

# SUPERFLUID HELIUM 3: LINK BETWEEN CONDENSED MATTER PHYSICS AND PARTICLE PHYSICS\*

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The discovery of the superfluid phases of Helium 3 in 1971 opened the door to one of the most fascinating systems known in condensed matter physics. Superfluidity of Helium 3, originating from pair condensation of Helium 3 atoms, turned out to be the ideal test ground for many fundamental concepts of modern physics, such as macroscopic quantum phenomena, (gauge-)symmetries and their spontaneous breakdown, topological defects, *etc.* Thereby the superfluid phases of Helium 3 enriched condensed matter physics enormously. In particular, they contributed significantly — and continue to do so — to our understanding of various other physical systems, from heavy fermion and high- $T_c$  superconductors all the way to neutron stars, particle physics, gravity and the early universe. A simple introduction into the basic concepts and questions is presented.

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## 1. The quantum liquids $^3\text{He}$ and $^4\text{He}$

There are two stable isotopes of the chemical element Helium: Helium 3 and Helium 4, conventionally denoted by  $^3\text{He}$  and  $^4\text{He}$ , respectively. From a microscopic point of view, Helium atoms are structureless, spherical particles interacting via a two-body potential that is well understood. The attractive part of the potential, arising from weak van der Waals-type dipole

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(and higher multipole) forces, causes Helium gas to condense into a liquid state at temperatures of 3.2 K and 4.2 K for  $^3\text{He}$  and  $^4\text{He}$ , respectively, at normal pressure. The pressure versus temperature phase diagrams of  $^3\text{He}$  and  $^4\text{He}$  are shown in Fig. 1. When the temperature is decreased even further one finds that the Helium liquids, unlike all other liquids, do not solidify unless a pressure of around 30 bar is applied. This is the first remarkable indication of macroscopic quantum effects in these systems. The origin of this unusual behaviour lies in the quantum-mechanical uncertainty principle, which requires that a quantum particle can never be completely at rest at given position, but rather performs a zero-point motion about the average position. The smaller the mass of the particle and the weaker the binding force, the stronger these oscillations are. In most solids the zero-point motion is confined to a small volume of only a fraction of the lattice-cell volume. In the case of Helium, however, two features combine to prevent the formation of a crystalline solid with a rigid lattice structure: (*i*) the strong zero-point motion arising from the small atomic mass (Helium is the second-lightest element in the periodic table); and (*ii*) the weakness of the attractive interaction due to the high symmetry of these simple atoms. It is this very property of Helium — of staying liquid — that makes it such a valuable system for observing quantum behaviour on a macroscopic scale. Quantum effects are also responsible for the strikingly different behaviours of  $^4\text{He}$  and  $^3\text{He}$  at even lower temperatures. Whereas  $^4\text{He}$  undergoes a second-order phase transition into a state later shown to be superfluid, *i.e.* where the liquid is capable of flowing through narrow capillaries or tiny pores without friction, no such transition is observed in liquid  $^3\text{He}$  in the same temperature range (Fig. 1). The properties of liquid  $^3\text{He}$  below 1 K are nevertheless found to be increasingly different from those of a classical liquid. It is only at a temperature roughly one thousandth of the transition temperature of  $^4\text{He}$  that  $^3\text{He}$  also becomes superfluid, and in fact forms *several* superfluid phases, each of which has a much richer structure than that of superfluid  $^4\text{He}$ .

The striking difference in the behaviours of  $^3\text{He}$  and  $^4\text{He}$  at low temperatures is a consequence of the laws of quantum theory as applied to systems of identical particles, *i.e.* the laws of quantum statistics. The  $^4\text{He}$  atom, being composed of an even number of electrons and nucleons, has spin zero and consequently obeys Bose–Einstein statistics. In contrast, the  $^3\text{He}$  nucleus consists of *three* nucleons, whose spins add up to give a total nuclear spin of  $I = 1/2$ , making the total spin of the entire  $^3\text{He}$  atom  $1/2$  as well. Consequently liquid  $^3\text{He}$  obeys Fermi–Dirac statistics. So it is the tiny nuclear spin, buried deep inside the Helium atom, that is responsible for all the differences of the macroscopic properties of the two isotopes.

