Universität Augsburg

Dynamic Epistemic Semirings

Kim Solin

Report 2006-17 June 2006

Institut für Informatik
D-86135 Augsburg

Dynamic Epistemic Semirings

Kim Solin[⋆]

Turku Centre for Computer Science
Lemminkäinengatan 14 A, FIN-20520 Åbo, Finland
kim.solin@utu.fi

Abstract. This paper proposes a semiring formulation for reasoning about an agent's changing beliefs: a *dynamic epistemic semiring* (DES). A DES is a modal semiring extended with a revision operator. The revision operator is given a relational interpretation and a basic calculus is developed – based on the revision operator a contraction operator is also defined. A DES only models actions on an agent's beliefs, whereas the *real dynamic epistemic semirings* also introduced in this paper facilitate actions on the world as well. To allow for iterated action both structures are extended with the Kleene star.

1 Introduction

Formal reasoning about epistemic and doxastic notions in the 20th century can be traced back to G.H. von Wright's *An Essay in Modal Logic* [13] and Hintikka's *Knowledge and Belief* [5]. These books were written as attempts to philosophically clarify our notions of knowledge and belief using modal logic. In the works of von Wright and Hintikka, the ones believing or knowing – the agents – cannot change their beliefs, *i.e.* only static aspects of an agent's beliefs can be reasoned about. Alchourron, Gärdenfors and Makinson presented a semiformal framework for reasoning about an agents changing beliefs [1] – this constituted the prelude for the vast amount of research on belief revision done in the last two decades. Segerberg was one of the first to make these streams flow together. In an array of papers (see e.g. [9–11]) he proposes and develops a fully formal framework for reasoning about an agent's changing beliefs: dynamic doxastic logic. Dynamic doxastic logic is, as the name discerns, a blend of dynamic and doxastic logic. In more recent times, formal reasoning about knowledge has been put to use in computer science, the standard references seem to be [4, 6, 12].[1]

Since Kozen published his axiomatisation of Kleene algebra (an idempotent semiring with the Kleene star as the least fixpoint with respect to the canonical order) [7] and also showed how to use an elaboration of it for program reasoning [8], there has been significant development and application of something

⋆ Work done while visiting Institut für Informatik, Universität Augsburg.

1 The epithet epistemic is usually used in computer science, whereas in philosophical logic one tends to differ between epistemic and doxastic. We will henceforth use 'epistemic', expect when referring to Segerberg's logic.

that could be called semiring structures (some form of semiring equipped with additional operators). The spirit in this development lies very much in the calculational prospect of abstract algebra – tedious model-theoretic reasoning can be reduced to simple, symbol-pushing calculations. One such development are the modal semirings by Desharnais, Möller and Struth [3]. A modal semiring is a semiring structure including a domain operator facilitating the definition of modal operators in the sense of dynamic logic.

Our intent is to let yet another stream run up to Segerberg's uniting work by viewing some aspects of dynamic doxastic logic from the point of modal semirings. In this paper we propose a modal semiring extended with a revision operator: a *dynamic epistemic semiring*. This structure allows us to reason about an agent's changing beliefs in an elegant, calculational fashion. The carrier elements of the algebra are viewed as epistemic actions – actions working on the agent's beliefs. To check whether the agent believes a proposition we introduce special actions: *epistemic tests*. Epistemic tests work like guards in program theory, *i.e.* programs that check if some predicate holds or not. The dynamic epistemic semirings do not allow actions on the world, or in other words, real action is excluded (we assume that the agent's beliefs are not a part of the world). As a first step towards a semiring incorporating also real action, we introduce *real dynamic epistemic semirings*. These are a combination of a modal semiring – representing the world – and an dynamic epistemic semiring – representing the agent's beliefs. To allow iterated action both the dynamic epistemic semirings and the real dynamic epistemic semirings are extended with the Kleene star.

The structure of the paper is: Section 2 defines modal semirings. In Section 3 the revision operator is introduced, in Section 4 we provide a relational model and in Section 5 we develop a basic calculus. Section 6 elaborates the dynamic epistemic semirings to the real dynamic epistemic semirings. The final section before the concluding discussion extends the semirings with Kleene star.

