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ISOTROPIC PLURIMINIMAL SUBMANIFOLDS

J.-H. Eschenburg **R. Tribuzy**

Abstract

Pluriminimal immersions and more generally pluriharmonic maps into symmetric spaces allow a one-parameter family of isometric deformations rotating the differential; in fact, they are characterized by this property. We investigate the "isotropic" case where these deformations are trivial.

Resumo

Imersões plurimínimas e mais geralmente, aplicações pluriharmônicas em espaços simétricos, admitem uma família a um parâmetro de deformações isométricas, obtidas pela rotação da diferencial; de fato, elas são caracterizadas por esta propriedade. Nós investigamos caso isotrópico onde essas deformações são triviais.

1. Introduction

Minimal surfaces in euclidean or constantly curved 3-space locally allow an associated one-parameter family of isometric deformations preserving the principal curvatures but rotating the second fundamental form in the parameter plane; the most prominent example is the deformation of the catenoid into the helicoid. This phenomenon is not restricted to ambient spaces of dimension three or of constant curvature. When we wrote our first joint paper [EGT] in 1982, we were very surprised to encounter it for minimal surfaces in $\mathbb{C}P^2$. Nowadays it is well known that harmonic maps of Riemann surfaces into arbitrary symmetric spaces allow such an associated family of deformations and in fact, harmonicity is equivalent to this property (e.g. cf. [U], [DPW], [BFPP]). Recall that minimal surfaces are nothing else but harmonic isometric immersions of surfaces; in fact, "isometric" can be weakened to "conformal".

If one wants to pass from surfaces to higher dimensional immersed submanifolds, then minimality or harmonicity is no longer sufficient; instead, one introduces the notions of pluriharmonicity and pluriminimality: If S is a Riemannian and M a Kähler manifold, then a smooth map $f : M \to S$ is called *pluriharmonic* if its restriction to any complex curve in M (complex one-dimensional submanifold) is harmonic or equivalently, the (1,1)-part of its hessian vanishes, i.e. $\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = 0$ for any holomorphic coordinate chart $(z_1, ..., z_m)$. An isometric pluriharmonic immersion is called *pluriminimal* or *(1,1)-geodesic*. Thus its restriction to any complex curve in M is a minimal surface or equivalently, the (1,1)-part of its second fundamental vanishes. As in the surface case, a pluriminimal immersion need not really to be isometric but only *pluriconformal*, i.e. conformal when restricted to a complex curve in M: in fact, an immersion $f : M \to S$ is pluriconformal iff the induced metric on M is Kähler and compatible to the complex structure (cf. [ET3], Thm. 1). Now the phenomenon mentioned above passes over to the higher dimensional case (cf. [OV], [BFPP], [ET4]), and it is one of our aims to give a simple direct proof for this fact:

Theorem 1. *If M is Kähler and S a Riemannian symmetric space of nonpositive or nonnegative curvature, then a smooth map $f : M \to S$ (in particular, a pluriconformal immersion) is pluriharmonic iff it allows an associated family of deformations.*

A cheap way to obtain an isometric deformation of an immersion or a smooth map is to compose it with a one-parameter family of isometries of the ambient space; we will call such a deformation *trivial*. But note that the associated deformation of a (non-planar) minimal surface in a 3-space form must be nontrivial since it rotates the principal curvature directions: If such a deformation is by extrinsic isometries, than every direction must be a principal curvature direction, hence such a surface is minimal and umbilic, i.e. totally geodesic. This is different if the codimension is bigger than one: There are minimal surfaces and pluriminimal immersions (or pluriharmonic maps) with trivial associated families which (following a notation in [EW]) will be called *isotropic*. In $\mathbb{R}^4$, these are locally the graphs of holomorphic functions (after a suitable identification of $\mathbb{R}^4$ and $\mathbb{C}^2$), and an analogue is true for arbitrary dimensions and codimensions:

Theorem 2. *The isotropic pluriharmonic maps $f : M \to \mathbb{R}^{2n} = \mathbb{C}^n$ are precisely the holomorphic maps, up to isometries of $\mathbb{R}^{2n}$. There are no full isotropic pluriharmonic maps into $\mathbb{R}^{2n+1}$.*

In S^4, the isotropic minimal surfaces were called *superminimal* and classified by R. Bryant ([B], cf. also [ET1]). Long time before, E.Calabi had already given a less explicit description of isotropic minimal surfaces in S^n for any n ([C], cf. [L]) which was later extended to $\mathbb{C}P^n$ by Chern, Wolfson ([Ch], [ChW]) and Eells and Wood [EW]. Similar constructions in Grassmannians and other symmetric spaces followed (e.g. [BS], [BR], [K1], [ET3]). All those are projections of holomorphic and ("super"-)horizontal maps into certain complex flag manifolds or flag domains fibering over the given symmetric space (cf. [BR]); in Bryant's S^4-case this fibration was the classical *twistor* map which is the natural projection $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$. We will show that isotropic pluriharmonic maps generally are characterized in this way; the flag manifold will arise naturally from the submanifold geometry:

Theorem 3. *Let S be an inner symmetric space of compact (resp. noncompact) type and $f : M \to S$ a full smooth map. Then f is isotropic pluriharmonic if and only if there is a flag manifold (resp. flag domain) Z over S with canonical projection $\tau : Z \to S$ and a holomorphic superhorizontal map $\phi : M \to Z$ such that $f = \tau \circ \phi$.*

What made the subject specially attractive for us was the intimate relationship between submanifold geometry and symmetric space theory. E.g. full isotropic pluriminimal submanifolds arise only in a symmetric space which is *inner*, i.e. a geodesic symmetry (point reflection) belongs to the identity component of the isotropy group. Among spheres and other spaces of constant curvature, these are precisely the even dimensional ones which explains the fact already noticed by Calabi [C] that there are no full isotropic pluriminimal immersions in odd-dimensional space forms.

