Isoparametric submanifolds and symmetric spaces

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ABSTRACT. We give a survey on a new proof of a theorem of Thorbergsson which says that a (complete, full, irreducible) isoparametric submanifold M of codimension $k \geq 3$ in euclidean space $\mathbb{R}^N$ is a principal orbit of an isotropy representation of a symmetric space. Using submanifold geometry, a Lie triple product on $\mathbb{R}^N$ is constructed such that M is a principal orbits of its orthogonal automorphism group.

1. Introduction

An immersed submanifold M in euclidean space $V = \mathbb{R}^N$ is called *isoparametric* if any normal vector extends to a normal vector field ξ which is parallel with respect to the covariant derivative on the normal bundle, and the corresponding shape operator (Weingarten map) A_ξ has constant eigenvalues on M. For hypersurfaces the definition just says that the principal curvatures are constant. The notion "isoparametric" for this case goes back to Levi-Civita [LC]; it means that M is a level hypersurface of a function F whose "differential parameters" ΔF and $|\nabla F|^2$ depend only on F. A similar characterization for higher codimension was given by Terng [Te]; in fact, F can be chosen to be a polynomial. We refer to [Th2] for a detailed survey. In the following, we shall always assume that $M \subset \mathbb{R}^N$ is a properly immersed isoparametric submanifold which is *full* (i.e. M is not contained in a hyperplane) and *irreducible* which means that M cannot be written as a product of lower dimensional isoparametric submanifolds; for short we will call M isoparametric if it has all these properties.

There are two classes of examples known. Class I (cf. [Te]) consists of homogeneous submanifolds, namely the principal orbits of isotropy representations of irreducible Riemannian symmetric spaces, so called *s-representations*. In other words, M is (up to connected components) a principal orbit of the group of orthogonal automorphisms for a Lie triple product R on $\mathbb{R}^N$ (see below). The first nontrivial examples in this class were discovered by E. Cartan [Ca], but he was not aware of the simple relationship to symmetric spaces. Class II in turn is related to representations of Clifford algebras; all of these examples have codimension 2, and those which are not in the intersection with Class I are inhomogeneous (cf. [FKM]). If M has codimension one, it is a round sphere as was shown by Segre

[S]. For codimension two a classification is not yet available, but G. Thorbergsson [Th1] has finished the remaining cases:

THEOREM 1.1 (Thorbergsson). *If M is isoparametric with codimension $k \geq 3$, than it belongs to Class I, i.e. it is a principal orbit of an s-representation.*

In the main step of the proof, Thorbergsson assigns to M a topological Tits building $\Delta(M)$ of rank k. A deep theorem of Burns and Spatzier states (generalizing a well known fact for projective spaces of dimension $k \geq 3$) that such a Tits building belongs to a semisimple Lie group G. Finally, M is shown to be an orbit of a maximal compact connected subgroup $K \subset G$ which acts orthogonally on $\mathbb{R}^N$. The representation must be *polar*, i.e. there exists a subspace $\Sigma \subset \mathbb{R}^N$ meeting any of the orbits perpendicularly [PT]. Now the proof is finished by applying a theorem of J. Dadok [D]:

THEOREM 1.2 (Dadok). *Any polar representation is orbit equivalent to the isotropy representation of a symmetric space.*

(Representations of two groups are called *orbit equivalent* if they have the same orbits after a suitable isometric identification of their representation spaces.)

However, the proof of Dadok's theorem is by classification: Using a necessary condition for polarity, Dadok obtains a finite list of possibly polar representations which are checked case by case. Theorem 1.2 follows from that list by inspection. It would be most desirable to finish the proof of Thorbergsson's theorem rather by constructing a Lie triple product R on $\mathbb{R}^N$ such that K acts as the group of orthogonal automorphisms of R. Moreover, since principal orbits of polar representations are isoparametric submanifolds, this would give a classification free proof of Dadok's theorem for cohomogeneity $k \geq 3$. Of course, the classification of all polar representations is much easier if Theorem 1.2 can be used (cf. [EH3], [B]).

Thorbergsson's homogeneity theorem was reproved in a completely different way by C. Olmos [O2] and extended to the infinite dimensional setting by Heintze and Liu [HL]. These authors applied submanifold geometry instead of Tits buildings, and the group $K \subset O(N)$ acting transitively on M was given in a more effective way. In a recent joint paper with E. Heintze [EH2] this could be used to construct the corresponding Lie triple product and thus also to give a conceptual proof of Theorem 1.2 for cohomogeneity $k \geq 3$:

THEOREM 1.3. *Let $M \subset V = \mathbb{R}^N$ be isoparametric with codimension ≥ 3. Then from the geometry of M one can construct a Lie triple product R on V such that M is a principal orbit of the connected orthogonal automorphism group of R.*

For completeness let us recall that a *Lie triple product* on a euclidean vector space V is a trilinear map ("triple product") $R : V \times V \times V \to V$ satisfying the algebraic curvature identities and moreover, the linear maps $z \mapsto R(x, y)z$ belong to the Lie algebra of the orthogonal automorphism group K of (V, R), for all $x, y \in V$. The curvature tensor of a symmetric space has this property, and a construction due to E. Cartan shows the converse: From a Lie triple (V, R) we obtain a symmetric space G/K where G has a Lie algebra $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{k} \oplus V$ as a vector space, and a Lie bracket extending that of $\mathfrak{k}$ is defined as follows: for all $A \in \mathfrak{k}$ and $x \in V$ we put $[A, x] = Ax \in V$ and $[x, y] = R(x, y) \in \mathfrak{k}$.

