, TD_1[(1)v_2^- v_2^+\rangle_{\mathsf{A}} \dots [(1)v_n^- v_n^+\rangle_{\mathsf{A}}\, TD_n[v^+(1)\rangle_{\mathsf{A}}$, where $TD_i = (M_i, C_i, \rho_i)$ with $\rho_i(t, e) \leq 1$ for all $(t, e) \in C_i$ (since $N_1$ untimed) and $v^+ \in \mathit{start}(M_n)$ can be chosen event-unique w.r.t. $v$.

Now $v' = v_0^+(1)v_1^-v_1^+(1)\dots(1)v_n^-v_n^+v^+(1)$ is wellformed and time-complete, hence $v' \in \mathsf{AFS}(N_1)$ and $w' = l_1(v') \in \mathsf{AL}(N_1) \subseteq \mathsf{AL}(N_2)$ by assumption. Then $w' = l_2(u)$ for some $u \in \mathsf{AFS}(N_2)$ of the form $u = u_0^+(1)u_1^-u_1^+(1)\dots(1)u_n^-u_n^+(1)$ for some $n \in \mathbb{N}_0$, such that $u_i^+ \in (T_2^+ \times E)^+$ for $i = 0,\dots,n-1$ and $u_i^- \in (T_2^- \times E)^+$ for $i = 1,\dots,n$ and $u_n^+ \in (T_2^+ \times E)^*$; furthermore, if $(M_i, C_i, \rho_i)$ is reached after $u_i^+$, then $M_i[T\rangle = \emptyset$ for $i = 0,\dots,n$, hence we have $seq(u) = u_0^+u_1^-u_1^+\dots u_n^-u_n^+ \in \mathsf{UL}(N_2)$, thus $l_2(seq(u)) = seq(l_2(u)) = seq(l_1(v')) = seq(w') \in \mathsf{UL}(N_2)$, and since $w$ is a prefix of $seq(w')$, we finally get $w \in \mathsf{UL}(N_2)$.

2. Consider $N_1$ and $N_2$ below.


We have $(a^+, e_a)(b^+, e_b)(1)(a^-, e_a)(1) \in \mathsf{AL}(N_1) \setminus \mathsf{AL}(N_2)$. ■ 3.12

## 3.3 Comparing Discrete and Continuous Behaviour

This subsection is devoted to the comparison of discrete and continuous behaviour; more precisely, we answer the following question for all $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$: given two timed nets $N_1$ and $N_2$, such that each discrete (continuous) X-trace of $N_1$ is also a discrete (continuous) X-trace of $N_2$; is then each continuous (discrete) X-trace of $N_1$ also a continuous (discrete) X-trace of $N_2$?

It will turn out that inclusion of continuous traces implies inclusion of discrete traces in all four variants for all timed nets; this is quite immediate from Definition 3.5: the discrete traces of a timed net form a syntactically decidable subset of its continuous traces. The reverse implication does not hold true in general. Altogether, we will show that continuous time can at most distinguish finer than discrete time.

In this respect, discrete time is exactly as distinctive as continuous time for all timed nets in the S-variant:

**Proposition 3.13**

For timed nets $N_1, N_2$, we have $\mathsf{SL}(N_1) \subseteq \mathsf{SL}(N_2)$ if and only if $\mathsf{SL}^c(N_1) \subseteq \mathsf{SL}^c(N_2)$.

*Proof:*

It suffices to show that $\mathsf{SFS}(N)$ can be constructed from $\mathsf{SFS}^c(N)$ and vice versa for any timed net $N$:

In $\mathsf{SFS}^c(N)$, an activated transition starts immediately or is deactivated before any time passes. If a transition is started at a discrete time, it ends at a discrete time by $\delta(t) \in \mathbb{N}$; hence, by induction, all starts and ends occur at discrete time which can be reached by (1)-steps in $\mathsf{SFS}(N)$ as well; note that a (time-complete) continuous trace

$w$ need not have discrete duration, but can be extended to some $w'$ by time-steps only, such that $dur(w') = \lceil dur(w) \rceil \in \mathbb{N}_0$.

Vice versa, replacing sequences of (1)-steps in $\mathsf{SFS}(N)$ by sequences of $(r)$-steps with the same duration (or a lesser duration, when considering the last coherent (1)-steps in a time-complete sequence) yields $\mathsf{SFS}^c(N)$. ■ 3.13

The situation is different for the A- and the L-variant: if we replace all time-steps by (1)-steps in a continuous L-trace of a timed net $N$, then the result obviously is a discrete L-trace of $N$, since the firing time of the underlying transitions can only increase; the same applies for A-behaviour, where we additionally observe that a sequence of time-steps occurs in a continuous A-trace only if no more start is possible, and that passage of time never activates new transitions or requires current transitions to finish. On the other hand, from a given discrete L- or A-trace of a timed net $N$, we can construct all 'corresponding' continuous L- or A-traces in general only if $N$ is untimed: this ensures that we can allow passage of only one time unit between an actions start and its end:

**Lemma 3.14**

For $w \in ((\Sigma^{\pm} \times E) \cup \{(r) \mid r \in (0;1]\})^*$ let $\hat{w}$ be $w$ with all $(r)$ replaced by (1).

Let $N$ be an untimed net and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{A}\}$. Then $w \in \mathsf{XL}^c(N)$ if and only if

1. $\hat{w} \in \mathsf{XL}(N)$ and
2. whenever $w = w_1(a^+, e)w_2(a^-, e)w_3$, then $dur(w_2) \geq 1$.

*Proof:*

By Definition 3.5, we may w.l.o.g. assume $w$ to be wellformed and time-complete, and then $\hat{w}$ must be wellformed and time-complete, too.

'only-if':

Let $w \in \mathsf{XL}^c(N)$; then $w = l(v)$ for some wellformed and time-complete $v \in \mathsf{XFS}^c(N)$; for this $v$ let $\hat{v}$ be $v$ with all $(r)$ replaced by (1), such that $\hat{w} = l(\hat{v})$.

1. Now $\hat{v}$ is wellformed and time-complete, too, and it suffices to show $\hat{v} \in \mathsf{XFS}(N)$. We show that $TD_N[v\rangle_{\mathsf{X}}(M, C, \rho)$ implies $TD_N[\hat{v}\rangle_{\mathsf{X}}(M, C, \hat{\rho})$ with $\hat{\rho} \leq \rho$ even for non-time-complete $v$ and $\hat{v}$ by induction on $|v|$. The base case $v = \lambda$ is clear, hence assume the claim to hold for some $v$ and consider $v' = v\varepsilon$.

$\varepsilon = (t^+, e)$ : then $TD_N[v\rangle_{\mathsf{X}}(M, C, \rho)[t^+, e\rangle_{\mathsf{X}}(M', C', \rho')$ for some $(M', C', \rho')$, hence by ind. also $TD_N[\hat{v}\rangle_{\mathsf{X}}(M, C, \hat{\rho})[t^+, e\rangle_{\mathsf{X}}(M'', C'', \hat{\rho})$, such that $(M'', C'') = (M', C')$ by Definition 3.2.1 and $\hat{\rho}'(t', e') = \hat{\rho}(t', e') \leq \rho(t', e') = \rho'(t', e')$ for $(t', e') \in C = \hat{C}$ by induction and $\hat{\rho}'(t, e) = \rho'(t, e) = 1$ since $N$ is untimed.

$\varepsilon = (t^-, e)$ : then $TD_N[v\rangle_{\mathsf{X}}(M, C, \rho)[t^-, e\rangle_{\mathsf{X}}(M', C', \rho')$ and $\rho(t, e) = 0$ by Definition 3.2.2, hence also $TD_N[\hat{v}\rangle_{\mathsf{X}}(M, C, \hat{\rho})[t^-, e\rangle_{\mathsf{X}}(M', C', \hat{\rho})$ by induction and Definition 3.2.2, since $\hat{\rho} \leq \rho$ by induction implies $\hat{\rho}(t, e) = 0$; furthermore, $\hat{\rho}'(t', e') = \hat{\rho}(t', e') \leq \rho(t', e') = \rho'(t', e')$ for $(t', e') \in C' = \hat{C}'$ by induction again.

$\varepsilon = (r)$ : then $TD_N[v\rangle_{\mathsf{X}}(M, C, \rho)[t^-, e\rangle_{\mathsf{X}}(M, C, \rho')$ by Definition 3.2.3, hence by induction also $TD_N[\hat{v}\rangle_{\mathsf{X}}(M, C, \hat{\rho})[t^-, e\rangle_{\mathsf{X}}(M, C, \hat{\rho})$ and $\rho = \rho \dot{-} r \geq \hat{\rho} \dot{-} 1 = \hat{\rho}'$ by induction and since $r \leq 1$.

2. If $w = w_1(a^+,e)w_2(a^-,e)w_3$, then $v = v_1(t^+,e)v_2(t^-,e)v_3$ for some $t$ with $l(t) = a$, such that $TD_N[v_1(t^+,e)\rangle_X(M_1,C_1,\rho_1)[v_2\rangle_X(M_2,C_2,\rho_2)$ with $(t,e) \in C_2$ and $\rho_1(t,e) = \delta(t) = 1$ since $N$ untimed and $\rho_2(t,e) = \rho_1(t,e) \dot{-} dur(v_2) = 0$ by Lemma 3.3.1 and Definition 3.2.2, hence $dur(v_2) \geq 1$, thus $dur(w_2) \geq 1$, since $dur(w_2) = dur(v_2)$ by $w = l(v)$.

'if':

Let 1. and 2. hold; then by 1. and the observation at the beginning of this proof, there is a wellformed and time-complete $u \in \mathsf{XFS}(N)$ with $\hat{w} = l(u)$; for this $u$, we show by induction on $|u|$ that there is a $v \in \mathsf{XFS}^c(N)$ with $\hat{v} = u$ and $w = l(v)$, such that $w \in \mathsf{XL}^c(N)$. Again, we show the claim even for non-time-complete $u$: then in the base case $u = \lambda$ we can clearly choose $v = \lambda$, hence assume that for $u$ we have constructed $v$ as desired and consider $u' = u\varepsilon$; we denote the $TD$'s reached after $u$ and $v$ by $TD_u$ and $TD_v$, which obviously coincide in their $M$- and $C$-component. Now:

$\varepsilon = (t^+,e)$ : we can choose $v' = v(t^+,e)$ by $M_v = M_u$ and $C_v = C_u$.