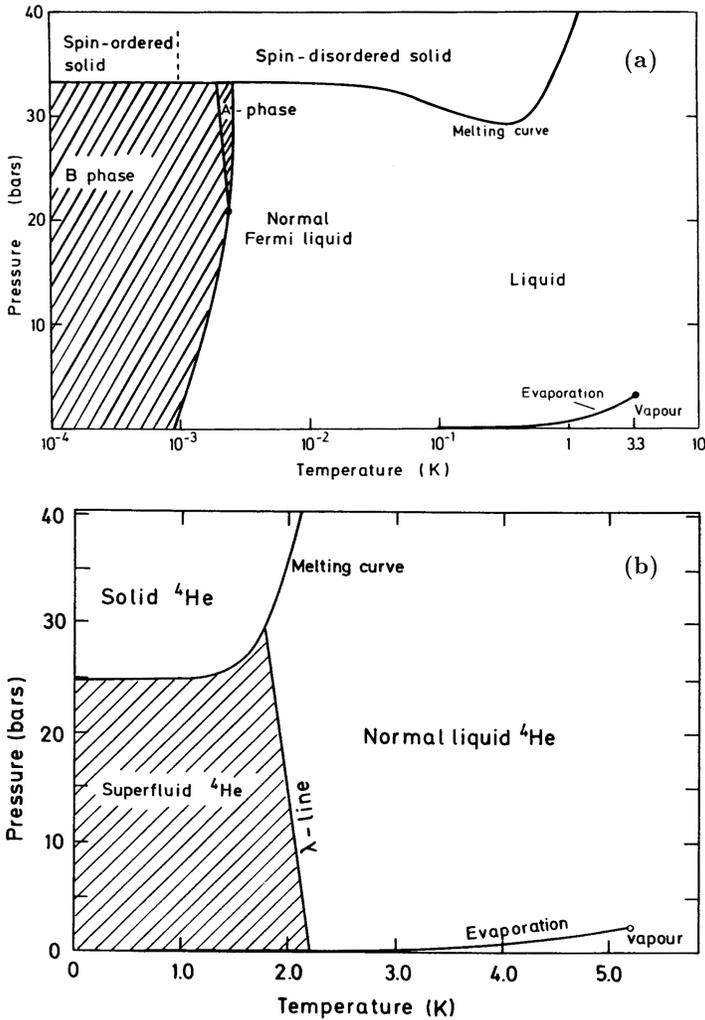


Fig. 1. Pressure vs temperature phase diagram for (a)  $^3\text{He}$  (logarithmic temperature scale); (b)  $^4\text{He}$  (linear temperature scale).

Since in a Bose system single-particle states may be multiply occupied, at low temperatures this system has a tendency to condense into the lowest-energy single-particle state (Bose-Einstein condensation). It is believed that the superfluid transition in  $^4\text{He}$  is a manifestation for Bose-Einstein condensation. The all-important qualitative feature of the Bose condensate is its phase rigidity, *i.e.* the fact that it is energetically favourable for the particles to condense into a single-particle state of fixed quantum-mechanical phase,

such that the global gauge symmetry spontaneously broken. As a consequence, macroscopic flow of the condensate is (meta-)stable, giving rise to the phenomenon of superfluidity. In a Fermi system, on the other hand, the Pauli exclusion principle allows only single occupation of fermion states. The ground state of the Fermi gas is therefore the one in which all single-particle states are filled up to a limiting energy, the Fermi energy  $E_F$ . As predicted by Landau [1] and later verified experimentally (for a review see [2]), the properties of  $^3\text{He}$  well below its Fermi temperature  $T_F = E_F/k_B \approx 1$  K are similar to those of a degenerate Fermi gas. In particular the formation of a phase-rigid condensate is not possible in this framework. Until the mid-1950s a superfluid phase of liquid  $^3\text{He}$  was therefore believed to be ruled out. On the other hand, it is most remarkable that the property of superfluidity [3,4] was indeed first discovered experimentally in a *Fermi* system, namely that of the “liquid” of conduction electrons in a superconducting metal [5]. The superfluidity of  $^4\text{He}$  was only found more than 25 years later.

## 2. Pair condensation in a Fermi liquid

As demonstrated by Bardeen, Cooper and Schrieffer (BCS) in 1957 [6] the key to the theory of superconductivity is the formation of “Cooper pairs”, *i.e.* pairs of electrons with opposite momentum  $\mathbf{k}$  and spin projection  $\sigma$ : ( $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$ ). In the case of conventional superconductors the Cooper pairs are structureless objects, *i.e.* the two partners form a spin-singlet state in a relative *s*-wave orbital state. These Cooper pairs have total spin zero and may therefore be looked upon in a way as composite bosons, which all have the same pair wave function and are all in the same quantum-mechanical state. In this picture the transition to the superconducting state corresponds to the Bose-condensation of Cooper pairs, the condensate being characterized by macroscopic quantum coherence. The concept of Bose–Einstein condensation is appealing since key features of superconductivity like the Meissner effect, flux quantization and superfluid mass currents in conventional superconductors are naturally implied. Nevertheless, since the theory of conventional superconductivity is firmly based on BCS theory, the concept of a Bose–Einstein condensation of Cooper pairs traditionally did not receive much attention (or was even considered to be downright wrong). Within the context of superfluid  $^3\text{He}$ , this notion was taken up again by Leggett [7,8], who argued that tightly bound Bose–Einstein-condensed molecules on the one hand and Cooper pairs on the other may be viewed as extreme limits of the same phenomenon. This approach, which was quite provocative at the time, is now well accepted [9–11]. However, the original idea that at  $T_c$  Cooper pairs form and automatically Bose-condense has been revised since then. Apparently Cooper-pair formation is not a separate phase transition

but is rather a matter of thermal equilibrium: for any finite coupling there exists a finite density of pairs even above  $T_c$ , although in conventional superconductors — and even in high  $T_c$  materials — their density is negligibly small. At weak coupling (BCS limit) the condensation temperature and the (not well-defined) temperature of pair formation practically coincide; they become different only at very strong coupling (Bose limit). Similar ideas are also implicit in several theoretical approaches to high- $T_c$  superconductivity.

While in free space an attractive force has to be sufficiently strong to bind two electrons, inside the metal the presence of the filled Fermi sea of conduction electrons blocks the decay of a Cooper pair, so that an *arbitrarily* small attractive interaction leads to the formation of stable Cooper pairs. The attractive interaction between the electrons of a Cooper pair in a conventional superconducting metal is due to the exchange of virtual phonons (electron–phonon interaction). If the phonon-mediated interaction is strong enough to overcome the repulsive Coulomb interactions between the two electrons then a transition into a superconducting state may occur. On the other hand, any other mechanism leading to attraction between electrons at the Fermi surface is equally well suited for producing superconductivity.

Given the success of the BCS theory in the case of superconductivity, it was natural to ask whether a similar mechanism might also work for liquid  $^3\text{He}$ . Since there is no underlying crystal lattice in the liquid that could mediate the attractive force, the attraction must clearly be an intrinsic property of the one-component  $^3\text{He}$  liquid itself. The main feature of the interatomic  $^3\text{He}$  potential is the strong repulsive component at short distances, and the weak van der Waals attraction at medium and long distances. It soon became clear that, in order to avoid the hard repulsive core and thus make optimal use of the attractive part of the potential, the  $^3\text{He}$  atoms would have to form Cooper pairs in a state of *nonzero* relative angular momentum  $l$ . In this case the Cooper-pair wave function vanishes at zero relative distance, thus cutting out the most strongly repulsive part of the potential. In a complementary classical picture one might imagine the partners of a Cooper pair revolving about their centre of gravity, thus being kept away from each other by the centrifugal force.