Various authors have contributed to the the proof of Theorem 1.3, and it is the aim of this survey to outline the full chain of arguments which was necessary to accomplish this proof. It is a pleasure for me to thank E. Heintze and A.L. Mare for many useful hints and discussion.

2. Isoparametric submanifolds

All details in this section can be found in [PT]. Let $M \subset V = \mathbb{R}^N$ be isoparametric. Let ν be the space of parallel normal fields on M; recall that a normal field ξ is *parallel* if its derivative $\partial\xi$ is always tangent to M and hence its shape operator is simply $A_\xi = -\partial\xi$. By the Ricci equation, two such operators commute, so they can be diagonalized simultaneously. Thus the tangent bundle has an orthogonal decomposition $TM = \sum_{\kappa\in\Lambda} E_\kappa$ for a finite set Λ of linear forms on ν such that $A_\xi = \kappa(\xi)\cdot I$ on E_κ. The E_κ are called *curvature distributions*, the linear forms $\kappa \in \Lambda$ are the *principal curvatures* and their dual vectors $n_\kappa \in \nu$ *principal curvature vectors.*

Isoparametric submanifolds come in families. For any $\xi \in \nu$ let $\pi_\xi : M \to V$ be the *end point map* $\pi_\xi(x) = x + \xi(x)$ and put $M_\xi = \pi_\xi(M)$. The differential $d\pi_\xi = I - A_\xi$ is zero precisely on all E_κ with $\kappa(\xi) = 1$, thus it has constant rank. Consequently, $M_\xi \subset V$ is again an immersed submanifold and $\pi_\xi : M \to M_\xi$ a submersion. If $\kappa(\xi) = 1$ for some $\kappa \in \Lambda$, then M_ξ drops dimension and is called a *focal manifold.* For such ξ the fibres of $\pi_\xi : M \to M_\xi$ are tangent to $\sum_{\kappa(\xi)=1} E_\kappa$. Hence these distributions are integrable. Moreover, the integral leaves (fibres) are totally geodesic since a fibre $F = \pi_\xi^{-1}(y)$ is the intersection of M with the affine subspace $y + \nu_y(M_\xi)$ containing all normal spaces $x + \nu_x(M)$, $x \in F$. In particular, if $\kappa(\xi) = 1$ for just one $\kappa \in \Lambda$, the fibre of π_ξ (the integral leaf of E_κ) through $x \in M$ is a sphere $S_\kappa(x)$ of radius $1/|n_\kappa|$ centered at $x + \xi_\kappa(x)$ where $\xi_\kappa = n_\kappa/|n_\kappa|^2$; it is called *curvature sphere.* For all other ξ (those with $\kappa(\xi) \neq 1$ for each $\kappa \in \Lambda$) the map π_ξ is regular with $T_{\pi_\xi(x)}M_\xi = T_xM$. Such M_ξ is again isoparametric since it has the same parallel normal fields, and its shape operator with respect to any $\eta \in \nu$ has constant eigenvalue $\kappa(\eta)/(1-\kappa(\xi))$ on E_κ. It is called a *parallel manifold* of M.

Thus the submanifolds M_ξ form a sort of foliation of $\mathbb{R}^N$ (at least locally, but according to [Te] even globally) with leaves of constant distance from each other, but some "singular" leaves (the focal manifolds) have lower dimension. The affine normal spaces $\nu_x = \{x + \xi(x);\ \xi \in \nu\}$ for $x \in M$ meet all leaves perpendicularly, and the singular leaves intersect ν_x in a subset S which is the union of the *focal hyperplanes* $l_\kappa(x) = \{x + \xi(x);\ \kappa(\xi) = 1\}$.

For some $\xi \in \nu$ it happens that $M_\xi = M$; in particular this is true for $\xi = 2\xi_\mu$ for any $\mu \in \Lambda$, since x and $x + 2\xi_\mu(x)$ are antipodal points in $S_\mu(x) \subset M$ for arbitrary $x \in M$. Those π_ξ form a group of diffeomorphisms of M, called the *Weyl group* W. They preserve the affine normal spaces, i.e. $\nu_{\phi(x)} = \nu_x$ for each $\phi \in W$, and consequently, the normal parallel translation from x to $\phi(x)$ defines an affine isometry ϕ^x of ν_x sending $x + \xi(x)$ onto $\phi(x) + \xi(\phi(x))$. This determines an affine action of W on ν_x keeping the singular set S invariant. For $\phi = \pi_{2\xi_\mu}$ the map ϕ^x is the reflection at the hyperplane l_μ. In fact, W is generated by the $\pi_{2\xi_\mu}$.

A priori, the principal curvature set Λ might contain the zero linear form, but in this case M would split off a euclidean factor which we have excluded by assumption. The reason for this splitting is that the eigenvalues of $\partial\xi = -A_\xi$ are bounded away from 0 for any $\xi \in \nu$ with $\kappa(\xi) \neq 0$ for all $\kappa \in \Lambda \setminus \{0\}$. Namely, the leaves $S_0(x)$ of the distribution E_0 on M are affine subspaces along which any $\xi \in \nu$ is constant; they are the fibres not of an endpoint map π_ξ but of the "Gauss map" $\xi : M \to S^{N-1}$. We claim that any two fibres $S_0(x)$ and $S_0(y)$ have bounded distance and hence are parallel affine subspaces which can be split off. This would be clear if the submersion ξ were Riemannian, i.e. if $d\xi$ preserved the length of "horizontal" curves in M (those perpendicular to E_0). But instead, $d\xi = -A_\xi$ multiplies each vector in E_κ by $-\kappa(\xi)$, hence the length of the Gauss image of any horizontal curve $\hat{c}$ in M is bounded from below: $L(\xi \circ \hat{c}) \geq m \cdot L(\hat{c})$ with $m = \min_{\kappa \neq 0} |\kappa(\xi)| > 0$. If in turn we start with a smooth curve c in S^{N-1} with length L from $\xi(x)$ to $\xi(y)$, we obtain a horizontal lift $\hat{c}$ between $S_0(x)$ and $S_0(y)$ starting at an arbitray $z \in S_0(x)$, and we have $L(\hat{c}) \leq L/m$ which shows that any point of $S_0(x)$ has distance $\leq L/m$ from $S_0(y)$, concluding the argument.