2 Modal semirings

By an *idempotent semiring* we shall understand a structure over the signature $(+, \cdot, 0, 1)$ such that the reduct over $(+, 0)$ is a commutative and idempotent monoid, and the reduct over $(\cdot, 1)$ is a monoid such that $\cdot$ distributes over $+$ and is strict with respect to 0 $(0 \cdot a = 0 = a \cdot 0)$. When no risk for confusion arises $\cdot$ will be left implicit. We define the *canonical order* $\leq$ on a semiring by $a \leq b \Leftrightarrow_{df} a + b = b$ for all a and b in the carrier set. With respect to the canonical order, 0 is the least element, $\cdot$ as well as $+$ are isotone and $+$ is join.

A *test semiring* [2, 8] is a two-sorted algebra

$$(S, \mathsf{test}(S), +, 0, \cdot, 1)$$

such that $(S, +, \cdot, \neg, 0, 1)$ is a semiring, $(\mathsf{test}(S), +, \cdot, \neg, 0, 1)$ is a Boolean algebra (BA) and $\mathsf{test}(S) \subseteq S$. Join and meet in $\mathsf{test}(S)$ are thus $+$ and $\cdot$, respectively, and the complement is denoted by $\neg$; 0 is the least and 1 is the greatest element.

We shall use $a, b, \ldots$ for general semiring elements and $p, q, \ldots$ for test elements. On a test semiring we axiomatise a domain operator $\ulcorner : S \to \mathsf{test}(S)$ by

$$a \leq \ulcorner a \cdot a, \tag{1}$$

$$\ulcorner(pa) \leq p \text{ and} \tag{2}$$

$$\ulcorner(a \ulcorner b) \leq \ulcorner(ab), \tag{3}$$

for all $a \in S$ and $p \in \mathsf{test}(S)$ [2]. Inequalities (d1) and (d3) can be strengthened to equalities. The domain operator satisfies stability of tests,

$$\ulcorner p \;=\; p \;, \tag{4}$$

and additivity,

$$\ulcorner(a+b) \;=\; \ulcorner a + \ulcorner b \;. \tag{5}$$

With the aid of the domain operator, we can define modal operators [3] by

$$[a]p =_{df} \ulcorner(ap) \quad \text{and} \quad [a]p =_{df} \neg\langle a\rangle\neg p \;. \tag{6}$$

Therefore we shall call a test semiring with a domain operator *modal*.[2] The diamond can be read “after some way of performing a it will be the case that p holds” and the box as “after every way of performing a it will be the case that p holds”. The following fact

$$\text{if } p \leq p' \text{ and } [a]p = 1, \text{ then } [a]p' = 1 \tag{7}$$

is not hard to prove and will be used later.

3 Dynamic epistemic semirings

When reasoning about an agent’s beliefs and changes in the agent’s beliefs a common way to go is to talk about the agent’s *belief set* and operators that work on this set. The framework of Alchourron, Gärdenfors and Makinson [1] had three such operators, namely

- addition: add a belief to the agents belief set,
- revision: add a belief to agents belief set and revise the set so that it is consistent, and
- contraction: remove a belief and ensure that it is not implied.

The addition operator can thus possibly add a belief such that it contradicts some other belief. Moreover, it is possible for the agent to be ignorant (or indifferent) regarding a certain proposition: the agent might not believe a proposition, but this does not mean the he believes the negation of the proposition either. In the present paper the agent will never hold contradicting beliefs and we will not

[2] A relational model for modal semirings is given in Section 4.

consider addition, we will only attend to revision and contraction. Our agent will also not have the possibility to be ignorant, he will be fully opinionated and have something to say about everything under his sun. We will therefore talk about the agent's *belief state* paralleling talking about the state of the world.

We shall now define the dynamic epistemic semiring, but first we provide an informal intuition. The carrier set should be seen as actions on the agents beliefs and upon these actions we shall work with four operators:

- $+$ is choice: do either the left or the right action,
- $;$ is sequential composition of actions,
- $^{\ulcorner}a$ denotes those states of belief from which it is possible to perform the action a, and
- $\circledast p$ revises the the agent's beliefs by p (it is always possible to perform this action).

We will have two distinguished actions, the fail action 0 which always fails and the idle action 1 that leaves everything as it is. The tests (defined according to the test semirings and named et) should be seen as actions that check whether the agent believes a proposition or not: epistemic tests. If the agent believes the proposition, the epistemic test will behave like the idle action, otherwise it will behave like the fail action.