When the superfluid phases of  $^3\text{He}$  were finally discovered in 1971 at temperatures of about 2.6 mK and 1.8 mK, respectively, by Osheroff, Richardson and Lee [12], in an experiment actually designed to observe a magnetic phase transition in solid  $^3\text{He}$ , the results came as a great surprise.

### 3. The superfluid phases of $^3\text{He}$

Soon after the discovery of the phase transitions by Osheroff *et al.* [12] it was possible to identify altogether *three* distinct stable superfluid phases of bulk  $^3\text{He}$ ; these are referred to as the A, B and  $A_1$  phases. In zero magnetic field only the A and B phases are stable. In particular, in zero field the A phase only exists within a finite range of temperatures, above a critical pressure of about 21 bar. Hence its region of stability in the pressure-temperature phase diagram has a roughly triangular shape as shown in Fig. 1(a). The B phase, on the other hand, occupies the largest part of this phase diagram and is found to be stable down to the lowest temperatures attained so far. Application of an external magnetic field has a strong influence on this phase diagram. First of all, the A phase is now stabilized down to zero pressure. Secondly, an entirely new phase, the  $A_1$  phase, appears as a narrow wedge between the normal state and the A and B phases.

Owing to the theoretical work on anisotropic superfluidity that had been carried out before the actual discovery of superfluid  $^3\text{He}$ , progress in understanding the detailed nature of the phases was very rapid. This was clearly also due to the excellent contact between experimentalists and theorists, which greatly helped to develop the right ideas at the right time; see the reviews by Leggett [14], Wheatley [15], and Lee and Richardson [16]. A comprehensive discussion of the superfluid phases of  $^3\text{He}$  with a very extensive list of references can be found in our book [13] which also provides most of the material presented in this article.

In particular, it fairly soon became possible to identify the A phase and the B phase as realizations of the states studied previously by Anderson and Morel [17] and Balian and Werthamer [18], respectively. Therefore the A phase is described by the so-called “Anderson–Brinkman–Morel” (ABM) state, while the B phase is described by the “Balian–Werthamer” (BW) state. Consequently, “A phase” and “ABM state” are now used as synonyms; the same is true in the case of “B phase” and “BW state”.

Although the three superfluid phases all have very different properties, they have one important thing in common: the Cooper pairs in all three phases are in a state with *parallel* spin ( $S = 1$ ) and relative orbital angular momentum  $l = 1$ . This kind of pairing is referred to as “spin-triplet *p*-wave pairing”. In contrast, prior to the discovery of the superfluid phases of  $^3\text{He}$ , Cooper pairing in superconductors was only known to occur in a state with opposite spins ( $S = 0$ ) and  $l = 1$ , *i.e.* in a “spin-singlet *s*-wave state”. It should be noted that Cooper pairs in a superconductor and in superfluid  $^3\text{He}$  are therefore very different entities: in the former case pairs are formed by point-like, structureless electrons and are spherically symmetric, while in the case of  $^3\text{He}$  Cooper pairs are made of actual atoms (or rather of quasiparticles involving  $^3\text{He}$  atoms) and have an internal structure themselves.

### 3.1. The Cooper-pair wave function

Quantum-mechanically, a spin-triplet configuration ( $S = 1$ ) of two particles has three substates with different spin projection  $S_z$ . They may be represented as  $|\uparrow\uparrow\rangle$  with  $S_z = +1$ ,  $2^{-1/2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$  with  $S_z = 0$ , and  $|\downarrow\downarrow\rangle$  with  $S_z = -1$ . The pair wave function  $\Psi$  is in general a linear superposition of all three substates, *i.e.*

$$\Psi = \psi_{1,+}(\mathbf{k}) |\uparrow\uparrow\rangle + \psi_{1,0}(\mathbf{k})(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + \psi_{1,-}(\mathbf{k}) |\downarrow\downarrow\rangle, \quad (1)$$

where  $\psi_{1,+}(\mathbf{k})$ ,  $\psi_{1,0}(\mathbf{k})$  and  $\psi_{1,-}(\mathbf{k})$  are the three complex-valued amplitudes of the respective substates. In the case of a superconductor, where  $S = 0$  and  $l = 0$ , the pair wave function is much simpler, *i.e.* it is given by only a single component

$$\Psi_{\text{sc}} = \psi_0(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad (2)$$

with a single amplitude  $\psi_0$ .

So far we have only taken into account that, since  $S = 1$ , there are three substates for the spin. The same is of course true for the relative orbital angular momentum  $l = 1$  of the Cooper pair, which also has three substates  $l_z = 0, \pm 1$ . This fact is important if we want to investigate the amplitudes  $\psi_{1,+}(\mathbf{k})$  *etc.* further. They still contain the complete information about the space (or momentum) dependence of  $\Psi$ . The pair wave function  $\Psi$  is therefore characterized by three spin substates and three orbital substates, *i.e.* by altogether  $3 \times 3 = 9$  substates with respect to the spin and orbital dependence. Each of these nine substates is connected with a complex-valued parameter. Here we see the essential difference between Cooper pairs with  $S = l = 0$  (conventional superconductors) and  $S = l = 1$  ( $^3\text{He}$ ): their pair wave functions are very different. In the former case a single complex-valued parameter is sufficient for its specification, in the latter case of superfluid  $^3\text{He}$  *nine* such parameters are required. This also expresses the fact that a Cooper pair in superfluid  $^3\text{He}$  has an internal structure, while that for a conventional superconductor does not: because  $l = 1$ , it is intrinsically *anisotropic*. This anisotropy may conveniently be described by specifying some direction with respect to a quantization axis both for the spin and the orbital component of the wave function.

