# Polar representations and symmetric spaces 

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#### Abstract

J. Dadok has shown by classification that any polar representation has the same orbits as the isotropy representation of some symmetric space. A conceptual proof of this result is given subject to some restriction.


## 0. Introduction

An orthogonal representation $\varrho: K \rightarrow O(V)$ of a compact Lie group on a Euclidean vector space is called polar if there exists a linear subspace (also called section) which intersects every orbit and which is perpendicular to the orbits at any point of intersection. Equivalently, a representation is polar if a normal space $v_{p}$ of a principal orbit intersects all orbits perpendicularly. Important examples are the adjoint representations of compact Lie groups (with maximal abelian subalgebras as sections) and more generally, the isotropy representations of symmetric spaces. By this we mean more precisely the isotropy representation of Riemannian symmetric spaces $X=G / K$ with $G$ being the identity component of the isometry group and $K$ connected. If $G$ is semisimple, this is also called an srepresentation while, in general, it splits into the direct sum of an s-representation and a trivial representation. By classifying irreducible polar representations, J. Dadok [D] has shown the following remarkable theorem:

[^0]Here representations of two groups are called orbit equivalent if they have the same orbits after a suitable isometric identification of their representation spaces.

There is an important application of Dadok's theorem to submanifold geometry. A submanifold $M^{n} \subset V=\mathbb{R}^{n+k}$ is called isoparametric (cf. [T]) if any normal vector $\xi_{0}$ extends to a normal vector field $\xi$ which is parallel with respect to the normal connection, and the shape operator $A_{\xi}$ has constant eigenvalues for any such parallel $\xi$. Principal orbits of polar representations are isoparametric since the normal vector fields one obtains by moving around normal vectors by the group are easily seen to be parallel in the normal bundle. Conversely, Thorbergsson [Th] has shown that any (irreducible, compact, connected, full) isoparametric submanifold of codimension $\geqq 3$ arises in this way and hence,

[^1]by Dadok's theorem, is an orbit of an s-representation. Thorbergsson's result has recently been extended to isoparametric submanifolds of Hilbert spaces ([HL]), and it is conjectured that Dadok's theorem also extends naturally to the infinite dimensional setting. But for this a better understanding of the finite dimensional case and in particular a more geometrically proof of Dadok's theorem seem necessary. Dadok obtained his classification by means of representation theory, in particular by studying an enormous number of special cases.

The main purpose of the present paper is to give (subject to a certain restriction) a classification-free proof of Dadok's theorem. This is done by making heavy use of the submanifold geometry of the orbits. Our proof also yields somewhat more, namely a characterization of s-representations as polar representations with additional geometric properties as we will now explain.

A subgroup $K \subset O(V)$ is called $\omega$-maximal if it coincides with the maximal connected subgroup of $O(V)$ having the same orbits as $K$, i.e. if $K$ is the connected component of the group $\{\phi \in O(V) ; \phi(K . x)=K . x \forall x \in V\}$. It is clear that any compact connected subgroup $K \subset O(V)$ can be replaced by a unique $\omega$-maximal one with the same orbits. Polar representations of $\omega$-maximal groups are called maximal polar.

Motivated by the case of s-representations and the work of Olmos [(O2)] we introduce another condition which we will call strong polarity. Recall that $K \subset O(V)$ (with Lie algebra $\mathfrak{l}$ ) is polar precisely if the normal space $v_{p}$ of a principal orbit $M=K . p$ intersects any orbit perpendicularly, i.e. $\mathfrak{f} . v_{p} \perp v_{p}$. We would like to extend this property to singular orbits. However, if a point $q \in v_{p}$ lies on a singular orbit K. $q$, we no longer have $\mathfrak{f} . v_{q} \perp v_{q}$ since the isotropy group $K_{q}$ (unlike $K_{p}$ ) does not act trivially on $v_{q}$. The best we can expect is to have a vector space decomposition $\mathfrak{f}=\mathfrak{f}_{q} \oplus \mathfrak{m}_{q}$ with $\mathfrak{m}_{q} \cdot v_{q} \perp v_{q}$. But we also need a certain compatibility of these decompositions for all $q \in v_{p}$. Therefore we require that there is a single decomposition $\mathfrak{f}=\mathfrak{f}_{p} \oplus \sum_{\alpha} \mathfrak{m}_{\alpha}$ such that for any $q \in v_{p}$ and any $\alpha$ either $\mathfrak{m}_{\alpha} \subset \mathfrak{f}_{q}$ or $m_{\alpha} \cdot v_{q} \perp v_{q}$ (and that the $m_{\alpha}$ 's are maximal with this property). This is what we call strong polarity (cf. Ch. 2). We can now state our main result:

Main Theorem. Let $V$ be a Euclidean vector space and $K \subset O(V)$ a compact connected subgroup which acts irreducibly and with cohomogeneity at least 3 on $V$. Then the following conditions are equivalent:
(S) The action of $K$ on $V$ is an s-representation.
(MP) The action of $K$ on $V$ is maximal polar.

## (SP) K acts strongly polar on $V$.

In the cohomogeneity 1 case, i.e. if $K$ acts transitively on the spheres centered at 0 , one only has the implications (MP) $\Rightarrow(\mathrm{S}) \Rightarrow(\mathrm{SP})$, but the reverse directions do not hold as the actions of $U(n)$ and $S U(n)$ on $V=\mathbb{C}^{n}$ show. In the cohomogeneity 2 case, the equivalence of $(\mathrm{S})$ and (MP) is still true ([D], cf. also [HsL]), but our proof does not work.

Dadok's theorem (for irreducible $K$ of cohomogeneity $\neq 2$ ) is of course contained in the implication (MP) $\Rightarrow(\mathrm{S})$. To obtain from this the full classification of all $K \subset O(V)$ whose action is irreducible and polar, one has for all s-representations $\tilde{K} \subset O(V)$ to classify all subgroups $K$ of $\tilde{K}$ having the same orbits. This is carried out in a subsequent paper ( $[\mathrm{EH}]$ ) which is also based on geometric ideas. Another conceptual proof of Dadok's theorem was given by $K$. Tetzlaff in his thesis [Tz], assuming that the codimension is at least 4.

The proof of the Main Theorem can be outlined as follows. It is rather easy to see that $(\mathrm{S})$ implies the other conditions. The step $(\mathrm{MP}) \Rightarrow(\mathrm{SP})$ is essentially contained in [O2] and [HL]. Hence the main work is to show $(\mathrm{SP}) \Rightarrow(\mathrm{S})$. To determine whether the action of a compact subgroup $K \subset O(V)$ is an s-representation we will construct (following Cartan and Kostant) a Lie bracket on the vector space $\mathfrak{g}:=\ddagger \oplus V$ extending the Lie structure on $\mathfrak{l}$ such that $(\mathfrak{g}, \mathfrak{f})$ becomes a symmetric pair. This is done by defining $[v, w] \in \mathfrak{l}$ with $\langle X,[v, w]\rangle_{\mathrm{t}}=\langle X . v, w\rangle$ for any $v, w \in V$ and $X \in f$. We have to use a scalar product $\langle,\rangle_{\mathrm{t}}$ on $\mathfrak{f}$ which imitates the restriction to $\mathfrak{f}$ of the Killing form metric on $\mathfrak{g}$ in the symmetric case and which we will call the ( $\mathfrak{f}, V$ )-trace metric (cf. Ch. 3). It is well known that the action of $K$ on $V$ is an s-representation if and only if this "Lie bracket" satisfies the Jacobi identity $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0$ for all $x, y, z \in V$ (Condition (J) in Ch. 4). But it might be surprising that the Jacobi identity is already equivalent to its special case where $[x, y]=0$ (Condition $\left(\mathrm{J}_{1}\right)$ ) or even $[x, y]=[y, z]=0$ (Condition $\left(\mathrm{J}_{2}\right)$ ). This turns out (Theorem 4.3) to be equivalent to (SP) with the extra condition that the subspaces $\mathrm{m}_{\alpha}$ above (including $\mathrm{m}_{0}:=\mathfrak{f}_{p}$ ) are orthogonal to each other (" $\left(\mathrm{SP}^{+}\right)$"). Therefore the essential step is to conclude ( $\mathrm{SP}^{+}$) from (SP) which is carried out in Theorem 5.2. The principal idea is that, by (SP), the slice representation of any orbit $K . q$ (the representation of $K_{q}$ on $v_{q}$ ) in fact coincides with the holonomy representation of the normal bundle of $K . q$ and is therefore an s-representation, due to a general result of Olmos [O1]. Since ( $\mathrm{SP}^{+}$) holds for s-representations, we in this way obtain the orthogonality of certain $m_{\alpha}$ 's with respect to the $\left(\mathfrak{f}_{q}, v_{q}\right)$-trace metric, which turns out to be sufficient information if the codimension is at least 3 .