$\varepsilon = (t^-,e)$ : then $u = u_1(t^+,e)u_2$ for some $u_1, u_2$ by Definition 3.2.1 and .2, hence $v = v_1(t^+,e)v_2$ for some $v_1, v_2$ by induction and $\hat{v} = u$ and also $w = w_1(l(t)^+,e)w_2$ by induction and $w = l(v)$, such that $dur(v_2) = dur(w_2) \geq 1$ by assumption 2., hence $\rho_v(t,e) = 1 \dot{-} dur(w_2) = 0$ by Lemma 3.3.1 and since $N$ untimed, thus we can choose $v' = v(t^-,e)$.

$\varepsilon = (1)$ : we can choose $v' = v(r)$ for any $r \in (0;1]$, since in the case $\mathsf{X} = \mathsf{A}$ we also have $M_v[T\rangle = M_u[T\rangle = \emptyset$. ■ 3.14

As a result, for L- and A-behaviour, continuous time is as distinctive as discrete time if untimed nets are compared only. In general, in the class of all timed nets, continuous time distinguishes finer than discrete time:

**Proposition 3.15**

Let $\mathsf{X} \in \{\mathsf{L}, \mathsf{A}\}$.

1. For timed nets $N_1, N_2$, if $\mathsf{XL}^c(N_1) \subseteq \mathsf{XL}^c(N_2)$, then $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$.
2. For untimed nets $N_1, N_2$, if $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$, then $\mathsf{XL}^c(N_1) \subseteq \mathsf{XL}^c(N_2)$.
3. There are timed nets $N_1, N_2$, such that $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$, but $\mathsf{XL}^c(N_1) \not\subseteq \mathsf{XL}^c(N_2)$.

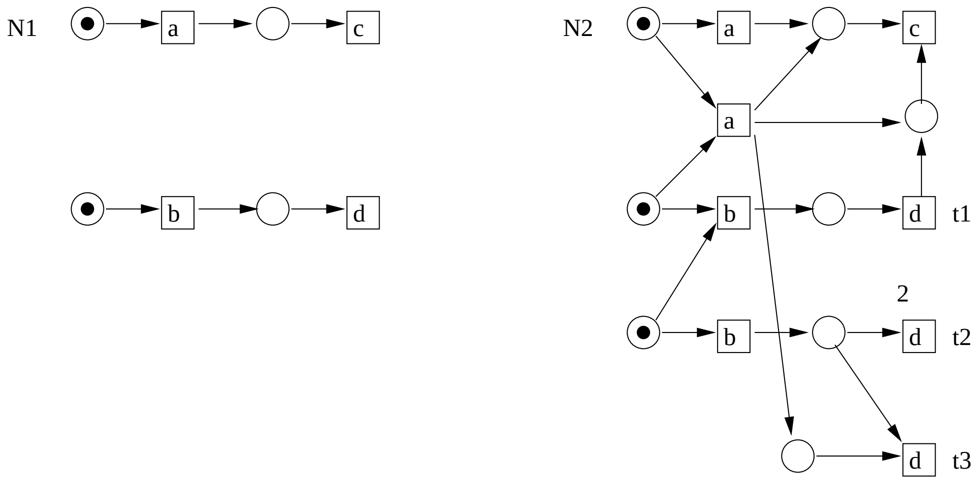
*Proof:*

1. Follows from Definition 3.5.

2. Assume $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$ and take some $w \in \mathsf{XL}^c(N_1)$; then by Lemma 3.14, we have $\hat{w} \in \mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$ and whenever $w = w_1(a^+,e)w_2(a^-,e)w_3$, then $dur(w_2) \geq 1$, hence $w \in \mathsf{XL}^c(N_2)$ by Lemma 3.14 again.

3. Consider $N_1$ and $N_2$ below.

We first argue that $\mathsf{LL}(N_1) \subseteq \mathsf{LL}(N_2)$ and $\mathsf{AL}(N_1) \subseteq \mathsf{AL}(N_2)$: if in a run of $N_1$ $d^-$ occurs not later than $a^-$, then $N_2$ can simulate this by choosing the upper $a$-transition, the




upper $b$-transition and $t_1$ for $d$. If $d^-$ occurs later than $a^-$, but also $a^-$ occurs later than $d^+$, then $d$ lasts at least 2 time units, hence $N_2$ can simulate this by choosing the lower $a$-transition, the lower $b$-transition and $t_2$ for $d$. If $a^-$ occurs not later than $d^+$ (in which case $d^-$ occurs later than $a^-$), then $N_2$ can simulate this by choosing the lower $a$-transition, the lower $b$-transition and $t_3$ for $d$.

Now $w = (a^+, e_a)(b^+, e_b)(1)(b^-, e_b)(d^+, e_a)(0.5)(a^-, e_a)(c^+, e_c)(0.5)(d^-, e_d) \in \mathsf{AL}^c(N_1)$ and $w \in \mathsf{LL}^c(N_1)$ by $\mathsf{AL}^c(N_1) \subseteq \mathsf{LL}^c(N_1)$, but neither $w \in \mathsf{AL}^c(N_2)$, nor $w \in \mathsf{LL}^c(N_2)$. Note that $d$ in this sequence lasts only time 1 and starts before the end of $a$; hence it could only correspond to $t_1$ in $N_2$, but this is would imply that $c$ could only start with or later than the end of $d$, which is not the case. ■ 3.15

Finally, in the E-variant, continuous time can be more distinctive than discrete time even for untimed nets:

**Proposition 3.16**

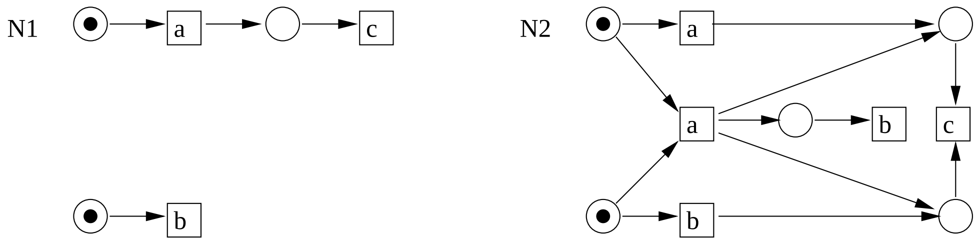
1. For timed nets $N_1, N_2$, if $\mathsf{EL}^c(N_1) \subseteq \mathsf{EL}^c(N_2)$, then $\mathsf{EL}(N_1) \subseteq \mathsf{EL}(N_2)$.
2. There are untimed nets $N_1, N_2$, such that $\mathsf{EL}(N_1) \subseteq \mathsf{EL}(N_2)$, but $\mathsf{EL}^c(N_1) \not\subseteq \mathsf{EL}^c(N_2)$.

*Proof:*

1. Follows from Definition 3.5.
2. Consider $N_1$ and $N_2$ below.

We first argue that $\mathsf{EL}(N_1) \subseteq \mathsf{EL}(N_2)$: if $b^+$ occurs not later than $a^+$ in $N_1$, then $b^-$ occurs not later than $a^-$ and $N_2$ can simulate this by choosing the upper $a$-transition. If $b^+$ occurs later than $a^+$, then it does not occur before $a^-$ and $N_2$ can simulate this by choosing the lower $a$-transition.

Now $(a^+, e_a)(0.5)(b^+, e_b)(0.5)(a^-, e_a)(c^+, e_c)(0.5)(b^-, e_d) \in \mathsf{EL}^c(N_1) \setminus \mathsf{EL}^c(N_2)$. ■ 3.16




The results of this subsection show that the discriminative power of continuous time vs. discrete time crucially depends on the chosen variant and/or the class of timed nets considered. As a – rather surprising – main result of the next section, testing in continuous time will *never* be more distinctive than testing in discrete time.

## 3.4 Comparing the Four Timed Variants of Behaviour

We finish this section by a comparison of the four timed variants of behaviour; more precisely, we answer the following question for all $\mathsf{X}, \mathsf{Y} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$ with $\mathsf{X} \neq \mathsf{Y}$: given two timed nets $N_1$ and $N_2$, such that each discrete (continuous) X-trace of $N_1$ is also a discrete (continuous) X-trace of $N_2$; is then each discrete (continuous) Y-trace of $N_1$ also a discrete (continuous) Y-trace of $N_2$? The results are gathered in a corollary at the end of this section.

Let us first restrict attention to the discrete variants of behaviour and untimed nets. In this matter, we are well supported by the relationship between discretely timed behaviour of untimed nets and their concurrent behaviour (established in Proposition 3.11) and the relationship between the four concurrent variants (stated in Proposition 2.7):

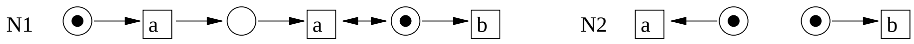
E.g. for untimed nets, LL-, EL- and SL-inclusion coincide with IL-, DL- and ML-inclusion resp. by Proposition 3.11. Hence, by Proposition 2.7.1 and .3, LL-inclusion implies EL-inclusion and not other implication holds in general between these three discrete timed variants for untimed nets. Since AL-inclusion implies UL-inclusion by Proposition 3.12.1, AL-inclusion also implies SL-inclusion by the above coincidence of SL- and ML-inclusion and Proposition 2.7.2. Furthermore, we can carry over the negative results of Proposition 2.7.3 for $\mathsf{Y} = \mathsf{U}$, i.e. for no $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{S}\}$ we have that XL-inclusion implies AL-inclusion in general; however, since AL-inclusion does not coincide with UL-inclusion in general (Proposition 3.12.2), we can not simply carry over the negative results of Proposition 2.7.3 for $\mathsf{X} = \mathsf{U}$ and $\mathsf{Y} \in \{\mathsf{I}, \mathsf{D}\}$; but it suffices to show:

**Proposition 3.17**

There are untimed nets $N_1, N_2$, such that $\mathsf{AL}(N_1) \subseteq \mathsf{AL}(N_2)$, but $\mathsf{EL}(N_1) \not\subseteq \mathsf{EL}(N_2)$. ■ 3.17

*Proof:* Consider

Now also AL-inclusion does not imply LL-inclusion in general, since LL-inclusion always implies EL-inclusion for untimed nets as shown above.




Hence, we have clarified the relationships between all discrete variants in the class of untimed nets. Furthermore, the negative results carry over to continuous behaviour as well by Proposition 3.13, Proposition 3.15.1 and Proposition 3.16.1; they also hold for the class of all timed nets, since untimed nets are timed nets.

Additionally, since $\mathsf{SL}^c$- and $\mathsf{AL}^c$-inclusion coincide for untimed nets with $\mathsf{SL}$- and $\mathsf{AL}$-inclusion, we immediately have that $\mathsf{SL}^c$-inclusion implies and $\mathsf{AL}^c$-inclusion for untimed nets but in general not vice versa (hence for all timed nets) by the above.