In order to understand the novel properties of superfluid  $^3\text{He}$ , it is therefore important to keep in mind that there are two characteristic directions that specify a Cooper pair. Here lies the substantial difference from a superconductor and the origin of the multitude of unusual phenomena occurring in superfluid  $^3\text{He}$ : the structure of the Cooper pair is characterized by *internal degrees of freedom*. Nevertheless, in both cases the superfluid/superconducting state can be viewed as the condensation of a macroscopic number of these Cooper pairs into the same quantum-mechanical state, similar to a Bose-Einstein condensation, as discussed above.

#### 4. Order parameter and broken symmetry

In the normal liquid state Cooper pair expectation value is zero. Obviously in the superfluid a new state of order appears, which spontaneously sets in at the critical temperature  $T_c$ . This particular transition from the normal fluid to the superfluid, *i.e.* into the ordered state, is called “continuous”, since the condensate — and hence the state of order — builds up continuously. This fact may be expressed quantitatively by introducing an “order-parameter” that is finite for  $T < T_c$  and zero for  $T \geq T_c$ .

**Ferromagnets:** A well-known example of such a transition is that from a paramagnetic to a ferromagnetic state of a metal when the system is cooled below the Curie temperature. In the paramagnetic regime the spins of the particles are disordered such that the average magnetization  $\langle \mathbf{M} \rangle$  of the system is zero. Mathematically speaking, the magnetization  $\mathbf{M}$  is a real three-component vector. Invariance in the paramagnetic phase under arbitrary global rotations in the three-dimensional spin space implies that the spin system is invariant under a symmetry group  $G=SO(3)$ . (For an introduction into the mathematical description of broken symmetries by group theory see Ref. [13]). By contrast, in the ferromagnetic phase the spins are more or less aligned and  $\langle \mathbf{M} \rangle$  is thus finite. In this case the system exhibits long-range order of the spins. Clearly, the existence of a preferred direction  $\mathbf{M}$  of the spins implies that the symmetry of the ferromagnet under spin rotations is reduced (“broken”) when compared with the paramagnet: the directions of the spins are no longer isotropically distributed, and the system will therefore no longer be invariant under the full spin rotation group  $G=SO(3)$ . So, in the ferromagnetic phase the system is only invariant under rotations *about* the axis  $\mathbf{M}$ , corresponding to the group of rotations  $H=U(1)$  — a subgroup of  $SO(3)$ ;  $H$  is called the *remaining symmetry* in the ferromagnetic phase. This phenomenon is called “spontaneously broken symmetry”; it is of fundamental importance in the theory of phase transitions. It describes the property of a macroscopic system (*i.e.*, a system in the thermodynamic limit) that is in a state that does not have the full symmetry of the microscopic dynamics. The degree of ordering is quantified by  $|\langle \mathbf{M} \rangle|$ , the magnitude of the magnetization. Hence  $|\langle \mathbf{M} \rangle|$  is called the “order parameter” of the ferromagnetic state.

**BCS superconductors and superfluid  $^4\text{He}$ :** The concept of spontaneously broken symmetry also applies to superconductors and superfluids. Here the order parameter measures the existence of Cooper pairs and is given by the probability amplitude for a pair to exist at a given temperature. In the case of conventional superconductors and superfluid  $^4\text{He}$  the order parameter is given by a single *complex* parameter  $\Delta = \Delta_0 e^{i\phi}$ , with amplitude  $\Delta_0$  and phase (“gauge”)  $\phi$ . Above  $T_c$ , in the normal phase, the

system is invariant under an arbitrary change of the phase  $\phi \rightarrow \phi'$ , *i.e.* under a gauge transformation. This gauge symmetry is due to the fact that the interparticle forces conserve particle number, and is equivalent to the symmetry group  $G=U(1)$ . Below  $T_c$ , when Cooper pairs have a finite expectation value such that particle number is no longer conserved, a particular value of  $\phi$  is spontaneously preferred and the  $U(1)$  symmetry is completely broken.

Gauge symmetry is spontaneously broken in any superfluid or superconductor. In addition, in an odd-parity pairing superfluid, as in the case of  $^3\text{He}$ , where  $l = 1$ , the pairs are necessarily in a spin-triplet state, implying that rotational symmetry in spin space is broken, just as in a magnet. At the same time, the anisotropy of the Cooper-pair wave function in orbital space calls for a spontaneous breakdown of orbital rotation symmetry, as in liquid crystals. All three symmetries are therefore simultaneously broken in superfluid  $^3\text{He}$ . What then are the order parameter and the spontaneously broken symmetries of superfluid  $^3\text{He}$ ?

#### 4.1. The order parameter and symmetry group describing superfluid $^3\text{He}$

Owing to the fact that the Cooper-pair wave function of superfluid  $^3\text{He}$  is complex and has a spin-triplet ( $S = 1$ ),  $p$ -wave ( $l = 1$ ) structure with  $3 \times 3 = 9$  substates the order parameter describing the superfluid phase may be written as a  $3 \times 3$  matrix  $A_{\mu j}$  with complex components. Here  $\mu = 1, 2, 3$  refers to spin space and  $j = 1, 2, 3$  to orbital (*i.e.*  $\mathbf{k}$ -)space. Obviously this order parameter has a considerably more complicated structure than conventional order parameters.