Now we can see that M lies itself in a sphere in $\mathbb{R}^N$; in particular, M is compact. In fact, the focal set $S \subset \nu_x$ consisting of the finitely many hyperplanes $l_\kappa(x)$ is invariant under the reflections at these hyperplanes (belonging to the Weyl group); this is possible only if all $l_\kappa(x)$ meet in a common point $x_o = x + \xi_o$. Then $\kappa(\xi_o) = 1$ for all $\kappa \in \Lambda$ which shows that M_{ξ_o} is the single point x_o, and M is contained in the sphere of radius $|\xi_o|$ around x_o. We may assume $x_o = 0$ and $M \subset S^{N-1}$. Then the position vector x is perpendicular to $T_x M$. Hence ν_x becomes the linear normal space $(T_x M)^\perp$, and $l_\kappa(x) \subset \nu_x$ is the linear hyperplane $n_\kappa(x)^\perp$; note that $\kappa(x) = -1$ for all $\kappa \in \Lambda$.

Examples of isoparametric submanifolds $M \subset V$ arise as principal orbits of subgroups $K \subset O(V)$ which are *polar*, i.e. there is a subspace $\Sigma \subset V$ ("section") meeting all orbits perpendicularly. For a principal orbit $M = K.x$ (we may assume $x \in \Sigma$), we have $\Sigma = \nu_x$. Since the isotropy group K_x of a principal orbit acts trivially on ν_x, any normal vector $\xi^o \in \nu_x$ can be extended to a unique K-invariant normal field ξ with $\xi(kx) = k\xi^o$. Then $\pi_\xi(M)$ (where $\pi_\xi(x) = x + \xi(x)$) is another orbit which by polarity is perpendicular to Σ. Hence ξ is parallel because $\partial_v \xi = \partial_v \pi_\xi - v \in \Sigma^\perp = T_x M$. Clearly, the singular orbits are the focal manifolds.

The most prominent examples of polar representations are the isotropy representations of symmetric spaces (s-representations): Let G/K be a Riemannian symmetric space of compact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its Cartan decomposition. Then the action of $Ad(K) \subset Ad(G)$ on $\mathfrak{p} \subset \mathfrak{g}$ is polar, and any maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{k}$ is a section: each $Ad(K)$-orbit meets $\mathfrak{a}$ (conjugacy of maximal abelian subalgebras of $\mathfrak{p}$, cf. [H]), and for any $x \in \mathfrak{a}$ we have $T_x(Ad(K)x) = ad(\mathfrak{k})x \perp \mathfrak{a}$ since $\langle ad(\mathfrak{k})x, y \rangle = \langle \mathfrak{k}, [x, y] \rangle = 0$ for all $y \in \mathfrak{k}$.

The principal curvatures in this example are expressed by the *roots* of the symmetric space. Recall that the common eigenspace decomposition of the $ad(x)$, $x \in \mathfrak{a}$ (which are skew symmetric linear maps of $\mathfrak{g}$) leads to orthogonal decompositions $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Delta} \mathfrak{k}_\alpha$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{p}_\alpha$, where Δ is a finite subset of $\mathfrak{a}^*$ (the set of *roots*), such that each $ad(x)$ maps $\mathfrak{p}_\alpha$ into $\mathfrak{k}_\alpha$ and vice versa while $ad(x)^2$ has eigenvalue $-\alpha(x)^2$ on $\mathfrak{k}_\alpha + \mathfrak{p}_\alpha$. Now let $M = Ad(K)x$ for some $x \in \mathfrak{a}$ be a principal orbit, i.e. $\alpha(x) \neq 0$ for all $\alpha \in \Delta$. Let $\xi \in \nu_x M = \mathfrak{a}$ and $v = ad(A)x = -ad(x)A \in T_x M$ for some $A \in \mathfrak{k}$. Then $A_\xi(v) = -\frac{d}{dt} Ad(e^{tA})\xi|_{t=0} = ad(\xi)A$. If $A \in \mathfrak{k}_\alpha$, then $ad(\xi)A = \frac{\alpha(\xi)}{\alpha(x)} ad(x)A$, thus A_ξ has eigenvalue $-\frac{\alpha(\xi)}{\alpha(x)}$ on $\mathfrak{p}_\alpha \subset T_x M$, and the principal curvatures are $\kappa = -\frac{\alpha}{\alpha(x)}$ for all $\alpha \in \Delta$. If also $2\alpha \in \Delta$, the corresponding curvature distribution is $\mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha}$.