Since even for untimed nets $\mathsf{EL}$-inclusion does not coincide with $\mathsf{EL}^c$-inclusion in general (Proposition 3.16.2), we cannot carry over Proposition 2.7.1 to continuous $\mathsf{E}$- and $\mathsf{L}$-behaviour of untimed nets. However, $\mathsf{LL}^c$-inclusion can at most imply $\mathsf{EL}^c$-inclusion for untimed – and, thus, for all timed – nets in general. In order to verify this implication for untimed nets, we first observe that $\mathsf{EL}^c(N)$ for untimed $N$ can be gained from $\mathsf{LL}^c(N)$ by taking all sequences with 'correctly' finishing actions:

**Lemma 3.18**

Let $N$ be an untimed net and let $w \in ((\Sigma^{\pm} \times E) \cup \{(r) \,|\, r \in (0;1]\})^*$. Then $w(r) \in \mathsf{EL}^c(N)$ if and only if $w(r) \in \mathsf{LL}^c(N)$ and whenever $w(r) = w_1(a^+, e)w_2(a^-, e)w_3$, then $dur(w_2) = 1$.

*Proof:*

We have $v(r) \in \mathsf{EFS}^c(N)$ iff $v(r) \in \mathsf{LFS}^c$ and whenever $v(r) = v_1(t^+, e)v_2(t^-, e)v_3$ and $TD[v_1(t^+, e)\rangle_{\mathsf{L}}(ID_1, \rho_1)[v_2\rangle_{\mathsf{L}}(ID_2, \rho_2)[(t^-, e)\rangle_{\mathsf{L}}$, then $\rho_1(t, e) = 1$ (since $N$ untimed) and $\rho_1(t, e) \geq dur(v_2)$ (by Definition 3.2.3, case $\mathsf{E}$) and $\rho_2(t, e) = \rho_1(t, e) \dot{-} dur(v_2) = 0$ (by Definition 3.2.2 and Lemma 3.3.1), hence iff $v(r) \in \mathsf{LFS}^c$ and whenever $v(r) = v_1(t^+, e)v_2(t^-, e)v_3$, then $dur(v_2) = 1$. Now $w(r) \in \mathsf{EL}^c(N)$ iff $w(r) = l(v(r))$ for some $v(r) \in \mathsf{LFS}^c$ and $v(r) = v_1(t^+, e)v_2(t^-, e)v_3$ implies $dur(v_2) = 1$, hence iff $w(r) \in \mathsf{LL}^c(N)$ and whenever $w = w_1(a^+, e)w_2(a^-, e)w_3$, then $dur(w_2) = 1$. ∎ 3.18

It remains to check that $\mathsf{EL}^c$-inclusion does not imply $\mathsf{XL}^c$-inclusion for any $\mathsf{X} \in \{\mathsf{L}, \mathsf{A}, \mathsf{S}\}$ and untimed – hence, all timed – nets in general.

**Proposition 3.19**

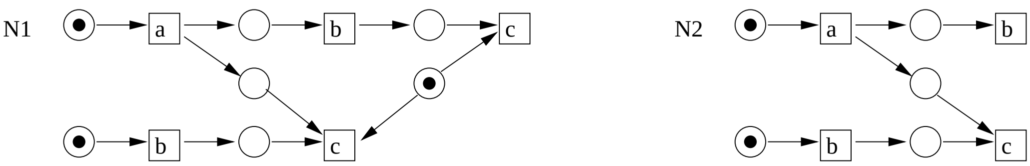
1. For untimed nets $N_1, N_2$, if $\mathsf{LL}^c(N_1) \subseteq \mathsf{LL}^c(N_2)$, then $\mathsf{EL}^c(N_1) \subseteq \mathsf{EL}^c(N_2)$.
2. Let $\mathsf{X} \in \{\mathsf{L}, \mathsf{A}, \mathsf{S}\}$. Then there are untimed nets $N_1, N_2$, such that $\mathsf{EL}^c(N_1) \subseteq \mathsf{EL}^c(N_2)$, but $\mathsf{XL}^c(N_1) \not\subseteq \mathsf{XL}^c(N_2)$.

*Proof:*

1. Assume $\mathsf{LL}^c(N_1) \subseteq \mathsf{LL}^c(N_2)$ and take some $w \in \mathsf{EL}^c(N_1)$; then $w$ is a prefix of some $w'(r) \in \mathsf{EL}^c(N_1) \subseteq \mathsf{LL}^c(N_1) \subseteq \mathsf{LL}^c(N_2)$ by Definition 3.5, Proposition 3.10 and assumption, such that whenever $w'(r) = w_1(a^+, e)w_2(a^-, e)w_3$, then $dur(w_2) = 1$ by

Lemma 3.18, hence $w'(r) \in \mathsf{EL}^c(N_2)$ by Lemma 3.18 again and $w \in \mathsf{EL}^c(N_2)$ by Definition 3.5.

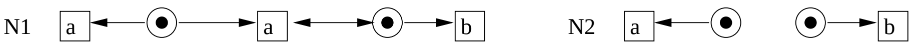
2. For $\mathsf{X} = \mathsf{L}$ consider




The reasoning for $\mathsf{EL}^c(N_1) \subseteq \mathsf{EL}^c(N_2)$ is a refinement of the one given in the proof of Proposition 2.7.3 case $(\mathsf{X},\mathsf{Y}) = (\mathsf{D},\mathsf{I})$, where we additionally note that the lower $b$ cannot overlap both $a$ and the upper $b$ in $\mathsf{EL}^c$.

We have $(a^+, e_a)(b^+, e_b)(1)(a^-, e_a)(b^+, e'_b)(1)(b^-, e'_b)(c^+, e_c) \in \mathsf{LL}^c(N_1) \setminus \mathsf{LL}^c(N_2)$.

For $\mathsf{X} \in \{\mathsf{A}, \mathsf{S}\}$ consider




Here, an analogous reasoning as in the proof of Proposition 2.7.3 case $\mathsf{X} = \mathsf{D}$ and $\mathsf{Y} \in \{\mathsf{U}, \mathsf{M}\}$ applies. ■ 3.19

Up to now, we have checked the relationship between all discrete and continuous variants for untimed nets; the negative results carry over to the class of all timed nets and it remains to check:

**Proposition 3.20**

There are timed nets $N_1, N_2$, such that $\mathsf{LL}^c(N_1) \subseteq \mathsf{LL}^c(N_2)$ and $\mathsf{AL}^c(N_1) \subseteq \mathsf{AL}^c(N_2)$, but $\mathsf{EL}(N_1) \not\subseteq \mathsf{EL}(N_2)$ and $\mathsf{SL}(N_1) \not\subseteq \mathsf{SL}(N_2)$.

*Proof:* Consider


■ 3.20

Altogether, we end up with the following map of implications between inclusion of (concurrent,) discrete and continuous behaviour in all four variants for the classes of untimed and all timed nets:

**Corollary 3.21**

The following and no other implications hold in general between inclusion of (concurrent,) discrete and continuous behaviour of two *untimed* nets:

$$
\begin{array}{ccccc}
\mathsf{LL}^c & \leftrightarrow & \mathsf{LL} & (\leftrightarrow & \mathsf{IL}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{EL}^c & \rightarrow & \mathsf{EL} & (\leftrightarrow & \mathsf{DL})
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\mathsf{AL}^c & \leftrightarrow & \mathsf{AL} & (\rightarrow & \mathsf{UL}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{SL}^c & \leftrightarrow & \mathsf{SL} & (\leftrightarrow & \mathsf{ML})
\end{array}
$$

The following and no other implications hold in general between inclusion of discrete and continuous behaviour of two *timed* nets:

$$
\begin{array}{ccc}
\mathsf{LL}^c & \rightarrow & \mathsf{LL} \\
 & & \\
\mathsf{EL}^c & \rightarrow & \mathsf{EL}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{AL}^c & \rightarrow & \mathsf{AL} \\
 & & \\
\mathsf{SL}^c & \leftrightarrow & \mathsf{SL}
\end{array}
$$

■ 3.21

# 4 Timed Testing in Discrete and Continuous Time

Timed testing ([Vog95, JV95]) is a modification of classical testing ([DNH84]): the qualitative problem 'may/must some behaviour occur ?' is quantitatively refined to 'may/must some behaviour occur in time?'.

Technically, the timed testing scenario is set-up as follows. A timed test consists of an *observer* $O$ and a *time bound* $r \in \mathbb{R}_0^+$. The observer is a timed net which is generally equipped with additional special actions, $\omega$ and $wait$; we let $\Sigma_t = \Sigma \cup \{\omega, wait\}$ and note that all developments and results in the previous sections did not require to exclude $\omega$ and $wait$ from the considered alphabet; hence, they hold as well if $\Sigma$ is replaced by $\Sigma_t$.

When testing a given timed net $N$ with observer $O$, we consider the traces of the parallel composition $N\|_\Sigma O$ with synchronization on all actions from $\Sigma$ (i.e. except for $\omega$ and $wait$) as defined below. Whenever a trace of the composition contains the start of some $\omega$, then success is signaled by $O$. The observer may also delay its activity explicitly by firing a $wait$-labelled transition, which is not synchronized with the activity of the tested net $N$.

**Definition 4.1**

Let $N_1, N_2$ be nets and $A \subseteq \Sigma$. The *parallel composition* $N = N_1\|_A N_2$ *with synchronization on* $A$ is defined as

$$\begin{aligned}
S &= S_1 \times \{*\} \cup \{*\} \times S_2 \\
T &= \{(t_1,t_2) \mid t_1 \in T_1, t_2 \in T_2, l_1(t_1) = l_2(t_2) \in A\} \cup \\
&\quad \{(t_1,*) \mid t_1 \in T_1, l_1(t_1) \notin A\} \cup \\
&\quad \{(*,t_2) \mid t_2 \in T_2, l_2(t_2) \notin A\} \\
F &= \{((s_1,s_2),(t_1,t_2)) \mid (s_1,t_1) \in F_1 \text{ or } (s_2,t_2) \in F_2\} \cup \\
&\quad \{((t_1,t_2),(s_1,s_2)) \mid (t_1,s_1) \in F_1 \text{ or } (t_2,s_2) \in F_2\} \\
l((t_1,t_2)) &= \begin{cases} l_1(t_1) & \text{if } t_1 \in T_1 \\ l_2(t_2) & \text{if } t_2 \in T_2 \end{cases} \\
\delta((t_1,t_2)) &= \max(\delta_1(t_1), \delta_2(t_2)), \quad \text{where } \delta_1(*) = \delta_2(*) = 0. \\
M_N &= M_{N_1} \dot{\cup} M_{N_2}, \text{ i.e. } M_N(s_1,s_2)) = \begin{cases} M_{N_1}(s_1) & \text{if } s_1 \in S_1 \\ M_{N_2}(s_2) & \text{if } s_2 \in S_2 \end{cases}
\end{aligned}$$

Here $*$ is a dummy-element with $* \notin (S_1 \cup S_2 \cup T_1 \cup T_2)$. ■ 4.1

Note that due to $\delta((t_1,t_2)) = \max(\delta_1(t_1), \delta_2(t_2))$ the *slower* transition determines the duration of the synchronized transition; by this, when testing an untimed net, the durations of all transitions can entirely be determined by the observer.

