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Jost-Hinrich Eschenburg, Ernst Heintze

Angaben zur Veröffentlichung / Publication details:

Eschenburg, Jost-Hinrich, and Ernst Heintze. 1999. "On the classification of polar representations." *Mathematische Zeitschrift* 232 (3): 391–98. https://doi.org/10.1007/PL00004763.

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J.-H. Eschenburg, E. Heintze

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany
(e-mail: eschenburg@math.uni-augsburg.de; heintze@math.uni-augsburg.de)

0. Introduction and statement of the result

Let V be a euclidean vector space and $K \subset O(V)$ a compact subgroup. The representation of K on V is said to be *polar*, if there exists a linear subspace Σ which meets every orbit and meets it perpendicularly at every point of intersection. Examples are the so called s-representations, i.e. the isotropy representations of symmetric spaces. Dadok ([D]) has classified almost explicitly all irreducible polar representations (cf. Theorem 9 and the subsequent remark on p. 129, together with the lists on pp. 133, 134 and 136 in [D]). As a consequence of his classification he gets that any polar representation is orbit equivalent to an s-representation, i.e. it has the same orbits as a suitable s-representation after an isometric identification of the vector spaces (cf. [D], Prop. 6). We will call this his *main result*. Dadok's proof for the classification consists in an ingeneous reduction of the set of all irreducible representations of all compact Lie groups to finitely many cases (many of which are left to the reader).

In [EH] we gave a classification free conceptual proof of the main result if the rank (i.e. the codimension of the principal orbits) is not two. In fact it turns out that for any polar representation ρ on V, the *maximal* subgroup of $O(V)$ with the same orbits as ρ is an s-representation. The purpose of the present paper is to derive the complete classification from the main result. More precisely, for any irreducible symmetric space $S = G/K$, we will determine all (connected) subgroups K' of K having the same orbits as K under the isotropy representation of S. The main idea of our proof consists in the observation that the principal K-orbits are isoparametric submanifolds which are foliated by curvature spheres on which K and also $K' \subset K$ act transitively. This is a very severe restriction on K' and implies in most cases immediately $K' = K$. But our proof also gives a geometric explanation for the occurence of those polar representation which are not s-representations. In this way, we can confirm Dadok's result and obtain the following explicit list:

Theorem *Let G/K be an irreducible symmetric space of rank at least two and with K connected. If $K' \subset K$ is a compact connected subgroup which has the same orbits as K under the isotropy representation, then $K' = K$ unless we have one of the following cases:*

(i) $G/K = SO(9)/SO(2) \times SO(7)$ *and* $K' = SO(2) \times G_2$
(ii) $G/K = SO(10)/SO(2) \times SO(8)$ *and* $K' = SO(2) \times Spin(7)$
(iii) $G/K = SO(11)/SO(3) \times SO(8)$ *and* $K' = SO(3) \times Spin(7)$
(iv) $G/K = SU(p+q)/S(U(p) \times U(q))$ *with* $p \neq q$ *and* $K' = SU(p) \times SU(q)$
(v) $G/K = SO(2n)/U(n)$, n *odd, and* $K' = SU(n)$
(vi) $G/K = E_6/SO(10) \cdot S^1$ *and* $K' = SO(10)$.

Remarks.

1. By the main result of Dadok and this theorem the only irreducible polar representations of rank at least two are the s-representations and the restrictions of s-representations to K' of the list above.
2. Reducible polar representations are recently classified by I. Bergmann in her thesis at Augsburg.
3. The rank one case of the theorem follows from the classification of compact connected subgroups K of $SO(m+1)$ which act transitively on S^m and which we state here for later use (cf. [B]):
   (i) $m = 1$: $K = SO(2)$
   (ii) $m = 2n, n \neq 3$: $K = SO(2n+1)$
   (iii) $m = 6$: $K = SO(7), G_2$
   (iv) $m = 2n-1, n$ odd, $n > 1$: $K = SO(2n), U(n), SU(n)$
   (v) $m = 4n - 1$:

$$K = SO(4n), U(2n), SU(2n), Sp(n), Sp(n) \cdot Sp(1), Sp(n) \cdot S^1,$$
$$Sp(n), Spin(9) \ \text{(if } n = 4), \ Spin(7) \ \text{(if } n = 2)$$

   with the only inclusions:
   $G_2 \subset SO(7)$
   $SU(n) \subset U(n) \subset SO(2n)$
   $Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n)$
   $Sp(n) \subset Sp(n) \cdot S^1 \subset Sp(n) \cdot Sp(1) \subset SO(4n)$
   $Sp(n) \cdot S^1 \subset U(2n)$
   $SU(4) \subset Spin(7) \subset SO(8)$
   $Spin(9) \subset SO(16)$.

1. Proof of the theorem

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Then the isotropy representation of G/K can be identified with the action of K on $\mathfrak{p}$ (by the adjoint representation). A section Σ for this polar action is provided by a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let $p_0 \in \mathfrak{a}$ be a regular element. Then the principal orbit $K.p_0$ is an isoparametric submanifold (cf. [T]). In particular the curvature distributions $E_1, \ldots, E_g$ are defined on $K.p_0$. These distributions are integrable with leaves being round spheres. Denote by S_i the leaf of E_i through p_0 and let $m_i := \dim S_i = \dim E_i, i = 1, \ldots, g$. Since the E_i are geometrically defined they are left invariant by K. Hence S_i is left invariant by an element $k \in K$ if only $k.p_0 \in S_i$. Since both K' and K act transitively on the orbit $K.p_0$ the groups $K_i := \{k \in K;\ k.S_i \subset S_i\}$ and $K_i' := K' \cap K_i$ act transitively on S_i. But in general these actions will not be effective. We decompose therefore the Lie algebra of K_i into ideals: $\mathfrak{k}_i = \mathfrak{n}_i \oplus \tilde{\mathfrak{k}}_i$ where $\mathfrak{n}_i$ is the Lie algebra of the kernel of the action and $\tilde{\mathfrak{k}}_i$ is the orthogonal complement with respect to a biinvariant inner product on $\mathfrak{k}$. Clearly we have $K_i \supset K_0$ for any i where K_0 is the isotropy group of K at p_0.

Now let $K_0' := K' \cap K_0$ and $\mathfrak{k}_0$ and $\mathfrak{k}_0'$ the Lie algebras of K_0 and K_0' respectively. Since $K/K_0 = K.p_0 = K'.p_0 = K'/K_0'$, we have $\dim K/K' = \dim K_0/K_0'$. In particular $K' = K$ if $\mathfrak{k}_0' = \mathfrak{k}_0$.

Let $(\tilde{\mathfrak{k}}_i)_0 = \tilde{\mathfrak{k}}_i \cap \mathfrak{k}_0$. Then $\mathfrak{k}_0 = \mathfrak{n}_i \oplus (\tilde{\mathfrak{k}}_i)_0$ is a direct sum of ideals (for any i) since $\mathfrak{n}_i \subset \mathfrak{k}_0 \subset \mathfrak{k}_i$. Furthermore $\mathfrak{k}_0 = \sum_{i=1}^{g} (\tilde{\mathfrak{k}}_i)_0$, since any $v \in \mathfrak{k}_0$ perpendicular to $(\tilde{\mathfrak{k}}_i)_0$ for all i lies in $\bigcap_{i=1}^{g} \mathfrak{n}_i$ and thus would give rise to a 1-parameter group acting trivially on all S_i and hence on $K.p_0$ and finally on $\mathfrak{p}$ since the action is irreducible. But this happens only if $v = 0$. To prove $K' = K$ it therefore suffices to show that $(\tilde{\mathfrak{k}}_i)_0 \subset \mathfrak{k}_0'$ for all i.

