Free, Isometric Circle Actions on Compact Symmetric Spaces

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Introduction

The existence of free circle actions (S^1-actions) on compact symmetric spaces is of importance in connection with constructing Riemannian manifolds of positive sectional curvature. Among the known compact, simply connected Riemannian manifolds of positive sectional curvature there are two infinite families occurring in dimensions 7 and 13 (cf. ([AW75, Esc82, Esc84, Baz96, Sha])). Both families are described as orbit spaces of free circle actions on symmetric spaces with Dynkin diagram A_2, namely as circle quotients of SU(3) and SU(6)/Sp(3). It has been shown by R. Bock in [Boc98] that the two other symmetric spaces of type A_2, namely SU(3)/SO(3) and E_6/F_4, do not admit free, isometric circle actions. In the present paper we ask: Which symmetric spaces allow free, isometric circle actions? We only need to answer this for the strongly irreducible, compact symmetric spaces i.e., those which are locally not Riemannian products.

Let S be a strongly irreducible, compact symmetric space and let $\tilde{G}$ denote the connected component of its isometry group. An *inner involution* is an element $u \in \tilde{G}$ such that $u^2 = \mathrm{id}$. Our main results are:

THEOREM 1. *Any inner involution on the symmetric space S without fixed points lies in a circle subgroup $U \subset \tilde{G}$ such that U acts freely on S.*

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This allows us to classify the irreducible symmetric spaces that admit free circle actions.

THEOREM 2. *Free circle actions exist precisely on the following compact, simply connected, strongly irreducible symmetric spaces:*

(1) *All simple compact Lie groups,*
(2) $\mathrm{SU}(2n)/\mathrm{SO}(2n)$ *for all* $n \geqslant 2$,
(3) $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ *for all* $n \geqslant 2$,
(4) $\mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$ *for all odd* $p, q \geqslant 1$.

THEOREM 3. *If S is not of group type, then the only free, isometric actions of* SU(2) *or of* SO(3) *are the usualHopf actions on the Grassmannians of odd dimensional subspaces in* R^{4m} (*Example* 4 *in Theorem* 2).

However, we show in Section 6 that the orbit space of a free circle action on a symmetric space $S = G/K$ with its induced metric has zero curvature 2-planes at every point unless S itself admits positive curvature, i.e., unless S is a compact, rank one, symmetric space. The above-mentioned metrics of positive curvature on the 7- and 13-dimensional orbit spaces are induced from nonsymmetric metrics on S. In fact, in both cases there is a subgroup $H \subset G$ of smaller dimension which still acts transitively on S, and S carries the normal homogeneous metric with respect to H. It remains open whether there exist nonsymmetric metrics on the Grassmannians for which the S^1-quotients (or the SU(2) or SO(3) quotients) admit positive curvature.

1. Preliminaries on Symmetric Spaces

Let S be a compact symmetric space. As usual we represent S as a coset space G/K where G is a compact, connected Lie group with an involutive automorphism σ (of order 2) called the *global symmetry* of S. Let $\hat{K} = \{g \in G; g^\sigma = g\}$ denote the fixed group of σ and let $\hat{K}^o$ denote its identity component. Then we have (cf. [Hel78] p. 212):

$$\hat{K}^o \subset K \subset \hat{K}, \tag{1.1}$$

and S may be viewed as a finite covering of the 'smallest' symmetric space $\hat{S} = G/\hat{K}$. The symmetry σ acts also on the Lie algebra $\mathfrak{g}$ of G and induces the Cartan decomposition $\mathfrak{g} = \boldsymbol{k} + \mathfrak{p}$ where $\boldsymbol{k}$ is the Lie algebra of K (and of $\hat{K}$). $\boldsymbol{k}$ is identified with the $(+1)$-eigenspace of σ_*, while $\mathfrak{p}$ is the (-1)-eigenspace which can also be viewed as the tangent space T_oS of S at the base point $o = eK$. As a consequence we have the Cartan relations:

$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k}, \quad [\boldsymbol{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \boldsymbol{k}. \tag{1.2}$$