Fortunately, in the isotropic case there is no difference between pluriharmonic maps and pluriminimal immersions: An isotropic pluriharmonic map is automatically pluriconformal (cf. Ch. 3) and hence locally (outside a sigular set of complex codimension one) a pluriminimal immersion of a complex submanifold of M (cf. [ET3]).

In this survey article, we will outline the ideas of the proofs; further details can be found in the references, mainly in [ET3], [ET4].

2. Associated families

Let M be a simply connected (but possibly incomplete) *Kähler manifold*, i.e. M is Riemannian with a compatible parallel almost complex structure J on TM. This determines a one-parameter family of tensors (called *canonical rotations*) $\mathcal{R}_\theta = (\cos\theta)I + (\sin\theta)J$ where I denotes the identity on TM. Further, let S be a Riemannian symmetric space with curvature tensor R_S which will be considered as a parallel algebraic structure (Lie triple product) on any tangent space of S. An *associated family* for a smooth map $f_o : M \to S$ will be a smooth family of maps $f_\theta : M \to S$ such that roughly $df_\theta = df_o \circ \mathcal{R}_\theta$, more precisely

$$\Phi_\theta \circ df_\theta = df_o \circ \mathcal{R}_\theta \tag{1}$$

for some parallel bundle isomorphism $\Phi_\theta : f_\theta^*TS \to f^*TS$ preserving R_S. Thus, to construct an associated family for f_o, we have somehow to integrate the "rotated differential" $F_\theta := df_o \circ \mathcal{R}_\theta$.

This can be put into a more general framework (cf. [ET2]): If some F *looks like* the differential of a map $f : M \to S$, when is it *really* a differential $F = df$? "Looks like" means that F is a linear map on TM with values in some vector bundle E over M which is a candidate for f^*TS, i.e. its fibres have the dimension of S, and any fibre carries an inner product $\langle\ ,\ \rangle$ and a Lie triple product R_S which makes it isometrically isomorphic to any tangent space of S, and these structures are parallel with respect to a covariant derivative D on E. In the simplest case $S = \mathbb{R}$ or $S = \mathbb{R}^n$, our F is just a one-form on M which is exact ($F = df$) if and only if $dF = 0$, i.e.

$$(D_X F)Y = (D_Y F)X \tag{2a}$$

for any two vector fields X, Y on M. (We will denote any covariant derivative by D if no confusion is possible; here it denotes the Levi-Civita connection on T^*M.) In the general case, (2a) is still necessary (where D now denotes the induced connection on $Hom(TM, E)$) but not yet sufficient; a second necessary condition is

$$R_E(X, Y)A = R_S(FX, FY)A \tag{2b}$$

for any section A of E, where $R_E : TM \otimes TM \to \mathrm{End}(E)$ denotes the curvature tensor of (E, D). These equations are equivalent to the Cartan structure equations and thus yield integrability (cf. [ET2]) which means precisely $\Phi \circ F = df$ for some smooth $f : M \to S$ and some parallel bundle isomorphism $\Phi : f^*TS \to E$ preserving the Lie triple product R_S. To our knowledge, Equations (2a) and (2b) were stated first in [GKM] (p. 48, (5) and (6)).

In the following, we shall have to complexify the real vector bundles and extend all linear maps complex linearly. The reason is that we need the splitting of the complexified tangent bundle $T^cM = TM \otimes \mathbb{C}$ into the $\pm i$ eigenbundles for J (where i always denotes $\sqrt{-1}$). Due to the parallelity of J, this splitting is parallel with respect to the (complex linearly extended) Levi-Civita connection. On these subbundles, denoted by $T'M$ and $T''M = \overline{T'M}$, the application of $\mathcal{R}_\theta$ is easy enough: it is just the multiplication by the constant factor $e^{\pm i\theta}$. Since (2a) holds for $F_o = df_o$, we obtain (2a) also for $F_\theta = F_o \circ \mathcal{R}_\theta$ provided that X and Y are complex vectors of the same type (both in $T'M$ or both in $T''M$). On the other hand, if X and Y are of different type, then (2a) holds for F_θ if and only if both sides vanish since they take up the different factors $e^{i\theta}$ and $e^{-i\theta}$.

But the vanishing of $(D_X df)Y$ for different type vectors X and Y is precisely the pluriharmonicity of f.

Equation (2b) is true for f_θ provided that both sides vanish for f_o if X and Y have the same type. This is a bit more complicated and needs the fact that the curvature operator of S is semi-definite. Let $F_o = df_o$ and put $DF_o(Y,Z) := (D_Y F_o)Z$ for $Y, Z \in T^cM$. An easy standard computation using $D_Y(F_oZ) = (D_Y F_o)Z + F_o(D_Y Z)$ leads to the substitute for the Gauss and Codazzi equations

$$R_E(X,Y)F_oZ = (D_X(DF_o))(Y,Z) - (D_Y(DF_o))(X,Z) + F_o(R(X,Y)Z) \quad (3)$$

for any $X, Y, Z \in TM$, where R_E denotes the curvature tensor of $E = f^*TS$ with the induced Levi-Civita connection of S. We apply this equality in the case where X and Y are complex vectors of the same type, say $X, Y \in T'M$, and $Z = \overline{Y} \in T''M$. Since M is Kähler, the expression $R(X,Y)$ vanishes; in fact, by parallelity of $T'M$ we have $R(U,V)X \in T'M$ for any $U, V \in T^cM$, but the complexified Riemannian metric vanishes on $T'M$ (consisting of vectors of the type $X - iJX$ with $X \in TM$), hence $\langle R(U,V)X, Y\rangle = 0$. Further, if f_o is pluriharmonic, then DF_o vanishes on vectors of different types and the same is true for a covariant derivative of DF_o. Thus from (2b) and (3) we get $R_S(F_oX, F_oY)F_o\overline{Y} = R_E(X,Y)F_o\overline{Y} = 0$. In particular, $\langle R_S(F_oX, F_oY)F_o\overline{Y}, F_o\overline{X}\rangle = 0$ and hence $R_S(F_oX, F_oY) = 0$ by semidefiniteness which in turn shows by (2b) that $R_E(X,Y)F_oZ = 0$ for any $Z \in T^cM$. Thus the proof of Theorem 1 is complete.