Lemma 1.1 *Let $n \in \mathbb{N}$, $n \neq 3$. If $m_i \in \{1, 2n\}$ for all i, then $K' = K$.*

Proof. By the classification of groups acting transitively on spheres, $\tilde{\mathfrak{k}}_i = \mathfrak{so}(2)$ if $m_i = 1$ and $\tilde{\mathfrak{k}}_i = \mathfrak{so}(2n+1)$ if $m_i = 2n$ respectively. In particular $\mathfrak{k}_0$ is a sum of ideals isomorphic to $\mathfrak{so}(2n)$. Let $m_i = 2n$. Since K_i' also acts transitively on S_i and there is only one compact connected group acting transitively and effectively on S_i, the projection of $\mathfrak{k}_i'$ onto the second factor $\tilde{\mathfrak{k}}_i$ of $\mathfrak{k}_i = \mathfrak{n}_i \oplus \tilde{\mathfrak{k}}_i$ is surjective. Thus $\mathfrak{k}_i'$ contains an ideal $\mathfrak{k}_i''$ which projects isomorphically onto $\tilde{\mathfrak{k}}_i \equiv \mathfrak{so}(2n+1)$. Since there are no nontrivial homomorphisms from $\mathfrak{so}(2n+1)$ into $\mathfrak{so}(2n)$, any homomorphism of $\mathfrak{k}_i''$ into $\mathfrak{n}_i \subset \mathfrak{k}_0$ must vanish. Therefore the projection of $\mathfrak{k}_i'' \subset \mathfrak{k}_i$ into the first factor $\mathfrak{n}_i$ of $\mathfrak{k}_i$ is zero, i.e. $\mathfrak{k}_i'' \subset \tilde{\mathfrak{k}}_i$ and thus $\mathfrak{k}_i'' = \tilde{\mathfrak{k}}_i$. This implies $(\tilde{\mathfrak{k}}_i)_0 \subset \mathfrak{k}_i'$ for all i, hence $\mathfrak{k}_0 = \mathfrak{k}_0'$. □

The assumption of Lemma 1.1 is satisfied in particular in the group case (all $m_i = 2$) but also for most other irreducible symmetric spaces. In [H], pp. 532-534 or [L], pp. 146-148, the multiplicities of the symmetric space are listed from which one gets the m_i by adding up the multiplicities of proportional roots. These lists show that the only exceptions are possibly the Grassmannians $SO(p+q)/SO(p) \times SO(q)$ with $2 \leq p < q$, $SU(p+q)/S(U(p) \times U(q))$ with $2 \leq p < q$, $Sp(p+q)/Sp(p) \times Sp(q)$ with $2 \leq p \leq q$, and further $SO(2n)/U(n)$ with n odd (D III) and $E_6/SO(10) \times S^1$ (E III). It is only at this point that we use the classification of symmetric spaces. The proof of the theorem is therefore completed by the following two lemmas and the consideration of the Grassmannians and the D III case for which we give direct proofs (cf. Sects. 2 and 3).

Lemma 1.2 *If G/K is hermitian symmetric, $K = K' \cdot S^1$, then K' has the same orbits under the isotropy representation as K provided that all $m_i \neq 1$.*

Proof. It is enough to show that all the K_i' act transitively on S_i since then the K'-orbit through p_0 has the same dimension as the K-orbit. Since $\dim K_i/K_i' \leq 1$, K_i' is a normal subgroup of codimension ≤ 1 in K_i and the same is true for the connected components. But it follows from the classification of groups acting transitively on the sphere S^{m_i} (or an easy homotopy argument) that such a normal subgroup acts also transitively on the sphere if $m_i > 1$. □

Lemma 1.3 *If $G/K = E_6/SO(10) \cdot S^1$ (type E III), then $K' \subset K$ has the same orbits as K if and only if $K' = K$ or $K' = SO(10)$.*

Proof. In this case the multiplicities are 6 and 9. Thus the second possibility for K' really occurs by Lemma 1.2. To prove that there are no more possibilities we have to show $\dim K_0/K_0' \leq 1$. Since $\dim K_0 = \dim K - \dim K/K_0 = \dim K - (\dim G/K - \operatorname{rank} G/K) = 46-(32-2) = 16$, this is equivalent to $\dim K_0' \geq 15$. The smallest group acting transitively on S^9 is $SU(5)$ with isotropy group $SU(4)$. Hence $\dim K_0' \geq \dim \mathfrak{su}(4) = 15$. □