The way the various conditions are related in the proof is shown by the following diagram (where " $\Rightarrow_{\omega}$ " or " $\Rightarrow_{3}$ " indicate that the conclusion is valid only in the case that $K$ is $\omega$-maximal or its cohomogeneity is $\geqq 3$, respectively, while all other conclusions hold if the cohomogeneity is $\geqq 2$ ).

$$
\left(\mathrm{J}_{2}\right) \Leftrightarrow_{\omega}(\mathrm{J}) \Leftrightarrow(\mathrm{S}) \Rightarrow(\mathrm{MP}) \Rightarrow_{3}(\mathrm{SP}) \Leftrightarrow_{3}\left(\mathrm{SP}^{+}\right) \Leftrightarrow\left(\mathrm{J}_{2}\right)
$$

A condition related to $\left(\mathrm{SP}^{+}\right)$(which will be denoted by ( Co )) was already considered by Conlon [C] many years ago; we will show in Theorem 4.3 that it is also equivalent to $\left(\mathrm{J}_{2}\right)$ and hence to $(\mathrm{S})$. This disproves his conjecture ([C], p. 151) saying that (Co) is true for all polar representations up to a finite number of exceptional cases. Note that there is even an infinite number of irreducible polar representations which are not s-representations ([D], cf. [EH]).

Finally, we would like to comment on the proof of $\left(\mathrm{J}_{2}\right) \Rightarrow_{\omega}(\mathrm{J})$ and a more algebraic approach to the Main Theorem which might be possible. As indicated above, after choosing
an invariant inner product on $\mathfrak{l}$ and a corresponding "Lie bracket" on $\mathfrak{f} \oplus V$, the (S)condition is equivalent to the vanishing of the Jacobi map

$$
J(x, y, z)=[x,[y, z]]+[y,[z, x]]+[z,[x, y]]
$$

where $x, y, z \in V$. Due to its symmetries, $J$ may be considered also as a symmetric linear map $J: \Lambda^{2} V \rightarrow \Lambda^{2} V$ which satisfies in addition $\langle J(x \wedge y), x \wedge z\rangle=0$ for all $x, y$, $z$ (i.e. $J \in \Lambda^{4} V$ ) and hence has trace zero. Since $\mathfrak{f} \subset \mathfrak{o}(V)=\Lambda^{2} V$ (natural identification), we have on orthogonal splitting $\Lambda^{2} V=\mathfrak{f}+\mathfrak{f}^{\perp}$ (with respect to the standard inner product on $\Lambda^{2} V$ ). The choice of the right inner product on $f$ yields precisely $\left.J\right|_{\mathrm{t}}=\lambda \cdot \mathrm{id}_{\mathrm{t}}$ for some $\lambda \in \mathbb{R}([\mathrm{HZ}]$, cf. Ch. 4). Hence $\left.J\right|_{\mathfrak{t}^{\perp}}=0$ already implies $J=0$ and thus (S). Now the $\omega$-maximality of $K$ is equivalent to the fact that $\mathfrak{f}^{\perp}$ is generated by decomposable elements $x \wedge y$. Since $x \wedge y \in \mathfrak{f}^{\perp}$ is equivalent to $[x, y]=0$, the condition $\left.J\right|_{\mathfrak{t}}{ }^{\perp}=0$ is equivalent to our condition $\left(\mathrm{J}_{1}\right)$ (and hence to $\left(\mathrm{J}_{2}\right)$ ), provided that $K$ is $\omega$-maximal. Moreover, the polarity too can be expressed easily in this language as $v_{p} \wedge v_{p} \subset \mathfrak{f}^{\perp}$ for some (equivalently for any) regular $p \in V$. Hence it seems conceivable that "(MP) $\Rightarrow(\mathrm{S})$ " can be proved by purely algebraic considerations. In this context it might be interesting to note that $\lambda=0$ (i.e. $\left.J\right|_{\mathrm{t}}=0$ ) also implies $J=0$ and hence (S). This follows essentially from a calculation made by E. Witt ([W], cf. [Db]) who constructed certain exceptional Lie algebras $\mathfrak{g}$ of the form $\mathfrak{g}=\mathfrak{f} \oplus V$ for suitable representations $\varrho: K \rightarrow O(V)$ by extending the Lie bracket from $\mathfrak{f}$ to g as above and who showed $J=0$ by computing trace $\left(J^{2}\right)$. We are grateful to P . Slodowy for bringing these papers to our attention.

## 1. Review of polar representations and isoparametric submanifolds

Let $V$ be a Euclidean vector space and $M \subset V$ a compact, connected isoparametric submanifold. Thus the normal bundle $\nu M$ of $M$ is flat and the eigenvalues of the shape operator $A_{\xi}$ are constant for any parallel normal field $\xi$. From the compactness it follows that $M$ is contained in a sphere which we may assume to have center in the origin (see [T] and [PT] for most details in this section).

By the Ricci equation, all $A_{\xi}$ with $\xi$ in a fixed normal space $v_{p}=v_{p} M$, commute. Hence we have an orthogonal simultaneous eigenspace decomposition

$$
T_{p} M=\sum_{\alpha \in \Pi} E_{\alpha}(p)
$$

where $\Pi$ is a finite subset of $\left(v_{p}\right)^{*}$ such that

$$
\left.A_{\xi}\right|_{E_{a}(p)}=\alpha(\xi) I_{E_{a}(p)}
$$

The linear forms $\alpha \in \Pi \subset\left(v_{p}\right)^{*}$ are called the principal curvatures at $p$ and the dual vectors $n_{\alpha} \in v_{p}$ are the principal curvature vectors or curvature normals; any two of them are linearly independent. Since the eigenvalues of $A_{\xi}$ are constant for any parallel $\xi$, each $n_{\alpha}$ can be considered as a parallel normal vector field $x \mapsto n_{\alpha}(x) \in v_{x} M$. The dimension of Span $\Pi$ is called the rank of the isoparametric submanifold. It coincides with the codimension if $M$ is full, i.e. not contained in any proper affine subspace. The subspaces $E_{\alpha}(p) \subset T_{p} M$ for
$p \in M$ form the so called curvature distributions $E_{\alpha}$. They are integrable, and the integral manifold $S_{\alpha}(p)$ of $E_{\alpha}$ through $p \in M$ is a called a curvature sphere; it is a round sphere of radius $1 /\left|n_{\alpha}\right|$ with center $p+n_{\alpha} /\left|n_{\alpha}\right|^{2}$.

Isoparametric submanifolds come in families. For any parallel normal field $\xi$ on $M$, the set

$$
M_{\xi}=\{p+\xi(p) ; p \in M\}
$$

is again a smooth submanifold (called parallel manifold) and the "endpoint mapping" $\pi_{\xi}: M \rightarrow M_{\xi}, p \mapsto p+\xi(p)$, is a fibration with totally geodesic fibres. If $p \in M$ and $q=p+\xi(p) \in M_{\xi}$ then

$$
\tau_{q}:=T_{q} M_{\xi}=\sum_{\alpha(\xi) \neq 1} E_{\alpha}(p), \quad v_{q}:=v_{q} M_{\xi}=v_{p}+\sum_{\beta(\xi)=1} E_{\beta}(p) .
$$

