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Classes of Dynamical Systems Being Equivalent to a Jerky Motion

Distinguishing between dynamical and topological equivalence, we discuss transformations of three-dimensional dynamical systems to jerky dynamics. We present several different classes of dynamical systems that possess an equivalent jerky dynamics and discuss relations between dynamical and topological equivalence.

Mechanically speaking, any temporal evolution of a real scalar variable $x(t)$ that is governed by an autonomous third-order ordinary differential equation $\dddot{x} = J(x, \dot{x}, \ddot{x})$ can be considered as a *jerky motion* or a *jerky dynamics* [1–5]. This is because $x(t)$ can be interpreted as the position, t as the time and $\dddot{x}$ as the time-rate of change of acceleration called the jerk. Jerky dynamics have recently attracted some attention in the context of minimal systems that can already exhibit many major features of regular and irregular or chaotic dynamical behavior [1–5]. Although any jerky dynamics can be rewritten as a three-dimensional dynamical system $\dot{\mathbf{u}} = \mathbf{W}(\mathbf{u})$ with $\mathbf{u} = (x, v = \dot{x}, a = \ddot{x})^{\mathsf{T}}$ and $\mathbf{W}(\mathbf{u}) = (v, a, J(x, v, a))^{\mathsf{T}}$ [6], an arbitrary three-dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ *cannot* generally be recast in form of a jerky dynamics. Here, $\mathbf{x} = (x, y, z)^{\mathsf{T}}$ denotes a point in a three-dimensional phase space $\Gamma \subseteq \mathbb{R}^3$ and $\mathbf{V}(\mathbf{x})$ the vector field of the dynamical system. Given the initial conditions $\mathbf{x}(t = 0) = \mathbf{x}_0$, the evolution of $\mathbf{x}$ with respect to time t represents a trajectory of the dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ in the phase space.

Obviously, jerky dynamics constitute only a subclass of all three-dimensional dynamical systems. Our contribution deals with the circumstances under which a dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ can be transformed to an *equivalent* jerky dynamics $\dddot{x} = J(x, \dot{x}, \ddot{x})$. Without loss of generality we consider mainly jerky dynamics for the variable x.

Definition 1. (i) *A jerky dynamics is* dynamically equivalent *to a dynamical system if, for the same initial conditions, both generate the same signal $x(t)$.* (ii) *A jerky dynamics is* topologically equivalent *to a dynamical system if, additionally, any trajectory of the dynamical system corresponds to exactly one trajectory of the jerky dynamics if interpreted as a dynamical system, and vice versa.*

From the Definition 1, it immediately follows that topological equivalence implies dynamical equivalence. The contrary, however, is not true in general. To discuss wether there is a jerky dynamics $\dddot{x} = J(x, \dot{x}, \ddot{x})$ belonging to a dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$, we consider the transformation of variables given by $\mathbf{T} = (T_1, T_2, T_3)^{\mathsf{T}} : \mathbf{x} \mapsto \mathbf{u}$ with $\mathbf{u} = \mathbf{T}(\mathbf{x})$. Under certain circumstances that we want to specify here, such a transformation converts the original dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ into the system $\dot{\mathbf{u}} = \mathbf{W}(\mathbf{u})$ which is then equivalent to the jerky dynamics $\dddot{x} = J(x, \dot{x}, \ddot{x})$. From $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ one obtains $T_1(\mathbf{x}) = x$, $T_2(\mathbf{x}) = V_1(\mathbf{x})$ and $T_3(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \cdot \nabla V_1(\mathbf{x})$ with $\nabla = (\partial_x, \partial_y, \partial_z)^{\mathsf{T}}$. From the third component $\ddot{x} = a = T_3(\mathbf{x})$, one can derive an equation for the jerk that reads $\dddot{x} = \mathbf{V}(\mathbf{x}) \cdot \nabla[\mathbf{V}(\mathbf{x}) \cdot \nabla V_1(\mathbf{x})]$. This expression, however, depends on y and z. To obtain a jerky dynamics, it must be possible to express these variables by functions of $\mathbf{u} = (x, \dot{x}, \ddot{x})^{\mathsf{T}}$. These functions can only be obtained from the transformation $\mathbf{T}$. In particular if $\mathbf{T}$ is globally invertible with an inverse $\mathbf{T}^{-1} = (T_1^{-1}, T_2^{-1}, T_3^{-1})^{\mathsf{T}} : \mathbf{u} \mapsto \mathbf{x}$ with $\mathbf{x} = \mathbf{T}^{-1}(\mathbf{u})$, the variables y and z can be substituted by $T_2^{-1}(\mathbf{u})$ and $T_3^{-1}(\mathbf{u})$. According to Definition 1, the resulting jerky dynamics is then topologically equivalent to the original dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$.