3. Isotropic pluriharmonic maps

As before, let M be a Kähler manifold and S a symmetric space. Now let us consider a pluriharmonic map $f : M \to S$ whose associated family is trivial, i.e. $f_\theta = f$ for all θ (we may neglect a possible extrinsic isometry since f_θ is determined only up to isometries of the ambient space anyway). We assume further that f is *full*, i.e. it does not take values in a totally geodesic subspace of S. By Theorem 1 and the definition (1) of an associated family in Ch.2 this holds if and only if $E := f^*TS$ has a one-parameter family of parallel bundle automorphisms $\Phi_\theta \in \mathrm{End}(E)$ preserving the Lie triple product R_S on each fibre of E such that

$$\Phi_\theta \circ df = df \circ \mathcal{R}_\theta. \tag{4}$$

An easy example is a holomorphic map $f : M \to S$ when S is also Kähler; in fact, (4) is satisfied with $\Phi_\theta = \mathcal{R}_\theta^S$ (the canonical rotations of S). The general situation is not too far from this case as we shall see.

Lemma 1.

(a) $(\Phi_\theta)_{\theta\in\mathbb{R}}$ *is a a one-parameter group of parallel automorphisms of* (E, R_S) *with* $\Phi_{2\pi}(p) = I$ *and* $\Phi_\pi(p) = -I$.

(b) There is a parallel subbundle $E_1 \subset E$ *containing the values of* df *and being invariant under all* Φ_θ *such that* Φ_θ *has only eigenvalues* $e^{i\theta}$ *on* E_1^c, *and* E_1 *generates* E *by the Lie triple product* R_S.

Proof. Let E_1 be the smallest parallel subbundle of $E = f^*TS$ containing the values of df. From (4) and the parallelity of Φ_θ we see that Φ_θ preserves E_1 with eigenvalues $e^{\pm i\theta}$. Moreover from the group law $\mathcal{R}_\theta \circ \mathcal{R}_{\theta'} = \mathcal{R}_{\theta+\theta'}$ we get the corresponding group law for $\Phi_\theta|_{E_1}$. Since all Φ_θ are automorphisms for the curvature tensor R_S, the same group law is true on the smallest R_S-stable subbundle E_0 containing E_1. (A subbundle $E_0 \subset E$ is called R_S-*stable* if $R_S(A,B)C \in E_0$ for any $A,B,C \in E_0$.) Since R_S is parallel, E_0 is also a parallel subbundle, and moreover E_0 is R_S-stable and contains $df(TM)$. Since f is full, we conclude $E_0 = E$ (cf. [ET2], Thm. 2), i.e. E_1 generates all of E. Further, $\Phi_{2\pi} = I$ on E_1 and hence on $E_0 = E$.

Now let $\theta = \pi$. Since $\Phi_\pi^2 = \Phi_{2\pi} = I$, the only eigenvalues of Φ_π are ± 1. Let $E_- \subset E$ be the (-1)-eigenbundle. This is parallel and contains $df(TM)$, and it is also R_S-stable since for any $A, B, C \in E_-$ we have

$$\Phi_\pi(R_S(A,B)C) = R_S(\Phi_\pi A, \Phi_\pi B)\Phi_\pi C = -R_S(A,B)C.$$

As before we conclude $E_- = E$ which finishes the proof.

$\square$

As an immediate consequence we see that

$$j_o := \Phi_{\pi/2}$$

is a parallel almost complex structure on E since $j_o^2 = \Phi_\pi = -I$. If $S = \mathbb{R}^n$, then $E_p = \mathbb{R}^n$ for any $p \in M$ and j is a constant complex structure of $\mathbb{R}^n$. Thus (4) shows that f is holomorphic which proves Theorem 2.

Another consequence is that any isotropic pluriharmonic map $f : M \to S$ is also *pluriconformal*, i.e. the complexified metric vanishes on $df(T'M)$ (a subspace with this property is called *isotropic*). In fact, $df(T'M)$ is contained in the isotropic subbundle

$$E' = \{W \in E^c;\ j_o(W) = iW\} = \{V - ij_o(V);\ V \in E\}.$$

Hence, if f is an immersion, the induced metric on M is a compatible Kähler metric (cf. [ET3]) and f is an isometric pluriminimal immersion with respect to this metric.

For any $p \in M$, the one-parameter subgroup $\Phi_\theta(p) \subset \mathrm{Aut}(T_{f(p)}S)$ has an infinitesimal generator $\phi(p) = \frac{d}{d\theta}\Phi_\theta(p)|_{\theta=0}$ which gives also a parallel endomorphism field ϕ of E. We will see below that ϕ can be re-interpreted as a holomorphic and horizontal map into some complex manifold Z fibering over S. In fact, the horizontality is just the parallelity of ϕ while the holomorphicity will be equivalent to (4). But before we have to recall some facts on inner symmetric spaces.