If $\alpha(\xi) \neq 1$ for all $\alpha \in \Pi$ then $M_{\xi}$ is diffeomorphic to $M$ and again an isoparametric submanifold. But if $\alpha(\xi)=1$ for some $\alpha$, then $M_{\xi}$ drops dimension. In this case one calls $M_{\xi}$ a focal manifold and one says that the $E_{\beta}$ with $\beta(\xi)=1$ focalize onto $M_{\xi}$. In particular we have $M_{-p}=\{0\}$ (where $p$ denotes the position vector field, considered here as parallel normal field) and hence $\alpha(p)=-1$ for all $\alpha$. The linear hyperplanes

$$
l_{\alpha}(p):=\left\{p+\xi ; \xi \in v_{p}, \alpha(\xi)=1\right\}=n_{\alpha}^{\perp} \subset v_{p}
$$

are therefore called focal hyperplanes. For any affine subspace $a \subset v_{p}$ we may focalize precisely the $E_{\alpha}$ with $n_{\alpha} \in a$ by choosing $\xi$ appropriately, namely such that

$$
p+\xi(p) \in \operatorname{Span}(\Pi \cap a)^{\perp} \backslash \bigcup_{\alpha \nsubseteq a} l_{\alpha}(p)
$$

In particular if we choose $a=\left\{n_{\alpha}\right\}$, we can just focalize $E_{\alpha}$ and in this case, the fibres of the end point map $\pi_{\xi}: M \rightarrow M_{\xi}$ are the curvature spheres $S_{\alpha}(q)$ with $q \in M$. In general the fibre of $\pi_{\xi}: M \rightarrow M_{\xi}$ through $q$ lies in the affine subspace $q+v_{q}$ (which actually coincides with $v_{q}$ ) and is an isoparametric submanifold therein. The curvature normals and distributions of the fibre are those $n_{\alpha}$ and $E_{\alpha}$ with $\alpha(\xi)=1$ or equivalently with $\alpha(x)=0$. If $M$ is irreducible (i.e. if the embedding of $M$ into $V$ does not split) and $M_{\xi} \neq\{0\}$, then it has been shown in [HOTh] that the fibres are extrinsically homogeneous. In fact the fibre over $x=p+\xi(p)$ is the orbit through $p$ of the normal holonomy group $\operatorname{Hol}_{x}^{\perp}$ of $M_{\xi}$ at $x$, which acts isometrically on $v_{x}$ by parallel translating normal vectors along closed curves in $M_{\xi}$. Furthermore the action of $\operatorname{Hol}_{x}^{\perp}$ on $v_{x}$ is equivalent to the direct sum of a trivial representation and an s-representation as follows from a general result of Olmos ([O1]) on the normal holonomy representation of submanifolds in Euclidean space. Hence any fibre different from $M$ is the orbit of an s-representation.

Examples of isoparametric submanifolds arise as principal orbits of polar representations. Let $K$ be a compact Lie group, f its Lie algebra, $\varrho: K \rightarrow O(V)$ a representation, and $M=K . p \subset V$ a principal orbit. (We will write shortly $k . p:=\varrho(k) p$ and $X . p:=\varrho_{*}(X) p$ for $k \in K$ and $X \in \mathcal{F}$.) The representation $\varrho$ is polar if the normal space $\nu_{p}=v_{p} M$ is a section, i.e. $v_{p} \subset v_{x}:=v_{x}(K . x)$ for any $x \in v_{p}$. Then a $K$-equivariant normal field $\xi: M \rightarrow V$ is parallel
with respect to the normal connection on $\nu M$. In fact, since $\xi(M)=K . \xi(p)$ is an orbit which intersects $v_{p}$ perpendicularly, the derivatives of $\xi$ have no normal components. Thus the shape operators $A_{\xi}$ have constant eigenvalues for any parallel normal field $\xi$ and $M$ is isoparametric (recall that the equivariant normal fields trivialize the normal bundle of a principal orbit). Furthermore, the parallel manifolds $M_{\xi}$ are precisely the other orbits; the principal orbits are isoparametric while the singular orbits are the focal manifolds. Moreover the isotropy group $K_{q}$ of any $q \in p+v_{p}=v_{p}$ acts transitively on the fibre over $q$ of the end point map $\pi_{\xi}: M \rightarrow M_{\xi}$ where $\xi$ is the parallel normal field with $p+\xi(p)=q$. The rank of a principal orbit, considered as an isoparametric submanifold, is also called the rank of the polar representation. It coincides with the cohomogeneity of the action (codimension of the principal orbits) if the representation does not contain trivial subrepresentations.

It has been shown by Thorbergsson [Th] that principal orbits of polar representations are in fact the only full and irreducible isoparametric submanifolds of codimension $\geqq 3$. More precisely, if $M \subset V$ is such an isoparametric submanifold, then the action of the group $K=\{\phi \in O(V) ; \phi(M)=M\}$ of extrinsic isometries is transitive on $M$ and polar on $V$. A completely different proof has been given by Olmos [O2] and an even simpler one follows from [HL] by specializing to the finite dimensional case. In addition these authors show a property of $K$ which will be important for our approach and which we will call Olmos condition: For any $x \in E_{\alpha}$ there exists a Killing field $X \in \mathfrak{f}$ with $X . p=x$ and $X . E_{\beta}(p) \perp E_{\beta}(p)$ for any $\beta \neq \alpha$. In fact, the one-parameter group of isometries corresponding to $X$ is constructed in [HL], Lemma 4.5. It may be interpreted as a transvection group of the connection introduced by Olmos [O2].

## 2. Olmos condition and strong polarity

Let $K \subset O(V)$ be a compact connected subgroup whose action on $V$ is polar. For any $x \in V$ let $K_{x}$ be the isotropy group with Lie algebra $\mathfrak{f}_{x}$. We denote the tangent and normal spaces of the orbit $K . x$ at $x$ by $\tau_{x}$ and $v_{x}$, respectively. Now fix a regular point $p \in V$ and let $M=K . p$. Recall that $\tau_{p}=\sum_{\alpha} E_{\alpha}(p)$. Hence for any vector space decomposition $\mathfrak{f}=\mathfrak{f}_{p} \oplus \mathfrak{m}$ there is a unique decomposition $\mathfrak{m}=\sum_{\alpha \in \Pi} \mathfrak{m}_{\alpha}$ such that $\mathfrak{m}_{\alpha} \cdot p=E_{\alpha}(p)$. For such
decompositions we have

Proposition 2.1. For any $q \in v_{p}$ we have
(i) $\mathfrak{f}_{q}=\mathfrak{f}_{p}+\sum_{\alpha(q)=0} \mathfrak{m}_{\alpha}$,
(ii) $v_{q}=v_{p}+\sum_{\alpha(q)=0} E_{\alpha}(p)$,
(iii) $\mathrm{m}_{\alpha} \cdot v_{p}=E_{\alpha}(p)$,
(iv) $\mathfrak{m}_{\alpha} \cdot E_{\beta}(p) \subset v_{p}+\sum_{\gamma \in(\alpha \beta)} E_{\gamma}(p)$,
(v) $\left[\mathfrak{m}_{\alpha}, \mathfrak{m}_{\beta}\right] \subset \mathfrak{f}_{p}+\sum_{\gamma \in(\alpha \beta)} \mathfrak{m}_{\gamma}$,
where $(\alpha \beta)$ denotes intersection of $\Pi$ with the affine span of $\alpha$ and $\beta$ (which are not necessary distinct).

Proof. For any $X \in \mathfrak{m}_{\alpha}$ and $\xi \in v_{p}$ we have $X \cdot \xi=-\alpha(\xi) X . p$. In fact, since $k . p \mapsto k . \xi$ is a parallel normal field on $M=K . p$, we obtain

$$
X \cdot \xi=\left.\frac{d}{d t}(\exp t X) \cdot \xi\right|_{t=0}=-A_{\xi}(X \cdot p)=-\alpha(\xi) X \cdot p
$$

This shows (iii), and it implies also that for any $X \in \mathfrak{m}_{\alpha}$ and $q \in v_{p}$ we have $X . q=0$ (i.e. $X \in \mathfrak{f}_{q}$ ) if and only if $\alpha(q)=0$. Moreover, $\mathfrak{f}_{p} \cdot q=0$ because $K_{p}$ acts trivially on $v_{p}$ since $K . p$ is a principal orbit. Thus we get (i). Assertion (ii) is true for any isoparametric submanifold, cf. Ch. 1 (recall $\alpha(q)=0$ for $q=p+\xi$ iff $\alpha(\xi)=1$ ). By choosing $q=p+\xi \in v_{p}$ appropriately we can focalize precisely the $E_{\gamma}$ with $\gamma \in(\alpha \beta)$ (cf. Ch. 1). Hence (iv) and (v) follow from the remark that $f_{q}$ is a subalgebra which moreover leaves $v_{q}=v_{p}+\sum_{\gamma \in(\alpha \beta)} E_{\gamma}(p)$ invariant.