Definition 1. A *dynamic epistemic semiring* is a two-sorted algebra

$$\mathcal{D} = (D, \mathsf{et}(D), ;\ , +, \neg, \circledast, {}^{\ulcorner}, 0, 1)$$

such that the reduct structure over

$$(D, \mathsf{et}(D), ;\ , +, \neg, {}^{\ulcorner}, 0, 1)$$

is a modal semiring and $\circledast : \mathsf{et}(D) \to D$ is a unary prefix operator such that

$$
\begin{aligned}
p &\leq \circledast p && (8)\\
\circledast p; \circledast q &\leq \circledast q, && (9)\\
\circledast(p+q); p &= \circledast p, && (10)\\
\circledast p \leq \circledast q &\Rightarrow p \leq q \text{ and} && (11)\\
{}^{\ulcorner}\circledast p &= 1,\ p \neq 0 && (12)
\end{aligned}
$$

hold for any $p \in \mathsf{et}(D)$ when $\leq$ is the canonical order. $\triangleleft$

We define a contraction operator $\ominus p : \mathsf{et}(D) \to D$ by

$$\ominus p =_{df} \circledast\neg p \qquad (13)$$

for which the following basic properties hold:

Proposition 1. *In any* DES *the equations*

$$\neg p \le \ominus p, \tag{14}$$
$$\ominus p; \ominus q \le \ominus q, \tag{15}$$
$$\ominus(p;q); \neg p = \ominus p, \tag{16}$$
$$\ominus p \le \ominus q \Rightarrow q \le p \text{ and} \tag{17}$$
$$\ulcorner\ominus p = 1,\ p \neq 1 \tag{18}$$

hold.

Proof. All statements are direct consequences of (13) and the axioms of $\circledast$. □

4 A relational model

We shall now give a relational interpretation satisfying the axioms of the above structure. Given two binary relations $S, T \subseteq \Sigma \times \Sigma$, define the binary operators ; and + and the unary operator $\ulcorner$ on the relations by the following:

$$(x,y) \in (S;T) \Leftrightarrow_{df} \exists z \in \Sigma \bullet (x,z) \in S \text{ and } (z,y) \in T, \tag{19}$$
$$(x,y) \in (S+T) \Leftrightarrow_{df} (x,y) \in S \text{ or } (x,y) \in T \text{ and} \tag{20}$$
$$(x,x) \in \ulcorner S \Leftrightarrow_{df} (x,y) \in S \text{ for some } y \in \Sigma. \tag{21}$$

Moreover, denote the identity relation by Δ and define the unary prefix operators

$$\circledast, \ominus : \wp(\Delta) \to \wp(\Sigma \times \Sigma)$$

by, for any given $P \subseteq \Delta$,

$$\circledast P =_{df} \{(x,p) \mid x \in \Sigma \text{ and } (p,p) \in P\} \text{ and} \tag{22}$$
$$\ominus P =_{df} \{(x,p) \mid x \in \Sigma \text{ and } (p,p) \in \Delta - P\}. \tag{23}$$

Finally, let $\neg$ denote the complement relative to Δ and let $\emptyset$ denote the empty relation. We then have the following proposition.

Proposition 2. *Let Σ be any set. The structure*

$$(\wp(\Sigma \times \Sigma), \wp(\Delta), ;, +, \neg, \circledast, \emptyset, \Delta)$$

is a DES *when the operators and the constants are interpreted as above. Moreover, $\ominus P = \circledast\neg P$.*

Proof. The only axioms that need to be verified are those concerning $\circledast$ and $\ominus$, the rest have been verified in [2]. The first axiom is verified by

$$\begin{array}{ll} & (p,p) \in P \\ \Rightarrow & \quad \{\!\![\text{ property of binary relations }]\!\!\} \\ & (p,p) \in \{(x,p) \mid x \in \Sigma \text{ and } (p,p) \in \Sigma\} \\ \Leftrightarrow & \quad \{\!\![\text{ definition of } \circledast \text{ }]\!\!\} \\ & (p,p) \in \circledast P, \end{array}$$

the second by

$$
\begin{array}{ll}
& (x, y) \in \circledast P; \circledast Q \\
\Leftrightarrow & \quad \{\![\text{ definition of ; }]\!\} \\
& \exists z \in \Sigma \bullet (x, z) \in \circledast P \text{ and } (z, y) \in \circledast Q \\
\Rightarrow & \quad \{\![\text{ definition of } \circledast \ ]\!\} \\
& (x, y) \in \circledast Q,
\end{array}
$$

the third by

$$
\begin{array}{ll}
& (x, y) \in \circledast (P \cup Q); P \\
\Leftrightarrow & \quad \{\![\text{ definition of ; and } P \subseteq \Delta \ ]\!\} \\
& (x, y) \in \circledast (P \cup Q) \text{ and } (y, y) \in P \\
\Leftrightarrow & \quad \{\![\text{ definition of } \circledast \ ]\!\} \\
& (x, y) \in \circledast P \ ,
\end{array}
$$