Theorem 1. *Consider a dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ with $\mathbf{V}(\mathbf{x}) = \mathsf{c} + \mathsf{B}\mathbf{x} + \mathbf{n}(\mathbf{x})$. Here, $\mathsf{c} \in \mathbb{R}^3$ are constants, $\mathsf{B} = (b_{ij}) \in \mathbb{R}^{3\times 3}$ is a matrix with constant coefficients and the components of $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x}), n_3(\mathbf{x}))^{\mathsf{T}}$ denote solely nonlinear functions that are at least twice differentiable. Suppose that the $\dot{x}$-equation depends only linearly on z. Then, $n_1(\mathbf{x})$ can be written as $n_1(\mathbf{x}) = n(x) + m(x, y)$ where $n(x)$ contains all nonlinear terms that solely depend on x. This system possesses a topologically equivalent jerky dynamics*

(a) *if $m(x, y) \not\equiv 0$ and the conditions $b_{13} \neq 0$, $b_{33} = 0$, $b_{12}b_{23} = 0$, $b_{12}b_{22} + b_{13}b_{32} + c \neq 0$ and*

$$V_1(\mathbf{x})\partial_x m(x, y) + V_2(\mathbf{x})\partial_y m(x, y) + b_{12}n_2(\mathbf{x}) + b_{13}n_3(\mathbf{x}) = cy + f(x, b_{12}y + b_{13}z + m) \tag{1}$$

hold where $c \in \mathbb{R}$ is a real constant and f an arbitrary function of the indicated arguments, or

(b) *if $m(x, y) \equiv 0$ and the conditions $b_{12}^2 b_{23} - b_{13}^2 b_{32} + b_{12}b_{13}(b_{33} - b_{22}) \neq 0$ and*

$$b_{12}n_2(\mathbf{x}) + b_{13}n_3(\mathbf{x}) = g(x, b_{12}y + b_{13}z) \tag{2}$$

hold where g is an arbitrary function of the indicated arguments.

Proof. In both cases, it must be shown that the transformation $\mathbf{T}$ is invertible. For part (a) a sketch of the construction of $\mathbf{T}^{-1}$ follows. Since $T_1(\mathbf{x}) = x$ holds, x has not to be transformed. Using the conditions of part (a), the second and third component $T_2(\mathbf{x}) = V_1(\mathbf{x})$ and $T_3(\mathbf{x}) = \mathbf{V}(\mathbf{x})\cdot\nabla V_1(\mathbf{x})$ of the transformation $\mathbf{T}$ can be written as $T_2(\mathbf{x}) = r(x)+b_{12}y+b_{13}z+m(x,y)$ and $T_3(\mathbf{x}) = s(x,v)+(b_{12}b_{22}+b_{13}b_{32}+c)y+f(x,b_{12}y+b_{13}z+m)$. Here, we have used the abbreviations $r(x) = c_1 + b_{11}x + n(x)$ and $s(x,v) = v(b_{11}+\partial_x n)+c_2b_{12}+c_3b_{13}+(b_{12}b_{21}+b_{13}b_{31})x$, where $V_1(\mathbf{x})$ in s has been substituted by $v = \dot{x}$. Taking into account $v = T_2(\mathbf{x})$, one can express the second argument of f in terms of x and v, $b_{12}y+b_{13}z+m = v - r(x)$. Now, T_3 depends solely on x and v and linearly on y. Therefore, using $a = T_3(\mathbf{x})$ we can solve it with respect to y and obtain $y = T_2^{-1}(\mathbf{u})$. Inserting this equation into T_2 and solving it with respect to z yields $z = T_3^{-1}(\mathbf{u})$. For part (b), the construction of $\mathbf{T}^{-1}$ has been discussed in Ref. [5].

As a simple application, consider the dynamical system $\dot{x} = 1+z+xy$, $\dot{y} = x - y^2$, $\dot{z} = y - yz$. This system fulfills the conditions (1) of Theorem 1(a) with $c = 1$ and $f(x, b_{12}y+b_{13}z+m) = x^2$. Therefore, the transformation $\mathbf{T}$, reading componentwise $T_1(\mathbf{x}) = x$, $T_2(\mathbf{x}) = 1+z+xy$ and $T_3(\mathbf{x}) = 2y+x^2$, is invertible with $T_1^{-1}(\mathbf{u}) = x$, $T_2^{-1}(\mathbf{u}) = \frac{1}{2}(\ddot{x}-x^2)$ and $T_3^{-1}(\mathbf{u}) = -1+\dot{x}-\frac{1}{2}x(\ddot{x}-x^2)$. The resulting topologically equivalent jerky dynamics reads $\dddot{x} = 2x\dot{x}+2x-\frac{1}{2}(\ddot{x}-x^2)^2$. For this model, one can also obtain a jerky dynamics in y (cf. Theorem 1(b)). Its functional form is given by $\dddot{y} = 2y-\dot{y}^2-2y\ddot{y}+2y^2\dot{y}+y^4$ and is functionally different from the jerky dynamics in x.