We will need the following totally geodesic submanifold of G:

$$P = \exp(\mathfrak{p}) = \{x \in G;\, x^\sigma = x^{-1}\}^o \subset G, \tag{1.3}$$

where $\{\cdot\}^o$ always denotes the identity component (the connected component of the unit element $e \in G$). This subspace is known as *Cartan embedding* of the symmetric space. In fact the map

$$\phi\colon G \to G, \quad \phi(g) = g^\sigma g^{-1},$$

takes values in P since $(g^\sigma g^{-1})^\sigma = g(g^\sigma)^{-1} = (g^\sigma g^{-1})^{-1}$ and descends to an embedding of $\hat{S} = G/\hat{K}$ (since $\tilde{g}^\sigma \tilde{g}^{-1} = g^\sigma g^{-1}$ if and only if $\tilde{g} = gk$ for some $k \in \hat{K}$).

The Riemannian metric on S is induced by a bi-invariant metric on G, and G acts on S by isometries. But the action of G need not be effective; its kernel consists of all $z \in G$ with $z(gK) = gK$ for all $g \in G$ which is equivalent to $g^{-1}zg \in K$ for all $g \in G$. In particular, $z \in K$ and, moreover, $g^{-1}zg = (g^{-1}zg)^\sigma = (g^{-1})^\sigma z g^\sigma$. Thus all $x = g^\sigma g^{-1} \in P$ commute with z, but since P generates G as a group (recall from the Cartan relations (1.2) that $\mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra), G must commute with z and hence z lies in the center Z of G. Conversely, $Z \cap K$ clearly acts trivially on G/K. We have shown:

PROPOSITION 1.1. *The identity component of the isometry group of the symmetric space $S = G/K$ is $\tilde{G} = G/(Z \cap K)$, where Z is the center of G.*

PROPOSITION 1.2. *Suppose that $S = G/K$ is irreducible where K is connected and $\mathbf{k}$ has no outer automorphisms. Then $\hat{K} = K \cdot Z_\sigma$ where Z is the center of G and $Z_\sigma = \{z \in Z : z^\sigma = z\}$.*

Proof. We need to determine $\hat{K}/K$. Any $\hat{k} \in \hat{K}$ determines an automorphism $i(\hat{k})\colon\ k \mapsto \hat{k}x\hat{k}^{-1}$ of $\hat{K}^o = K$. This must be inner by assumption i.e., $i(\hat{k}) = i(k)$ for some $k \in K$. Hence, the coset $\hat{k}K$ contains some representative $\hat{k}^* = \hat{k}k^{-1}$ commuting with all of K. Without loss of generality we may assume $\hat{k} = \hat{k}^*$.

Now recall that the adjoint representations of K and $\hat{K}$ on $\mathfrak{g}$ leave $\mathfrak{p}$ invariant and their restrictions to $\mathfrak{p}$ are the isotropy representations of $S = G/K$ and $\hat{S} = G/\hat{K}$ respectively. By irreducibility of the isotropy representation of K we obtain $\mathrm{Ad}(\hat{k}) = \pm\mathrm{I}$ on $\mathfrak{p}$ unless this representation is complex and $\mathrm{Ad}(\hat{k})$ acts as a complex scalar. But in the latter case S is hermitian symmetric and $\hat{k}$ lies in the identity component K (in fact in the central S^1-factor). If $\mathrm{Ad}(\hat{k})$ is the identity on $\mathfrak{p}$, then $\hat{k}$ acts trivially on $\hat{S}$, hence $\hat{k} \in Z$. If $\mathrm{Ad}(\hat{k}) = -\mathrm{I}$ on $\mathfrak{p}$, then $\hat{k}$ acts as the symmetry σ on $\hat{S}$. Thus $\hat{k}$ belongs to G i.e., $\hat{S}$ is an *inner symmetric space* which is equivalent to saying that $\hat{K}$ contains a maximal torus T with $\hat{k} \in T$ (cf. [He178], p. 424f). This implies $\hat{k} \in \hat{K}^o = K$ which finishes the proof. □

2. Involutions

Let G be a compact, connected Lie group. Given $g \in G$, conjugation by the element g yields an automorphism of G, $i(g)(x) = gxg^{-1}$, which is called an *inner* by the set of automorphism. The fixed group of $i(g)$, elements of G commuting with g, evidently contains a maximal torus: the torus containing g. In particular, the fixed point groups of two inner involutions have conjugate maximal tori. The last statement can be generalized to arbitrary involutions.