If the Olmos condition holds (which is true if in addition $K$ is $\omega$-maximal, cf. Ch. 1), we may choose a particular complement $\mathrm{m}^{\circ}$ as follows (we will call it Olmos complement). Let

$$
\mathfrak{m}_{\alpha}^{o}:=\left\{X \in \mathfrak{f} ; X \cdot p \in E_{\alpha}, X . E_{\beta}(p) \perp E_{\beta}(p) \forall \beta \neq \alpha\right\}
$$

and put $\mathfrak{m}^{o}=\sum_{\alpha} \mathfrak{m}_{\alpha}^{o}$.

Proposition 2.2. Suppose that the action of $K \subset O(V)$ is irreducible and polar with rank $\geqq 2$ and the Olmos condition is satisfied. Then $\mathfrak{m}^{o}=\sum_{\alpha} \mathfrak{m}_{\alpha}^{o} \subset \mathfrak{l}$ is a complement of $\mathfrak{1}_{p}$ and the sum is direct. Furthermore, each $\mathfrak{m}_{\alpha}^{o}$ is invariant under $\operatorname{Ad}\left(K_{p}\right)$.

Proof. Since all $E_{\alpha}$ and $E_{\beta}$ are invariant under $K_{p}$, the last statement is clear by definition of $\mathfrak{m}_{\alpha}^{o}$. The other statements follow by dimension reasons if we can show that the linear map $\phi: \mathfrak{m}_{\alpha}^{o} \rightarrow E_{\alpha}, \phi(X)=X . p$ is a vector space isomorphism. In fact, it is onto by the Olmos property. Moreover, if $X_{0} \in \operatorname{ker} \phi=\mathfrak{m}_{\alpha}^{o} \cap \mathfrak{f}_{p}$ then $X_{0} \cdot E_{\beta}(p)=0$ for any $\beta \neq \alpha$ since $\mathrm{m}_{\alpha}^{o} \cdot E_{\beta}(p) \perp E_{\beta}(p)$ and $f_{p} \cdot E_{\beta}(p) \subset E_{\beta}(p)$. Let $q \in l_{\alpha}(p) \backslash\{0\}$. Then the orbit $K . q$ which is a focal manifold of $K . p$ has tangent space at $q$ contained in $\sum_{\alpha \neq \beta} E_{\beta}(p)$. Thus the 1-parameter group $\exp t X_{0}$ fixes $q$ and $T_{q}(K . q)$, and so it acts trivially on $K . q$ and its linear span. But since $K$ acts irreducibly, $K . q$ spans $V$, hence $X_{0}=0$.

For the Olmos complement we can improve Proposition 2.1 as follows:
Proposition 2.1'. Let $K \subset O(V)$ be a compact polar subgroup satisfying the Olmos condition, and let $\mathrm{m}^{o}=\sum_{\alpha} \mathfrak{m}_{\alpha}^{o}$ be the Olmos complement. Then we have for $\alpha \neq \beta$
$\left(\mathrm{iv}^{\prime}\right) \mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}(p) \subset \sum_{\gamma \in(\alpha \beta)^{*}} E_{\gamma}(p)$,
( $\left.\mathrm{v}^{\prime}\right)\left[\mathfrak{m}_{\alpha}^{o}, \mathfrak{m}_{\beta}^{o}\right] \subset \mathfrak{f}_{p}+\sum_{\gamma \in(\alpha \beta)^{*}} \mathfrak{m}_{\gamma}^{o}$,
where $(\alpha \beta)^{*}=(\alpha \beta) \backslash\{\alpha, \beta\}$.
Proof. If $\alpha \neq \beta$, then $\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}(p) \perp E_{\beta}(p)$ by definition of the Olmos complement, and $\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}(p) \perp E_{\alpha}(p)$ because $\left\langle\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}(p), E_{\alpha}(p)\right\rangle=\left\langle E_{\beta}(p), \mathfrak{m}_{\alpha}^{o} \cdot E_{\alpha}(p)\right\rangle$ and $\mathfrak{m}_{\alpha}^{o} \cdot E_{\alpha}(p) \subset v_{p}+E_{\alpha}(p)$ by 2.1 (iv). Thus $\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}(p)$ and $\left[\mathfrak{m}_{\alpha}^{o}, \mathfrak{m}_{\beta}^{o}\right] . p=\mathfrak{m}_{\alpha}^{o} \mathfrak{m}_{\beta}^{0} . p-\mathfrak{m}_{\beta}^{o} \mathfrak{m}_{\alpha}^{o} \cdot p$ are both perpendicular to $E_{\alpha}(p)$ and $E_{\beta}(p)$ and hence $(\alpha \beta)$ may be replaced by $(\alpha \beta)^{*}$ in 2.1 (iv), (v). Further, by 2.1 (iii) we have $\left\langle\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}, v_{p}\right\rangle=\left\langle E_{\beta}, E_{\alpha}\right\rangle=0$, hence $\mathfrak{m}_{\alpha}^{o} . E_{\beta}$ has no $v_{p}$-component.

Recall that $K \subset O(V)$ is strongly polar if there is a decomposition $\mathfrak{f}=\mathfrak{f}_{p}+\sum_{\alpha} \mathfrak{m}_{\alpha}^{s}$ such that for any $q \in v_{p}$ and any index $\alpha$ either $\mathfrak{m}_{\alpha}^{s} \subset \mathcal{f}_{q}$ or $\mathfrak{m}_{\alpha}^{s} \cdot v_{q} \perp v_{q}$, and the $\mathfrak{m}_{\alpha}^{s}$ are maximal with this property. A priori, the index set is arbitrary. But if we put $\mathfrak{m}_{\alpha}:=\sum_{\mu} \mathfrak{m}_{\mu}^{s}$ where the sum has to be taken over all $\mu$ with $\mathfrak{m}_{\mu}^{s} . p \subset E_{\alpha}(p)$, then we obtain from (SP) that $\mathfrak{f}_{q}=\mathfrak{f}_{p}+\mathfrak{m}_{\alpha}$ for all $q \in v_{p}$ with $v_{q}=v_{p}+E_{\alpha}$, and hence $\mathfrak{m}_{\alpha} \cdot p=E_{\alpha}(p)$. Thus $\mathfrak{f}=\mathfrak{f}_{p}+\sum_{\alpha \in \Pi} \mathfrak{m}_{\alpha}$ is a direct vector space decomposition which also satisfies (SP) and thus coincides with the given one, due to the maximality assumption. In other words, if a decomposition $\mathfrak{f}=\mathfrak{f}_{p}+\sum_{\alpha} \mathrm{m}_{\alpha}^{s}$ satisfies (SP), then the index set necessarily coincides with $\Pi$ and $m_{\alpha}^{s} \cdot p=E_{\alpha}(p)$ (after a possible reordering of the indices).

Proposition 2.3. For an irreducible polar representation of rank $\geqq 2$ the Olmos condition and the strong polarity $(\mathrm{SP})$ are equivalent and $\mathfrak{m}^{o}=\mathfrak{m}^{s}$.

Proof. By the above discussion, the Olmos condition is the special case of the strong polarity condition for those $q$ with $v_{q}=v_{p}+E_{\alpha}$ for some single $\alpha$. Vice versa, if the Olmos condition holds, let $\mathfrak{m}^{o}=\sum_{\alpha} \mathfrak{m}_{\alpha}^{o} \subset \mathfrak{f}$ be the Olmos complement. Let $q \in v_{p}$ and $\alpha \in \Pi$. If $\alpha(q)=0$, then $\mathfrak{m}_{\alpha}^{o} \subset \mathfrak{f}_{q}$ by 2.1 (i). Otherwise, if $\alpha(q) \neq 0$, we have by 2.1 (ii), (iii)

$$
\mathfrak{m}_{\alpha}^{o} \cdot v_{q} \subset E_{\alpha}+\sum_{\beta(q)=0} \mathfrak{m}_{\alpha}^{o} \cdot E_{\beta} .
$$

This is perpendicular to $v_{q}$ by 2.1 (ii) and $2.1^{\prime}\left(\mathrm{iv}^{\prime}\right)$; note that $\gamma(q) \neq 0$ for all $\gamma \in(\alpha \beta)^{*}$. Thus $\mathrm{m}^{o}$ satisfies ( SP ).