Under certain circumstances, a dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ can possess a jerky dynamics also if $\mathbf{T}$ is not invertible. In that case, the jerky dynamics is dynamically equivalent. Then, it is possible to separate $\mathbf{T}$ into two transformations $\mathbf{S} : \mathbf{x} \mapsto (x,\eta,\zeta)^{\mathsf{T}}$ and $\Theta : (x,\eta,\zeta)^{\mathsf{T}} \mapsto \mathbf{u}$ such that $\mathbf{T} = \Theta \circ \mathbf{S}$ is valid. The non-invertible transformation $\mathbf{S}$ can be chosen such that it converts the original dynamical system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ into another dynamical system (e.g. of the form considered in Theorem 1) that possesses a *topologically* equivalent jerky dynamics. From the latter dynamical system the jerky dynamics is obtained by the invertible transformation Θ. This transformation is, at least in general, not the identity transformation.

Theorem 2. *Consider a dynamical system* $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ *with* $V_1(\mathbf{x}) = c_1 + b_{11}x + b_{12}y^k + b_{13}z^l + n(x) + m(x,y^k)$, $V_2(\mathbf{x}) = \frac{1}{k}y[b_{22}+\tilde{n}_2(x,y^k,z^l)]$, $V_3(\mathbf{x}) = \frac{1}{l}z[b_{33}+\tilde{n}_3(x,y^k,z^l)]$ *where* $c_1, b_{11}, b_{12}, b_{13}, b_{22}, b_{33}$ *are real constants and* $n(x), m(x,y^k), \tilde{n}_2(x,y^k,z^l), \tilde{n}_3(x,y^k,z^l)$ *denote arbitrary (at least twice differentiable) functions of the indicated arguments and* $k,l \in \mathbb{N}$ *with* $k>1$ *even or/and* $l>1$ *even. The nonlinearity* $n(x)$ *contains all nonlinear terms of* V_1 *that solely depend on* x. *This system has a dynamically equivalent jerky dynamics*

(a) *if* $m(x,y^k) \not\equiv 0$ *and the conditions* $b_{13}\neq 0, b_{33} = 0$, $b_{12}b_{22}+c \neq 0$ *and*

$$V_1\partial_x m(x,y^k) + V_2\partial_{y^k}m(x,y^k) + b_{12}y^k\tilde{n}_2(x,y^k,z^l) + b_{13}z^l\tilde{n}_3(x,y^k,z^l) = cy^k + f(x, b_{12}y^k + b_{13}z^l + m) \quad (3)$$

hold where $c \in \mathbb{R}$ *is a real constant and* f *an arbitrary function of the indicated arguments, or*

(b) *if* $m(x,y^k) \equiv 0$ *and the conditions* $b_{12}b_{13}(b_{33}-b_{22}) \neq 0$ *and*

$$b_{12}y^k\tilde{n}_2(x,y^k,z^l) + b_{13}z^l\tilde{n}_3(x,y^k,z^l) = g(x, b_{12}y^k + b_{13}z^l) \quad (4)$$

hold where g *is an arbitrary function of the indicated arguments.*

Proof. Setting $\eta = y^k$ and $\zeta = z^l$ as the second and third component of the transformation $\mathbf{S}$, one obtains the dynamical system $\dot{x} = c_1 + b_{11}x + b_{12}\eta + b_{13}\zeta + n(x) + m(x,\eta)$, $\dot{\eta} = b_{22}\eta + \eta\tilde{n}_2(x,\eta,\zeta)$, $\dot{\zeta} = b_{33}\zeta + \zeta\tilde{n}_3(x,\eta,\zeta)$. This system is of the form considered in Theorem 1. The conditions (3) and (4) correspond to (1) and (2), respectively. Therefore, there exists the transformation Θ and it is invertible.

As a simple application, consider the dynamical system $\dot{x} = x+y$, $\dot{y} = z^2$ and $\dot{z} = xz$. Choosing $\eta = y$ and $\zeta = z^2$, this system converts into $\dot{x} = x+\eta$, $\dot{\eta} = \zeta$ and $\dot{\zeta} = 2x\zeta$. The jerky dynamics reads $\dddot{x} = \ddot{x} + 2x(\ddot{x}-\dot{x})$.

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