4. Flag manifolds and flag domains

In this chapter, we consider a symmetric space S of compact or noncompact type. Let G be the identity component of its isometry group. For some fixed $o \in S$ let $K = \{g \in G;\ g.o = o\}$ be the isotropy group. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, i.e. $\mathfrak{k}, \mathfrak{p} \subset \mathfrak{g}$ are the (± 1)-eigenspaces of the involution $Ad(\sigma)$ on $\mathfrak{g}$ where σ is the geodesic symmetry of

S at o. Recall that the identity components of K and of the fixed group under the conjugation by σ agree, hence $\mathfrak{k}$ is the Lie algebra of K.

The symmetric space S is called *inner* if σ lies in the identity component of K, i.e. there exists $\xi \in \mathfrak{k}$ with $\sigma = \exp(\pi\xi)$ and in particular

$$Ad(\sigma) = \exp(\pi \cdot ad(\xi)). \tag{5}$$

Since $ad(\xi)$ is skew symmetric with respect to an $Ad(K)$-invariant inner product on $\mathfrak{g}$, the eigenvalues λ of $ad(\xi)$ are purely imaginary, and from (5) we see $\lambda = ik$ where $i = \sqrt{-1}$ and k is an integer. Let $\mathfrak{u}_k \subset \mathfrak{g}^c := \mathfrak{g} \otimes \mathbb{C}$ be the corresponding eigenspace. From the eigenspace decomposition $\mathfrak{g}^c = \sum_k \mathfrak{u}_k$ we get the real decomposition $\mathfrak{g} = \sum_{k \geq 0} \mathfrak{g}_k$ where

$$\mathfrak{g}_k = (\mathfrak{u}_k q + \overline{\mathfrak{u}_k}) \cap \mathfrak{g};$$

observe that $\overline{\mathfrak{u}_k} = \mathfrak{u}_{-k}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ are the (± 1)-eigenspaces of $Ad(\sigma)$, we see from (5) that

$$\mathfrak{k} = \sum_{k \text{ even}} \mathfrak{g}_k, \quad \mathfrak{p} = \sum_{k \text{ odd}} \mathfrak{g}_k.$$

The generating element $\xi \in \mathfrak{g}$ will be called *canonical* if $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra (cf. [BR] for further details throughout this chapter).

Now let us consider the adjoint orbit $Z = Ad(G)\xi \subset \mathfrak{g}$. It can be identified with the coset space $\tilde{Z} = G/H$ where $H = \{h \in G;\ Ad(h)\xi = \xi\}$ is the centralizer of ξ with Lie algebra $\mathfrak{h} = \mathfrak{g}_0$. We call $\tilde{Z}$ a *flag manifold* over S if S is of compact type (i.e. G is compact) and otherwise a *flag domain*. The canonical element $\xi \in \mathfrak{g}$ is uniquely determined (up to conjugacy) by $\tilde{Z}$ (cf [BR]). Any $\zeta \in Z$ shares the property of ξ that $\tilde{\sigma} := \exp(\pi\zeta)$ is a geodesic symmetry; in fact, if $\zeta = Ad(g)\xi$, then $\tilde{\sigma}$ is the reflection at the point $g.o$.

Lemma 2. *There is an equivariant fibration $\tau : Z \to S$ with $\tau(Ad(g)\xi) = g.o$.*

Proof. We must show that τ is well defined. In fact, if $Ad(g)\xi = Ad(g')\xi$, then $g' = gh$ for some $h \in H$. Any such h commutes with $\sigma = \exp(\pi\xi)$ and thus preserves its fixed point set. If S has noncompact type, a geodesic symmetry has only one fixed point. If S is of compact type, then G is compact and H (being the centralizer of a torus) is connected (cf. [H], [E]). In both cases, each $h \in H$ preserves the isolated fixed point o of σ which finishes the proof. In other words, we have shown $H \subset K$, and τ is the canonical projection $G/H \to G/K$.

The tangent space of $Z = Ad(G)\xi$ is $\mathfrak{z} := T_\xi Z = ad(\mathfrak{g})\xi = \sum_{k>0} \mathfrak{g}_k$. It contains the $Ad(H)$-invariant subspaces $\mathfrak{p}$ and $\mathfrak{q} := \mathfrak{z} \cap \mathfrak{k}$ which define two transversal G-invariant distributions $\mathcal{H}$ and $\mathcal{V}$ on Z, called *horizontal* and *vertical* distributions: for any $\zeta = Ad(g)\xi \in Z$ we put $\mathcal{H}(\zeta) = Ad(g)\mathfrak{p}$ and $\mathcal{V}(\zeta) = Ad(g)\mathfrak{q}$ (which is well defined since $\mathfrak{p}$ and $\mathfrak{q}$ are $Ad(H)$-invariant). Since $\mathfrak{q} = \mathfrak{z} \cap \mathfrak{k}$, the vertical distribution consists of the tangent spaces of the fibres of $\tau : Z \to S$. The $Ad(H)$-invariant subspace $\mathfrak{g}_1 \subset \mathfrak{p}$ defines a subdistribution $\mathcal{H}_1$ of $\mathcal{H}$ (where $\mathcal{H}_1(\zeta) = Ad(g)\mathfrak{p}_1$) which is called *superhorizontal.*

Remark. An important special case is when $ad(\xi)$ has only eigenvalues 0 and $\pm i$. Thus $H = K$ and $\mathfrak{p} = \mathfrak{z}$, and $ad(\xi)$ gives a parallel almost complex structure on S turning S into a Kähler manifold. In this case $\tau : Z \to S$ is an isometry; in fact, $S \to Z \subset \mathfrak{g}$ is the extrinsic symmetric standard embedding of the hermitean symmetric space S into $\mathfrak{g}$ (cf. [F]).