the fourth by

$$
\begin{array}{ll}
& \circledast P \subseteq \circledast Q \\
\Leftrightarrow & \quad \{\![\text{ definition of } \circledast \ ]\!\} \\
& \{(x, p) \mid x \in \Sigma \text{ and } (p, p) \in P\} \subseteq \{(x, q) \mid x \in \Sigma \text{ and } (q, q) \in Q\} \\
\Rightarrow & \quad \{\![\text{ property of binary relations }]\!\} \\
& P \subseteq Q
\end{array}
$$

and, finally,

$$
\begin{array}{ll}
& \ulcorner \circledast P \\
\Leftrightarrow & \quad \{\![\text{ definition of } \circledast \ ]\!\} \\
& \ulcorner \{(x, p) \mid x \in \Sigma \text{ and } (p, p) \in P\} \\
\Leftrightarrow & \quad \{\![\text{ definition of } \ulcorner \text{and } P \neq \emptyset \ ]\!\} \\
& \Delta
\end{array}
$$

verifies the last axiom. That $\ominus P = \circledast \neg P$ is immediate from the definitions. □

5 Basic properties

In this section we develop a basic calculus for DESs. The first two propositions are helpful tools.

Proposition 3. *In any* DES *the equations*

$$\circledast p; p = \circledast p, \tag{24}$$

$$p \leq p; \circledast p, \tag{25}$$

$$\circledast p; \neg p = 0, \tag{26}$$

$$\ominus p; \neg p = \ominus p, \tag{27}$$

$$\neg p \leq \neg p; \ominus p \text{ and} \tag{28}$$

$$\ominus p; p = 0 \tag{29}$$

hold.

Proof. The first statement follows from axiom (10) setting $q = 0$. The second statement follows from epistemic tests being a BA, isotony of ; and axiom (8). The third statement is proved by

$$
\begin{array}{ll}
 & \circledast p \neg p \\
= & \quad \{\![\ (24) \ ]\!\} \\
 & \circledast p p \neg p \\
= & \quad \{\![\ \text{epistemic tests BA} \ ]\!\} \\
 & \circledast p 0 \\
= & \quad \{\![\ 0 \text{ strict} \ ]\!\} \\
 & 0,
\end{array}
$$

and the remaning statements follow directly from the first two and the definition of contraction (13). □

Proposition 4. *In any* DES *the equations*

$$\circledast p; \circledast p = \circledast p \text{ and} \tag{30}$$

$$\ominus p; \ominus p = \ominus p \tag{31}$$

hold.

Proof. The first statement is proved by

$$
\begin{array}{ll}
 & \circledast p \\
= & \quad \{\![\ (24) \ ]\!\} \\
 & \circledast p p \\
\leq & \quad \{\![\ (8) \ ]\!\} \\
 & \circledast p \circledast p
\end{array}
$$

and axiom (9) with $q = p$. The second statement follows directly from the first and the definition of contraction (13). □

The next proposition settles additivity of $\circledast$ and isotony of $\circledast$ and $\ominus$.

Proposition 5. *In any* DES *the equations*

$$\circledast(p + q) = \circledast p + \circledast q \tag{32}$$

$$p \leq q \Rightarrow \circledast p \leq \circledast q \tag{33}$$

$$p \leq q \Rightarrow \ominus q \leq \ominus p \tag{34}$$

hold. However, $\ominus$ is not additive, i.e.

$$\ominus(p + q) = \ominus p + \ominus q \tag{35}$$

does not hold in general.

Proof. The first statement (additivity) is proved by

$$
\begin{array}{ll}
 & \circledast(p+q) \\
= & \quad \{\![\ (24)\ ]\!\} \\
 & \circledast(p+q)(p+q) \\
= & \quad \{\![\ ;\ \text{distributive over}\ +\ ]\!\} \\
 & \circledast(p+q)p + \circledast(p+q)q \\
= & \quad \{\![\ (10)\ \text{and commutativity of}\ +\ ]\!\} \\
 & \circledast p + \circledast q\ ,
\end{array}
$$

the second one (isotony) is straightforward from the first and the third statement is a direct consequence of the second statement, the definition of contraction (13) and contraposition. For the last statement, consider any binary relation $P \neq \Delta$. Then $\ominus(P \cup \Delta) = \emptyset \neq \ominus P \cup \ominus\Delta$. □

Corollary 1. *In any* DES *the sets*

$$\{\circledast p \mid p \in \mathsf{et}(D)\} \tag{36}$$

and $\mathsf{et}(D)$ *are order-isomorphic. The set (36) is thus a* BA *with* $\circledast 0$ *and* $\circledast 1$ *as its least and greatest element, respectively.*

Proof. This follows directly from the isotony of star and axiom (11). □

The following proposition settles some relations between the constants, the domain operator and the epistemic operators.