PROPOSITION 2.1. *Let G be a compact, connected Lie group and let $\sigma, \bar{\sigma}$ be two involutions of G such that $\bar{\sigma} = \sigma\gamma$ for some inner automorphism $\gamma = i(g)$ i.e., $\gamma(x) = gxg^{-1}$ for some $g \in G$. Let K and $\tilde{K}$ be the identity components of the fixed groups of σ and $\tilde{\sigma}$. Then the maximaltori of K and $\tilde{K}$ are conjugate in G.*

Proof. We start by proving the statement for two special cases for $\bar{\sigma}$. Suppose first $\bar{\sigma} = \sigma\kappa$ with $\kappa = i(k)$ for some $k \in K$. Let $T_o \subset K$ be a maximal torus with $k \in T_o$. Extend T_o to a maximal torus T of G. Since T_o is maximal Abelian in K and contained in $K \cap T$ which is abelian, we get $T_o = K \cap T$. On the other hand

$$\tilde{K} \cap T = \{t \in T; \sigma\kappa t = t\} = \{t \in T; \sigma t = t\} = K \cap T.$$

Therefore the maximal torus T_o of K is contained in a maximal tours of $\tilde{K}$ (which extends $\tilde{K} \cap T$), and in particular we have also $k \in \tilde{K}$. Reversing now the roles of σ and $\bar{\sigma}$ we see that K and $\tilde{K}$ have in fact the same maximal torus T_o.

The assertion remains true if $\tilde{\sigma}$ is only conjugate to $\sigma\kappa$, say $\tilde{\sigma} = \alpha\sigma\kappa\alpha^{-1}$ for some $\alpha = i(a)$ for $a \in G$. But note that $\alpha\sigma = \sigma\alpha^{\sigma}$, where $\alpha^{\sigma} = \sigma\alpha\sigma = i(a^{\sigma})$. So we have in this case,

$$\bar{\sigma} = \sigma\alpha^{\sigma}\kappa\alpha^{-1} = \sigma\gamma,$$

for $\gamma = i(g)$ with $g = a^{\sigma}ka^{-1}$. To complete the proof we will show that any $g \in G$ can be represented as $g = a^{\sigma}ka^{-1}$ for some $k \in K$ and $a \in G$.

The group G acts on itself isometrically by $a \cdot x := a^{\sigma}xa^{-1}$ (for any $a, x \in G$). The orbit $G \cdot e$ is just the Cartan embedding of the symmetric space G/K into G, and its normal space $\nu_e(G \cdot e)$ at the unit element e is the Lie algebra $\boldsymbol{k} \subset \mathfrak{g}$ of K. It follows from a straightforward argument that any $g \in G$ is of the form $g = a \cdot k$ for some $a \in G$ and $k \in K = \exp(\boldsymbol{k})$. In fact, there is a shortest geodesic from g to the closed set $G \cdot e \subset G$. This geodesic meets $G \cdot e$ perpendicularly at some point $a \cdot e$. Thus $g = \exp_{a \cdot e}(\xi)$ for some normal vector $\xi \in \nu_{a \cdot e}(G \cdot e)$. Let $\xi_a = a^{-1} \cdot \xi \in \nu_e(G \cdot e) = \boldsymbol{k}$. Then $g = \exp_{a \cdot e}(a \cdot \xi_o) = a \cdot \exp_e(\xi_o) = a \cdot k$ for $k = \exp(\xi_o) \in K$, and we are done. $\square$

Remark 2.2. In the terminology of [HPTT94] and [Kol02], the conjugacy of the maximal tori of the fixed point groups K and $\hat{K}$ can be seen from the fact that these maximal tori are sections of the above-mentioned polar action of G on itself; the polar action is given by $a \cdot x := a^{\sigma}xa^{-1}$ (see also [dS56], Chapter II).

Remark 2.3. Given any connected Dynkin diagram, there is, up to conjugacy, at most one diagram automorphism of order 2. Since the full automorphism group is a (split) extension of the inner automorphism group, any two outer involutions differ by an inner automorphism.