2. Isotropy representation of Grassmannians

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and consider the Grassmannian $S = G_p(\mathbb{K}^{p+q})$ of all oriented p-planes in $\mathbb{K}^{p+q}$, where $2 \leq p \leq q$. Hence $S = G/K$ with

$$\begin{aligned} G &= SO(p+q), & K &= SO(p) \times SO(q) & &\text{if } \mathbb{K} = \mathbb{R}, \\ G &= SU(p+q), & K &= S(U(p) \times U(q)) & &\text{if } \mathbb{K} = \mathbb{C}, \\ G &= Sp(p+q), & K &= Sp(p) \times Sp(q) & &\text{if } \mathbb{K} = \mathbb{H}. \end{aligned}$$

S is a symmetric space of rank p. Its tangent space can be viewed as

$$\mathfrak{p} = \mathrm{Hom}(\mathbb{K}^p, \mathbb{K}^q),$$

and the isotropy representation ρ then is given by

$$\rho(A, B)X = BXA^{-1}$$

for $(A, B) \in K$ and $X \in \mathfrak{p}$. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$, the group K splits as

$$K = K_1 \times K_2.$$

If $\mathbb{K} = \mathbb{C}$, we replace K by the locally isomorphic group $SU(p) \times U(q)$ which has the same image under ρ. This has the advantage that in all three cases, K splits as $K = K_1 \times K_2$. The projection onto the factor K_i will be denoted by π_i.

Now let $K' \subset K$ be a closed connected subgroup acting with the same orbits. Applying K to any $X \in \mathfrak{p}$, we can transform the subspaces $\ker X \subset \mathbb{K}^p$ and $\mathrm{im}\ X = X(\mathbb{K}^p) \subset \mathbb{K}^q$ into arbitrary subspaces of the same dimension. Therefore, the two projections $\pi_i(K')$ of $K' \subset K$ must act transitively on the set of subspaces of any fixed dimension between 1 and p.

Lemma 2.1 *If $p = q$, then $K' = K$.*

Proof. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, this follows from Lemma 1.1 (cf. [H], p.532 for the multiplicities). If $\mathbb{K} = \mathbb{H}$, then by Lemma A1 of the appendix, $\pi_i(K') = Sp(p)$ for $i = 1, 2$. Hence, if K' is a proper subgroup of K, it must be the diagonal embedding of $Sp(p)$, but this has dimension too small to act transitively on a principal K-orbit M, since $\dim M = \dim S - \mathrm{rk}\ S = p(4p - 1) > p(2p + 1) = \dim(Sp(p))$. □

Lemma 2.2 *If $2 \leq p < q$, then $K' \subset K$ acts transitively on the K-orbits if and only if*

$$K' = K_1 \times K_2'$$

where $K_2' \subset K_2$ acts transitively on the Stiefel manifold of (real) orthonormal or (complex or quaternionic) unitary p-frames $\{e_1, ..., e_p\}$ in $\mathbb{K}^q$.

Proof. Suppose that $K' \subset K$ acts with the same orbits on $\mathfrak{p}$. Consider the subspace $\bar{\mathfrak{p}} = \{X \in \mathfrak{p};\ X(\mathbb{K}^p) \subset \mathbb{K}^p \subset \mathbb{K}^q\}$ of $\mathfrak{p}$. This space is invariant under the action of the subgroup $\bar{K} = K \cap (GL(\mathbb{K}^p) \times GL(\mathbb{K}^p))$. The group $\bar{K}' := K' \cap \bar{K}$ acts transitively on the sub-orbits $\bar{K}.X$ for any $X \in \bar{\mathfrak{p}}$. (In fact, if $X(\mathbb{K}^p) = \mathbb{K}^p$, then any $k \in K$ with $k.X \in \bar{\mathfrak{p}}$ must lie in $\bar{K}$.) Now we are back to the case $p = q$, and from Lemma 2.1 we may conclude $\bar{K}' = \bar{K}$. In particular, $K' \supset \bar{K}$ contains $K_1 \times 1$, hence $K' = K_1 \times K_2'$ for some subgroup $K_2' \subset K_2$ which acts transitively on the set of p-dimensional subspaces in $\mathbb{K}^q$ and contains K_1. Thus, K_2' acts transitively on the set of unitary p-frames in $\mathbb{K}^q$.