Proposition 6. *In any* DES *the equations*

$$\circledast 0 = 0 = \ominus 1 \tag{37}$$

$$\ulcorner\circledast 0 = 0 = \ulcorner\ominus 1 \tag{38}$$

$$\ulcorner\ominus 0 = 1 = \ulcorner\circledast 1 \tag{39}$$

hold.

Proof. The first part is proved by

$$
\begin{array}{ll}
 & \circledast 0 \\
= & \quad \{\![\ (24)\ ]\!\} \\
 & \circledast 0 0 \\
= & \quad \{\![\ 0\ \text{strict}\ ]\!\} \\
 & 0 \\
= & \quad \{\![\ 0\ \text{strict}\ ]\!\} \\
 & \ominus 1 0 \\
= & \quad \{\![\ \text{epistemic tests}\ \mathsf{BA}\ ]\!\} \\
 & \ominus\neg 0 0
\end{array}
$$

$$
\begin{aligned}
&= \quad \{\!\![\ (27) \text{ and epistemic tests } \mathsf{BA}\ ]\!\!\} \\
&\quad \ominus\neg 0 \\
&= \quad \{\!\![\ \text{epistemic tests } \mathsf{BA}\ ]\!\!\} \\
&\quad \ominus 1.
\end{aligned}
$$

The second part follows from the first statement and the fact that $\ulcorner 0 = 0$ and the third statement follows from axiom (12) and the definition of contraction (13). □

Our last propositions settles that our semiring works as intended viewed in the light of the reasoning-about-knowledge tradition. After revising with p the agent's belief state should be p. This is implicit already in (24), but it can also be expressed more traditionally using the modal operators.

Proposition 7. *In any* DES *the equations*

$$[\circledast p]p = 1 \ \textit{and} \tag{40}$$

$$[\ominus p]\neg p = 1 \tag{41}$$

hold.

Proof. For the first part, calculate

$$
\begin{aligned}
& [\circledast p]p = 1 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
& \neg\langle\circledast p\rangle\neg p = 1 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
& \neg\ulcorner(\circledast p\neg p) = 1 \\
\Leftrightarrow & \quad \{\!\![\ (26)\ ]\!\!\} \\
& \neg\ulcorner 0 = 1 \\
\Leftrightarrow & \quad \{\!\![\ \text{modal semiring properties}\ ]\!\!\} \\
& \mathsf{true}.
\end{aligned}
$$

The second part follows directly from the first and the definition of contraction (13). □

Slightly misleading, one can say that after revising with a "rational" proposition (not the contradiction), the agent should not believe its negation. Similarly, after contracting with a "rational" proposition (not the tautology), the agent should not believe it.

Proposition 8. *In any* DES *the equations*

$$[\circledast p]\neg p = 0, \ \ p \neq 0 \ \textit{and} \tag{42}$$

$$[\ominus p]p = 0, \ \ p \neq 1 \tag{43}$$

hold.

Proof. The first part follows from

$$
\begin{array}{ll}
 & [\circledast p]\neg p = 0 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
 & \neg\langle \circledast p\rangle p = 0 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
 & \neg\ulcorner(\circledast p p) = 0 \\
\Leftrightarrow & \quad \{\!\![\ (24)\ ]\!\!\} \\
 & \neg\ulcorner \circledast\, p = 0 \\
\Leftrightarrow & \quad \{\!\![\ (12) \text{ and } (4)\ ]\!\!\} \\
 & \neg 1 = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{epistemic tests } \mathsf{BA}\ ]\!\!\} \\
 & \mathsf{true}.
\end{array}
$$

and the second one is a direct consequence of the first and the definition of contraction (13). □

If the agent performs an "irrational" epistemic action, he might as well believe anything.

Proposition 9. *In any* DES *the equations*

$$[\circledast 0]p = 1 \ \textit{and} \tag{44}$$

$$[\ominus 1]p = 1 \tag{45}$$

hold.