In order to see the complex structure on Z in the general case, we consider the complexified Lie algebra $\mathfrak{g}^c$ and the corresponding complex Lie group G^c containing G. We have seen already that $\mathfrak{g}^c$ decomposes into a finite sum of eigenspaces of $ad(\xi)$:

$$\mathfrak{g}^c = \sum_k \mathfrak{u}_k$$

where $\mathfrak{u}_k$ is the eigenspace corresponding to the eigenvalue ik. We have $[\mathfrak{u}_k, \mathfrak{u}_l] \subset \mathfrak{u}_{k+l}$ since by the Jacobi identity

$$[\xi, [X_k, X_l]] = [[\xi, X_k], X_l] + [X_k, [\xi, X_l]] = i(k+l)[X_k, X_l]$$

where $X_j \in \mathfrak{u}_j$. Thus $\mathfrak{n}_+ = \sum_{k>0} \mathfrak{u}_k$ and $\mathfrak{n}_- = \sum_{l<0} \mathfrak{u}_l$ are nilpotent complex subalgebras. Also, $\mathfrak{p}_+ = \sum_{k \geq 0} \mathfrak{u}_k = \mathfrak{n}_+ + \mathfrak{h}^c$ is a complex subalgebra (called *parabolic*), and we have a vector space decomposition $\mathfrak{g}^c = \mathfrak{p}_+ \oplus \mathfrak{n}_-$. There is a closed complex subgroup of G^c with Lie algebra $\mathfrak{p}_+$, namely

$$P_+ = \{g \in G^c;\ Ad(g)\xi \in \xi + \mathfrak{n}_+\}.$$

In fact, on the one hand, exp $\mathfrak{p}_+$ is contained in P_+ since for all $X = \sum_{k \geq 0} X_k \in \mathfrak{p}_+$ with $X_k \in \mathfrak{u}_k$, the expression $\exp(ad(X))\xi - \xi$ is a sum of iterated Lie brackets

$$ad(X_{k_1})...ad(X_{k_r})\xi \subset \mathfrak{u}_{k_1+...+k_r} \subset \mathfrak{n}_+$$

(note that this expression vanishes if all $k_i = 0$), and on the other hand, the Lie algebra of P_+ consists of $X = \sum_k X_k$ such that $[\xi, X] = \sum ikX_k \in \mathfrak{n}_+$ which implies that $X_k = 0$ for $k < 0$, i.e. $X \in \mathfrak{p}_+$.

We claim that $\tilde{Z} = G/H$ is essentially the coset space G^c/P_+ of complex Lie groups and hence a complex manifold. In fact, both manifolds have the same dimension since the $\mathbb{R}$-linear map $r : \mathfrak{g}^c \to \mathfrak{g}$ with $r(X) = X + \bar{X}$ maps $\mathfrak{n}_-$ isomorphically onto $\mathfrak{z}$. On the other hand, $G^c/P_+ \supset G/(P_+ \cap G)$, and $P_+ \cap G = H$ since $g \in G$ satisfies $Ad(g)\xi \in (\xi + \mathfrak{n}_+) \cap \mathfrak{g} = \{\xi\}$ iff $g \in H$. Thus G/H is an open subset of G^c/P_+; in the compact case it is also closed and the two spaces agree. In any case, G/H inherits a G-invariant almost complex structure from the complex manifold G^c/P_+.

It remains to compute how the almost complex structure on $\tilde{Z}$ is transferred to $Z \subset \mathfrak{g}$. We have $T_\xi Z = ad(\mathfrak{g})\xi = \mathfrak{z}$. On the other hand, the tangent spaces of G/H and G^c/P_+ at their basepoints are viewed as $\mathfrak{z}$ and $\mathfrak{n}_-$, using the decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ and $\mathfrak{g}^c = \mathfrak{p}_+ \oplus \mathfrak{n}_-$. The G-equivariant diffeomorphisms $G/H \to Z$, $gH \mapsto Ad(g)\xi$, and $G/H \to G^c/P_+$, $gH \mapsto gP_+$ induce the linear maps of tangent spaces $-ad(\xi) : \mathfrak{z} \to \mathfrak{z}$ and $r^{-1} : \mathfrak{z} \to \mathfrak{n}_-$; the latter holds since $X \in \mathfrak{u}_{-k}$ and $r(X) = X + \bar{X} \in \mathfrak{g}_k$ are congruent mod $\mathfrak{p}_+$, for any $k > 0$. The almost complex structure on $\mathfrak{n}_-$ is given by the multiplication by i on any of the $\mathfrak{u}_{-k}$; this is transferred to a 90^o-rotation on $\mathfrak{g}_k$ which commutes with $ad(\xi)$. Thus we have established the almost complex structure j on $\mathfrak{z} = T_\xi Z$. On $\mathfrak{g}_1$, it agrees to the action of $ad(\xi)$ and of $Ad(\exp(\frac{\pi}{2}\xi))$. Thus we have shown:

Lemma 3. *$\mathcal{H}_1$ is a complex subbundle of TZ, and on $\mathcal{H}_1(\zeta)$ we have $j = Ad(\exp(\frac{\pi}{2}\zeta)) = ad(\zeta)$, for any $\zeta \in Z$.*

5. Proof of Theorem 3

As before, let $S = G/K$ be a symmetric space of compact or noncompact type. For any $x = g.o \in S$ let $G_x = \{g \in G;\ gx = x\} = gKg^{-1}$ be the isotropy group at x. The isotropy representation ρ_x which sends each $g \in G_x$ to its differential $dg_x \in \mathrm{End}(T_xS)$ is faithful and takes values in the group $\mathrm{Aut}(T_xS)$ of linear maps on T_xS which preserve the inner product and the Lie triple product R_S on T_xS; in fact, the connected components of $\rho_x(G_x)$ and $\mathrm{Aut}(T_xS)$ agree (cf. [H], [E]). In other words, any such automorphism of (T_xS, R_S) is the differential at x of a unique isometry of S fixing x. In the sequel, we will not distinguish between $\rho_x(G_x)$ and G_x.