3. Homogeneous slice theorem and normal holonomy

We want to prove that an isoparametric submanifold $M \subset V$ is the orbit of an s-representation. As a first main step we show that many totally geodesic submanifolds of M satisfy this property, namely the fibres of the focal projections (Homogeneous Slice Theorem, cf. [HOTh]). Let $\xi \in \nu$ be a *focal* normal field, i.e. $\kappa(\xi) = 1$ for at least one $\kappa \in \Lambda$. Consider the submersion $\pi = \pi_\xi : M \to F = M_\xi$ with fibres $S(z) = \pi^{-1}z \subset M$ (also called *slices*) for all $z \in F$; these are the integral leaves of the distribution $E_F := \sum_{\kappa(\xi)=1} E_\kappa$ called *vertical bundle*. From the invariance of E_F under the shape operators A_η it follows easily that $S(x)$ is totally geodesic in M and again isoparametric in its linear span (cf. [HPT]). But more is true: there is a large group of isometries acting on $S(x)$. This is due to a general fact for a submersion $\pi : M \to F$ with totally geodesic fibres: any piecewise smooth curve c in F from z_0 to z_1 determines an isometry $h_c : S(z_0) \to S(z_1)$ where $h_c(x)$ for any $x \in S(z_0)$ is the end point of the horizontal lift of c starting from x. In particular, for a closed curve ($z_0 = z_1 = z$) we obtain an isometry of $S(z)$. Since the horizontal distribution is totally non-integrable (i.e. its iterated Lie brackets generate all vector fields), the group of isometries generated by the h_c acts transitively on $S(z)$ (cf. [HOTh]).

In our case, $S(z)$ is not only a subset of M but also of the normal space $z + \nu_z F = \nu_z F$ because for $z = \pi(x)$ we have $x - z = -\xi(x) \in \nu_x M \subset \nu_z F$. If $x(t)$ is a curve in M with projection $z(t) = \pi(x(t)) = x(t) + \xi(x(t))$, then $x(t)$ is a *horizontal* lift of $z(t)$ iff $x'(t) \in E_F(x(t))^\perp = T_{z(t)}F$. This means that $\frac{d}{dt}\xi(x(t)) \in T_{z(t)}F$ or in other words, $\xi(t) = \xi(x(t))$ considered as a normal field of F along the curve $z(t)$ is parallel in the normal bundle of F. Thus the isometries h_c for *closed* curves $c(t) = z(t)$ in F are precisely the elements of the *holonomy group* of νF, so each $S(z)$ is an orbit of the normal holonomy group at z.

This completes the proof of the Homogeneous Slice Theorem since a few years before, Olmos [O1] had shown that the restricted normal holonomy group of *any* submanifold $F \subset \mathbb{R}^N$ is an s-representation. The idea of this remarkable theorem is easy to understand: Starting from the normal curvature tensor $R^\nu : TF \otimes TF \to \mathrm{End}(\nu F)$ an algebraic curvature tensor $\mathcal{R}$ on each $\nu_z F$ is obtained by putting

$$\mathcal{R}(\xi,\eta)\zeta = \sum_i R^\nu(A_\xi(e_i), A_\eta(e_i))\zeta$$

where (e_i) is an orthonormal basis of T_zF. By the Ricci equation, the "sectional curvature" of $\mathcal{R}$ is $\langle \mathcal{R}(\xi,\eta)\eta, \xi\rangle = -\frac{1}{2}|[A_\xi, A_\eta]|^2$; in particular, $\mathcal{R}$ has nonpositive scalar curvature which is zero only if $\mathcal{R} = 0$. Let H denote the restricted holonomy group of νF at z (a compact subgroup of $O(\nu_z F)$) and $\mathfrak{h}$ its Lie algebra. Averaging $\mathcal{R}$ over H, i.e. putting $R = \int_H (h.\mathcal{R})dh$ where $(h.\mathcal{R})(\xi,\eta) = h\mathcal{R}(h^{-1}\xi, h^{-1}\eta)h^{-1}$, one gets an H-invariant curvature tensor which is nontrivial since averaging does not change the scalar curvature. Now from $R^\nu(v,w) \in \mathfrak{h}$ for all $v, w \in T_zF$ we obtain $\mathcal{R}(\xi,\eta) \in \mathfrak{h}$ and consequently $R(\xi,\eta) \in \mathfrak{h}$ which shows that R is a (semisimple) Lie triple product on $\nu_z F$; recall that $\mathfrak{h}$ belongs to the Lie algebra of the automorphism group of R. This argument was already used by Simons [Si].

4. The homogeneous structure

A compact submanifold $M \subset S^{N-1} \subset V = \mathbb{R}^N$ is called *extrinsic homogeneous* if it is the orbit of a compact subgroup $K \subset O(N)$. This global property can also be expressed in local terms which was observed first by Ferus [F] and Strübing [St] in a special case: M is *extrinsic symmetric* (i.e. invariant under the reflections at all normal spaces) if and only if it has parallel second fundamental form α for the Levi-Civita connection ∇ on TM and νM. More generally, M is extrinsic homogeneous if and only if α is parallel for a connection D on $TM \oplus \nu M$ preserving this splitting such that $D - \nabla$ is D-parallel ([OS], [E]).