Proposition 2.4. Let $K \subset O(V)$ be a compact polar subgroup satisfying (SP) for some principal orbit $M=K . p$. Let $q \in v_{p}$. Then $K_{q}$ leaves $v_{q}$ invariant, and the image of $K_{q}$ in $O\left(v_{q}\right)$ contains the holonomy group of the normal bundle of $N=K . q \subset V$.

Proof. The rank one case is trivial, so we may assume rank $\geqq 2$. By 2.1 we have a reductive decomposition $\mathfrak{f}=\mathfrak{f}_{q}+\mathfrak{m}_{q}$ where $\mathfrak{m}_{q}=\sum_{\alpha(q) \neq 0} \mathfrak{m}_{\alpha}^{o}$. This subspace $\mathfrak{m}_{q} \subset \mathfrak{f}$ extends to a left invariant distribution on $K$ which is "horizontal", i.e. transversal to the fibres of the projection $K \rightarrow K . q$ (in other words, this distribution is a canonical connection on the principal bundle $K \rightarrow K . q)$. Any curve $q(t)$ in $N=K . q$ has a horizontal lift $k(t)$ in $K$, i.e. $k(t) \cdot q=q(t)$ and $k^{\prime}(t) \in k(t) \cdot \mathrm{m}_{q}$. Moreover, for any $\xi \in v_{q}$, the normal field $\xi(t)=k(t) \cdot \xi$ is parallel along the curve $q(t)$ since

$$
\xi^{\prime}(t)=k^{\prime}(t) \cdot \xi \in k(t) \cdot \mathfrak{m}_{q} \cdot v_{q} \subset k(t) \cdot\left(v_{q}\right)^{\perp}=\left(v_{q(t)}\right)^{\perp}
$$

Thus $k(t)$ is the normal parallel transport along $q(t)$. If the curve $q(t), t \in[0,1]$ starts and ends at $q$, then $k(1) \in K_{q}$. Hence the normal holonomy group at $q$ is contained in $K_{q}$.

## 3. Trace metrics

Let $\varrho: K \rightarrow O(V)$ be a (not necessary faithful) orthogonal representation of a compact Lie group $K$ with Lie algebra f . We will use the following $\operatorname{Ad}(K)$-invariant inner product on $\mathfrak{f}$ which we will call ( $\mathfrak{f}, V$ )-trace metric or simply trace metric: For any $X, Y \in \mathfrak{f}$ we put

$$
\langle X, Y\rangle_{\mathrm{t}}=-\operatorname{trace} \operatorname{ad}(X) \operatorname{ad}(Y)-\operatorname{trace} \varrho_{*}(X) \varrho_{*}(Y)=-\operatorname{trace} \hat{\varrho}_{*}(X) \hat{\varrho}_{*}(Y)
$$

where $\varrho=\mathrm{Ad} \oplus \varrho: K \rightarrow \mathrm{Gl}(V \oplus \mathrm{f})$. If $\varrho$ is an s-representation, i.e. if the Lie structure on $\mathfrak{f}$ extends to a Lie algebra structure with Cartan decomposition on $\mathrm{g}:=\mathfrak{f} \oplus V$ with corresponding Lie group $G$, and $\varrho$ is the $V$-part of the adjoint representation of $K$, then $\varrho_{*}$ is the restriction to $K$ of the adjoint representation of $G$ and $-\langle,\rangle_{\mathrm{t}}$ is the Killing form of g restricted to f .

Strictly speaking, $\langle,\rangle_{\mathrm{t}}$ is not always a positive definite inner product. In fact, it has a kernel which may be nonzero, namely ker $\varrho_{*} \cap \mathfrak{\jmath}$ where $\mathfrak{z}$ is the center of $\mathfrak{f}$.

We are now going to investigate the relation between various trace metrics.
Lemma 3.1. Let $\phi: K \rightarrow \bar{K}$ be a surjective homomorphism between compact Lie groups, $\bar{\varrho}: K \rightarrow O(V)$ a faithful representation and $\varrho=\bar{\varrho} \circ \phi$. For some $p \in V$ let $\mathfrak{n} \subset \mathfrak{f}$ be an $\operatorname{Ad}\left(K_{p}\right)-$ invariant linear subspace with $\mathfrak{n} \cap \mathfrak{f}_{p}=0$. Then for any $X \in \mathfrak{n}$ and $Y \in \mathfrak{f}$ we have

$$
\left\langle\phi * X, \phi_{*} Y\right\rangle_{\mathbf{t}}=\langle X, Y\rangle_{\mathbf{r}} .
$$

Proof. Let $f^{\prime} \subset\left(\operatorname{ker} \varrho_{*}\right)^{\perp}$ be a complementary ideal to $\operatorname{ker} \varrho_{*}=\operatorname{ker} \phi *$ in $f$. Then $\phi_{*} \|_{\mathrm{t}^{\prime}}$ is an isomorphism and an isometry, and moreover one sees easily that

$$
\left\langle\phi_{*} X^{\prime}, \phi * Y\right\rangle_{\overline{\mathrm{t}}}=\left\langle X^{\prime}, Y\right\rangle_{\mathrm{I}}
$$

for any $X^{\prime} \in \mathfrak{f}^{\prime}$ and $Y \in \mathfrak{f}$. Let $\mathfrak{n}^{\prime}=\pi^{\prime}(\mathfrak{n})$ where $\pi^{\prime}: \mathfrak{f} \rightarrow \mathfrak{i}^{\prime}$ is the projection along ker $\varrho_{\approx .}$. Since $\operatorname{ker} \varrho_{*} \subset \mathfrak{f}_{p}$, we have $\left[\operatorname{ker} \varrho_{*}, \mathfrak{n}\right] \subset \operatorname{ker} \varrho_{*} \cap \mathfrak{n}=0$. Thus ker $\varrho_{*}$ commutes with $\mathfrak{n}$ and $\mathfrak{n}^{\prime}$, hence $n \subset \mathfrak{n}^{\prime} \oplus 3_{0}$ where $3_{0}$ is the center of ker $\varrho_{*}$. But recall that $3_{0}=\operatorname{ker} \varrho_{*} \cap \mathfrak{z}$ is the kernel of the trace metric on $\mathfrak{f}$. Thus for any $X \in \mathfrak{n}$ and $Y \in \mathfrak{f}$ we have

$$
\langle X, Y\rangle_{\mathrm{t}}=\left\langle\pi^{\prime}(X), Y\right\rangle_{\mathrm{I}}=\left\langle\phi * \pi^{\prime}(X), \phi * Y\right\rangle_{\mathrm{I}}=\langle\phi * X, \phi * Y\rangle_{\bar{I}}
$$

which finishes the proof.
Lemma 3.2. Let $K \subset O(V)$ be strongly polar, $M=K . p$ a principal orbit and $q \in v_{p}$. Then the components $\mathfrak{m}_{\alpha}^{o}$ of $\mathfrak{f}$ which are contained in $\mathfrak{f}_{q}\left(\right.$ including $\left.\mathfrak{m}_{0}^{o}:=\mathfrak{f}_{p}\right)$ are mutually perpendicular with respect to the $(\mathfrak{f}, V)$-trace metric on $\mathfrak{f}$ if and only if they are perpendicular with respect to the $\left(\mathfrak{f}_{q}, v_{q}\right)$-trace metric on $\mathfrak{f}_{q}$.