Now a timed net $N$ *may* satisfy the timed test $(O,r)$, if *there is* a trace of $N\|_\Sigma O$ with a duration *at most* $r$ which contains the start of a $\omega$. $N$ *must* satisfy $(O,r)$, if *all* traces of $N\|_\Sigma O$ with a duration *greater than* $r$ contain the start of a $\omega$. Obviously, must-testing is only reasonable for the strict (S-) variant, since all three other variants allow actions to be delayed and/or to last arbitrarily long, hence no timed test must be satisfied in general; for may-testing, we distinguish all four timed variants.

Based on test satisfaction, a preorder on timed nets can be naturally defined: for $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$, timed nets $N_1$ and $N_2$ are in relation $N_1 \succeq_{\mathsf{X}} (\sqsupseteq) N_2$ if whenever a timed test $(O, r)$ may (must) be satisfied by $N_2$, then it may (must) be satisfied by $N_1$ as well. In general, $N_1$ not only performs successful with more observers $O$ than $N_2$, but also with lower time bounds for the same $O$; this justifies to see $\succeq_{\mathsf{X}}$ and $\sqsupseteq$ as *efficiency preorders*, where $\succeq_{\mathsf{X}}$ compares the *best-case* efficiency and $\sqsupseteq$ compares the *worst-case* efficiency.

In [Vog95], the discrete sub-setting has already been treated for test-equivalences (rather than preorders) and $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{S}\}$: time bounds are natural numbers and only discrete traces are considered. Of course, we also investigate the relation between the discrete and continuous preorders. The following definition gathers the ideas formally:

**Definition 4.2**

A timed net is *testable*, if the special actions $\omega$ and *wait* do not occur as transition labels. A *continuous(ly timed) test* $(O, r)$ consists of

- a timed net $O$ (called *observer*) labelled with actions from $\Sigma_t$ and
- a time bound $r \in \mathbb{R}_0^+$.

$(O, r)$ is a *discrete(ly timed) test*, if $r \in \mathbb{N}_0$.

Let $N$ be a testable timed net, let $O$ be an observer, let $r \in \mathbb{R}_0^+$, let $d \in \mathbb{N}_0$ and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$. We write

$N \; may_{\mathsf{X}}^c \; (O, r)$ if there is *some* $w \in \mathsf{XL}^c(N \|_\Sigma O)$ containing $(\omega^+, e)$ and $dur(w) \leq r$,
$N \; may_{\mathsf{X}} \; (O, d)$ if there is *some* $w \in \mathsf{XL}(N \|_\Sigma O)$ containing $(\omega^+, e)$ and $dur(w) \leq d$.

For testable timed nets $N_1$, $N_2$ and $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$ we write

$N_1 \succeq_{\mathsf{X}}^c N_2$ if $N_2 \; may_{\mathsf{X}}^c \; (O, r)$ implies $N_1 \; may_{\mathsf{X}}^c \; (O, r)$ for all continuous tests $(O, r)$,
$N_1 \succeq_{\mathsf{X}} N_2$ if $N_2 \; may_{\mathsf{X}} \; (O, d)$ implies $N_1 \; may_{\mathsf{X}} \; (O, d)$ for all discrete tests $(O, d)$.

Finally, we write for a testable timed net $N$, observer $O$, $r \in \mathbb{R}_0^+$ and $d \in \mathbb{N}_0$:

$N \; must^c \; (O, r)$ if *all* $w \in \mathsf{SL}^c(N \|_\Sigma O)$ with $dur(w) > r$ contain $(\omega^+, e)$,
$N \; must \; (O, d)$ if *all* $w \in \mathsf{SL}(N \|_\Sigma O)$ with $dur(w) > d$ contain $(\omega^+, e)$.

and

$N_1 \sqsupseteq^c N_2$ if $N_2 \; must^c \; (O, r)$ implies $N_1 \; must^c \; (O, r)$ for all continuous tests $(O, r)$,
$N_1 \sqsupseteq N_2$ if $N_2 \; must \; (O, d)$ implies $N_1 \; must \; (O, d)$ for all discrete tests $(O, d)$.

■ 4.2

The usefulness of this definition of timed testing depends (even crucially in the must-case) on the following property of all four timed variants:

**Proposition 4.3**

Let $N$ be a timed net, let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$ and let $v \in X\mathsf{FS}^c(N)$. Then $v$ is the prefix of some $v' \in X\mathsf{FS}^c(N)$ with $dur(v') \geq dur(v) + 1$.

*Proof:*

Let $v$ be the prefix of wellformed and time-complete $u \in X\mathsf{FS}^c(N)$ with $TD_N[u\rangle_{\mathsf{X}} TD$ and $dur(u) \geq dur(v)$. If $\min_{(t,e)\in C} \rho(t,e) \geq 1$, then $TD[u(1)\rangle_{\mathsf{X}}$, since $u$ time-complete implies $M[T\rangle = \emptyset$ for $\mathsf{X} \in \{\mathsf{A}, \mathsf{S}\}$, and we have wellformed and time-complete $v' = u(1) \in X\mathsf{FS}^c(N)$ with $dur(v') = dur(u) + 1 \geq dur(v) + 1$.

Now let $\min_{(t,e)\in C} \rho(t,e) < 1$ and let $\{r_1, \ldots, r_n\} = \{\rho(t,e) < 1 \mid (t,e) \in C\}$ be the finite set of residual times lesser than 1 of transitions in $C$, such that $r_i < r_{i+1}$ for all $i = 1, \ldots, n-1$. Then $TD[(r_1)\rangle_{\mathsf{X}} TD_1 [u_1^-\rangle_{\mathsf{X}} TD_1^- [u_1^+\rangle_{\mathsf{X}} TD_1^+ [(r_2 - r_1)\rangle_{\mathsf{X}} TD_2 \ldots [(r_n - r_{n-1})\rangle_{\mathsf{X}} TD_n$
where $u_i^- \in end(C_i)$ and $u_i^+ \in start(M_i^-)$: if $\rho_i(t,e) \geq 1$ for some $i$ and $(t,e) \in C_i$, then $\rho_j(t,e) \geq 1 - (r_j - r_i) > 0$ for all $i \leq j \leq n$, since $\rho_j(t,e) = \rho_i(t,e) \dot{-} (r_j - r_i)$ by Lemma 3.3.1 and $0 \leq r_i < r_j < 1$. In particular, $\rho_n(t,e) \geq 1 - (r_n - r_1) > 0$ for all $(t,e) \in C_n$, $v' = u(r_1) u_1^- u_1^+ (r_2 - r_1) \ldots (r_n - r_{n-1})(1 - (r_n - r_1)) \in X\mathsf{FS}^c(N)$, since $v'$ is time-complete and can be assumed event-unique, hence wellformed, and we have $dur(v') = dur(u) + r_n + (1 - (r_n - r_1)) = dur(u) + 1 + r_1 \geq dur(v) + 1$.

This property ensures, that in none of the variants $N \|_\Sigma O$ can reach a 'time-stop'. Otherwise, all traces of $N \|_\Sigma O$ might have a duration less than $r$, but none of them contains $\omega$; hence $N$ $must^c$ $(O, r)$ although success is never reached. In other words, by Proposition 4.3 a system can always be oberseved up to an arbitrary time.

## 4.1 Comparing Continuous and Discrete Testing

In the end of section 3 we have seen that continuous time distinguishes finer than discrete time in general; only for the class of untimed nets in three of the four variants continuous time is as discriminating as discrete time. We now examine this topic for the testing preorders.

By Proposition 3.13, the coincidence of discrete and continuous testing preorders is quite straightforward for the S-variant:

**Proposition 4.4**

On testable timed nets, $\succeq_{\mathsf{S}}^c$ coincides with $\succeq_{\mathsf{S}}$ and $\sqsupseteq^c$ coincides with $\sqsupseteq$.

*Proof:*

Let $N$ be a timed net and $(O, r)$ be a continuously timed test. We first show that that $N$ $may_{\mathsf{S}}^c$ $(O, r)$ iff $N$ $may_{\mathsf{S}}$ $(O, \lfloor r \rfloor)$ and that $N$ $must^c$ $(O, r)$ iff $N$ $must$ $(O, \lfloor r \rfloor)$:

If $N$ $may_{\mathsf{S}}^c$ $(O, r)$, then there is w.l.o.g. a $w(\omega^+, e) \in \mathsf{SL}^c(N \|_\Sigma O)$ with $dur(w) \leq r$; now by the proof of Proposition 3.13, we have $dur(w) \in \mathbb{N}_0$, hence $dur(w) \leq \lfloor r \rfloor$, and we can construct a $w'(\omega^+, e) \in \mathsf{SL}(N \|_\Sigma O)$ with $dur(w') = dur(w) \leq \lfloor r \rfloor$, thus $N$ $may_{\mathsf{S}}$ $(O, \lfloor r \rfloor)$. If $N$ $may_{\mathsf{S}}$ $(O, \lfloor r \rfloor)$, then there is w.l.o.g. a $w(\omega^+, e) \in \mathsf{SL}(N \|_\Sigma O) \subseteq \mathsf{SL}^c(N \|_\Sigma O)$ with $dur(w) \leq \lfloor r \rfloor \leq r$ by Proposition 3.10, hence also $N$ $may_{\mathsf{S}}^c$ $(O, r)$.

Let $N$ $must^c$ $(O, r)$ and take some $w \in \mathsf{SL}(N\|_\Sigma O) \subseteq \mathsf{SL}^c(N\|_\Sigma O)$ with $dur(w) > \lfloor r \rfloor$; then also $dur(w) > r$ by $dur(w) \in \mathbb{N}_0$, thus $w$ contains some $(\omega^+, e)$ by assumption, hence $N$ $must$ $(O, \lfloor r \rfloor)$. Finally, let $N$ $must$ $(O, \lfloor r \rfloor)$ and take some $w \in \mathsf{SL}^c(N\|_\Sigma O)$ with $dur(w) > r$; then by the proof of Proposition 3.13, there is a $w' \in \mathsf{SL}(N\|_\Sigma O)$ with $dur(w') \geq dur(w) > r \geq \lfloor r \rfloor$ and $seq(w') = seq(w)$, hence $w'$ contains some $(\omega^+, e)$ by assumption, thus $w$ does and we conclude $N$ $must^c$ $(O, r)$.