Proof. The first part is proved by

$$
\begin{array}{ll}
 & [\circledast 0]p = 1 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
 & \neg\langle \circledast 0\rangle \neg p = 1 \\
\Leftrightarrow & \quad \{\!\![\ (6)\ ]\!\!\} \\
 & \neg\ulcorner(\circledast 0 \neg p) = 1 \\
\Leftrightarrow & \quad \{\!\![\ (37)\ ]\!\!\} \\
 & \neg\ulcorner(0 \neg p) = 1 \\
\Leftrightarrow & \quad \{\!\![\ 0 \text{ strict}\ ]\!\!\} \\
 & \neg\ulcorner 0 = 1 \\
\Leftrightarrow & \quad \{\!\![\ (4) \text{ and epistemic tests } \mathsf{BA}\ ]\!\!\} \\
 & \mathsf{true}
\end{array}
$$

and the second part is a direct consequence of the first and the definition of contraction (13). □

6 Real dynamic epistemic semirings

In order to be able to express also propositions about the world and actions on the world we introduce the *real dynamic epistemic semirings*. The basic idea is that we work with two different kinds of actions: the real actions and the epistemic actions. Real actions act on the world, and epistemic actions act on the agent's belief state. World tests check whether some proposition holds in the current state of the world, whereas epistemic test play the same rôle as above: determining if the agent beliefs a certain proposition or not. There need be no connection between the state of the world and the belief state of the agent.

Definition 2. A *real dynamic epistemic semiring* is a two-sorted algebra

$$\mathcal{E} = (R \times E, \mathsf{wt}(R) \times \mathsf{et}(E), ;\ , +, \neg, \circledast, ^{\ulcorner}, (0_R, 0_E), (1_R, 1_E)) \tag{46}$$

such that

- the structures $(R, \mathsf{wt}(R), ;_R\ , +_R, \neg_R, {}^{R\ulcorner}, 0_R, 1_R)$ and $(E, \mathsf{et}(E), ;_E\ , +_E, \neg_E, \circledast_E, {}^{E\ulcorner}, 0_E, 1_E)$ constitute a modal and a dynamic epistemic semiring, respectively, with the canonical orderings $\leq_R$ and $\leq_E$;
- $\mathsf{wt}(R)$ is isomorphic to $\mathsf{et}(E)$ and
- the operators are defined by, for any $r, r' \in R, e, e' \in E, p \in \mathsf{wt}(R)$ and $u \in \mathsf{et}(E)$,

$$(r, e); (r', e') = (r;_R r', e;_E e') \tag{47}$$
$$(r, e) + (r', e') = (r +_R r', e +_E e') \tag{48}$$
$$\neg(p, u) = (\neg_R p, \neg_E u) \tag{49}$$
$$^{\ulcorner}(r, e) = ({}^{R\ulcorner}r, {}^{E\ulcorner}e) \tag{50}$$
$$\circledast(r, u) = (r, \circledast_E u) \tag{51}$$
$$\tag{52}$$

where $\circledast : R \times \mathsf{et}(E) \to R \times E$. The canoncial order $\leq$ is defined by $(r, e) \leq (r', e') \Leftrightarrow r \leq_R r'$ and $e \leq_E e'$. $\lhd$

In the sequel we will omit the subscripts, no confusion need arise. We define the contraction operator by

$$\ominus(r, p) =_{df} \circledast(r, \neg p)$$

for any $r \in R$ and $p \in \mathsf{et}(R)$.

The actions in in an RDES

$$\mathcal{E} = (R \times E, \mathsf{wt}(R) \times \mathsf{et}(E), ;\ , +, \neg, \circledast, ^{\ulcorner}, (0, 0), (1, 1))$$

can be divided in to three groups. The *pure real actions* $\{(r,1) \mid r \in R\}$ only affect the state of the world, whereas the *pure epistemic actions* $\{(1,e) \mid e \in E\}$ only affect the beliefs of the agent. An action is *mixed* when it belongs to the set $\{(r,e) \mid r,e \neq 1, r \in R$ and $e \in E\}$ and thus acts upon both the world and the agent's beliefs. Similarly the tests can be divided into three categories: *pure real tests* $\{(p,1) \mid p \in \mathsf{wt}(R)\}$, *pure epistemic tests* $\{(1,p) \mid p \in \mathsf{et}(D)\}$ and *mixed tests* $\{(p,p') \mid p,p' \neq 1, p \in \mathsf{wt}(D)$ and $p' \in \mathsf{et}(R)\}$. We also have special fail actions. For any $e \in D$, $(0,e)$ is a *paralyze action* – the agent fails to change the world after having performed this action. For any $r \in R$, $(r,0)$ is a *brain-killer action* – after having performed this action the agent can no longer hold any "rational" beliefs. The bottom element $(0,0)$ is the ultimate fail action – the agent cannot change the world nor his beliefs.