Proof. Let $\alpha \neq \beta \in \Pi \cup\{0\}$ such that $\mathfrak{m}_{\alpha}^{o}+\mathfrak{m}_{\beta}^{o} \subset \mathfrak{f}_{q}$, which means $\alpha(q)=\beta(q)=0$. Choose $X \in \mathfrak{m}_{\alpha}^{o}$ and $Y \in \mathfrak{m}_{\beta}^{o}$. It suffices to show that the traces of $\operatorname{ad}(X) \circ \operatorname{ad}(Y)$ and $X \circ Y$ do not change when the mappings are restricted to ${ }_{f}{ }_{q}$ and $v_{q}$, respectively; in other words, there are no contributions to the traces coming from $\mathfrak{m}_{\gamma}^{o}$ and $E_{\gamma}$, respectively, for any $\gamma \in \Pi$ with $\gamma(q) \neq 0$. But this is a consequence of Prop. 2.1': We have

$$
\operatorname{ad}(X) \circ \operatorname{ad}(Y)\left(\mathfrak{m}_{\gamma}^{o}\right) \subset \operatorname{ad}(X)\left(\mathfrak{f}_{p}+\sum_{\delta \in(\beta \gamma)^{*}} \mathfrak{m}_{\delta}^{o}\right) \subset \mathfrak{f}_{p}+\sum_{\delta \in(\beta \gamma)^{*}} \sum_{\varepsilon \in(\alpha \delta)^{*}} \mathfrak{m}_{\varepsilon}^{o} .
$$

A nonzero contribution to the trace can occur only if $\mathfrak{m}_{\gamma}^{o}=\mathfrak{m}_{\varepsilon}^{o}$ for some $\varepsilon \in(\alpha \delta)^{*}$ and some $\delta \in(\beta \gamma)^{*}$. In other words, we would have $\gamma \in(\alpha \delta)^{*}$ and $\delta \in(\beta \gamma)^{*}$ which implies $\gamma \in(\alpha \beta)$. But this is impossible since $\alpha(q)=\beta(q)=0$, but $\gamma(q) \neq 0$. In the same way we see that $(X \circ Y) . E_{\gamma}$ does not contribute to trace $X \circ Y$ which finishes the proof.

## 4. Detecting s-representations

Let $V$ be a euclidean vector space of dimension at least two and $K \subset O(V)$ a compact connected subgroup, which acts irreducibly on $V$. In order to detect whether the action of $K$ on $V$ is an s-representation we extend the bracket of the Lie algebra of $K$ to the vector space direct sum

$$
\mathfrak{g}:=\mathfrak{f}+V
$$

by putting

$$
\begin{aligned}
{[A, v] } & :=A v=-[v, A], \\
\langle A,[v, w]\rangle_{\mathrm{t}} & :=\langle A v, w\rangle, \\
{[v, w] } & \in \mathfrak{f}
\end{aligned}
$$

for all $A \in \mathfrak{f}$ and $v, w \in V$ where $\langle,\rangle_{\mathrm{t}}$ is the $(\mathfrak{f}, V)$-trace metric defined above (cf. Ch. 3). Let $J: V \times V \times V \rightarrow V$ be defined by

$$
J(x, y, z):=[[x, y], z]+[[y, z], x]+[[z, x], y] .
$$

Then $J$ is $K$-equivariant and $J=0$ is equivalent to the Jacobi identity of $\mathfrak{g}$, since the $K$-equivariance of $[x, y]$ implies the Jacobi identity whenever one entry lies in $\mathfrak{f}$ (by differentiating $[(\exp t A) \cdot x,(\exp t A) \cdot y], A \in \mathfrak{f})$.

Lemma 4.1. The following two conditions are equivalent:
(S) The action of $K$ on $V$ is an s-representation.
(J) $J=0$.

Proof. " $(\mathrm{S}) \Rightarrow(\mathrm{J})$ ": If $K$ on $V$ is an s-representation then there exists a compact symmetric pair $(G, K)$ whose isotropy representation is the given action. The Lie algebra g of $G$ decomposes as above and the negative of the Killing form is a biinvariant metric.

This implies $\langle A,[v, w]\rangle_{\mathrm{t}}=-\mathfrak{B}_{\mathrm{g}}(A v, w)$ for all $A \in \mathfrak{f}$ and $v, w \in V$. Note that $-\mathfrak{B}_{\mathrm{g}}$ and the given inner product on $V$ coincide up to a constant factor, due to the irreducibility of the $K$-action. This in turn implies that the bracket on $\mathfrak{g}$ coincides with the one described above on $V \times V$ up to a constant factor. Therefore $J=0$ follows from the Jacobi identity of $g$.
$"(\mathrm{~J}) \Rightarrow(\mathrm{S}) "$ is well known and follows almost immediately from the remark that $(\mathfrak{g}, \mathfrak{f})$ is a symmetric pair (cf. [HZ], Lemma 1).

Remark. Notice that $[x, y]=0$ for $x, y \in V$ is equivalent to $y \in v_{x}(K . x)$, by the definition of the Lie bracket. In particular, $\left[v_{p}, v_{p}\right]=0$.

Lemma 4.2. If $K$ is $\omega$-maximal, i.e. maximal among the connected subgroups having the same orbits, then $(\mathrm{J})$ is equivalent to

$$
\begin{equation*}
J(x, y, z)=0 \text { for all } x, y, z \in V \text { with }[x, y]=0 \tag{1}
\end{equation*}
$$

Proof. $\langle J(x, y, z), u\rangle$ defines a skew-symmetric 4-form on $V$; in fact

$$
\langle J(x, y, z), u\rangle=\langle J(u, x, y), z\rangle
$$

which shows the skew symmetry also in the arguments $x, y, u$. Thus $J$ may be interpreted as a symmetric endomorphism $\tilde{J}: \Lambda^{2} V \rightarrow \Lambda^{2} V$ of trace 0 . Since $\Lambda^{2} V$ is canonically isomorphic to the Liealgebra $\mathfrak{o}(V)$ of skewsymmetric endomorphisms, we have an orthogonal splitting $\Lambda^{2} V=\mathfrak{f}+\mathfrak{f}^{\perp}$. It follows from the choice of the inner product on $\mathfrak{f}$ that $\left.\widetilde{J}\right|_{\mathfrak{t}}=\lambda \cdot I$ for some $\lambda \in \mathbb{R}$ (Lemma 2 of $[\mathrm{HZ}]$ ). Hence ( J ) is equivalent to $\left.\widetilde{J}\right|_{\mathfrak{t}}{ }^{\perp}=0$. Now by the remark above, $\left(\mathrm{J}_{1}\right)$ is equivalent to $\widetilde{J}(x \wedge y)=0$ for all $x \in V$ and all $y \in v_{x}$, i. e. for all $x \wedge y \in \mathfrak{f}^{\perp}$, since $\langle A, x \wedge y\rangle=\langle A x, y\rangle$. Thus (J) follows from ( $\mathrm{J}_{1}$ ), if these decomposable elements span $\mathfrak{l}^{\perp}$, i. e. if any $A \in \mathfrak{o}(V)=\Lambda^{2} V$ with $A \perp x \wedge y$ for all $x \wedge y \in \mathfrak{l}^{\perp}$ belongs to $\mathfrak{l}$, or in other words if any $A \in \mathfrak{o}(V)$ with $A x \in T_{x}(K \cdot x)$ for any $x \in V$ is in $f$. But this last condition is satisfied precisely if $K$ is $\omega$-maximal.

Theorem 4.3. Let $K \subset O(V)$ be a compact connected subgroup acting irreducibly on V. Choose a regular point $p \in V$, i.e. a point lying on a principal orbit. Then the following conditions are equivalent:
$\left(\mathrm{J}_{1}\right) \quad J(x, y, z)=0$ for all $x, y, z \in V$ with $[x, y]=0$.
$\left(\mathrm{J}_{2}\right) \quad J(x, y, z)=0$ for all $x, y, z \in V$ with $[x, y]=[x, z]=0$.
$\left(\mathrm{J}_{3}\right) \quad \mathfrak{f}_{x}^{\perp} \cdot v_{x} \perp v_{x}$ for all $x \in V$ where $\mathfrak{f}_{x}^{\perp} \subset \mathfrak{f}$ denotes the orthogonal complement of $\mathfrak{f}_{x}$ with respect to the $(\mathfrak{f}, V)$-trace metric.
$\left(\mathrm{SP}^{+}\right) \quad$ The $K$-action is strongly polar, and the $\mathrm{m}_{\alpha}^{o}$ are mutually orthogonal (including $\mathrm{m}_{0}^{o}:=\mathfrak{f}_{p}$ ).
(Co) The $K$-action is polar and the subspaces $\overline{\mathrm{m}}_{\alpha}$ of $\mathfrak{f}$ with $\overline{\mathrm{m}}_{\alpha} \perp \mathfrak{f}_{p}$ and $\overline{\mathrm{m}}_{\alpha} \cdot p=E_{\alpha}(p)$ (for all $\alpha \in \Pi$ ) are mutually orthogonal.