Let us briefly recall the idea of the proof. We consider a connection D on the trivial bundle $E = M \times V = TM \oplus \nu M$ for which TM and νM are parallel subbundles and α is D-parallel, i.e. $\nabla - \partial$ is D-parallel, where ∂ is the trivial connection on E. Suppose further that $\gamma := \partial - D$ is D-parallel (which is equivalent to the D-parallelity of $D - \nabla$). Then a D-parallel basis $b = (b_1, ..., b_N)$ of E along a D-geodesic γ (we may assume $\gamma' = b_1$) solves $b' = \gamma_{\gamma'} b = b \cdot A$ for some constant matrix A (the matrix of the parallel endomorphism $\gamma_{\gamma'}$ with respect to the parallel basis b), hence $b(t) = b(0)e^{tA}$. Now consider $k \in O(N)$ sending $x = \gamma(0)$ onto $kx = y \in M$ such that $k(T_xM) = T_yM$ and $k.(\gamma(x)) = \gamma(y)$ (where $k.\gamma_v w = k\gamma_{k^{-1}v}k^{-1}w$). Then $\bar{b} = kb$ solves the same ODE as b, so it is a parallel basis along the D-geodesic $\bar{\gamma}$ starting at y with initial direction $k\gamma'(0) = kb_1(0)$. In particular $\bar{b}_1 = kb_1$ and by integration $\bar{\gamma} = k\gamma$. Applying the same argument to *all* geodesics γ starting at x and even to all geodesic polygons we see that $k(M) = M$. In particular we can choose k to be a parallel displacement from x to y along any curve connecting these points. So we see that the group $K = \{k \in O(N);\ k(M) = M\}$ acts transitively on M; in fact this hold even for the subgroup generated by the *transvections* of D (i.e. isometries being parallel displacements along some geodesic).

Vice versa, if $M \subset V$ is extrinsic homogeneous then $M \cong K/K_x$ where K_x is the isotropy group at $x \in M$. An $Ad(K_x)$-invariant decomposition $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{m}$ determines a left invariant horizontal distribution $\mathcal{H}$ (a connection) on the principal bundle $K \to K/K_x = M$ with $\mathcal{H}_k = k.\mathfrak{m} \subset T_kK$ and hence a K-invariant covariant derivative D on the associated bundle E: a section $v(t)$ of E along a curve $x(t)$ in M is by definition D-parallel iff $v(t) = k(t).v$ for some $v \in V$ and a horizontal curve $k(t)$ in K (a horizontal lift of $x(t)$). Thus the K-invariant tensors α and $\gamma := \partial - D$ are D-parallel. It remains to compute γ at x. Let $x(t)$ be a D-geodesic starting at x. Then $x(t) = e^{tA}x$ for some $A \in \mathfrak{m}$ and $k(t) = e^{tA}$ is a horizontal lift, hence $v(t) = k(t)v$ is parallel along $x(t)$. Thus $\gamma_{Ax}v = v'(0) = Av$, so $\gamma_{Ax} = A \in \mathrm{End}(V)$. We will call such connections *canonical* (cf. [K]).

The aim in [O2] was to show the homogeneity of an isoparametric submanifold M by constructing such a connection D (now called *Olmos connection*) which imitates the canonical connection when $M = Ad(K)x =: K.x$ is a principal orbit of an s-representation. In that case which we will call *standard*, the canonical connection given by $\mathfrak{m} = \sum_{\alpha \in \Delta} \mathfrak{k}_\alpha$ (a reductive complement of $\mathfrak{k}_x$) has an important additional property: If we pass to a singular orbit $F = K.z$ with $z - x \in \nu_x$, we get a decomposition $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ where $\mathfrak{m}_1$ is a reductive complement of $\mathfrak{k}_z$ in $\mathfrak{k}$ and $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{k}_z$ is a reductive complement of $\mathfrak{k}_x$ in $\mathfrak{k}_z$. In fact, we put $\mathfrak{m}_0 = \sum_{\alpha(z)=0} \mathfrak{k}_\alpha$ and $\mathfrak{m}_1 = \sum_{\beta(z)\neq 0} \mathfrak{k}_\beta$. Hence the canonical connection for M also determines a canonical connection for F and further $\mathfrak{m}_0.x \perp \mathfrak{m}_1.x$, and $\mathfrak{m}_1.x = \mathfrak{m}_1.z = T_zF$.

Let us translate this property into the language of the covariant derivative D. The $Ad(K_x)$-invariant splitting $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ determines a D-parallel splitting $TM = E_0 + E_1$ with $E_i(kx) := k.\mathfrak{m}_i.x$ for $i = 0, 1$. Recall that $E_1(x) = T_zF$ is invariant under $\mathfrak{k}_z$ and hence under $\mathfrak{m}_0$. Further note that $\mathfrak{m}_1$ is a $\mathfrak{k}_z$-module, thus $[\mathfrak{m}_0, \mathfrak{m}_1] \subset \mathfrak{m}_1$. This implies $\gamma_{E_1} E_0 \subset E_1$ since at the base point x we have

$$\langle \gamma_{E_1} E_0, E_0 \rangle = \langle \mathfrak{m}_1.\mathfrak{m}_0.x, E_0 \rangle = \langle \mathfrak{m}_0.\mathfrak{m}_1.x, E_0 \rangle + \langle [\mathfrak{m}_1, \mathfrak{m}_0].x, E_0 \rangle = \langle E_1, E_0 \rangle = 0.$$

Consequently, from $D = \partial - \gamma$ we obtain

$$D_{E_1} E_0 = \pi_0(\partial_{E_1} E_0) \tag{1}$$

(where π_0 denotes the projection onto E_0) while $D_{E_0} E_0$ is the canonical connection of the fibre $S = K_z.x$ belonging to the focal projection $\pi : M \to F$.