3. Fixed Point Free Involutions

Since the circle group $S^1 \subset \mathbf{C}$ contains the element -1 of order 2, a free isometric circle action on a symmetric space $S = G/K$ can occur only if there is an involution in $\tilde{G}$ acting on S without fixed points, where $\tilde{G}$ is the identity component of the isometry group. So we ask when this is possible.

LEMMA 3.1. *An element $u \in G$ acting on $S = G/K$ has a fixed point if and only if u is conjugate to some element of K.*

Proof. An element $u \in G$ has a fixed point on $S = G/K$ if and only if $ugk = gK$ or $g^{-1}ug \in K$ for some $g \in G$ i.e., if and only if u is conjugate to some element of K. $\square$

If K is connected, then the above condition is equivalent to saying the the conjugacy class of u intersects a maximal torus of K. In particular, if S is an *inner symmetric space* i.e., if G and K have the same rank and hence share a maximal torus, then any conjugacy class meets K. Then any $u \in G$ acting on S has a fixed point. This fact rules out most symmetric spaces on Helgason's list ([Hel78], p. 518, 532f). The remaining types are AI, AII, DI, EI and EIV.

Moreover, by Proposition 2.1 and Remark 2.3, it suffices to consider one type of symmetric space for a given group G as long as the fixed groups are connected. The most convenient type is the so called *normalform* ; $S = G/K$ is said to be of normal form if a maximal torus T of G is contained in P, the Cartan embedding. (This means that the Satake diagram of S has only white points and no arrows; see [Hel78], pp. 426, 531, 532ff.)

LEMMA 3.2. *If $S = G/K$ is of normalform and if $\hat{K}\backslash K$ contains no elements of order 2, then any $u \in G$ with $u^2 = e$ is conjugate to an element of K and hence, has a fixed point on S.*

Proof. We may find $v \in T$ which is conjugate to u, and since $T \subset P$ we have $v^\sigma = v^{-1} = v$. Therefore $v \in \hat{K}$ and hence, $v \in K$ since v has order 2. $\square$

We now apply this lemma to the normal form spaces of types EI, AI and DI. Let us start with EI, the symmetric space $S = \mathrm{E}_6/\mathrm{PSp}(4)$ where $\mathrm{PSp}(4) = \mathrm{Sp}(4)/\{\pm \mathrm{I}\}$. From Proposition 1.2 we see that $\hat{K}\backslash K$ contains no order 2 elements since the center $Z(\mathrm{E}_6)$ is $\mathbf{Z}_3$ (cf. [He178], p. 516). Moreover the Dynkin diagram of $K = \mathrm{PSp}(4)$ is of type C_4, allowing no diagram automorphisms; hence K has no outer automorphism. So Lemma 3.2 implies that any $u \in G$ with $u^2 = e$ has a fixed point on S. By Proposition 2.1 the same is true for $G/\tilde{K}$ where $\tilde{K} = \mathrm{F}_4$ (note that $G/\tilde{K}$ is of type EIV). We have proved:

PROPOSITION 3.3. $\mathrm{E}_6/\mathrm{Sp}(4)$ *and* $\mathrm{E}_6/\mathrm{F}_4$ *do not admit any fixed point free, inner involutions.*

Next we consider the case AI where $S = \mathrm{SU}(n)/\mathrm{SO}(n)$. Here the symmetry σ is complex conjugation of matrices and therefore $\hat{K}$ is the set of all real matrices in $\mathrm{SU}(n)$ i.e., $\hat{K} = \mathrm{SO}(n) = K$. Lemma 3.2 shows that any order 2 element of $\mathrm{SU}(n)$ has a fixed point. If n is odd, $\mathrm{SO}(n)$ has no center and $\mathrm{SU}(n)$ acts effectively. We have shown

PROPOSITION 3.4. $\mathrm{SU}(n)/\mathrm{SO}(n)$ *admits no fixed point free involutions if n is odd.*