Proof. $\left(\mathrm{J}_{1}\right)$ implies $\left(\mathrm{J}_{2}\right)$ trivially, while $\left(\mathrm{J}_{2}\right)$ is equivalent to $[y, z] \cdot x=0$ for all $x \in V$ and $y, z \in v_{x}$ (cf. Remark preceding 4.2), i.e. to $\left[\nu_{x}, v_{x}\right] \subset \mathfrak{f}_{x}$ or equivalently to $\mathfrak{f}_{x}^{\perp} \perp\left[\nu_{x}, v_{x}\right]$ and thus to $\left(\mathrm{J}_{3}\right)$. If $x$ is regular then $\mathrm{f}_{x} \cdot v_{x}=0$, since $K_{x}$ acts trivially on $v_{x}$. Therefore $\left(\mathrm{J}_{3}\right)$ implies $\mathrm{f} . v_{x} \perp v_{x}$ for all regular $x$. Hence it follows from any condition above that the $K$-action is polar.

The sections of a polar action are the normal spaces of a principal orbit, and the normal space of any point is a union of sections. Therefore for any $x, y \in V$ with $[x, y]=0$, i.e. $y \in v_{x}$, we can find an element in $K$, which maps $x$ and $y$ to $v_{p}$. Since $J$ is $K$-equivariant, $\left(\mathrm{J}_{1}\right)$ is therefore equivalent to $J\left(v_{p}, v_{p}, V\right)=0$ and thus to $J\left(v_{p}, v_{p}, E_{\alpha}\right)=0$ for all $\alpha$, where $E_{\alpha}=E_{\alpha}(p)$. (Note that $J\left(v_{p}, v_{p}, v_{p}\right)=0$ since $\left[v_{p}, v_{p}\right]=0$, cf. Remark above.) Let $l_{\alpha} \subset v_{p}$ be the focal hyperplane $n_{\alpha}^{\perp}$. Then $J\left(v_{p}, v_{p}, E_{\alpha}\right)=0$ if and only if $J\left(l_{\alpha}, v_{p}, E_{\alpha}\right)=0$, because $J\left(n_{\alpha}, n_{\alpha}, E_{\alpha}\right)=0$. Since $E_{\alpha}(p)$ focalizes under the map $\pi_{x-p}$ for $x \in l_{\alpha}$, we have that $E_{\alpha}=E_{\alpha}(p)$ lies in the normal space $v_{x}$ for any $x \in l_{\alpha}$, i.e. $\left[E_{\alpha}, l_{\alpha}\right]=0$ by the remark above. Hence ( $\mathrm{J}_{1}$ ) follows from $\left(\mathrm{J}_{2}\right)$, and we have proved the equivalence of $\left(\mathrm{J}_{1}\right),\left(\mathrm{J}_{2}\right)$ and $\left(\mathrm{J}_{3}\right)$. Furthermore, $\left(\mathrm{J}_{1}\right)$ is equivalent to
(*)

$$
\left[v_{p}, E_{\alpha}\right] \cdot l_{\alpha}=0
$$

From ( $\mathrm{J}_{3}$ ) we obtain $\left(\mathrm{SP}^{+}\right)$as follows. Let $\mathfrak{m}=\mathrm{f}_{p}^{\perp}$ and $\mathfrak{m}_{\alpha}=\left\{X \in \mathfrak{m} ; X . p \in E_{\alpha}(p)\right\}$. Now ( $\mathrm{I}_{3}$ ) implies that for all $x \in \nu_{p}$

$$
\left(f_{x}\right)^{\perp} \cdot p \subset\left(v_{x}\right)^{\perp}=\sum_{\beta(x) \neq 0} E_{\beta},
$$

and by reasons of dimension we have in fact equality. Thus $\left(f_{x}\right)^{\perp}=\sum_{\beta(x) \neq 0} \mathfrak{m}_{\beta}$ while $\mathfrak{f}_{x}=\mathfrak{f}_{p}+\sum_{\alpha(x)=0} \mathfrak{m}_{\alpha}$ by Lemma 1. Thus $\mathfrak{m}=\mathfrak{m}^{s}=\mathfrak{m}^{o}$. Since for any $\alpha \neq \beta$ we find $x \in v_{p}$ with $\alpha(x)=0$ and $\beta(x) \neq 0$, the decomposition $\mathfrak{m}=\sum_{\alpha} \mathfrak{m}_{\alpha}$ is orthogonal.

Clearly, $\left(\mathrm{SP}^{+}\right)$implies ( Co ) and $\overline{\mathrm{m}}=\mathrm{m}^{0}$. It remains to show that (Co) implies (*), hence $\left(\mathrm{J}_{1}\right)$. From $\left\langle\mathfrak{f}_{p},\left[\nu_{p}, E_{\alpha}\right]\right\rangle=\left\langle\mathfrak{f}_{p} \cdot v_{p}, E_{\alpha}\right\rangle=0$ and $\left\langle\bar{m}_{\beta},\left[v_{p}, E_{\alpha}\right]\right\rangle=\left\langle E_{\beta}, E_{\alpha}\right\rangle=0$ for all $\beta \neq \alpha$, we conclude that (Co) implies $\left[v_{p}, E_{\alpha}\right] \subset \overline{\mathrm{m}}_{\alpha}$. But $\overline{\mathrm{m}}_{\alpha} \subset \mathfrak{f}_{x}$ for any $x \in l_{\alpha}$ (cf. Prop. 2.1), hence $\overline{\mathrm{m}}_{\alpha} \cdot l_{\alpha}=0$ and we obtain (*).

## 5. Proof of the Main Theorem

In this section let $K \subset O(V)$ be a compact connected polar subgroup which acts irreducibly on $V$. We have already seen that maximal polarity of $K$ implies the Olmos Condition (Ch. 1) and hence the strong polarity (Ch. 2). Furthermore, (S) is equivalent to $\left(\mathrm{SP}^{+}\right)$if $K$ is $\omega$-maximal (4.1, 4.2, 4.3). To finish the proof of the Main Theorem stated in the introduction it therefore suffices to show that (S) implies (MP), and (SP) implies $\left(\mathrm{SP}^{+}\right)$as well as the $\omega$-maximality of $K$.

Proposition 5.1. (S) implies (MP) if the rank is at least 2; in particular, any irreducible s -representation of rank $\geqq 2$ is $\omega$-maximal.

Proof. By 4.1, 4.2, 4.3, $K$ satisfies (SP) with certain subspaces $m_{\alpha}^{o}$ of $\neq$ which are perpendicular to $\mathcal{f}_{p}$. Since $\mathfrak{m}^{o}=\sum_{\alpha} \mathfrak{m}_{\alpha}^{o}$ is ad $\left(f_{p}\right)$-invariant, the subalgebra generated by $\boldsymbol{m}^{o}$ is an ideal of $\mathfrak{l}$. Its orthogonal complement is therefore also an ideal of $\neq$ which is contained in $\mathfrak{f}_{p}$ and thus trivial by the effectiveness of the action of $K$ on $K . p$. Hence $\mathfrak{m}^{o}$ generates $\mathfrak{f}$ as a Lie algebra. If $\tilde{K} \subset O(V)$ is a compact subgroup containing $K$ and with the same orbits as $K$, then $\mathfrak{m}_{\alpha}^{o} \subset \tilde{\mathrm{f}}$ and thus $\widetilde{K}$ satisfies (SP) with the same $\mathrm{m}_{\alpha}^{o}$ (but a priori not necessarily $\left(\mathrm{SP}^{+}\right)$). Therefore, $\tilde{\mathrm{f}}=\tilde{\mathrm{f}}_{p}+\mathrm{m}^{o}$ is a reductive decomposition as well by Proposition 2.2. As before we conclude that the subalgebra generated by $\boldsymbol{m}^{o}$ is an ideal of $\bar{f}$. Hence $\mathfrak{f}$ is an ideal of $\mathfrak{f}$. If $\mathfrak{f} \neq \mathfrak{f}$ then there exists a nontrivial element $A$ in a complementary ideal. Since $A$ is skew symmetric and $A^{2}$ commutes with $K$ which acts irreducibly, we may assume $A^{2}=-I$. But since $K$ acts as an s-representation, $A$ belongs to $\neq$ (the symmetric space is Kählerian and thus $A$ commutes with the curvature tensor). This contradicts $\mathfrak{f} \neq \tilde{f}$ and thus $K$ is $\omega$-maximal.