To change the agent's beliefs, we have the pure epistemic action $\circledast(1,p)$ – this action moves the agent to belief state p, but leaves the world unchanged. To illustrate, we prove the generalised case of (40). The box operator is defined according to (6).

Proposition 10. *In any* RDES *the equation*

$$[\circledast(1,p)](1,p) = (1,1) \tag{53}$$

holds.

Proof. The calculation

$$
\begin{aligned}
& [\circledast(1,p)](1,p) \\
= & \quad \{\!\![\ (6)\ ]\!\!\} \\
& \neg^{\ulcorner}(\circledast(1,p)\neg(1,p)) \\
= & \quad \{\!\![\ (49)\ ]\!\!\} \\
& \neg^{\ulcorner}(\circledast(1,p)(0,\neg p)) \\
= & \quad \{\!\![\ (51)\ ]\!\!\} \\
& \neg^{\ulcorner}((1,\circledast p)(0,\neg p)) \\
= & \quad \{\!\![\ (47) \text{ and } 0 \text{ strict}\ ]\!\!\} \\
& \neg^{\ulcorner}((0,\circledast p;\neg p)) \\
= & \quad \{\!\![\ (26)\ ]\!\!\} \\
& \neg^{\ulcorner}(0,0) \\
= & \quad \{\!\![\ (50)\ ]\!\!\} \\
& \neg({}^{\ulcorner}0,{}^{\ulcorner}0) \\
= & \quad \{\!\![\ (4)\ ]\!\!\} \\
& \neg(0,0) \\
= & \quad \{\!\![\ (49) \text{ and epistemic tests } \mathsf{BA}\ ]\!\!\} \\
& (1,1)
\end{aligned}
$$

proves the statement. □

To relate the the world and the agent's beliefs, conditions can be imposed on the elements. We let ϕ denote the bijective mapping between $\mathsf{wt}(R)$ and $\mathsf{et}(E)$ (written postfix). The state of the world p is thus reflected by the agent's belief state $p\phi$. To express "if p is the state of the world, then the agent's believe state should be $p\phi$" we can say that

$$(p, 1) \leq (p, p\phi). \tag{54}$$

If we assume (54) for some special $p \in \mathsf{wt}(R)$ we have the following result.

Proposition 11. *Let $p \in \mathsf{wt}(R)$ of an* RDES *and let $(p, 1) \leq (p, p\phi)$. Then*

$$[(a, 1)](p, 1) = (1, 1) \;\Rightarrow\; [(a, 1)](p, p\phi) = (1, 1) \tag{55}$$

holds.

Proof. The claim follows directly from (7). □

Any RDES has a relational interpretation as the following proposition shows. It is not hard to prove, but quite tedious and we therefore leave the calculations to the keener readers.

Proposition 12. *Any* RDES *satisfies the axioms of a* DES *and any* RDES *thus has a relational model.*

The relational model presented for the DES may seem rather crude when RDESs are considered. Another way of giving a relational model is to use the product of the two components' relational models – the operators are then defined in the classical way according to the axiomatisation above, but taking into account the special definition of $\circledast$. We thus have the following proposition.

Proposition 13. *Any* RDES *can be given a relational interpretation as a combination of the relational interpretations of the two underlying structures.*

7 Dynamic epistemic Kleene algebra

To incorporate iterated action into our framework we extend the semirings with Kleene star. A *Kleene algebra* is a structure over $(+, \cdot, ^*, 0, 1)$ such that the reduct $(+, \cdot, 0, 1)$ is an idempotent semiring and the star * satisfies the axioms

$$1 + aa^* \leq a^* \;, \qquad\qquad 1 + a^*a \leq a^* \;, \tag{56}$$

$$b + ac \leq c \Rightarrow a^*b \leq c \text{ and} \qquad\qquad b + ca \leq c \Rightarrow ba^* \leq c \;, \tag{57}$$

for a, b and c in the carrier set. When the semiring reduct has a test set and a domain operator the whole structure is called a modal Kleene algebra [3].