If we only consider the interactions responsible for the formation of the condensed state then the free energy of the system has to be invariant under separate three-dimensional rotations in spin space, in orbital space and under a gauge transformation. Hence the symmetry group that allows for these symmetries is given by a product of three independent symmetries [14]

$$G = \text{SO}(3)_L \times \text{SO}(3)_S \times U(1)_\phi. \quad (3)$$

Here the indices  $L$ ,  $S$ , and  $\phi$  indicate orbital space, spin space and gauge, respectively.

Clearly, this symmetry group is much richer in structure than those considered before. In fact, it incorporates the order parameter symmetries of liquid crystals,  $\text{SO}(3)_L$ , as well as of magnets,  $\text{SO}(3)_S$ , and of isotropic superfluids,  $U(1)_\phi$ , all at the same time. This implies that the *broken* symmetries resulting from (3) will also be much more intricate than, say, in the case of the ferromagnet, where only a single degree of freedom is relevant.

One might think that a study of the above mentioned broken symmetries could be performed much more easily by investigating them separately,

*i.e.* within the isotropic superfluid, the magnet, the liquid crystal, *etc.*, itself. However, the combination of several *simultaneously* broken continuous symmetries is much more than just the simple sum of the properties of all these known systems. Namely, some of the symmetries of superfluid  ${}^3\text{He}$  are jointly broken, leading to what is called “relative” symmetries. For example, the spin–orbit rotation symmetry, *i.e.*, the  $\text{SO}(3)_L \times \text{SO}(3)_S$ -part of (3), may be simultaneously broken to become  $\text{SO}(3)_{L+S}$ , *i.e.* a (sub-)group describing a *linear combination* of the individual groups  $\text{SO}(3)_L$  and  $\text{SO}(3)_S$ . This situation is realized in the B phase of superfluid  ${}^3\text{He}$  (see below). Another example is the gauge-orbit symmetry, *i.e.*, the  $\text{SO}(3)_L \times \text{U}(1)_\phi$ -part of (3), which may be simultaneously broken to a single subgroup  $\text{U}(1)_{L_z+\phi}$  [19,20]. This is the case in the A phase of superfluid  ${}^3\text{He}$  (see below). In both cases a rigid connection is established between the previously independent degrees of freedom, leading to long-range order in the condensate only in the *combined* (and not in the individual) degrees of freedom.

#### 4.2. The structure of the superfluid phases of ${}^3\text{He}$

It is clear that in principle the internal degrees of freedom of a spin-triplet  $p$ -wave state allow for many different Cooper-pair states and hence superfluid phases. (This is again different from ordinary superconductivity with  $S = 0$ ,  $l = 0$  pairing, where only a *single* phase is possible). Of these different states, the one with the lowest energy for given external parameters will be realized.

##### 4.2.1. B phase

In fact, Balian and Werthamer [18] showed, that, within a conventional “weak-coupling” approach, of all possible states there is precisely one state (the “BW state”) that has the lowest energy at *all* temperatures. This state is the one that describes the B phase of superfluid  ${}^3\text{He}$ . The state originally discussed by these authors is one in which the orbital angular momentum  $\mathbf{l}$  and spin  $\mathbf{S}$  of a Cooper-pair couple to a total angular momentum  $\mathbf{J} = \mathbf{l} + \mathbf{S} = 0$ . This  ${}^3P_0$  state is, however, only a special case of a more general one with the same energy (in the absence of spin–orbit interaction), obtained by an arbitrary rotation of the spin axes relative to the orbital axes of the Cooper-pair wave function. Such a rotation may be described mathematically by specifying a rotation axis  $\hat{\mathbf{n}}$  and a rotation angle  $\theta$ . In the B phase all three spin substates in (1) occur with equal measure. This state has a rather surprising property: in spite of the intrinsic anisotropy, the state has an *isotropic* energy gap. (The energy gap is the amount by which the system lowers its energy in the condensation process, *i.e.* it is the minimum energy required for the excitation of a single particle out of the condensate.)

Therefore the B phase resembles ordinary superconductors in several ways. On the other hand, even though the energy gap is isotropic, the B phase is intrinsically anisotropic. This is clearly seen in dynamic experiments in which the Cooper-pair structure is distorted. For this reason the B phase is sometimes referred to as “pseudo-isotropic”. Owing to the quantum coherence of the superfluid state, the rotation axis  $\hat{n}$  and angle  $\theta$  characterizing a Cooper pair in the B phase are macroscopically defined degrees of freedom, whose variation is physically measurable.

In the B phase gauge symmetry is completely broken (as in conventional superconductors or superfluid  $^4\text{He}$ ) and a linear combination of the spin-orbit symmetry is broken; the remaining symmetry of the order parameter of that phase corresponds to the group  $\text{SO}(3)_{L+S}$ .

#### 4.2.2. A phase

Since in weak-coupling theory the B phase always has the lowest energy, an explanation of the existence of the A phase of superfluid  $^3\text{He}$  obviously requires one to go beyond such an approach and to include “strong-coupling effects” [21,22]; for a review of a systematic approach see [23]. In view of the fact that, at present, microscopic theories are not capable of computing transition temperatures for  $^3\text{He}$ , it is helpful to single out a particular effect that can explain the stabilization of the A phase over the B phase at least qualitatively. As demonstrated by Anderson and Brinkman [21] such a conceptually simple effect — a feedback mechanism — indeed exists: the pair correlations in the condensed state change the pairing interaction between the  $^3\text{He}$  quasiparticles, the modification depending on the actual state itself. As a specific mechanism, these authors considered the role of spin fluctuations and showed that a stabilization of the state first considered by Anderson and Morel [17,24] is indeed possible. This only happens at somewhat elevated pressures, since spin fluctuations become more pronounced only at higher pressures. This “ABM state” does indeed describe the A phase. It has the property that, in contrast with  $^3\text{He-B}$ , its magnetic susceptibility is essentially the same as that of the normal liquid. This is a clear indication that in this phase the spin substate with  $S_z = 0$ , which is the only one that can be reduced appreciably by an external magnetic field, is absent. Therefore  $^3\text{He-A}$  is composed only of  $|\uparrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$  Cooper pairs. This implies that the anisotropy axis of the spin part of the Cooper-pair wave function, called  $\hat{\mathbf{d}}$ , has the same fixed direction in every pair. (More precisely,  $\hat{\mathbf{d}}$  is the direction along which the total spin of the Cooper pair vanishes:  $\hat{\mathbf{d}} \cdot \mathbf{S} = 0$ .) Likewise, the direction of the relative orbital angular momentum  $\hat{\mathbf{l}}$  is the same for all Cooper pairs. Therefore in the A phase the anisotropy axes  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  of the Cooper-pair wave function are long-range-