This gives us the idea how to define the connection D in the general case where $M \subset V = \mathbb{R}^N$ is an isoparametric submanifold: On νM we let $D = \nabla$ (normal Levi-Civita connection), and for any two tangent vector fields X in E_κ and Y in E_λ we put

$$\begin{aligned} D_X Y &= D^\kappa_X Y \qquad \text{if } \kappa = \lambda, \\ D_X Y &= \pi_\lambda(\partial_X Y) \quad \text{if } \kappa \neq \lambda, \end{aligned} \tag{2}$$

where π_λ denotes the projection onto E_λ and D^κ the canonical connection on the curvature sphere $S_\kappa \subset M$. Then the curvature distributions E_κ are D-parallel which implies the D-parallelity of α. Moreover, the fibres S of any focal projections $\pi : M \to F$ (which are already "standard", cf. Ch.2) are totally geodesic with respect to D, and the induced connection D^S is the canonical one, according to (1).

However, only if $k \geq 3$ we can also show that $\gamma = \partial - D$ is parallel. In fact, it suffices to compute $(D_X \gamma)_Y Z$ for $X \in E_\kappa$, $Y \in E_\lambda$ and $Z \in E_\mu$ at an arbitrary point $x \in M$. If we have $k \geq 4$, then all three vectors are tangent to a fibre S of some focal projection $\pi = \pi_\xi$ (where $\kappa(\xi) = \lambda(\xi) = \mu(\xi) = 1$), thus $(D_X \gamma)_Y Z = (D^S_X \gamma^S)_Y Z = 0$. But if $k = 3$, this argument works only if κ, λ, μ are linearly dependent. If they are *independent*, we find a focal projection $\pi : M \to F$ such that $E_\lambda + E_\mu \subset E_F$ (being the vertical bundle, cf. Ch.2) and $E_\kappa \subset (E_F)^\perp$. Let $x(t)$ be π-horizontal curve with $x'(0) = X$ and put $z(t) = \pi(x(t))$. By (2), the connection D restricted to the subbundle E_F is closely related to the normal connection of F. In fact, the vectors $Y, Z \in E_F(x)$ can be extended to parallel normal fields of F along $z(t)$; then $Y'(t), Z'(t)$ are perpendicular to E_F and hence also D-parallel along the curve $x(t)$, according to (2). We have to show that $\gamma_{Y(t)} Z(t)$ is D-parallel. But the parallel displacements in νF along the curve $z(t)$ conjugate the actions of the normal holonomy groups at $z(t)$ for different t, thus $\gamma_{Y(t)} Z(t) = \gamma^t_{Y(t)} Z(t)$ (where $\gamma^t := \gamma^{S(z(t))}$) is parallel with respect to the normal connection on F and hence with respect to D.

Thus we have shown that the Olmos connection is canonical; in particular, the group $K = \{k \in O(N);\ k(M) = M\}$ acts transitively on M. But much more is true: the Olmos connection corresponds to a very special reductive decomposition $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{m}$ (*Olmos decomposition*) where $\mathfrak{m}$ is the space of infinitesimal transvections at x for the connection D. We shall see that $\mathfrak{m}$ has the same properties as in the standard case. First of all, when we restrict D to a fibre S of a focal projection $\pi_\xi : M \to F$, it becomes the standard canonical connection on S (corresponding to an s-representation) and $\mathfrak{m}_0 := \mathfrak{m} \cap \mathfrak{k}_z$ (for $z = \pi_\xi(x)$) is the corresponding space of infinitesimal transvections which generate the standard group action on S. In other words, the isotropy group K_z acts on $E_0(x) \subset \nu_z F$ as an s-representation. Using the isomorphism $\phi_x : \mathfrak{m} \to T_xM$, $\phi_x(A) = A.x$ we get a decomposition $\mathfrak{m} = \sum \mathfrak{m}_\kappa$ with $\mathfrak{m}_\kappa.x = E_\kappa(x)$. Then $\mathfrak{m}_0 = \sum_{\kappa(\xi)=1} \mathfrak{m}_\kappa$ since $\mathfrak{m}_\kappa.\pi_\xi(x) = \pi_\xi(\mathfrak{m}_\kappa.x) = \pi_\xi(E_\kappa(x))$ for all $\kappa \in \Lambda$.

For any κ, λ we find $\xi \in \nu$ with $\kappa(\xi) = \lambda(\xi) = 1$ (using $k \geq 3$), hence for $F = \pi_\xi(M)$ and $z = \pi_\xi(x)$ we have $E_\kappa + E_\lambda \subset \nu_z F$ and $\mathfrak{m}_\kappa + \mathfrak{m}_\lambda \subset \mathfrak{k}_z$. From the standard case we can see

$$\mathfrak{m}_\kappa.E_\lambda(x) \subset \sum_{\mu\in(\kappa,\lambda)^*} E_\mu(x), \quad [\mathfrak{m}_\kappa, \mathfrak{m}_\lambda] \subset \sum_{\mu\in(\kappa,\lambda)^*} \mathfrak{m}_\mu, \tag{3}$$

where (κ, λ) denotes the intersection of the affine span of κ and λ with Λ and $(\kappa, \lambda)^* = (\kappa, \lambda) \setminus \{\kappa, \lambda\}$.

More generally, let $F = \pi_\xi(M)$ an *arbitrary* focal manifold and $z = \pi_\xi(x)$. We get a decomposition $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ with $\mathfrak{m}_1 = \sum_{\lambda(\xi)\neq 1} \mathfrak{m}_\lambda$. By (3),

$$\mathfrak{m}_1.\nu_z F \perp \nu_z F, \quad [\mathfrak{m}_0, \mathfrak{m}_1] \subset \mathfrak{m}_1 \tag{4}$$

because for $\kappa(\xi) = 1$ and $\lambda(\xi) \neq 1$ we have $\mu(\xi) \neq 1$ for all $\mu \in (\kappa, \lambda)^*$. In particular, $\mathfrak{k} = \mathfrak{k}_z + \mathfrak{m}_1$ is a reductive decomposition.