The converse statement is obvious: In order to transform $X \in \mathfrak{p} = \mathrm{Hom}(\mathbb{K}^p, \mathbb{K}^q)$ into BX for some $B \in K_2$, we only need to transform an orthonormal or unitary basis of the subspace $\mathrm{im}\ X$ into its image under B which is possible by some $B' \in K_2'$. $\square$

Lemma 2.3 *If $2 \leq p < q$, then precisely the following connected proper subgroups $K' \subset K$ act transitively on the K-orbits:*

$\mathbb{K} = \mathbb{R}$*:*

$$K' = SO(2) \times G_2 \qquad (p = 2,\ q = 7)$$
$$K' = SO(2) \times Spin(7)\ (p = 2,\ q = 8)$$
$$K' = SO(3) \times Spin(7)\ (p = 3,\ q = 8)$$

$\mathbb{K} = \mathbb{C}$*:* $\quad K' = SU(p) \times SU(q)$

$\mathbb{K} = \mathbb{H}$*:* $\quad$ *None.*

Proof. It follows from the lemmas 2.2, A1 and A2 (cf. appendix) that $K_2' = Sp(q)$ if $\mathbb{K} = \mathbb{H}$ and $K_2' \supset SU(q)$ if $\mathbb{K} = \mathbb{C}$. This settles the last two cases. If $\mathbb{K} = \mathbb{R}$, then inspection of the groups acting transitively on the sphere shows that no proper subgroup of $SO(q)$ acts transitively on the 4-frames in $\mathbb{R}^q$, only $Spin(7) \subset SO(8)$ acts transitively on the 3-frames, and only $G_2 \subset SO(7)$ and $Spin(7) \subset SO(8)$ act transitively on the 2-frames. In fact, G_2 acts transitively on S^6 with isotropy group $SU(3)$ which is still transitive on S^5 (but its isotropy group $SU(2)$ does not act transitively on S^4), so G_2 preserves the 2-frames (but not the 3-frames), and $Spin(7)$ preserves the 3-frames (but not the 4-frames) since it acts transitively on S^7 with isotropy group G_2. All other groups acting transitively on the sphere preserve Hopf fibrations, so they do not act transitively on the real 2-frames. By Lemma 2.2, the proof is complete. $\square$

3. The D III case

Lemma 3. *If $G/K = SO(2n)/U(n)$ with n odd, then $K' = SU(n)$ is the only proper connected subgroup with the same orbits as K.*

Proof. Since the multiplicities in this case are 4 and 5 (cf. [H], p. 533), $K' = SU(n)$ acts transitively on the K-orbits by Lemma 1.2. Now let us show that there are no other such groups K'. Let $\mathfrak{g}$ denote the Lie algebra of $G = SO(2n)$ and $\mathfrak{k}$ the Lie algebra of $K = U(n)$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with

$$\mathfrak{k} = \{X \in \mathfrak{g};\ XJ = JX\},\ \ \mathfrak{p} = \{X \in \mathfrak{g};\ XJ = -JX\},$$

where J denotes the usual complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. We are interested in the principal K-orbits on $\mathfrak{p}$.