Now we consider an isotropic pluriharmonic map $f : M \to S$ on some Kähler manifold M. Let $\Phi_\theta = \exp(\theta\phi)$ be the associated parallel bundle automorphisms of $E = f^*TS$ satisfying (4). Since $\Phi_\theta(p)$ for any $p \in M$ and any θ is an orthogonal automorphism of $(T_{f(p)}S, R_S)$, it can be viewed as an element of the isotropy group $G_{f(p)} \subset G$ (using $\rho_{f(p)}$). In particular, $\Phi_\pi(p) = -I$ becomes the geodesic symmetry at $f(p)$. Thus S is an inner symmetric space.

Since Φ_θ is viewed now as a map $M \to G$, the generator $\phi = \frac{d}{d\theta}\Phi_\theta|_{\theta=0}$ becomes a map $\phi : M \to \mathfrak{g}$. Fix some $p_o \in M$ and consider $o = f(p_o)$ as the base point of S, i.e. put $K = G_o$. Let $\xi = \phi(p_o) \in \mathfrak{k}$. Then ξ satisfies (5), hence $Z := Ad(G)\xi$ is a flag manifold or a flag domain over S. From Lemma 1 we see that $\mathfrak{g}_1 \subset \mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra since it generates $\mathfrak{p}$ by the Lie triple product $R_S(X,Y)Z = \pm[[X,Y],Z]$. Thus ξ is canonical.

Lemma 4. *The map $\phi : M \to \mathfrak{g}$ takes values in Z and is a superhorizontal lift of f, i.e. $d\phi$ takes values in $\phi^*\mathcal{H}_1$, and $\tau \circ \phi = f$.*

Proof. Recall that ϕ is a parallel endomorphism of E. Parallel translations in S are given by horizontal curves in G, where a curve $g(t)$ in G is called horizontal if $g'(t) \in dL_{g(t)}\mathfrak{p}$ for any t (as usual, $L_g : G \to G$ denotes the left translation $L_g(h) = gh$). Parallelity of the endomorphism field ϕ means: For any curve $p(t)$ in M with $p(0) = p_o$ there is a horizontal curve $g(t)$ in G with $f(p(t)) = g(t).o$ such that $\phi(p(t))$ is conjugate to $\phi(p_o) = \xi$ by $dg(t)_o$ or (using the above identification) $\phi(p(t)) = Ad(g(t))\xi$. This shows $\phi(M) \subset Z$ and $\tau \circ \phi = f$ (using Lemma 2), and moreover $\frac{d}{dt}\phi(p(t)) \in Ad(g(t))ad(\mathfrak{p})\xi \subset Ad(g(t))\mathfrak{p} = \mathcal{H}(f(p(t))$, hence ϕ is even a *horizontal* lift of f. But by Lemma 1, df takes values in the subbundle $E_1 \subset f^*TS$ whose preimage in $\mathcal{H}$ under $d\tau$ is the superhorizontal bundle $\mathcal{H}_1$. Thus ϕ is in fact superhorizontal.

Now we can show that $\phi : M \to Z$ is holomorphic. By Lemma 3, the complex structure j on $\mathcal{H}_1(\phi(p)) \subset T_{\phi(p)}Z$ is given by $Ad(\Phi_{\pi/2}(p)$. Further recall that τ is equivariant, so $d\tau_\xi$ interchanges the isotropy actions of H on $T_\xi Z$ and $T_o S$, hence it commutes with the adjoint action of $H \subset K$ on $\mathfrak{p} \subset \mathfrak{z}$. In particular,

$$d\tau_{\phi(p)} \circ Ad(\Phi_\theta(p)) = \Phi_\theta(p) \circ d\tau_{\phi(p)}. \tag{6}$$

If we denote by $d\tau^{-1}$ the inverse map of $d\tau|_{\mathcal{H}} : \mathcal{H} \to \tau^*TS$, then we get the holomorphicity of ϕ using (4) and (6):

$$\begin{aligned} d\phi \circ J &= d\phi \circ \mathcal{R}_{\pi/2} \\ &= d\tau^{-1} \circ df \circ \mathcal{R}_{\pi/2} \\ &= d\tau^{-1} \circ \Phi_{\pi/2} \circ df \\ &= Ad(\Phi_{\pi/2}) \circ d\tau^{-1} \circ df \\ &= j \circ d\phi. \end{aligned}$$

Vice versa, if a superhorizontal holomorphic map $\phi : M \to Z \subset \mathfrak{g}$ is given, where Z is a flag manifold or a flag domain over S, then $f = \tau \circ \phi : M \to S$ is an isotropic pluriharmonic map. In fact, since ϕ is horizontal, it can be viewed as a parallel derivation (infinitesimal automorphism) of (E, R_S) where $E = f^*TS$ (see proof of Lemma 4). Then $\Phi_\theta(p) = \exp(\theta\phi(p))$ defines a one-parameter group of parallel automorphisms of (E, R_S). Since the almost complex structure j of Z agrees with $ad(\phi(p))$ on the complex subspace $\mathcal{H}_1(\phi(p)) \subset T_{f(p)}Z$ (Lemma

3), we get from the holomorphicity of ϕ:

$$d\phi_p \circ \mathcal{R}_\theta = \exp\left(\theta ad(\phi(p))\right) = Ad(\Phi_\theta(p)) \circ d\phi_p \tag{7}$$

Now (7) and (6) yield equation (4) which finishes the proof of Theorem 3:

$$\begin{aligned} df_p \circ \mathcal{R}_\theta &= d\tau_{\phi(p)} \circ d\phi_p \circ \mathcal{R}_\theta \\ &= d\tau_{\phi(p)} \circ Ad(\Phi_\theta(p)) \circ d\phi_p \\ &= \Phi_\theta(p) \circ d\tau_{\phi(p)} \circ d\phi_p \\ &= \Phi_\theta(p) \circ df_p. \end{aligned}$$