Theorem 5.2. (SP) implies $\left(\mathrm{SP}^{+}\right)$provided that the rank is $\geqq 3$. More precisely, if the action of $K \subset O(V)$ is strongly polar with rank $\geqq 3$, then $\mathfrak{f}=\mathfrak{f}_{p}+\sum_{\alpha} \mathrm{m}_{\alpha}^{o}$ is an orthogonal decomposition.

Proof. Fix some regular point $p \in V$ and let $M=K . p$. Let $\alpha \neq \beta \in \Pi$. Recall that $(\alpha \beta)$ is the set of $\gamma \in \Pi$ which lie on the line through $\alpha$ and $\beta$. We can find $q \in v_{p}$ such that precisely the $E_{\gamma}(p)=: E_{\gamma}$ with $\gamma \in(\alpha \beta)$ focalize onto $K . q$. This implies $v_{q}=v_{p}+\sum_{\gamma \in(\alpha \beta)} E_{\gamma}$ and
$T_{p} L=\sum E_{\gamma}$, where $L$ is the fibre through $p$ of the endpoint mapping $T_{p} L=\sum_{\gamma \in(\alpha \beta)} E_{\gamma}$, where $L$ is the fibre through $p$ of the endpoint mapping

$$
\pi_{\xi}: M=K \cdot p \rightarrow M_{\xi}=K \cdot q
$$

and $\xi$ is the parallel normal field with $p+\xi(p)=q$. The fibre $L$ is a rank 2 isoparametric submanifold of $v_{q}$ and actually the orbit of the normal holonomy group $\mathrm{Hol}_{q}^{v}$ of $K . q$ through $p$, whose action on $v_{q}$ is equivalent to the direct sum of an s-representation and a trivial representation (cf. Ch. 1). But $L$ is also an orbit of the isotropy group $K_{q}$ acting on $v_{q}$ by the normal isotropy representation $\varrho_{q}: K_{q} \rightarrow O\left(v_{q}\right)$. This implies that $\mathrm{Hol}_{q}^{v}$ and $K_{q}$ have the same orbits. Moreover, by Prop. 2.4, $\mathrm{Hol}_{q}^{v}$ is contained in $\varrho_{q}\left(K_{q}\right)$.

Case 1. $L$ is irreducible, i.e. $(\alpha \beta) \neq\{\alpha, \beta\}$. Then we conclude from the maximality of s-representations (cf. 5.1) that the connected components containing the identity of $\mathrm{Hol}_{q}^{v}$ and $\varrho_{q}\left(K_{q}\right)$ coincide. Since $\mathrm{Hol}_{q}^{v}$ acts as an s-representation, it satisfies ( $\mathrm{SP}^{+}$) (cf. 4.1, 4.2, 4.3). Thus it follows from Lemma 3.1 that $\mathfrak{F}_{q}=\mathfrak{F}_{p}+\sum_{\gamma \in(\alpha \beta)} \mathfrak{m}_{\gamma}^{o}$ is an orthogonal decomposition with respect to the $\left(\mathfrak{f}_{q}, v_{q}\right)$-trace metric and hence also with respect to the $(\mathfrak{f}, V)$-trace metric by Lemma 3.2. In particular we get $\mathfrak{f}_{p} \perp \mathfrak{m}_{\alpha}^{o} \perp \mathfrak{m}_{\beta}^{o}$ in this case.

Case 2. $L$ is reducible, i.e. $(\alpha \beta)^{*}:=(\alpha \beta) \backslash\{\alpha, \beta\}=\emptyset$. Then we show by a direct argument that $\mathfrak{m}_{\alpha}^{o} \perp \mathfrak{m}_{\beta}^{o}$ with respect to the $\left(\mathfrak{f}_{q}, v_{q}\right)$-trace metric (and hence with respect to the ( $\mathfrak{f}, V$ )-trace metric by Lemma 3.2). In fact, in this case $\mathfrak{f}_{q}=\mathfrak{f}_{p}+\mathfrak{m}_{\alpha}^{o}+\mathfrak{m}_{\beta}^{o}$ and $v_{q}=v_{p}+E_{\alpha}+E_{\beta}$, and further $v_{p}$ is the orthogonal sum of $l_{\alpha}$ and $l_{\beta}$. From 2.1' we obtain $\mathfrak{m}_{\alpha}^{o} \cdot E_{\beta}=\mathfrak{m}_{\beta}^{o} \cdot E_{\alpha}=0$. Thus $\mathfrak{m}_{\beta}^{o} \cdot v_{q} \subset E_{\beta}+l_{\alpha}$ (recall that $\left\langle\mathfrak{m}_{\beta}^{o} \cdot E_{\beta}, l_{\beta}\right\rangle=\left\langle E_{\beta}, \mathfrak{m}_{\beta}^{o} \cdot l_{\beta}\right\rangle=0$ ) which implies $\mathfrak{m}_{\alpha}^{o} \cdot \mathfrak{m}_{\beta}^{o} \cdot v_{q}=0$. In particular, $\left[\mathfrak{m}_{\alpha}^{o}, \mathfrak{m}_{\beta}^{o}\right] \subset \operatorname{ker} \varrho_{q^{*}}$ which is an ideal of $\mathfrak{f}_{q}$ being contained in $\mathfrak{E}_{p}$. Now for $X \in \mathfrak{m}_{\alpha}^{o}$ and $Y \in \mathfrak{m}_{\beta}^{o}$ we have $\varrho_{q^{*}}(X) \varrho_{q^{*}}(Y)=0$. Further, $\operatorname{ad}(X) \circ \operatorname{ad}(Y) \operatorname{maps}$ $\mathfrak{m}_{\beta}^{o}$ into $\mathfrak{m}_{\alpha}^{o}+\operatorname{ker} \varrho_{q^{*}}$ and $\mathfrak{f}_{p}+\mathfrak{m}_{\alpha}^{o}$ into $\operatorname{ker} \varrho_{q^{*}} \subset \mathfrak{f}_{p}$ while ker $\varrho_{q^{*}}$ itself is mapped to 0 (recall that $\left[\mathfrak{m}_{\beta}^{o}, \operatorname{ker} \varrho_{q^{*}}\right] \subset \mathfrak{m}_{\beta}^{o} \cap \operatorname{ker} \varrho_{q^{*}}=0$ ). This shows $\operatorname{trace}{\underset{t_{q}}{ }} \operatorname{ad}(X) \operatorname{ad}(Y)=0$.

Since for any $\alpha \in \Pi$ we can find some $\beta \in \Pi$ with $\left\langle n_{\alpha}, n_{\beta}\right\rangle \neq 0$ (which implies that the corresponding $L$ is irreducible), we get $\mathfrak{f}_{p} \perp \mathfrak{m}_{\alpha}^{o}$ for all $\alpha$ from Case 1 .

Corollary 5.3. (SP) implies (MP) if the rank is at least 3.
Proof. Let $\tilde{K} \subset O(V)$ be a compact, connected subgroup which contains $K$ and has the same orbits. Then also $\tilde{K}$ satisfies (SP) with the same $\mathfrak{m}_{\alpha}^{o} \subset \mathfrak{f} \subset \tilde{f}$. By Theorem 5.2, the decomposition $\tilde{\mathfrak{f}}=\tilde{f}_{p}+\sum_{\alpha} \mathfrak{m}_{\alpha}^{o}$ is orthogonal (with respect to the ( $\mathfrak{f}, V$ )-trace metric). From the effectiveness of the $\tilde{K}$-action on $V$ it follows therefore that $\sum_{\alpha} \mathfrak{m}_{\alpha}^{o}$ generates $\tilde{f}$ as a Lie algebra (cf. the proof of Proposition 5.1) and by the same argument also $\mathfrak{f}$. This implies $\tilde{f}=\mathfrak{f}$.

Combining the results of this and the previous section we finally get the following corollary, which includes in particular the theorem of the introduction.

Corollary 5.4. For any compact, connected subgroup of $O(V)$ which acts irreducibly and with cohomogeneity $\geqq 3$ on $V$, the conditions (S), (MP), (SP), ( J$),\left(\mathrm{J}_{1}\right),\left(\mathrm{J}_{2}\right),\left(\mathrm{J}_{3}\right),(\mathrm{Co})$, $\left(\mathrm{SP}^{+}\right)$and the Olmos condition are all equivalent.

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[^0]:    Theorem (Dadok). Any polar representation is orbit equivalent to the isotropy representation of a symmetric space.

[^1]:    7 Journal für Mathematik. Band 507