Now assume $N_1 \succeq_S N_2$ for some testable nets $N_1$, $N_2$; then $N_2$ $may_S$ $(O, \lfloor r \rfloor)$ for some observer $O$ and $r \in \mathbb{R}_0^+$ implies $N_2$ $may_S^c$ $(O, r)$ by the above, hence also $N_1$ $may_S^c$ $(O, r)$ by assumption and $N_1$ $may_S$ $(O, \lfloor r \rfloor)$ by the above again, and we conclude $N_1 \succeq_S N_2$. The reverse direction and the *must*-case is analogous. ■ 4.4

For the other three variants, we first show how to construct from a successful continuous trace a 'faster' successful discrete trace:

**Proposition 4.5**

Let $N$ be a net and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}\}$. For each $w \in \mathsf{XFS}^c(N)$ there is a $u \in \mathsf{XFS}(N)$ with $dur(u) \leq dur(w)$ and all $\varepsilon$ in $seq(w)$ are also in $seq(u)$.

*Proof:*

We first construct for each $w \in \mathsf{XFS}^c(N)$ a $v$ with only (1)-time-steps, such that $seq(v) = seq(w)$ and $TD_N[v\rangle_\mathsf{X}$; note that $seq(v) = seq(w)$ and $w \in \mathsf{XFS}^c(N)$ implies that $v$ is event-unique, but $v$ will in general neither be max-caused nor be time-complete, hence we will transform $v$ to the desired $u \in \mathsf{XFS}(N)$ at the end of this proof, such that $dur(u) = dur(v)$ and and all $\varepsilon$ in $seq(w)$ are also in $seq(u)$.

We denote the $TD$'s reached after $w$ and $v$ by $TD_w$ and $TD_v$; note that, as a consequence of $seq(v) = seq(w)$, $TD_w$ and $TD_v$ coincide in their $M$- and $C$-component, hence we will denote both $M_w$ and $M_v$ by $M$ and both $C_w$ and $C_v$ by $C$. Furthermore, we have $\rho_v(t, e) \in \mathbb{N}_0$ for all $(t, e) \in C$ by Lemma 3.3.1, since $v$ has only (1)-time-steps and $\delta(t) \in \mathbb{N}_0$. We let $\Delta = dur(w) - dur(v)$ and show that $\Delta - 1 < \rho_v - \rho_w \leq \Delta$ and $0 \leq \Delta < 1$, which in particular implies $dur(v) \leq dur(w)$.

The proof is by induction on $|w|$, where for $w = \lambda$ we can choose $v = \lambda$, yielding $\Delta = 0$ and $\rho_v = \rho_w = \emptyset$. Hence, assume that for $w \in \mathsf{XFS}^c(N)$ we have constructed $v$ as desired and consider $w' = w\varepsilon$. We denote the $TD$'s reached after $w'$ and the corresponding $v'$ $TD_{w'}$ and $TD_{v'}$ with common marking $M'$ and current transitions $C'$.

If $\varepsilon = (t^+, e)$, we choose $v' = v\varepsilon$; then $seq(v') = seq(w')$ and $TD_v[(t^+, e)\rangle_\mathsf{X}$ by Definition 3.2.1 and by induction, since $w' = w(t^+, e) \in \mathsf{XFS}^c(N)$ by assumption. Furthermore, $dur(w') = dur(w)$ and $dur(v') = dur(v)$ implies $\Delta' = \Delta$, hence $0 \leq \Delta' < 1$ by induction, and the residual times $\rho_{v'}$ and $\rho_{w'}$ coincide with $\rho_v$ and $\rho_w$ on $C$ and are both equal $\delta(t)$ for $(t, e)$, such that by the above also in this case $\Delta' - 1 < 0 = \delta(t) - \delta(t) = 0 \leq \Delta'$.

If $\varepsilon = (t^-, e)$, then we must have $\rho_w(t, e) = 0$ by Definition 3.2.2, hence by induction $\rho_v(t, e) = \rho_v(t, e) - 0 \leq \Delta < 1$, thus $\rho_v(t, e) = 0$ since $\rho_v(t, e) \in \mathbb{N}_0$; then by Definition 3.2.2, we can choose $v' = v\varepsilon$, yielding $seq(v') = seq(w')$ by induction. Furthermore,

$\Delta' = \Delta$, $\rho_{w'} = \rho_w|_{C'}$ and $\rho_{v'} = \rho_v|_{C'}$, hence $\Delta' - 1 < \rho_{v'} - \rho_{w'} \leq \Delta'$ and $0 \leq \Delta' < 1$ follow directly by induction, too.

Now let $\varepsilon = (r)$ with $r \in (0;1]$. If $\Delta + r < 1$, then we choose $v' = v$; obviously, $seq(v') = seq(w')$ and $0 \leq \Delta < \Delta + r = \Delta' < 1$ by induction and assumption. Furthermore, $\rho_{v'} - \rho_{w'} = \rho_v - (\rho_w \dot{-} r) \leq \rho_v - (\rho_w - r) = \rho_v - \rho_w + r \leq \Delta + r = \Delta'$ by induction. Now if $r > \rho_w$ (meaning $r > \rho_w(t,e)$ for *some* $(t,e) \in C$, by abuse of notation), then $\rho_{w'} = 0$ (meaning $\rho_{w'}(t,e) = 0$ for the *same* $(t,e) \in C$), hence $\Delta' - 1 < 0 \leq \rho_v = \rho_{v'} - \rho_{w'}$, and if $r \leq \rho_w$, then $\rho_{w'} = \rho_w - r$ and $\Delta - 1 + r < \rho_v - \rho_w + r$ by induction, thus $\Delta' - 1 < \rho_{v'} - \rho_{w'}$.

If, on the other hand, $\Delta + r \geq 1$, we choose $v' = v(1)$; then $seq(v') = seq(w')$ and $0 \leq \Delta + r - 1 = \Delta'$ by assumption and $\Delta' = \Delta + r - 1 \leq \Delta < 1$ by induction since $r \leq 1$. Furthermore, if $\rho_v = 0$ (with the same abuse of notation as above), then $\rho_{v'} - \rho_{w'} = -(\rho_w \dot{-} r) \leq 0 \leq \Delta + r - 1 = \Delta'$ by assumption, and if $\rho_v > 0$, then $\rho_v \dot{-} 1 = \rho_v - 1$ (by $\rho_v \in \mathbb{N}_0$) and $\rho_w \dot{-} r \geq \rho_w - r$ yield $\rho_{v'} - \rho_{w'} \leq \rho_v - 1 - \rho_w + r \leq \Delta - 1 + r = \Delta'$ by induction. Finally, if $r \leq \rho_w$, then $\rho_{v'} - \rho_{w'} = (\rho_v \dot{-} 1) - \rho_w + r \geq \rho_v - 1 - \rho_w + r > \Delta - 1 - 1 + r = \Delta' - 1$ by induction, and if $r > \rho_w$, then $\rho_{v'} - \rho_{w'} = \rho_{v'} \geq 0 > \Delta' - 1$ by $\Delta' < 1$.

Now $TD_v[(1)\rangle_{\mathsf{L}}$ by Definition 3.2.3, and it remains to check the other two cases:

$\mathsf{X} = \mathsf{E}$: $w' = w(r) \in \mathsf{EFS}^c(N)$ implies $r \leq \rho_w$, and we have $\rho_w < \rho_v + 1 - \Delta$ by the additional property and $1 - \Delta \leq r$ by assumption, thus $r < \rho_v + 1 - \Delta \leq \rho_v + r$, yielding $\rho_v > 0$, hence $\rho_v \geq 1$ since $\rho_v \in \mathbb{N}_0$, and we conclude $TD_v[(1)\rangle_{\mathsf{E}}$.

$\mathsf{X} = \mathsf{A}$: $w' = w(r) \in \mathsf{AFS}^c(N)$ implies $M_v[T\rangle = M_w[T\rangle = \emptyset$, hence $TD_v[(1)\rangle_{\mathsf{A}}$.

For $w \in \mathsf{XFS}^c(N)$, we have constructed an event-unique $v$ with only (1)-time-steps, such that $seq(v) = seq(w)$ and $dur(v) \leq dur(w)$ and $TD_N[v\rangle_{\mathsf{X}}(M, C, \rho_v)$; now let $\tilde{u} = vv^-v^+$, where $v^- \in end(C)$, such that $(M, C, \rho_v)[v^-\rangle_{\mathsf{X}}(M^-, C^-, \rho_v^-)$ and $v^+ \in start(M^-)$, such that $(M^-, C^-, \rho_v^-)[v^+\rangle_{\mathsf{X}}(M^+, C^+, \rho_v^+)[(1)\rangle_{\mathsf{X}}$ with $M^+[T\rangle = \emptyset$ and $\rho_v^+(t,e) \geq 1$ for all $(t, e, \in C^+)$ by Lemma 3.7. We infer $TD_N[\tilde{u}(1)\rangle_{\mathsf{X}}$ and may assume $\tilde{u}(1)$ to be event-unique. Now by Lemma 3.3.2, moving transition ends in front of transition starts that happen at the same time in $\tilde{u}(1)$ (i.e. are between the same two successive (1)-time-steps) yields a max-caused (hence wellformed) and time-complete $u(1)$ with $dur(u) = dur(v) \leq dur(w)$ and all $\varepsilon$ in $seq(w)$ are also in $seq(u)$ and $TD_N[u(1)\rangle_{\mathsf{X}}$, thus $u \in \mathsf{XFS}(N)$, and we are done. ■ 4.5

The coincidence of discrete and continuous may-testing now follows directly:

**Theorem 4.6**

For all $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}\}$, the relations $\succeq_{\mathsf{X}}^{\mathsf{c}}$ and $\succeq_{\mathsf{X}}$ coincide on timed nets.