However, if $n = 2m$ is even, then $Z(\mathrm{SU}(2m)) \cap K = \{\pm \mathrm{I}\}$. So, we pass to the effective groups $G = \mathrm{SU}(2m)/\{\pm \mathrm{I}\}$ and $K = \mathrm{SO}(2m)/\{\pm \mathrm{I}\}$. Now $\hat{K}$ modulo sign consists of all matrices $A \in \mathrm{SU}(2m)$ which are either real or purely imaginary; in the latter case we have $\sigma(A) = \bar{A} = -A \equiv A \bmod \pm \mathrm{I}$. Since the product of two purely imaginary matrices is real, $\hat{K}$ is a degree 2 extension of K. An imaginary diagonal matrix $\hat{k}$ is of order 2 in $\mathrm{SU}(2m)/\{\pm \mathrm{I}\}$ if and only if $\hat{k} = i \cdot \mathrm{diag}(-\mathrm{I}_j, \mathrm{I}_{2m-j})$ with $m + j$ even (in order to have $\det \hat{k} = 1$). If $j \neq m$, then the eigenvalues of $\hat{k}$ do not come in conjugate pairs, thus $\hat{k}$ is not conjugate to a real matrix, i.e., not conjugate to an element of K. This defines a fixed point free involution on S. Hence, there are free $\mathbf{Z}_2$-actions on $S = \mathrm{SU}(2m)/\mathrm{SO}(2m)$, but not on $\hat{S}$ (by Lemma 3.2). By Proposition 3.1, the same holds for the spaces $\mathrm{SU}(2m)/\mathrm{Sp}(m)$, i.e., there are free $\mathbf{Z}_2$-actions on $S = \mathrm{SU}(2m)/\mathrm{Sp}(m)$, but not on $\hat{S} = G/\hat{K}$.

Finally we consider the case DI or more precisely the normal form symmetric space $S = \mathrm{SO}(2n)/\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n))$ for odd n (the Grassmannian of n-planes in $\mathbf{R}^{2n}$). The kernel of the action of $\mathrm{SO}(2n)$ on S is $\{\pm \mathrm{I}\}$; the symmetry σ on $G = \mathrm{SO}(2n)/\{\pm \mathrm{I}\}$ is given by conjugation with the matrix $\mathrm{diag}(-\mathrm{I}_n, \mathrm{I}_n)$ (up to sign). Note that $K = \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(n))/\{\pm \mathrm{I}\} \cong \mathrm{SO}(n) \times \mathrm{SO}(n)$ is connected while $\hat{K} = \mathrm{Fix}(\sigma)$ has another connected component formed by the off-diagonal block matrices $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ in $\mathrm{SO}(2n)$ with $\sigma(A) = -A \equiv A \bmod \pm \mathrm{I}$. Thus $\hat{K} = K \cup K \cdot J$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. We conclude from Lemma 3.2 that the space $\hat{S} = G/\hat{K}$ admits no fixed point free involutions, but S does (as we will see in Section 4).

4. Circle Actions

To complete the proof of Theorem 1, we now show that all remaining spaces admit free circle actions. Let us start with $\mathrm{SU}(2n)/\mathrm{SO}(2n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$. As we have seen, these can be treated together since the maximal tori of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(n)$ are conjugate within $\mathrm{SU}(2n)$. Recall from Section 3 that each $u \in G = \mathrm{SU}(2n)$ with $u^2 = e$ has a fixed point on $S = G/K$, where $K = \mathrm{SO}(2n)$ or $\mathrm{Sp}(n)$. Among these, only $u = -\mathrm{I}$ belongs to the kernel $Z \cap K$ of the action of G on S. Any circle subgroup $U \subset T \subset G$ contains an element of order 2. Thus the effective action of U can be free only if $-\mathrm{I} \in U$. Since $U \subset G$ is abelian, it can be extended to a maximal torus T of G, and by conjugacy we may assume that T is the torus of diagonal matrices. Hence U is of the form,

$$U = \{u(z) = \mathrm{diag}(z^{k_1}, \ldots, z^{k_{2n}});\ z \in S^1\}, \tag{4.1}$$

where all the k_i are odd integers with $\sum k_i = 0$; then $u(-1) = -\mathrm{I}$. To avoid ineffective coverings of S^1 we assume that the k_i are relatively prime. U acts freely if and only if the entries z^{k_i} for all $z \in S^1 \backslash \{\pm 1\}$ do not come in conjugate pairs. So for U to act freely, we must ensure (after possibly reordering) that we never have $z^{k_1+k_2} = \cdots = z^{k_{2n-1}+k_{2n}} = 1$ for any $z \neq \pm 1$. The failure of this condition would mean that the exponents have a common divisor not equal to 2. We have shown:

PROPOSITION 4.1. *Let* $U \subset \mathrm{SU}(2n)$ *be conjugate to a subgroup of the form* (4.1). *Then* U *acts freely on* $\mathrm{SU}(2n)/\mathrm{SO}(2n)$ *or an* $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ *if and only if all the integers* k_i *are odd and for any permutation* π *of the set* $\{1, \ldots, 2n\}$, *the greatest common divisor of the numbers* $k_{\pi(1)} + k_{\pi(2)}, k_{\pi(3)} + k_{\pi(4)}, \ldots, k_{\pi(2n-1)} + k_{\pi(2n)}$ *is* 2.

Remark 4.2. For the case SU(6)/Sp(3) we encounter the Bazaikin spaces ([Baz96], see also [Zil]). These were originally constructed as quotients of the homogeneous space $S' = \mathrm{SU}(5)/\mathrm{Sp}(2)$ by a circle group U. Here any $u \in U$ is a composition of certain left translations in SU(5) with the right translation by $u_0(z) = \mathrm{diag}(z, z, z, z, \bar{z}^4) = \mathrm{diag}(z \cdot \mathrm{I}_4, \bar{z}^4)$ (for $z \in S^1$) with commutes with the right action of Sp(2). However, note (by counting dimensions, for instance) that $\mathrm{SU}(5) \subset \mathrm{SU}(6)$ acts transitively on $S = \mathrm{SU}(6)/\mathrm{Sp}(3)$ with stabilizer $\mathrm{SU}(5) \cap \mathrm{Sp}(3) = \mathrm{Sp}(2)$ which implies $S' = S$. Moreover, the above right action of $u_0 \in \mathrm{SU}(5)$ on S is the same as the left action by $u_1 = \mathrm{diag}(z \cdot \mathrm{I}_5, \bar{z}^5) \in \mathrm{SU}(6)$. In fact, both transformations commute with the transitive action of SU(5), and applied to the base point $o = e \cdot \mathrm{Sp}(2) = e \cdot \mathrm{Sp}(3)$ we have $u_o \cdot o = u_1 \cdot o$ since $u_1 u_0^{-1} = \mathrm{diag}(z \cdot \mathrm{I}_4, z, \bar{z}^5) \cdot \mathrm{diag}(z \cdot \mathrm{I}_4, \bar{z}^4, 1)^{-1} = \mathrm{diag}(\mathrm{I}_4, z^5, \bar{z}^5) \in \mathrm{Sp}(3)$. Therefore, the two sided action of U on can be replaced with a left action of some $U' \subset \mathrm{SU}(6)$.

Remark 4.3. None of the subgroups $U' \subset \mathrm{SU}(2n)$ of Proposition 4.1 can be extended to a group isomorphic to SU(2) or SO(3). In fact it follows from the representation theory of SU(2) that a circle group $U \subset \mathrm{SU}(2n)$ can be extended if and only if, up to conjugation, its Lie algebra is generated by an orthogonal sum of vectors of the type $i \cdot \mathrm{diag}(-m, -m+2, \ldots, m-2, m)$ with $m \leqslant n$. But the eigenvalues of such matrices evidently come in conjugate pairs, so they violate the condition of Proposition 4.1.

It remains to consider the real, odd-dimensional, oriented Grassmannians $S = G/K$ with $G = \mathrm{SO}(p+q)$ and $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ for p,q odd (of type DI). The symmetry σ is conjugation by $\mathrm{diag}(-\mathrm{I}_p, \mathrm{I}_q)$. Its fixed group $\hat{K}$ consists of all block diagonal matrices $\mathrm{diag}(a, b) \in \mathrm{SO}(2n)$, where a is a $p \times p$ and b is a $q \times q$ matrix. Then $\hat{K} = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$ and $\hat{S} = G/\hat{K}$ is the Grassmannian of un-oriented p-planes in $\mathbf{R}^{p+q}$. The group G acts effectively on S, but not on $\hat{S}$; the kernel of the latter action is $\{\pm \mathrm{I}\}$. At the end of Section 3 we saw that the fixed group of the action of σ on $\tilde{G} = G/\{\pm \mathrm{I}\}$ is a proper extension of $\hat{K}/\{\pm \mathrm{I}\}$ in the case $p = q = n$. However, if $p \neq q$, then there is no such extension since there are no orthogonal off diagonal block matrices.