Definition 3. A *dynamic epistemic Kleene algebra* is a two-sorted algebra

$$\mathcal{D} = (D, \mathsf{et}(D), ;\ , +, ^*, \neg, \circledast, ^\ulcorner, 0, 1)$$

such that the reduct structure over

$$(D, \mathsf{et}(D), +, ;\ , \neg, \circledast, ^\ulcorner, 0, 1)$$

is a DES and the reduct structure over

$$(D, +, ;\ , ^*, 0, 1)$$

is a Kleene algebra. $\lhd$

Iteration a^* of an action a should be understood as a repetition of a of any finite length. A relational interpretation of star is the reflexive transitive closure: given a relation $R \subseteq \Sigma \times \Sigma$ we have

$$R^* = \bigcup_{i \in \mathsf{N}} R^i,$$

where $R^0 = \Delta$ and $R^{i+1} = R; R^i$. Intuitively, repeating $\circledast p$ more than once has no more effect than performing the action once. Indeed, it is not hard to prove that for any $p \in \mathsf{et}(D)$

$$\circledast p; (\circledast p)^* = (\circledast p)^*; \circledast p = \circledast p \tag{58}$$

holds.

A real dynamic epistemic Kleene algebra is defined analogously to an RDES having a modal Kleene algbra and a dynamic epistemic Kleene algebra as the underlying structures and letting $(r, e)^* = (r^*, e^*)$.

8 Concluding discussion

This paper brings about, so we believe, a resolute first step towards epistemic reasoning with semirings. But the theme is by no means played out, a lot remains to be done. It is an interesting task to provide a semiring formulation where the agent can be ignorant (or indifferent) regarding an issue: he does not believe a proposition, but he does not believe its negation either. One step further would be to also render contradictory beliefs possible – then the addition operator could be introduced. Yet another possibility is to extend the structure with a top element so that the revision operator could be defined – this might ease calculation. In the relational model such a top element does exist. The completeness of the revision operator with respect to the relational interpretation (in the sense that everything that can be proved in the relational interpretation can be proved in the axiomatisation) is a question that should be settled.

The most important question concerning the dynamic epistemic semirings is their usefulness in applications involving epistemic notions. A good place to start would be to redo old applications, in order to determine the practical value of our formulation.

We hope to be able to pursue these issues in future papers.

Acknowledgements. The author has benefited significantly from discussions with Bernhard Möller.

References

1. Alchourrón, C.E., Gärdenfors, P., Makinson, D.: On the logic of theory change: partial meet contraction and revision functions. *Journal of Symbolic Logic* 50(2):510-530, 1985.
2. Desharnais, J., Möller, B., Struth, G.: Kleene algebra with domain. *ACM Transactions on Computational Logic* (to appear 2006). Preliminary version: Universität Augsburg, Institut für Informatik, Report No. 2003-07, June 2003.
3. Desharnais, J., Möller, B., Struth, G.: Modal Kleene Algebra and Applications. A Survey. *Journal on Relational Methods in Computer Science.* 1(1):93 - 131, 2004.
4. Fagin, R., Halpern, J., Moses, Y., Vardi, M. Y.: *Reasoning About Knowledge.* MIT Press, 1995.
5. Hintikka, J.: *Knowledge and Belief: an Introduction to the Logic of the Two Notions.* Ithaca, N. Y. : Cornell University Press, 1962.
6. Meyer, J.-J.Ch., van der Hoek, W.: *Epistemic Logic for Computer Science and Artificial Intelligence.* Cambridge Tracts in Theoretical Computer Science 41, Cambridge University Press, 1995.
7. Kozen, D: A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366-390, 1994.
8. Kozen, D.: Kleene algebra with tests. *ACM Transactions on Programming Languages and Systems*, 19(3):427-443, 1997.
9. Segerberg, K: Proposal for a theory of belief revision along the lines of Lindström and Rabinowicz. *Fundamenta Informatica*, 32:183-191, 1997.
10. Segerberg, K: Two traditions in the logic of belief: bringing them together. In H.J. Ohlback and U. Reyle (eds.), *Logic, Language and Reasoning*, pp. 134-147, Kluwer, the Netherlands, 1999.
11. Segerberg, K: The basic dynamic doxastic logic of AGM. In M.-A. Williams and H. Rott (eds.), *Frontiers in Belief Revision*, pp. 57-84, Kluwer, the Netherlands, 1999.
12. Wooldridge, M.: *An Introduction to Multiagent Systems.* John Wiley & Sons, England, 2002.
13. Wright, G.H. von: *An Essay in Modal Logic.* Amsterdam : North-Holland, 1951.