*Proof:*

Let $N$ be a testable net, let $(O, r)$ be a continuously timed test and let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}\}$. By the proof of Proposition 4.4, it suffices to show that $N$ $may_{\mathsf{X}}^{\mathsf{c}}$ $(O, r)$ if and only if $N$ $may_{\mathsf{X}}$ $(O, \lfloor r \rfloor)$:

If $N$ $may^c_X$ $(O,r)$, then there is a $w \in \mathsf{XL}^c(N)$ with $dur(w) \leq r$, that contains an $(\omega^+, e)$ and $w = l(v)$ for some $v \in \mathsf{XFS}^c(N)$ with $dur(v) \leq r$, that contains an $(\omega^+, e)$; now by Proposition 4.5, there is a $u \in \mathsf{XFS}(N)$, with $dur(u) \leq dur(v) \leq r$ and $seq(u)$ is a permutation of $seq(v)$, hence $dur(u) \leq \lfloor r \rfloor$ (since $dur(u) \in \mathbb{N}_0$) and $u$ contains an $(\omega^+, e)$, thus $w' = l(u) \in \mathsf{XL}(N)$ with $dur(w') \leq \lfloor r \rfloor$ contains an $(\omega^+, e)$ and we conclude $N$ $may_X$ $(O, \lfloor r \rfloor)$. If, on the other hand, $N$ $may_X$ $(O, \lfloor r \rfloor)$, then there is $w \in \mathsf{XL}(N)$ with $dur(w) \leq \lfloor r \rfloor$, that contains an $(\omega^+, e)$, and for this $w$ also $w \in \mathsf{XL}^c(N)$, since $\mathsf{XL}(N) \subseteq \mathsf{XL}^c(N)$, such that $N$ $may^c_X$ $(O,r)$, too. ■ 4.6

As a result of this subsection, we can restrict attention to the discrete testing preorders in the remainder of this section, since checking $N_1 \sqsupseteq^c N_2$ or $N_1 \succeq^c_X N_2$ now reduces to checking $N_1 \sqsupseteq N_2$ or $N_1 \succeq_X N_2$ resp.

## 4.2 Characterizing Discrete May-Testing

At this point, it is by no means clear how to check $N_1 \sqsupseteq N_2$ or $N_1 \succeq_X N_2$ for given testable $N_1$ and $N_2$. Obviously, it is impossible even in the discrete variants to apply the definition of timed testing directly, since there are infinitely many timed tests to apply.

Hence, in this subsection we look for the just necessary refinements of the four basic semantics that are precongruences for parallel composition. The corresponding testing preorders are then characterized by inclusion of these refined languages. Most the developments are already carried out in [Vog95], but are presented also here for convenient reading.

We first decompose the timed states of a composition into timed states of the components:

**Definition 4.7**

Let $N_1$ and $N_2$ be timed nets, let $A \subseteq \Sigma$ and $N = N_1 \|_A N_2$. Let $TD$, $TD_1$ and $TD_2$ be reachable timed states of $N$, $N_1$, $N_2$ resp. We say that $TD$ is the *combination* of $TD_1$ and $TD_2$, if

$$\begin{aligned} M_1 &= \{s_1 \mid (s_1, *) \in M\} \\ M_2 &= \{s_2 \mid (*, s_2) \in M\} \\ C_1 &= \{(t_1, e) \mid ((t_1, t_2), e) \in C, t_1 \in T_1\} \\ C_2 &= \{(t_2, e) \mid ((t_1, t_2), e) \in C, t_2 \in T_2\} \\ \rho((t_1, t_2), e) &= \max(\rho_1(t_1, e), \rho_2(t_2, e)), \text{ where } \rho_i(*, e) = 0 \text{ for } i = 1, 2. \end{aligned}$$

■ 4.7

Now the operational behaviour of a parallel composition can be decomposed into operational behaviour of the components:

**Lemma 4.8**

Let $N_1$ and $N_2$ be timed nets, let $A \subseteq \Sigma$ and $N = N_1 \|_A N_2$. Let $TD = (M, C, \rho)$, $TD_1 = (M_1, C_1, \rho_1)$ and $TD_2 = (M_2, C_2, \rho_2)$ be corresponding $TD$'s, such that $TD$ is the combination von $TD_1$ and $TD_2$. Let $\mathsf{X} \in \{\mathsf{L}, \mathsf{E}, \mathsf{A}, \mathsf{S}\}$. Then $TD[\varepsilon\rangle_X$ in $N$ if and only if $TD_1[\varepsilon_1\rangle_L$ in $N_1$, $TD_2[\varepsilon_2\rangle_L$ in $N_2$ and one of the following cases applies:

a) $\varepsilon = ((t_1, t_2)^+, e)$, $\varepsilon_1 = (t_1^+, e)$, $\varepsilon_2 = (t_2^+, e)$ and $l_1(t_1) = l_2(t_2) \in A$.

b) $\varepsilon = ((t_1, *)^+, e)$, $\varepsilon_1 = (t_1^+, e)$ and $\varepsilon_2 = \lambda$, $l_1(t_1) \notin A$.

c) analogously to b) for $\varepsilon = ((*, t_2)^+, e)$.

d) $\varepsilon = ((t_1, t_2)^-, e)$, $\varepsilon_1 = (t_1^-, e)$, $\varepsilon_2 = (t_2^-, e)$ and $l_1(t_1) = l_2(t_2) \in A$.

e) $\varepsilon = ((t_1, *)^-, e)$, $\varepsilon_1 = (t_1^-, e)$, $\varepsilon_2 = \lambda$, $l_1(t_1) \notin A$.

f) analogously to e) for $\varepsilon = ((*, t_2)^-, e)$.

g) $\varepsilon = \varepsilon_1 = \varepsilon_2 = (1)$ and

- If $\mathsf{X} = \mathsf{E}$, then additionally $(*)$ below.
- If $\mathsf{X} = \mathsf{A}$, then additionally $(**)$ below.
- If $\mathsf{X} = \mathsf{S}$, then additionally $(*)$ and $(**)$ below.

The conditions $(*)$ and $(**)$ are as follows:

$(*)$ for all $((t_1, t_2), e) \in C$ we have $\rho_1(t_1, e) + \rho_2(t_2, e) > 0$ (where $\rho_i(*, e) = 0$).

$(**)$ $M_1[A\rangle\rangle \cap M_2[A\rangle\rangle = \emptyset$ and $M_1[\Sigma - A\rangle\rangle \cup M_2[\Sigma - A\rangle\rangle = \emptyset$.

In all cases, if $TD[\varepsilon\rangle_{\mathsf{X}} TD'$, $TD_1[\varepsilon\rangle_{\mathsf{L}} TD_1'$ and $TD_2[\varepsilon\rangle_{\mathsf{L}} TD_2'$, then $TD'$ is the combination of $TD_1'$ and $TD_2'$.

*Proof:* Easy but tedious (cf. [Vog95]) ■ 4.8

Furthermore, for the $\mathsf{L}$-variant the traces of the parallel composition can be calculated from the traces of the components without further refinement via the shuffle $\|_A$:

**Definition 4.9**

Let $u, v \in ((\Sigma \times E) \cup \{(1)\})^*$ be wellformed and let $A \subseteq \Sigma$. Define

$$u\|_A v = \{\ w \in ((\Sigma \times E) \cup \{(1)\})^* \mid w \text{ is wellformed, und and we can write:} \\ u = u_1 \dots u_n,\ v = v_1 \dots v_n,\ w = w_1 \dots w_n \\ \text{with } n \geq 0 \text{, such that for all } i = 1, \dots, n \\ \text{either } w_i = u_i = v_i \in A^\pm \times E \cup \{(1)\} \\ \text{or } w_i = u_i \in (\Sigma - A)^\pm \times E \text{ and } v_i = \lambda \\ \text{or } w_i = v_i \in (\Sigma - A)^\pm \times E \text{ and } u_i = \lambda\ \}$$

■ 4.9

**Proposition 4.10**

Let $N_1$ and $N_2$ be timed nets and $A \subseteq \Sigma$.
Then $\mathsf{LL}(N_1 \|_A N_2) = \bigcup \{u\|_A v \mid u \in \mathsf{LL}(N_1),\ v \in \mathsf{LL}(N_2)\}$.

*Proof:* Using Lemma 4.8 and Definition 4.9 (cf. [Vog95]). ■ 4.10

In words: inclusion of $\mathsf{LL}$-semantics is a precongruence for parallel composition. This is enough for the characterization of $\succeq_{\mathsf{L}}$:

**Theorem 4.11**

Let $N_1$ and $N_2$ be timed testable nets. Then $N_1 \succeq_{\mathsf{L}} N_2$ if and only if $\mathsf{LL}(N_1) \supseteq \mathsf{LL}(N_2)$.

*Proof:*

'if': Let $(O, d)$ be a discrete test and assume $\mathsf{LL}(N_1) \supseteq \mathsf{LL}(N_2)$; then $\mathsf{LL}(N_1 \|_\Sigma O) \supseteq \mathsf{LL}(N_2 \|_\Sigma O)$ by Proposition 4.10, hence if $N_2$ $may_{\mathsf{L}}$ $(O, d)$ by some $w \in \mathsf{LL}(N_2 \|_\Sigma O)$, then also $N_1$ $may_{\mathsf{L}}$ $(O, d)$ by $w \in \mathsf{LL}(N_1 \|_\Sigma O)$, thus $N_1 \succeq_{\mathsf{L}} N_2$.

'only-if': Let $N$ be any testable timed net; then by [Vog95], for each $w \in (\Sigma^\pm \times E) \cup \{(1)\})^*$ there exists a discrete test $(O, d)_w$, such that $N$ $may_{\mathsf{L}}$ $(O, d)_w$ if and only if $w \in \mathsf{LL}(N)$. Now let $N_1 \succeq_{\mathsf{L}} N_2$; then $w \in \mathsf{LL}(N_2)$ implies $N_2$ $may_{\mathsf{L}}$ $(O, d)_w$, hence $N_1$ $may_{\mathsf{L}}$ $(O, d)_w$ by assumption and $w \in \mathsf{LL}(N_1)$ by the above, too. ■ 4.11

For the characterization of $\succeq_{\mathsf{L}}$ we did not have to refine $\mathsf{LL}$, since it is already a precongruence for parallel composition. This is not the case in the three other variants, where a refinement is necessary:

**Definition 4.12**

Let $N$ be a timed net and let $w = w_1(1)w_2 \dots (1)w_n(1) \in \mathsf{LL}(N)$ for some $n$ with $w_i \in (\Sigma^\pm \times E)^*$, such that $TD_N[w_1\rangle\rangle_{\mathsf{L}} TD_1[(1)w_2\rangle\rangle_{\mathsf{L}} TD_2 \dots [(1)w_n\rangle\rangle_{\mathsf{L}} TD_n[(1)\rangle\rangle_{\mathsf{L}}$ for some $TD_i$ for all $i = 1, \dots, n$. Then $w = w_1 X_1 w_2 X_2 \dots w_n X_n$ with $X_i \subseteq (\Sigma \cup (\Sigma^- \times E))$ is a *timed refusal trace* of $N$ if for all $i = 1, \dots, n$ the following conditions hold:

- $M_i[X_i \cap \Sigma\rangle\rangle = \emptyset$
- If $(a^-, e) \in X_i$, then $\neg TD_i[(a^-, e)\rangle\rangle_{\mathsf{L}}$ and:
  - either $(a^+, e)$ occurs in $w_i$, or $i > 1$ and $(a^-, e) \in X_{i-1}$
  - $i < n$ implies: $(a^-, e)$ occurs in $w_{i+1}$ or $(a^-, e) \in X_{i+1}$

The set of timed refusal traces of $N$ is denoted $\mathsf{SRT}(N)$. We additionally define

$$\mathsf{ERT}(N) = \{w \in \mathsf{SRT}(N) \mid X \subseteq (\Sigma^- \times E) \text{ for all } X \text{ in } w\}$$
$$\mathsf{ART}(N) = \{w \in \mathsf{SRT}(N) \mid X \subseteq \Sigma \text{ for all } X \text{ in } w\}$$

The sets $X$ in a timed refusal trace are called *refusal sets* and are sometimes referred to as *time steps*. Termination is defined accordingly for refusal-traces, too. ■ 4.12

Refusal traces refine the corresponding traces ...