A maximal torus T of G consists of all block diagonal matrices $A = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_i = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix}$. Since p, q are odd, say $p = 2m - 1$, the subgroup $T \cap K$ is a maximal torus of K and consists of all matrices $A \in T$ with $a_m = \mathrm{I}_2$. Furthermore, $A \in T \cap \hat{K}$ if and only if $a_m = \pm\mathrm{I}_2$.

Up to conjugation, any circle subgroup $U \subset \mathrm{SO}(2n)$ with $2n = p + q$ lies in T and consists of elements of the form

$$u(z) = \mathrm{diag}(a(z)^{k_1}, \ldots, a(z)^{k_n}) \tag{4.2}$$

where $a(\mathrm{e}^{i\alpha}) := \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $k_1, \ldots, k_n$ are relatively prime integers. An element $u(z)$ is conjugate to an element of K if and only if $a(z)^{k_i} = \mathrm{I}_2$ for some i. To ensure freeness, we wish to avoid such a possibility and this is achieved for all $z \neq 1$ if and only if $k_i = \pm 1$ for all i. So, for a free action, the elements of U must be of the form

$$u(z) = \mathrm{diag}(a(z), \ldots, a(z)) \tag{4.3}$$

up to conjugacy in $\mathrm{O}(2n)$. This is simply the Hopf circle action on $\mathbf{R}^{2n} = \mathbf{C}^n$ and it descends to a free action on $\hat{S}$. There are no other free circle actions on $\hat{S}$ since any such action would lift to a free circle action on S. We have shown:

PROPOSITION 4.4. *The only free circle actions on* $S = \mathrm{SO}(2n)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$ *and on* $\hat{S} = \mathrm{SO}(2n)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$*, where p, q odd and $p + q = 2n$, are by subgroups conjugate to the type described in* (4.3).

It is well known that the Hopf circle action extends to a Hopf SU(2)-action on $\mathbf{R}^{2n}$ if n is even, yielding a free SU(2) action on S and a free SO(3)-action on $\hat{S}$. (Note that the action of $H \subset G$ on S is free if and only if the action of the maximal torus of H is free.)

We conclude that these are the only free actions of connected groups on strongly irreducible symmetric spaces that are not group type. This completes the proof of Theorem 1.

5. Reducible Symmetric Spaces

PROPOSITION 5.1. *Let $S = S_1 \times S_2$ be a Riemannian product of compact symmetric spaces. Then S admits a fixed point free inner involution if and only if either of S_1 or S_2 do.*

Proof. Let u be any inner involution on S. So $u = (u_1, u_2)$ where u_i are inner involutions on S_i. Then $\mathrm{Fix}(u) = \mathrm{Fix}(u_1) \times \mathrm{Fix}(u_2) \subset S$ is empty if and only if $\mathrm{Fix}(u_1) = \emptyset$ or $\mathrm{Fix}(u_2) = \emptyset$. $\square$

COROLLARY 5.2. *$S = S_1 \times S_2$ admits a free, isometric circle action if and only if S_1 or S_2 admit such an action.*

Proof. A free circle action of S_1 or S_2 can easily be extended to a free circle action of S. The converse follows from Proposition 5.1 and Theorem 1. $\square$

Remark 5.3. Of course there are many ways to extend a free circle action on S_1 to $S_1 \times S_2$; we may use, for instance, any circle action (not necessarily free) on S_2 and then act via the diagonal. An instructive example is that of the *Berger spheres*, $(\mathrm{S}^{2n-1} \times_{s^1} S^1)$.