5. The Lie triple product

Now we must define the Lie triple product R on $V = \mathbb{R}^N$. Again we get the idea from the standard case where $V = \mathfrak{p}$ is part of a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. There we have $R(x,y) = [x,y] \in \mathfrak{k}$ for any $x, y \in \mathfrak{p}$ and $\langle A, [x,y]\rangle = \langle ad(A)x, y\rangle$ for any $A \in \mathfrak{k}$, if the inner product on $\mathfrak{g}$ is the negative Killing form. Likewise, in the general case we may define $R(x,y) \in \mathfrak{k}$ by putting

$$\langle A, R(x,y)\rangle = \langle Ax, y\rangle,$$

for all $A \in \mathfrak{k}$, where the *canonical* inner product on $\mathfrak{k}$ is given by

$$\langle A, B\rangle = -\operatorname{trace}_V(AB) - \operatorname{trace}_{\mathfrak{k}}(ad(A)ad(B)), \tag{5}$$

imitating the restriction to $\mathfrak{k}$ of the negative Killing form on $\mathfrak{g}$. This defines a K-equivariant linear map $R : V \wedge V \to \mathfrak{k}$. To finish the proof of Theorem 1.3, it remains to prove the Bianchi or Jacobi identity

$$J(x,y,z) := R(x,y)z + R(y,z)x + R(z,x)y \overset{!}{=} 0. \tag{J}$$

The same method was used before frequently (e.g. cf. [C], [EH1], [Tz], [W]). E. Witt [W] has applied it for the construction of exceptional Lie algebras.

It turns out (see below) that the proof of (J) can be reduced to the case where $R(x,z) = R(y,z) = 0$. Let us consider this case first. We have $R(x,z) = 0$ if and only if $x \in \nu_z := \nu_z(K.z)$ since it means $0 = \langle \mathfrak{k}, R(x,z)\rangle = \langle \mathfrak{k}.x, z\rangle$, i.e. $z \perp \mathfrak{k}.x = T_xM$ for $M = K.x$. Thus this special case of (J) is $R(x,y)z = 0$ or $R(x,y) \in \mathfrak{k}_z$ or $0 = \langle \mathfrak{k}_z^\perp, R(x,y)\rangle = \langle \mathfrak{k}_z^\perp.x, y\rangle$, for all $x, y \in \nu_z$. So for any $z \in V$ we have to show

$$\mathfrak{k}_z^\perp.\nu_z \perp \nu_z. \tag{J'}$$

We may assume $z \in \nu_x M$ for some $x \in M$. Let $F = K.z$ and let $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ denote the corresponding splitting of the Olmos complement. Now (J') follows from (4) if we show that $\mathfrak{k}_z^\perp = \mathfrak{m}_1$. Since $\mathfrak{k}_z = \mathfrak{k}_x + \mathfrak{m}_0$, this will follow if all $\mathfrak{m}_\kappa$ are perpendicular to each other and to $\mathfrak{k}_x$.

To show this we have to use again $k \geq 3$: Any two different $\mathfrak{m}_\kappa, \mathfrak{m}_\lambda$ are contained in an isotropy subalgebra $\mathfrak{k}_z$ (for $z = x + \xi(x)$ with $\kappa(\xi) = \lambda(\xi) = 1$) which is standard. Thus $\mathfrak{m}_\kappa \perp \mathfrak{m}_\lambda \perp \mathfrak{k}_x$ with respect to the canonical (Killing) inner product on $\mathfrak{k}_z$ since in the standard case, these spaces are sums of root spaces (cf. Ch.1) which are perpendicular. It remains to show this orthogonality also for the canonical inner product on $\mathfrak{k}$. But the canonical inner products on $\mathfrak{k}$ and $\mathfrak{k}_z$ differ by the contributions to the traces coming from $\mathfrak{k}_z^\perp$ and $\nu_z^\perp$ (cf. (5)). But these vanish since for any $\mu \in \Lambda$ with $\mu(\xi) \neq 1$ we obtain from (3)

$$\langle \mathfrak{m}_\kappa.\mathfrak{m}_\lambda.E_\mu(x), E_\mu(x)\rangle = \langle \mathfrak{m}_\lambda.E_\mu(x), \mathfrak{m}_\kappa.E_\mu(x)\rangle = 0,$$
$$\langle ad(\mathfrak{m}_\kappa)ad(\mathfrak{m}_\lambda)\mathfrak{m}_\mu, \mathfrak{m}_\mu\rangle = \langle [\mathfrak{m}_\lambda, \mathfrak{m}_\mu], [\mathfrak{m}_\kappa, \mathfrak{m}_\mu]\rangle = 0.$$

In fact, if $\mu \notin (\kappa, \lambda)$, then $(\lambda, \mu) \cap (\kappa, \mu) = \{\mu\}$ and $(\lambda, \mu)^* \cap (\kappa, \mu)^* = \emptyset$ which shows the above orthogonality. A similar argument holds for $\kappa = 0$ or $\lambda = 0$ if we put $\mathfrak{m}_0 := \mathfrak{k}_x$.