**Proposition 4.13**

Let $N_1$ and $N_2$ be timed nets and $\mathsf{X} \in \{\mathsf{E}, \mathsf{A}, \mathsf{S}\}$.
Then $\mathsf{XRT}(N_1) \subseteq \mathsf{XRT}(N_2)$ implies $\mathsf{XL}(N_1) \subseteq \mathsf{XL}(N_2)$.

*Proof:*

Let $w = w_1 X_1 \dots w_n X_n \in \mathsf{SRT}(N)$ for some timed net $N$; then we say (only for this proof) that $w$ is $\mathsf{E}$-maximal, if for all $i = 1, \dots, n$ each occurrence of some $(a^+, e)$ in $w_i$ implies $(a^-, e) \in X_i$; we say that $w$ is $\mathsf{A}$-maximal if $\Sigma \subseteq X_i$ for all $i = 1, \dots, n$; finally, $w$ is $\mathsf{S}$-maximal if it is both $\mathsf{E}$-maximal and $\mathsf{A}$-maximal. Now for $\mathsf{X} \in \{\mathsf{E}, \mathsf{A}, \mathsf{S}\}$ we have $\mathsf{XL}(N) = \{w \mid w$ is a $\mathsf{X}$-maximal $v \in \mathsf{XRT}(N)$ with all refusal sets replaced by $(1)\}$. From this, the claimed implication follows quite directly. ■ 4.13

... and their inclusion is a precongruence for parallel composition:

**Definition 4.14**

Let $u = u_1 X_1 u_2 X_2 \dots u_n X_n$ and $v = v_1 X_1 v_2 X_2 \dots v_n X_n$ be timed refusal traces with $n \in \mathbb{N}_0$ and $A \subseteq \Sigma$. Define

$$
\begin{aligned}
u \|_A^r v = \{\; & w = w_1 Z_1 w_2 Z_2 \dots w_n Z_n \mid \\
& w_1(1) w_2(1) \dots w_n(1) \in (u \|_A v) \text{ and for all } i = 1, \dots, n \text{ we have:} \\
& Z_i \cap A \subseteq (X_i \cup Y_i) \cap A \text{ and} \\
& Z_i \cap (\Sigma - A) \subseteq (X_i \cap Y_i) \cap (\Sigma - A) \text{ and} \\
& Z_i \cap (\Sigma^- \times E) = (X_i \cup Y_i) \cap (\Sigma^- \times E). \;\}
\end{aligned}
$$

■ 4.14

**Proposition 4.15**

Let $N_1$ and $N_2$ be timed nets, $A \subseteq \Sigma$ and $\mathsf{X} \in \{\mathsf{E}, \mathsf{A}, \mathsf{S}\}$.
Then $\mathsf{XRT}(N_1 \|_A N_2) = \bigcup \{u \|_A v \mid u \in \mathsf{XRT}(N_1),\, v \in \mathsf{XRT}(N_2)\}$.

*Proof:*

In [Vog95], the result is shown for $\mathsf{X} \in \{\mathsf{E}, \mathsf{S}\}$, and applies for $\mathsf{X} = \mathsf{A}$ by similar arguments. ■ 4.15

For the A- and the S-variant, they also characterize the may-testing preorders:

**Theorem 4.16**

Let $N_1$ and $N_2$ be timed testable nets. Then $N_1 \succeq_{\mathsf{S}} N_2$ if and only if $\mathsf{SRT}(N_1) \supseteq \mathsf{SRT}(N_2)$.

*Proof:*

Analogously to the proof of Theorem 4.11, where for the 'if'-direction additionally Proposition 4.13 is applied. ■ 4.16

**Theorem 4.17**

Let $N_1$ and $N_2$ be timed testable nets. Then $N_1 \succeq_{\mathsf{A}} N_2$ if and only if $\mathsf{ART}(N_1) \supseteq \mathsf{ART}(N_2)$.

*Proof:*

'if':

Analogously to the 'if'-direction in the proof of Theorem 4.11, where additionally Proposition 4.13 is applied.

'only-if':

Let $N_1 \succeq_{\mathsf{A}} N_2$ and take some $w = w_1 Z_1 \dots w_n Z_n \in \mathsf{ART}(N_2)$. We may assume that

1. $w$ is terminated by Proposition 3.9.1 and .3 and
2. $w_1$ is the empty sequence by Proposition 3.9.1 and .2 and
3. $Z_i \subseteq A = l_1(T_1) \cup l_2(T_2)$ for all $i$, since we may assume the alphabet $\Sigma$ to be restricted to $A$.

We have to show that $w \in \mathsf{ART}(N_1)$ and will therefore construct a discrete test $(O, n)$, such that $N$ $may_{\mathsf{A}}$ $(O, n)$ for some testable timed net $N$ if and only if $w \in \mathsf{ART}(N)$; then $N_2$ $may_{\mathsf{A}}$ $(O, n)$, hence $N_1$ $may_{\mathsf{A}}$ $(O, n)$ and $w \in \mathsf{ART}(N_1)$.

We construct $(O, n)$ as follows: $O$ consists of two parts. The first part contains the places $s_i$ and $s_{a,i}$ and the transitions $t_i$ and $t_{a,i}$. The $s_i$ and $t_i$ model something like a clock, while the $s_{a,i}$ and $t_{a,i}$ test for refusal of $Z_i \cap \Sigma$. The second part consists of $(e, 1), \ldots, (e, 7)$ for each $e$ in $w$, which test for the correctly timed occurrence of the action attached by $e$.

$$\begin{aligned} S_O &= \{s_i \mid i = 1, \ldots, n+1\} \cup \{s_{a,i} \mid i = 1, \ldots, n \text{ and } a \in Z_i\} \\ &\quad \cup \{(e, i) \mid i = 1, 3, 5, 7 \text{ and } (a^+, e) \text{ occurs in } w \text{ for some } a \in \Sigma\} \end{aligned}$$

$$\begin{aligned} T_O &= \{\mathit{success}\} \cup \{t_i \mid i = 1, \ldots, n\} \cup \{t_{a,i} \mid i = 1, \ldots, n \text{ and } a \in Z_i\} \\ &\quad \cup \{(e, i) \mid i = 2, 4, 6 \text{ and } (a^+, e) \text{ occurs in } w \text{ for some } a \in \Sigma\} \end{aligned}$$

$F_O$ contains exactly the following pairs:

$(s_i, t_i), (t_i, s_{i+1}),\ \ i = 1, \ldots, n$
$(s_{n+1}, \mathit{success})$
$(s_{a,i}, t_{a,i}),\ \ i = 1, \ldots, n,\ a \in Z_i$
$(s_{a,i}, t_{i+1}),\ \ i = 1, \ldots, n-1,\ a \in Z_i$
$(s_{a,n}, \mathit{success}),\ \ a \in Z_n$
$(t_i, s_{a,i+1}),\ \ i = 1, \ldots, n-1,\ a \in Z_i$
$((e, i), (e, i+1)),\ \ i = 1, \ldots, 6$
$((e, 7), \mathit{success})$

The labelling $l_O(t)$, $t \in T_O$, is *wait* except for the following cases:

$l_O(\mathit{success}) = \omega$
$l_O(t_{a,i}) = a$
$l_O(e, 4) = a$ if $(a^+, e)$ occurs in $w$

The duration $\delta_O(t)$, $t \in T_O$, is 1 except for the following cases: if $(a^+, e)$ occurs in $w$ then

$\delta_O((e, 2))$ is the number of time steps in $w$ before $(a^+, e)$ and
$\delta_O((e, 4))$ is the number of time steps in $w$ between $(a^+, e)$ and $(a^-, e)$ and
$\delta_O((e, 6))$ is the number of time steps in $w$ after $(a^-, e)$.

The marking $M_O(s)$, $s \in S_O$, is 0 except for: $M_O(s_1) = M_O((e, 1)) = M_O(s_{a,1}) = 1$.

Assume $N$ $may_{\mathsf{A}}$ $(O, n)$ due to some $w'(1) = w_1(1) \ldots w_n(1) w_{n+1}(1) \in \mathsf{AL}(N \|_{\Sigma} O)$ for any testable timed net $N$ (i.e. some $(\omega^+, e)$ occurs in $w'$ and $\mathit{dur}(w') = n$). We can regard $w'(1)$ as a timed refusal trace $w''$ in replacing (1)-steps by $\Sigma$'s. Then by Proposition 4.15 we have $w'' \in u \|_{\Sigma}^{r} v$ for some $u \in \mathsf{ART}(N)$ and $v \in \mathsf{ART}(O)$. We consider the different parts of $O$ and draw conclusions for $u = u_1 X_1 \ldots u_n X_n u_{n+1} X_{n+1}$ and $v = v_1 Y_1 \ldots v_n Y_n v_{n+1} Y_{n+1}$.

Let us first have a look at the part of $O$ containing $s_i$, $t_i$, $s_{a,i}$, $t_{a,i}$ and *success*. In order to reach *success* in time it is necessary to fire the sequence $t_1 \dots t_n$; more precisely, $t_i$ has to start immediately before and has to end immediately after the $i$-th time step. Thus, $s_{a,i}$ is marked before the $i$-th time step and $t_{a,i}$ might empty $s_{a,i}$; to prevent this in A-behaviour, $N$ must refuse $a$ at this moment, i.e. we must have $a \in X_i$. If this is the case, $s_{a,i}$ can be emptied after the $i$-th time step by $t_{i+1}$ (or by *success* if $i = n$). Hence, $t_1 \dots t_n$ is fired in $N \|_\Sigma O$ if and only if $Z_i \cap \Sigma \subseteq X_i \cap \Sigma$ for $i = 1, \dots, n$. These inclusions hold if $u_1 X_1 \dots u_n X_n = w$.