ordered, *i.e.* are preferred directions in the whole macroscopic sample. This implies a pronounced anisotropy of this phase in all its properties. In particular, the value of the energy gap now explicitly depends on the direction in  $\mathbf{k}$  space on the Fermi sphere and takes the form

$$\Delta_{\hat{\mathbf{k}}}(T) = \Delta_0(T) \left[ 1 - (\hat{\mathbf{k}} \cdot \hat{\mathbf{l}})^2 \right]^{1/2}. \quad (4)$$

Hence the gap vanishes at two points on the Fermi sphere, namely along  $\pm \hat{\mathbf{l}}$ . Because of the existence of an axis  $\hat{\mathbf{l}}$ , this state is also called the “axial state”. The existence of nodes implies that in general quasi-particle excitations may take place at arbitrarily low temperatures. Therefore, in contrast with  $^3\text{He-B}$  or ordinary superconductors, there is a finite density of states for excitations with energies below the average gap energy, leading for example to a specific heat proportional to  $T^3$  at low temperatures. The existence of point nodes (“Fermi points”) of the A phase order parameter allows one to draw detailed analogies with relativistic quantum field theory [25,26], as will be discussed in Section 6.

In the A phase the spin rotation symmetry is broken (as in a ferromagnet), as well as a linear combination of the gauge-orbit symmetry. Neglecting an additional discrete  $Z_2$  symmetry, the remaining symmetry of the order parameter of that phase therefore corresponds to the group  $U(1)_{S_z} \times U(1)_{L_z + \phi}$ .

#### 4.2.3. $A_1$ phase

The third experimentally observable superfluid phase of  $^3\text{He}$ , the  $A_1$  phase, is only stable in the presence of an external magnetic field. In this phase Cooper pairs are all in a single spin substate, the  $|\uparrow\uparrow\rangle$  state, corresponding to  $S_z = +1$ ; the components with  $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$  states are missing. It is therefore a “magnetic” superfluid — the first homogeneous magnetic liquid ever observed in nature. The remaining symmetry of that phase corresponds to  $U(1)_{S_z + \phi} \times U(1)_{L_z + \phi}$ .

#### 4.3. Broken symmetries in high-energy physics

Symmetries and symmetry breaking are of fundamental importance in high-energy physics. This has to do with the close connection between symmetries and conserved quantities (Noether’s theorem). If, experimentally, a conserved quantity is identified, the question arises as to what symmetry is responsible for it. For example, a one-parameter quantity (a charge, say) involves an underlying  $U(1)$  symmetry. One then has to reconstruct the initial underlying symmetry that by symmetry breaking leads to the experimentally observable lower symmetries. In this way high-energy physics may be understood as a sequence of symmetry breakings. We should note,

however, that in high-energy physics the relevant symmetries are usually *local* (*i.e.* gauge-) symmetries rather than the global symmetries considered in condensed-matter physics. (This implies an Anderson–Higgs mechanism rather than Goldstone modes [27, 28]). On the other hand, even in high-energy physics, namely in the theory of strong interactions, there is an example of a globally broken symmetry that is very similar to those considered here.

For this let us consider an isodoublet of quarks with only two flavours, namely an up- and a down-quark ( $u, d$ ). Initially there is a chiral invariance between left-handed (L) and right-handed (R) states (we neglect the weak interaction):

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} u \\ d \end{pmatrix}_R. \quad (5)$$

The corresponding symmetry, *i.e.* separate rotations with respect to left and right, is represented by

$$G = \text{SU}(2)_L \times \text{SU}(2)_R. \quad (6)$$

(For simplicity, we have neglected two additional U(1) symmetries in (6): one related to baryon-number conservation and another connected with an axial symmetry; the latter is not realized in nature and is believed to be broken by strong interactions (“the U(1) problem” [29]).) Experimentally, however, we know that the invariance under the full symmetry group (6) is not observed — a fact attributed to a broken symmetry. Indeed, one imagines a condensation between quarks ( $u, d$ ) and antiquarks ( $\bar{u}, \bar{d}$ ) to take place — very similar to Cooper pairing. In this way the symmetry (6) is broken down to a remaining  $\text{SU}(2)_{L+R}$ , in analogy to the B phase. Here we again encounter a relative broken symmetry where now the corresponding Goldstone bosons are the three pions.

A different example, now involving gauge symmetries in the theory of electroweak interactions, is a model due to Pati and Salam [30], an extension of the Weinberg–Salam model. Again we consider left/right isodoublets of  $u, d$  quarks, and, in addition, a hypercharge. The left/right symmetries are both described by an  $\text{SU}(2)$ , the latter one by a U(1) symmetry. Hence the underlying symmetry is

$$G = \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_{Y/2} \quad (7)$$

(note that this group is essentially identical with the symmetry group (3) describing superfluid  $^3\text{He}$ !). On an experimental level at low energies only the subgroup  $H = \text{U}(1)_{L_z + Y/2}$ , *i.e.* ordinary QED, is realized. It involves the  $z$  components of the left-handed isospin (the  $\text{SU}(2)_R$  is assumed to be

completely broken) and the hypercharge, whose generator is the ordinary electric charge  $Q = L_z + Y/2$ . This implies a broken “relative” symmetry involving two symmetries.