6. Constructions of isotropic pluriharmonic maps

Let $S = G_r(\mathbb{C}^n) = U(n)/U(r) \times U(n-r)$ be the Grassmannian of all r-dimensional subspaces in $\mathbb{C}^n$ and let Z be a classical flag manifold, namely the set of all flags

$$0 = W_0 \subset W_1 \subset W_2 \subset ... \subset W_h = \mathbb{C}^n$$

where the W_k are subspaces of a fixed dimension d_k with $d_{k-1} < d_k$. Equivalently, such a flag is an orthogonal decomposition $\mathbb{C}^n = \sum_k E_k$ where $E_k = W_k \ominus W_{k-1}$ has dimension $n_k = d_k - d_{k-1}$; we then have $W_k = \sum_{j=1}^k E_j$. Assume that $r = \sum_{k \text{ odd}} n_k$ and consider the projection $\tau : Z \to S$ sending the above flag onto the r-plane $\sum_{k \text{ odd}} E_k$. The corresponding canonical element $\xi \in \mathfrak{g} = \mathfrak{u}(\mathfrak{n})$ is $\xi = \sum_k ik \cdot p_k$ where p_k denotes the projection onto E_k. We have $T_E^c Z = \sum_{k,l} Hom(E_k, E_l)$, and $ad(\xi)$ has eigenvalue $i(l-k)$ on $Hom(E_k, E_l)$. Thus

$$\mathcal{H}^c(E) = \sum_{k-l \text{ odd}} Hom(E_k, E_l), \quad \mathcal{H}_1^c(E) = \sum_{|k-l|=1} Hom(E_k, E_l).$$

Now let M be a Kähler manifold and $\phi : M \to Z$ be a smooth map. Thus ϕ assigns to each $p \in M$ a flag $W(p)$ or equivalently a decomposition $E(p)$ of the above type, so ϕ determines complex vector bundles $W_k, E_k \subset M \times \mathbb{C}^n$. For any $X \in T_pM$ we let $\partial_X W_k = \text{Span}\,(\partial_X f_1, ..., \partial_X f_{d_k})$ for any smooth local basis

$f_1, ..., f_{d_k}$ of W_k; this is a well determined subspace modulo $W_k(p)$. We assume further that ϕ is holomorphic, i.e. $d\phi$ maps T'_pM into $T'_{\phi(p)}Z$ where

$$T'_E Z = \sum_{k<l} Hom(E_k, E_l).$$

Let ∂' denote the restriction of the differential d on M to $T'M$. In the following we will omit the vector $X \in T'M$ with respect to which we differentiate.

Proposition. (F.Burstall) *Let M be a Kähler manifold. A holomorphic map $\phi : M \to Z$ is superhorizontal iff $\partial' W_k \subset W_{k+1}$.*

Proof. Since the map ϕ is holomorphic, $A'E_k := \partial' E_k \cap E_k^\perp$ takes values in $\sum_{l>k} E_l$. Thus ϕ is superhorizontal iff $A'E_k \subset E_{k+1}$. By induction, this implies

$$\partial' W_k = \partial'(W_{k-1} + E_k) \subset W_k + E_{k+1} = W_{k+1}.$$

Vice versa, if $\partial' W_k \subset W_{k+1}$, then $\partial' E_1 \subset E_1 + E_2$, thus $A'E_1 \subset E_2$, further $\partial'(E_1 + E_2) \subset E_1 + E_2 + E_3$, thus $A'E_2 \subset E_3$ etc. which shows that ϕ is superhorizontal.

□

This construction gives a vast number of explicit local examples: We start with pointwise linear independent holomorphic maps $f_i : M \to \mathbb{C}^n$ for $i = 1, ..., d_1$ (for large enough n) where M is an open subset of $\mathbb{C}^m$, and let W_1 be the span of the f_i, further W_2 the span of the f_i and their first derivative $\partial'_j f_i$ and W_k the span of f_i and their first $k-1$ partial derivatives, as long as those are pointwise linearly independent. This defines a horizontal holomorphic map $\phi : M \to Z$, and $f = \tau \circ \phi$ is isotropic pluriharmonic.

The construction was first introduced by Eells and Wood [EW] for the case $h = 3$, i.e. each element of Z is an orthogonal decomposition (E_1, E_2, E_3) of $\mathbb{C}^n$ which is mapped by τ to $E_1 + E_3 \in S = G_r(\mathbb{C}^n)$. The horizontal and the superhorizontal bundles agree in this case. A map $\phi = (E_1, E_2, E_3) : M \to Z$ is holomorphic iff E_1 is a holomorphic subbundle of $M \times \mathbb{C}^n$ (generated by holomorphic maps $M \to \mathbb{C}^n$) and E_3 an antiholomorphic one, and a holomorphic ϕ is horizontal iff $\partial' E_1 \perp E_3$. This was invented to descibe minimal surfaces in $\mathbb{C}P^n = G_1(\mathbb{C}^{n+1})$; in fact it was shown that minimal 2-spheres in $\mathbb{C}P^n$ are precisely of this type; in another language, this result was also obtained by Chern and Wolfson ([Ch], [ChW], [Ws]). The construction was later extended to pluriharmonic maps into complex Grassmannians (cf. [ErW], [OU]).