Secondly, let us consider some $(a^+, e)$ appearing in $w$; let $\tau_1$ be the number of time steps in $w$ before $(a^+, e)$, $\tau_2$ be the number of those between $(a^+, e)$ and $(a^-, e)$, and $\tau_3$ be the number of those after $(a^-, e)$; observe that $\tau_1 \geq 1$ since $w_1 = \lambda$. In order to mark $(e, 7)$ in time, $(e, 4)$ must start after $\tau_1$ time steps and end after $\tau_1 + \tau_2$ time steps. This is possible if and only if $u_{\tau_1+1}$ contains the start of some $a$ that ends in $u_{\tau_1+\tau_2+1}$. Without loss of generality, we may assume that $u_{\tau_1+1}$ contains $(a^+, e)$ just as $w_{\tau_1+1}$ does and that $u_{\tau_1+\tau_2+1}$ contains $(a^-, e)$ just as $w_{\tau_1+\tau_2+1}$ does.

We conclude that $N$ $may_{\mathsf{A}}$ $(O, n)$ by the above $w'(1)$ only if $u_i$ is essentially $w_i$ (i.e. up to permutations within some $u_i$'s) and $Z_i \subseteq X_i$ for all $i = 1, \dots, n$. On the other hand, our considerations also show that $N$ $may_{\mathsf{A}}$ $(O, n)$ if there is some $u = u_1 X_1 \dots u_n X_n \in \mathsf{ART}(N)$ of this form. We conclude $N$ $may_{\mathsf{A}}$ $(O, n)$ if and only if $w \in \mathsf{ART}(N)$ and are done. ■ 4.17

## 4.3 Comparing the Four Variants of Discrete May-Testing

That ERT-inclusion is actually finer than $\succeq_{\mathsf{E}}$ is shown by the following results:

**Theorem 4.18**

For testable timed nets, the relations $\succeq_{\mathsf{E}}$ and $\succeq_{\mathsf{L}}$ coincide.

*Proof:*
By Proposition 3.8, LL- and EL-semantics coincide with liberal- and mixed-behaviour resp. defined in [Vog95]; there it is shown that for any testable timed net $N$ and any discrete test $(O, d)$ we have $N$ $may_{\mathsf{L}}$ $(O, d)$ if and only if $N$ $may_{\mathsf{E}}$ $(O, d)$; the coincidence of $\succeq_{\mathsf{E}}$ and $\succeq_{\mathsf{L}}$ follows directly. ■ 4.18

**Proposition 4.19**

Let $N_1$ and $N_2$ be timed nets. Then

1. $\mathsf{SRT}(N_1) \subseteq \mathsf{SRT}(N_2)$ implies $\mathsf{ERT}(N_1) \subseteq \mathsf{ERT}(N_2)$ and $\mathsf{ART}(N_1) \subseteq \mathsf{ART}(N_2)$.
2. $\mathsf{ERT}(N_1) \subseteq \mathsf{ERT}(N_2)$ or $\mathsf{ART}(N_1) \subseteq \mathsf{ART}(N_2)$ implies $\mathsf{LL}(N_1) \subseteq \mathsf{LL}(N_2)$.

*Proof:*
1. Straightforward with Definition 4.12, since $\mathsf{ERT}(N)$ and $\mathsf{ART}(N)$ are syntactically decidable subsets of $\mathsf{SRT}(N)$ for any timed net $N$.

2. Straightforward with Definition 4.12: take some $w \in \mathsf{LL}(N_1)$ and replace all (1)-steps in $w$ by $\emptyset$; this yields a refusal trace $w'$ in $\mathsf{ERT}(N_1)$ and $\mathsf{ART}(N_2)$, hence $w' \in \mathsf{ERT}(N_2)$ or $w' \in \mathsf{ART}(N_2)$, thus $w \in \mathsf{LL}(N_2)$. ■ 4.19

Hence, it turns out that there are much more relations between the four variants of may-testing than between their corresponding basic semantics; on the class of untimed nets even $\succeq_{\mathsf{S}}$ coincides with $\succeq_{\mathsf{A}}$, thus there are only two preorders and one of them refines the other one:

**Proposition 4.20**

Let $N_1$ and $N_2$ be untimed nets. Then

1. $\mathsf{LL}(N_1) \subseteq \mathsf{LL}(N_2)$ implies $\mathsf{ERT}(N_1) \subseteq \mathsf{ERT}(N_2)$.
2. $\mathsf{ART}(N_1) \subseteq \mathsf{ART}(N_2)$ implies $\mathsf{SRT}(N_1) \subseteq \mathsf{SRT}(N_2)$.

*Proof:* Let $N$ be an untimed net.

1. Then $\mathsf{ERT}(N) = \{w_1X_1 \ldots w_nX_n \mid w_1(1) \ldots w_n(1) \in \mathsf{LL}(N)$ and
for all $i = 1, \ldots, n-1$: $(a^+, e)$ occurs in $w_i$ only if $(a^-, e)$ occurs in $w_{i+1}$ and
for all $i = 1, \ldots, n$: $(a^-, e) \in X_i$ only if $(a^+, e)$ occurs in $w_i\}$.
2. Then $\mathsf{SRT}(N) = \{w_1X_1 \ldots w_nX_n \mid w_1Y_1 \ldots w_nY_n \in \mathsf{ART}(N)$ for some $Y_i$, such that
for all $i = 1, \ldots, n-1$: $(a^+, e)$ occurs in $w_i$ only if $(a^-, e)$ occurs in $w_{i+1}$ and
for all $i = 1, \ldots, n$: $Y_i \subseteq X_i$ and $(a^-, e) \in X_i$ only if $(a^+, e)$ occurs in $w_i\}$.

From this, the claimed implications follows quite directly. ■ 4.20

**Corollary 4.21**

The following implications and no other hold in general between the discrete may-testing preorders for *untimed (left)* and *all timed (right)* nets.

$$\begin{array}{ccccc} & & \succeq_{\mathsf{L}} & & \\ & \swarrow\!\!\!\!\nearrow & & \nwarrow & \\ \succeq_{\mathsf{E}} & & & & \succeq_{\mathsf{A}} \\ & \nwarrow & & \swarrow\!\!\!\!\nearrow & \\ & & \succeq_{\mathsf{S}} & & \end{array} \qquad\qquad \begin{array}{ccccc} & & \succeq_{\mathsf{L}} & & \\ & \swarrow\!\!\!\!\nearrow & & \nwarrow & \\ \succeq_{\mathsf{E}} & & & & \succeq_{\mathsf{A}} \\ & \nwarrow & & \nearrow & \\ & & \succeq_{\mathsf{S}} & & \end{array}$$

*Proof:*

For all timed nets, and hence for untimed nets, too, the positive results follow from Theorem 4.18 ($\succeq_{\mathsf{L}} \leftrightarrow \succeq_{\mathsf{E}}$), Proposition 4.19.1 ($\succeq_{\mathsf{E}} \leftarrow \succeq_{\mathsf{S}} \rightarrow \succeq_{\mathsf{A}}$) and Proposition 4.19.2 ($\succeq_{\mathsf{L}} \leftarrow \succeq_{\mathsf{A}}$), and for untimed nets additionally $\succeq_{\mathsf{S}} \leftarrow \succeq_{\mathsf{A}}$ by Proposition 4.20.2. We cannot have $\succeq_{\mathsf{L}} \rightarrow \succeq_{\mathsf{S}}$ (and hence not $\succeq_{\mathsf{L}} \rightarrow \succeq_{\mathsf{A}}$) for untimed (and hence all timed) nets in general, since this would imply that $\mathsf{LL}$-inclusion yields $\mathsf{SL}$-inclusion for untimed nets in general, a contradiction to Proposition 3.20 with Proposition 3.15.1 and .2.

For the additional negative result concerning timed nets consider:


We have $\mathsf{ART}(N_1) \subseteq \mathsf{ART}(N_2)$, but $(a^+, e_a)\{(a^-, e_a)\}\{(a^-, e_a)\} \in \mathsf{SRT}(N_1) \setminus \mathsf{SRT}(N_2)$.

■ 4.21

## 4.4 Comparing Strict May- and Must-Testing

Finally, must-testing is also characterized by $\mathsf{SRT}$-inclusion, but in reverse direction: whereas for may-testing the chance to perform successful in a test was increased with the number of refusal traces, for must-testing the number of *failable* tests increases with the number of refusal traces:

**Theorem 4.22**

Let $N_1$ and $N_2$ be timed testable nets. Then $N_1 \sqsupseteq N_2$ if and only if $\mathsf{SRT}(N_1) \subseteq \mathsf{SRT}(N_2)$.

*Proof:*

'if': Assume $\mathsf{SRT}(N_1) \subseteq \mathsf{SRT}(N_2)$ and let $(O, d)$ be a timed test. Then $\mathsf{SRT}(N_1) \subseteq \mathsf{SRT}(N_2)$ implies $\mathsf{SL}(N_1 \|_\Sigma O) \subseteq \mathsf{SL}(N_2 \|_\Sigma O)$ by Proposition 4.15 and Proposition 4.13. Thus, if $N_1$ fails the test due to some $w \in \mathsf{SL}(N_1 \|_\Sigma O)$, then so does $N_2$.

'only-if': Let $N$ be any testable timed net; then by [Vog95], for each $w \in (\Sigma^{\pm} \times E) \cup \{(1)\})^*$ there exists a discrete test $(O, d)_w$, such that $N$ *m̸ust* $(O, d)_w$ if and only if $w \in \mathsf{SRT}(N)$. Now let $N_1 \sqsupseteq N_2$; then $w \in \mathsf{SRT}(N_1)$ implies $N_1$ *m̸ust* $(O, d)_w$, hence $N_2$ *m̸ust* $(O, d)_w$ by assumption and $w \in \mathsf{SRT}(N_1)$ by the above, too. ■ 4.22

**Corollary 4.23**

Let $N_1$ and $N_2$ be timed testable nets. Then $N_1 \sqsupseteq N_2$ if and only if $N_2 \succeq_{\mathsf{S}} N_1$.

*Proof:* Follows from Theorem 4.22 and Theorem 4.16. ■ 4.23

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