### 5. Orientational effects on the order parameter

For a pair-correlated superfluid, the pairing interaction is the most important interaction, since it is responsible for the formation of the condensate itself. Nevertheless, there also exist other, much weaker, interactions, which may not be important for the actual transition to the pair-condensed state, but which do become important if their symmetry differs from the aforementioned. In particular, they may be able to break remaining degeneracies.

**The dipole–dipole interaction:** The dipole–dipole interaction between the nuclear spins of the  $^3\text{He}$  atoms leads to a very weak, spatially strongly anisotropic, coupling. The relevant coupling constant  $g_D(T)$  is given by

$$g_D(T) \approx \frac{\mu_0^2}{a^3} \left( \frac{\Delta(T)}{E_F} \right)^2 n. \quad (8)$$

Here  $\mu_0$  is the nuclear magnetic moment, such that  $\mu_0^2/a^3$  is the average dipole energy of two particles at relative distance  $a$  (the average atomic distance), while the second factor measures the probability for these two particles to form a Cooper pair and  $n$  is the overall particle density. Since  $\mu_0^2/a^3$  corresponds to about  $10^{-7}\text{K}$ , this energy is extremely small and the resulting interaction of quasiparticles at temperatures of the order of  $10^{-3}\text{K}$  might be expected to be completely swamped by thermal fluctuations. This is indeed true in a normal system. However, the dipole–dipole interaction implies a spin–orbit coupling and thereby has a symmetry different from that of the pairing interaction. In the condensate the symmetries with respect to a rotation in spin and orbital space are spontaneously broken, leading to long-range order (for example of  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  in the case of  $^3\text{He-A}$ ). Nevertheless, the pairing interaction does not fix the *relative* orientation of these preferred directions, leaving a continuous degeneracy. As pointed out by Leggett [14,31–33], in this situation the tiny dipole interaction is able to lift the degeneracy, namely by choosing that particular relative orientation of the long-range ordered preferred directions for which the dipolar energy is minimal. Thereby this interaction becomes of *macroscopic* importance. One may also view this effect as a *permanent* local magnetic field of about 3 mT at any point in the superfluid (in a liquid!). In  $^3\text{He-A}$  the dipolar interaction is minimized by a parallel orientation of  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$ .

**Effect of a magnetic field:** An external magnetic field acts on the nuclear spins and thereby leads to an orientation of the preferred direction in spin

space. In the case of  ${}^3\text{He-A}$  the orientation energy is minimal if  $\hat{\mathbf{d}}$  is perpendicular to the field  $\mathbf{H}$ , since (taking into account  $\hat{\mathbf{d}} \cdot \mathbf{S} = 0$ ) this orientation guarantees  $\mathbf{S} \parallel \mathbf{H}$ .

**Walls:** Every experiment is performed in a volume of finite size. Clearly, the walls will have some effect on the liquid inside. In superfluid  ${}^3\text{He}$  this effect may readily be understood by using a simple picture. Let us view the Cooper pair as a kind of giant “molecule” of two  ${}^3\text{He}$  quasiparticles orbiting around each other. For a pair not to bump into a wall, this rotation will have to take place in a plane parallel to the wall. In the case of  ${}^3\text{He-A}$ , where the orbital angular momentum  $\hat{\mathbf{l}}$  has the same direction in all Cooper pairs (standing perpendicular on the plane of rotation), this means that  $\hat{\mathbf{l}}$  has to be oriented perpendicular to the wall. So there exists a strict orientation of  $\hat{\mathbf{l}}$  caused by the walls [34]. In the B phase, with its (pseudo) isotropic order parameter, the orientational effect is not as pronounced, but there are qualitatively similar boundary conditions.

### 5.1. Textures and defects

From the above discussion, it is clear that the preferred directions  $\hat{\mathbf{l}}$  and  $\hat{\mathbf{d}}$  in  ${}^3\text{He-A}$  are in general subject to different, often competing, orientational effects (for simplicity, we shall limit our description to  ${}^3\text{He-A}$ ). At the same time, the condensate will oppose any spatial variation of its long-range order. Any “bending” of the order-parameter field will therefore increase the energy, thus giving an internal stiffness or rigidity to the system. While the orientational effects might want  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  to adjust on the smallest possible length-scale, the bending energy wants to keep the configuration as uniform as possible. Altogether, the competition between these two opposing effects will lead to a smooth spatial variation of  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  throughout the sample, called a “texture”. This nomenclature is borrowed from the physics of liquid crystals, where similar orientational effects of the preferred directions occur.

The bending energy and all quantitatively important orientational energies are invariant under the replacement  $\hat{\mathbf{d}} \rightarrow -\hat{\mathbf{d}}, \hat{\mathbf{l}} \rightarrow -\hat{\mathbf{l}}$ . A state where  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  are parallel therefore has the same energy as one where  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{l}}$  are antiparallel. This leads to two different, degenerate ground states. There is then the possibility that in one part of the sample the system is in one ground state and in the other in a different ground state. Where the two configurations meet they form a planar “defect” in the texture, called a “domain wall” [35]. This is in close analogy to the situation in a ferromagnet composed of domains with different orientations of the magnetization. Domain walls are spatially localized and are quite stable against external perturbations. In fact, their stability is guaranteed by the specific nature of the

order-parameter structure of  $^3\text{He-A}$ . Mathematically, this structure may be analysed according to its topological properties; for reviews see the articles by Mermin [36] and Mineev [37]. The stability of a domain wall can then be traced back to the existence of a conserved “topological charge”. Using the same mathematical approach, one can show that the order-parameter fields of the superfluid phases of  $^3\text{He}$  not only allow for planar defects but also for point and line defects, called “monopoles” and “vortices”, respectively. Defects can be “non-singular” or “singular”, depending on whether the core of the defect remains superfluid or whether it is forced to become normal liquid. The concept of vortices is, of course, well-known from superfluid  $^4\text{He}$ . However, since the order-parameter structure of superfluid  $^3\text{He}$  is so much richer than that of superfluid  $^4\text{He}$ , there exist a wide variety of different vortices in these phases. Their detailed structure has been the subject of intensive investigation, in particular in the context of experiments on rotating superfluid  $^3\text{He}$ , where they play a central role [38,39].