It remains to show how (J) for general x, y, z follows from the special case (J') where $R(x,z) = R(y,z) = 0$. The mapping $J \in \Lambda^4 V$ can be considered as a trace free symmetric linear map on $\Lambda^2 V = \mathfrak{o}(V) = \mathfrak{k} + \mathfrak{k}^\perp$; we use the trace scalar product $\langle X, Y\rangle_0 = -\operatorname{trace}(XY)$ for $X, Y \in \mathfrak{o}(V)$. In [HZ] it was shown that $J|_\mathfrak{k}$ is a multiple of the identity (see below). Thus (J) is equivalent to $J|_{\mathfrak{k}^\perp} = 0$ (recall that $\operatorname{trace} J = 0$). Now $\mathfrak{k}^\perp$ is spanned by decomposable elements $x \wedge y$. In fact, $x \wedge y \perp \mathfrak{k}$ iff $0 = \langle \mathfrak{k}, x \wedge y\rangle_0 = \langle \mathfrak{k}x, y\rangle$, hence iff $y \in \nu_x$, and an arbitrary $A \in \mathfrak{o}(V)$ is perpendicular to all $x \wedge y \in \mathfrak{k}^\perp$ iff $A.x \perp \nu_x$ for all x, hence iff $A.x$ is tangent to the orbit $K.x$ for all $x \in V$ which shows $A \in \mathfrak{k}$. Thus it remains to show $J(x \wedge y) = 0$ for $y \in \nu_x$.

The sections of a polar representation are the normal spaces of a principal orbit, and the normal spaces of any orbit are unions of sections. Thus $x, y \in \nu_p M$ for some $p \in M$, and we have to show $J(\nu_p, \nu_p, V) = 0$, i.e. $J(\nu_p, \nu_p, E_\kappa(p)) = 0$ for any $\kappa \in \Lambda$ (note that $J(\nu_p, \nu_p, \nu_p) = 0$ due to $R(\nu_q, \nu_q) = 0$). Since $J(n_\kappa(p) \wedge n_\kappa(p)) = 0$, this is equivalent to $J(l_\kappa(p), \nu_p, E_\kappa(p)) = 0$. But $E_\kappa(p) \subset \nu_{x_\kappa}$ for any $x_\kappa \in l_\kappa(p)$, or in other words, $R(l_\kappa(p), E_\kappa(p)) = 0$. Since also $R(l_\kappa(p), \nu_p) = 0$, we are done by the special case (J').

It remains to prove that $J|_\mathfrak{k}$ is a multiple of the identity (cf. [HZ]). Consider the vector space $\mathfrak{g} = \mathfrak{k} \oplus V$. We imitate the way how the Lie structure on $\mathfrak{g}$ is defined by the Lie triple product $R(x,y) = [x,y]$ on $\mathfrak{p} = V$ in the symmetric case: First of all, $\mathfrak{g}$ is a K-module (where K acts on $\mathfrak{k}$ by Ad), hence we have a map $ad(A) : \mathfrak{g} \to \mathfrak{g}$ for any $A \in \mathfrak{k}$. Further, for any $x \in V$ we let $ad(x)$ be the skew symmetric linear map on $\mathfrak{g}$ interchanging the subspaces $\mathfrak{k}$ and V with $ad(x)y = R(x,y)$ and $ad(x)A = -A.x$ for $y \in V$ and $A \in \mathfrak{k}$. This defines a K-equivariant map $ad : V \to \mathfrak{o}(\mathfrak{g})$. Now $J(x,y,z) = ad(R(x,y))z - [ad(x), ad(y)]z$ for all $x, y, z \in V$ and hence

$$J(x \wedge y) = ad(R(x,y)) - [ad(x), ad(y)] \in \mathfrak{o}(V).$$

We consider $J(x \wedge y)$ as an element of $\mathfrak{o}(\mathfrak{g})$ rather than of $\mathfrak{o}(V)$. On $\mathfrak{o}(\mathfrak{g})$ we use the inner product $\langle P, Q\rangle_1 = -\operatorname{trace}_\mathfrak{g} PQ$. The canonical inner product on $\mathfrak{k}$ was chosen so that $\langle A, B\rangle = \langle ad(A), ad(B)\rangle_1$. Further $\langle ad(x), ad(y)\rangle_1 = \lambda \cdot \langle x, y\rangle$ for all $x, y \in V$ since V is an irreducible K-representation (otherwise, M would split extrinsically). Now we have for any $A \in \mathfrak{k} \subset \mathfrak{o}(V)$:

$$\begin{aligned}\langle ad(A), ad(R(x,y))\rangle_1 &= \langle A, R(x,y)\rangle = \langle A.x, y\rangle, \\ \langle ad(A), [ad(x), ad(y)]\rangle_1 &= \langle [A, ad(x)], ad(y)\rangle_1 \\ &= \langle ad(Ax), ad(y)\rangle_1 = \lambda\langle A.x, y\rangle\end{aligned}$$

Thus $\langle ad(A), J(x \wedge y)\rangle_1 = (1+\lambda)\langle A.x, y\rangle = \frac{1}{2}(1+\lambda)\langle A, x\wedge y\rangle_0$. On the other hand, since A is a derivation of R we get $ad(R(x,y))A = [ad(x), ad(y)]A$. Hence the linear map $J(x\wedge y)$ vanishes on $\mathfrak{k}$, and therefore the trace in $\langle ad(A), J(x\wedge y)\rangle_1$ has to be taken only over V. Hence we obtain

$$\langle A, J(x\wedge y)\rangle_0 = \langle ad(A), J(x\wedge y)\rangle_1 = \mu \cdot \langle A, x\wedge y\rangle_0$$

where $\mu = \frac{1}{2}(1+\lambda)$. Thus $J|_{\mathfrak{k}} = \mu \cdot I$. The proof of Theorem 1.3 is now complete.

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