Since each $X \in \mathfrak{p}$ is an antisymmetric real matrix, it has imaginary eigenvalues $i\alpha$ with $\alpha \in \mathbb{R}$. Let $E_\alpha \subset \mathbb{C}^{2n}$ be the corresponding eigenspace. From $JX = -XJ$ we get $JE_\alpha = E_{-\alpha}$. Thus the corresponding real subspace

$$V_\alpha = (E_\alpha + E_{-\alpha}) \cap \mathbb{R}^{2n}$$

is invariant under J, hence a complex subspace of $\mathbb{R}^{2n} = \mathbb{C}^n$. For nonzero α, the complex dimension of V_α must be at least 2 since all antilinear maps $\mathbb{C} \to \mathbb{C}$ are of the type $z \mapsto \overline{az}$ for some $a \in \mathbb{C}$, but this map is symmetric, not antisymmetric. If $\dim_{\mathbb{C}} V_\alpha = 2$, then $X|_{V_\alpha} = \alpha \cdot j$ where j is a complex structure on V_α with $Jj = -jJ$, i.e. a *quaternionic structure*. $X \in \mathfrak{p}$ lies in a principal orbit with respect to the action of K by conjugation if these invariant subspaces V_α are as small as possible, namely $\dim_{\mathbb{C}} V_\alpha = 2$ for all nonzero eigenvalues, and $\dim_{\mathbb{C}} \ker X \leq 1$. Thus, if $n = 2m + 1$ is odd, a principal orbit $M \subset \mathfrak{p}$ is characterized by m pairwise distinct positive numbers $\alpha_1, ..., \alpha_m$, and any $X \in M$ is determined by an orthogonal decomposition

$$\mathbb{C}^n = V_1 \oplus ... \oplus V_m \oplus L \tag{$*$}$$

where $V_1, ..., V_m$ are complex planes and L a complex line in $\mathbb{C}^n$, together with quaternionic structures j_p on V_p for $p = 1, ..., m$; in fact, $X = \sum_p \alpha_p j_p$. Any $k \in K$ conjugates the decomposition $(*)$ and the quaternionic structures. In particular, any one-dimensional subspace $L \subset \mathbb{C}^n$ occurs in $(*)$ as the kernel for some $X \in M$. Thus a subgroup $K' \subset K$ which acts transitively on M must also act transitively on $\mathbb{C}P^{2m}$. By Lemma A2 (see Appendix) this shows that $K' \supset SU(2m)$. $\square$

Appendix: Transitive groups on projective spaces

For the convenience of the reader we add the proofs of the following well known lemmas which were needed above.

Lemma A1 *For $n \geq 2$, let $H \subset Sp(n)$ be a connected closed subgroup acting transitively on the set of one-dimensional quaternionic subspaces of $\mathbb{H}^n$. Then $H = Sp(n)$.*

Proof. We extend H to $H \cdot Sp(1)$ where $Sp(1) = S^3 \subset \mathbb{H}$ acts by right scalar multiplication on $\mathbb{H}^n$. Then $H \cdot Sp(1)$ acts transitively on the unit sphere in $\mathbb{H}^n$. The only proper subgroups of $Sp(n) \cdot Sp(1)$ acting transitively on the sphere are $Sp(n)$ and $Sp(n) \cdot S^1$ which are both not of the type $H \cdot Sp(1)$ if $n > 1$. Hence $H \cdot Sp(1) = Sp(n) \cdot Sp(1)$ which shows $H = Sp(n)$. $\square$

Lemma A2 *For $n \geq 2$, let $H \subset U(n)$ be a connected closed subgroup acting transitively on the set of one-dimensional and, if n is even, also on the two dimensional complex subspaces of $\mathbb{C}^n$. Then $H \supset SU(n)$.*

Proof. We extend H to $H \cdot S^1$ where $S^1 = S^1 \subset \mathbb{C}$ acts by scalar multiplication on $\mathbb{C}^n$. Then $H \cdot S^1$ acts transitively on the unit sphere and, if n is even, on the 2-dimensional complex subspaces in $\mathbb{C}^n$. The only proper subgroups of $U(n)$ with this property is $SU(n)$ since $Sp(\frac{n}{2}) \cdot Sp(1)$ preserves the set of quaternionic one-dimensional subspaces which are special 2-dimensional complex subspaces. Thus $H \cdot S^1 \supset SU(n)$ which shows that $H \supset SU(n)$ since $SU(n)$ is a simple group. $\square$

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