Many years before, Calabi ([C], cf. [L]) gave a similar description of minimal 2-spheres in S^n. In fact, this can be viewed as a special case of the construction of Eells and Wood if we immerse the n-sphere as $\mathbb{R}P^n$ into $\mathbb{C}P^n$. More generally, we consider the *real* Grassmannian $G_r(\mathbb{R}^n)$ as the submanifold of $G_r(\mathbb{C}^n)$ containing the r-dimensional subspaces which are invariant under complex conjugation. Thus we need holomorphic and antiholomorphic bundles E_1 and E_3 such that $E_1 + E_3$ is invariant under conjugation; this is satisfied if $E_3 = \overline{E_1}$. In other words, the map ϕ can be viewed as a holomorphic bundle $E = E_1 \subset M \times \mathbb{C}^n$ such that $\overline{E}$ is perpendicular to E with respect to the *hermitean* inner product $\bar{x}^T y$ on $\mathbb{C}^n$, i.e. the *biinvariant* inner product $x^T y$ vanishes on E (it is isotropic). Thus the corresponding flag manifold Z over $S = G_r(\mathbb{R}^n)$ is the set of isotropic n_1-planes in $\mathbb{C}^n$, where $2n_1 + r = n$, and $\tau : Z \to S$ is the realification $E \mapsto (E + \overline{E}) \cap \mathbb{R}^n$. As coset spaces, we have $Z = SO(n)/SO(r) \times U(n_1)$ and $S = SO(n)/SO(r) \times SO(2n_1)$ (we have passed to the *oriented* Grassmannian). The canonical element ξ in the Lie algebra of $SO(n)$ is the antisymmetric real matrix with eigenvalue i on E and $-i$ on $\overline{E}$, being 0 on the complement. The case of Calabi is $r = 1$.

However, this construction is less explicit then the previous one since it is nontrivial to find a holomorphic isotropic bundle E with $\partial' E \perp \overline{E}$. In [ET3] we gave an example of such a construction by choosing E to be the (1,0)-Gauss map of an immersion $F : M \to \mathbb{R}^n$ which is either pluriminimal or the standard embedding of a compact hermitean symmetric space. Examples of pluriminimal submanifolds of $\mathbb{R}^n$ have been constructed by Dacjzer and Gromoll [DG].

But long time before, R.Bryant [B] gave an explicit and complete description in the Calabi case $S = S^4$. This was re-interpreted by Lawson [L] and generalized by Burstall [Bs] as follows. As before, let $S = G/K$ be a symmetric space and $\tau : Z \to S$ a flag fibration where $Z = Ad(G)\xi$ for a canonical element $\xi \in \mathfrak{g}$. Recall form Ch.4 the decomposition $\mathfrak{g}^c = \mathfrak{p}_+ \oplus \mathfrak{n}_-$. Thus the nilpotent group $N_- = \exp \mathfrak{n}_- \subset G^c$ acts on $Z = G^c/P_+$ with an open dense orbit. Let us consider the case where $ad(\xi)/i$ has only two negative eigenvalues -1 and -2 and the (-2)-eigenspace has complex dimension one. This is precisely the case when Z is the twistor space over a quaternionic symmetric space S (a so called *Wolf space*). It means that the tangent space $\mathfrak{p}$ of S carries the structure of a quaternionic vector space, and $K = K' \cdot Sp(1)$ where K' acts quaternionic linearly and $Sp(1)$ by quaternionic scalars, and $Z = G/K' \cdot U(1)$ is the twistor bundle (cf. [K1]). In this case, $\mathfrak{n}_- = \mathfrak{u}_{-1} + \mathfrak{u}_{-2}$ where $\mathfrak{u}_{-1}$ determines the (super)-horizontal subbundle (which is left invariant) and $\mathfrak{u}_{-2} = [\mathfrak{n}_-, \mathfrak{n}_-]$ is one-dimensional. It follows easily that between any two such Lie algebras of the same dimension there is an isomorphism preserving this splitting, and this extends to a polynomial isomorphism of the corresponding Lie groups. Thus, if S' is another Wolf space of the same dimension with twistor bundle Z', we find a holomorphic map F between open dense subsets of Z and Z' preserving the horizontal structure. (In fact, F extends to a birational map between Z and Z'.) Hence we may transform any holomorphic and horizontal map $\phi : M \to Z$ into another holomorphic and horizontal map $\phi' = F \circ \phi : M \to Z'$. In this way, the isotropic pluriharmonic maps with values in S and S' are transformed into each other. But among the Wolf spaces there are the Grassmannians $S = G_2(\mathbb{C}^n)$ of real dimension $4(n-2)$, and in this case Z is the 3-step flag manifold with $n_1 = n_3 = 1$ (notation as above). There we have the Eells-Wood construction which now can be transferred to any of the Wolf spaces. Bryant treated the case $S' = S^4 = \mathbb{H}P^1$ with $Z' = \mathbb{C}P^3$ and used the transformation to the full flag manifold Z over $S = \mathbb{C}P^2$.

The question whether there are similar correspondences of flag spaces in other cases has been answered negatively by Kobak [K1], [K2].

7. Further problems

In euclidean 3-space, not only minimal surfaces allow isometric deformations rotating the second fundamental form, but also constant mean curvature surfaces. In fact, this can also be generalized to Kähler submanifolds in symmetric spaces using a more general type of associated families, cf. [ET4]. A similar characterization of the *isotropic* submanifolds of this type (those with trivial associated families) has still to be given. Another open question is if other special surface classes have interesting generalizations to higher dimension and codimensions.

Siu [S] and Carlson and Toledo [CT] have given bounds for the rank of pluriharmonic maps into a symmetric space S. In the case $S = G_2(\mathbb{C}^n)$, this upper bound could be realized by isotropic pluriharmonic maps (cf. [ET3]). We do not know if the same is true for other Wolf spaces and more generally for other symmetric spaces.

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J.H. Eschenburg
Institut für Mathematik
Universität Augsburg
Universitätsstraße 14
D - 86135 Augsburg
Germany

R. Tribuzy
Departamento de Matematica
Universidade do Amazonas
ICE
69000 Manaus, AM
Brasil