## 6. Relation to other fields

Why spend so much effort on sorting out the strange behaviour of states of matter only found at temperatures so low that they are even outside the reach of most low-temperature laboratories? Partly, of course, “because it is there”, and because — like any other system — superfluid  $^3\text{He}$  deserves to be studied in its own right. However, what is even more important is that superfluid  $^3\text{He}$  is a model system that exemplifies many of the concepts of modern theoretical physics and, as such, has given us, and will further provide us, with new insights into the functioning of quantum-mechanical many-body systems close to their ground state.

**Quantum amplifier:** The macroscopic coherence of superfluid  $^3\text{He}$  may serve as a “quantum amplifier” of weak and ultraweak interaction effects. One example (see Section 5) is the dipole–dipole interaction between the Helium nuclei. Even more spectacular is the prediction by Leggett [40] of a permanent electric dipole moment of the Cooper pairs in the B phase due to the amplification of the parity-violating part of the electron–proton interaction (“weak neutral currents”) mediated by the exchange of  $Z^0$  bosons; for a detailed discussion see [13]. The permanent parity-violating dipole moment in the B phase is quite small but is not hopelessly outside the experimentally accessible range. The weak interaction between electrons and hadrons also has a  $T$ - (*i.e.*  $CP$ -)violating part. A measurement of the electric dipole moment of the B phase would then allow one to determine the strength of the  $T$ -odd interaction [41].

**Unconventional superconductors:** Anisotropic superconducting states are particularly exciting. It is now well established that superconductivity in the high- $T_c$  cuprates and, at least in some cases, in the so-called “heavy-fermion” systems, is due to the formation of anisotropic pairs with  $d$ -wave symmetry [42]. Many of the concepts and ideas developed for superfluid  $^3\text{He}$  have been adapted to these systems.

**Neutron stars:** There are several other physical systems for which the ideas developed in the context of superfluid  $^3\text{He}$  are relevant. One of them is an anisotropic superfluid system that already exists in nature but is not accessible for laboratory experiments: this is the nuclear matter forming the cores of neutron stars. There the pairing of neutrons has been calculated to be of  $p$ -wave symmetry. Because of the strong spin-orbit nuclear force, the total angular momentum of the Cooper pairs is predicted to be  $J = 2$  [43,44].

**Cosmology and early universe:** Recently superfluid  $^3\text{He}$  has been used as a test system for the creation of “cosmic strings” in the early stages of the universe. According to Kibble [45] and Zurek [46] the observed inhomogeneity of matter in the universe may be understood as the result of the creation of defects generated by a rapid cooling through second-order phase transitions, which led to the present symmetry-broken state of the universe. In two different experiments with superfluid  $^3\text{He}$ , performed at the low-temperature laboratories in Grenoble [47] and Helsinki [48], a nuclear reaction in the superfluid, induced by neutron radiation, caused a local heating of the liquid into the normal state. During the subsequent, rapid cooling back into the superfluid state the creation of a vortex tangle was observed. The experimentally determined density of this defect state was found to be consistent with Zurek’s estimate and thus gives important support to this cosmological model; see also [49]. Furthermore, a recent experimental verification of momentogenesis in  $^3\text{He-A}$  by Bevan *et al.* [50] was found to support current ideas on cosmological baryogenesis. (Baryogenesis during phase transitions in the early universe is believed to be responsible for the observed excess of matter over antimatter.) In view of these exciting new developments it may become possible in the future to model and study cosmological problems in the low-temperature laboratory in much more detail.

**Particle physics and gravity:** As discussed in Section 4, the key to understanding superfluid  $^3\text{He}$  is “spontaneously broken symmetry”. Volovik showed that the existence of point nodes (“Fermi points”) in the A phase order parameter allows one to draw detailed analogies with relativistic quantum field theory [25,26,51–53]. Thereby fundamental connections with particle physics exist which derive from the interpretation of the order-parameter field as a quantum field with a rich group structure. The collective modes of the order parameter as well as the localized topological defects in a given

ground-state configuration are the particles of this quantum field theory. Various anomalies (*e.g.* the axial anomaly) known from particle physics can be identified in the  $^3\text{He}$  model system, and it appears that fundamental insight gained from the experimental study of superfluid  $^3\text{He}$  will be useful in particle theory. Indeed, both the A phase of superfluid  $^3\text{He}$  and the electroweak vacuum belong to the class of gapless Fermi systems with topologically stable point nodes [25]. The fermionic (quasi-)particles in this class are chiral, and the collective bosonic modes are the effective gauge and gravitational fields [26, 52]. This provides a direct link between the dynamics of quantum liquids and gravity [53]. In particular, in the limit of very low energies these condensed matter systems reproduce all symmetries known today in high energy physics, *e.g.* Lorentz invariance, gauge invariance and general covariance.

Clearly, the unique richness of the structure of the superfluid phases of  $^3\text{He}$  continues to lead to new aspects whose investigation provides unexpected insights.

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