6. Existence of Zero Curvature 2-Planes

PROPOSITION 6.1. *Let $S = G/K$ be any compact Riemannian symmetric space of rank $r \geqslant 2$ and $U \subset G$ a circle subgroup acting freely on S. Then the induced metric on the orbit space S/U has a zero curvature plane at every point.*

Proof. It suffices to show that at any point $o \in S$ there is a zero curvature 2-plane in T_oS which is horizontal i.e., perpendicular to the fiber $U(o)$ of the Riemannian submersion $\pi\colon S \to S/U$. The O'Neill tensor does not increase the curvature in this situation (cf. [Esc84, Esc92 and GM74]). Let $o \in S$ be arbitrary and let K be the identity component of the group of isometries of S which fixes o and hence acts linearly on $\mathfrak{p} = T_oS$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing a non-zero tangent vector $v \in T_o(U(o)) \subset \mathfrak{p}$.

Now we use a well known convexity theorem (see for instance [PT88]): The orthogonal projection of the orbit $K(v) \subset \mathfrak{p}$ onto $\mathfrak{a}$ is the convex hull of the finite set $W(v) \subset \mathfrak{a}$ where W is the Weyl group of S acting on $\mathfrak{a}$. We only need that the convex hull of $W(v)$ contains the origin 0. Then the $\mathfrak{a}$-projection of some $k(v)$ is 0. In other words, $k(v) \perp \mathfrak{a}$ or $v \perp \mathfrak{a}' := k^{-1} \cdot \mathfrak{a}$. Thus $T_o(U(o)) = \mathbf{R}v$ is perpendicular to $\mathfrak{a}'$ which means that $\mathfrak{a}'$ is a horizontal flat subspace of $\mathfrak{p}$ of dimension $r \geqslant 2$. $\square$

Remark 6.2. The only space (up to coverings) where the above proposition does not apply is the Grassmannian $S = G/K$, where $G = \mathrm{SO}(4m)$ and $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ for p, q odd and $p + q = 4m$. This space admits a free, isometric action of SU(2). However, the tangent space of the SU(2)-orbit at the base point $o = eK \in S$ is perpendicular to a maximal flat subspace of T_oS, and so $\mathrm{SU}(2)\backslash G/\mathrm{K}$ cannot have positive curvature at all points.

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References

[AW75] Aloff, S. and Wallach, N.: An infinite family of distinct 7-manifolds admitting positively curved riemannian structures, *Bull. Amer. Math. Soc.* **81** (1975), 93–97.

[Baz96] Bazaĭkin, Ya. V̄: On a family of 13-dimensional closed Riemannian manifolds of positive curvature, *Sibirsk. Mat. Zh.* **37**(6) (1996), 1219–1237, ii.

[Boc98] Bock, R.: Doppelquotienten ungerader Dimension und positive Schnittkūmmung, PhD thesis, Universität Augsburg, 1998.

[dS56] de Siebenthal, J.: Surles goupes de Lie compacts non connexes, *Comm. Math. Helv.* **31** (1956), 41–89.

[Esc82] Eschenburg, J.-H.: New examples of manifolds with strictly positive curvature, *Invent. Math.* **66**(3) (1982), 469–480.

[Esc84] Eschenburg, J.-H.: Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrūmmten Orbiträumen. Universität Münster, Schriftenreihe des Mathematisches Institut, Münster, 1984.

[Esc92] Eschenburg, J.-H.: Inhomogeneous spaces of positive curvature, *Differential Geom. Appl.* **2**(2) (1992), 123–132.

[GM74] Gromoll, D. and Meyer, W.: An exotic sphere with nonnegative sectional curvature, *Ann. of Math. (2)* **100** (1974), 401–406.

[Hel78] Helgason, S.: *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[HPTT94] Heintze, E., Palais, R., Terng, C.-L., and Thorbergsson, G.: Hyperpolar actions on symmetric spaces, In: S.-T. Yau (ed.), *Geometry, Topology and Physics for Raoul Bott*, International Press, Cambridge, MA, 1994.

[Ko102] Kollross, A.: A classification of hyperpolar and cohomogeneity one actions, *Trans. Amer. Math. Soc.* **354** (2002), 571–612.

[PT88] Palais, R. S. and Terng, C.-L.: *Critical Point Theory and Submanifold Geometry*, Springer-Verlag, Berlin, 1988.

[Sha] Shankar, K.: Strong inhomogeneity of Eschenburg spaces (with an Appendix by M. Dickinson and K. Shankar), *Michigan Math. J.* **50** (2002), 125–142.

[Zil] Ziller, W.: Lecture notes. University of Bonn, Summer 2001.