Pluriharmonic Maps, Loop Groups and Twistor Theory

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0. Introduction

Any simply connected minimal surface in 3-space allows a one-parameter family ('*associated family*') of isometric deformations preserving the principal curvatures while rotating the principal curvature directions; the most famous example is the deformation of the catenoid into the helicoid. This property holds also in a much more general situation: It is valid for harmonic maps of a surface M into a symmetric space $P = G/K$. If M is simply connected, any harmonic map $f\colon M \to G/K$ determines a smooth family $f_\lambda\colon \tilde{M} \to G/K$ of harmonic maps defined on the universal cover $\tilde{M}$ of M, parametrized by $\lambda = e^{-i\theta} \in S^1$, such that the differential $\mathrm{d}f_\lambda$ is obtained essentially from a rotation of $\mathrm{d}f$ by the angle θ, and this property characterizes harmonic maps. It may happen that this associated family is trivial, i.e. f_λ is congruent to f for all λ, in which case the harmonic map is called *isotropic*. In particular this happens for $M = S^2$ if the target space is an n-sphere or a complex projective space. This case was studied by many authors, starting 1967 with Calabi's work [6]. One application was the explicit description of all minimal

2-spheres in the 4-sphere by Bryant [1]. A description of all isotropic harmonic maps into arbitrary symmetric spaces was given in [5]: They arise as projections from holomorphic and 'superhorizontal' maps $\hat{f}$ into a so called *twistor space* Z which is an adjoint G-orbit fibering over G/K where *superhorizontal* means that $d\hat{f}$ takes values in a certain distribution on Z.

If the target space is a compact Lie group or a compact symmetric space other than S^n or $\mathbb{C}P^n$, harmonic spheres are no longer isotropic. They were investigated by Uhlenbeck [22] for $U(n)$ and Burstall and Guest [4] in the general case. Then the associated family f_λ could be considered as a map from M into the loop group ΛG consisting of all sufficiently regular maps $\gamma\colon S^1 \to G$. Using the Cartan embedding these results then extend to harmonic maps into arbitrary compact symmetric spaces. However, in these papers the associated family f_λ is differently defined and not harmonic for most values of λ.

Ideas contained in [22] and in the work of Pohlmeyer [20] and the Russian soliton school were used in [8] to generate harmonic maps from certain almost arbitrarily chosen meromorphic $(1, 0)$-forms, so called *normalized potentials*, on a simply connected domain $M \subset \hat{\mathbb{C}}$; like in the case of minimal surfaces in Euclidean space this construction was called a *Weierstrass representation* for harmonic maps. As an application of this one can change the setting in [22] or [4] slightly and does, indeed, obtain an associated family of harmonic maps from a given harmonic map into an arbitrary compact symmetric space G/K, without using the Cartan embedding.

Of course one also wants to replace the surface M by a simply connected manifold of higher dimension. Since complex analysis plays an essential rôle in the theory, one assumes that M (like a surface) is a complex manifold. But harmonicity is too weak in higher dimensions; instead one assumes that f is *pluriharmonic* which means harmonicity along any complex curve in M. It has been shown first by Ohnita and Valli (cf. [11, 19]) that these maps are also characterized by associated families. The twistor theory for *isotropic* pluriharmonic maps was developed in [11].

In the present paper we show that the method of [8] essentially can be extended to the pluriharmonic setting. However, a few new features arise. Since we are dealing with several variables, holomorphic $(1, 0)$-forms are not automatically locally integrable. As a matter of fact, while the normalized potentials are trivially integrable in the one-dimensional case their higher-dimensional analogues are closely related to the 'curved flats' of Ferus and Pedit [12], thus permitting to construct arbitrary pluriharmonic maps from certain given curved flats. Also, while in the one-dimensional case a simply connected Riemann surface (other than $\hat{\mathbb{C}}$) is contractible and can be realized as an open subset of $\mathbb{C}^1$, the analogous result is not true for higher-dimensional complex manifolds. Thus globally defined framings do not exist automatically, opposite to the one-dimensional case. Of course, locally extended framings $F\colon M_o \to \Lambda$ exist, where M_o is a contractible open subset of M and Λ is a group of (twisted) loops $g\colon S^1 \to G$.

It turns out, however, that after projecting the local extended framings from Λ to $\mathcal{Z} = \Lambda/K$ (where K is embedded into Λ as constant loops) our approach works globally. Since $\mathcal{Z}$ fibres naturally over G/K, our description can be viewed as a generalized twistor theory where the twistor space Z is replaced with $\mathcal{Z}$ which is an infinite-dimensional adjoint orbit. As in the isotropic case, every pluriharmonic map $f\colon M \to G/K$ arises by projecting a holomorphic and 'superhorizontal' map $\hat{f}\colon M \to \mathcal{Z}$ into G/K. The classical finite-dimensional twistor theory embeds nicely.

We start our paper with a geometric foundation of the basic notions. In Section 1, given a simply connected manifold M, a vector bundle E over M and an E-valued 1-form $\delta \in \mathrm{Hom}(TM, E)$, we investigate when there does exist a smooth map $f\colon M \to P = G/K$ such that $E \cong f^*TP$ and $\delta \cong \mathrm{d}f$; such δ will be called *integrable*. In Section 2 we introduce the rotations $R_\lambda = (\cos\theta)I + (\sin\theta)J$, where $\lambda = e^{-i\theta}$ and J denotes the complex structure of M. We show that for any map $f\colon M \to P$ the 1-form $\delta_\lambda = \mathrm{d}f \circ R_\lambda$ is integrable if and only if f is pluriharmonic. Thus we have introduced the loop parameter in a geometric fashion.

In Section 3 we consider locally extended framings $F\colon M_o \to \Lambda$ for pluriharmonic maps $f\colon M \to G/K$ from M to a symmetric space G/K. We show that F can be defined globally as a map from M to Λ/K. The following section, Section 4, discusses the Maurer–Cartan forms of extended framings. Here we show in particular that our geometrically defined associated family coincides with the one introduced in [8].

The next two sections present for pluriharmonic maps constructions related to the techniques of [8]. In particular, we introduce locally the notion of holomorphic framings (Section 5) and globally the notion of normalized framings (Section 6). In this section the contrast with the one-dimensional theory is best visible. Formulas (38) and (39) are trivial in the one-dimensional case. In general they define a curved flat (see e.g. [12]).

Section 7 briefly presents the notion of dressing, both from the point of view of extended framings with values in $\mathcal{Z}$ and the usual 'group splitting' point of view. In Section 8 we discuss, as an example, the isotropic case. With formula (53) we make also contact with [14], §2(1), and [7]. The last section (Section 9) discusses pluriharmonic maps into Lie groups. In particular we draw the connection of our approach to the slightly different one of Uhlenbeck.

1. Integrability of 1-Forms

Let M and P be smooth manifolds. Suppose that P carries a Riemannian metric. On any Riemannian manifold there is a canonical differentiation of tangent vector fields, the *Levi-Civita covariant derivative* D which in turn defines *parallel displacements* of vectors along curves. Let $f\colon M \to P$ be a smooth map. Its differential is considered as a bundle homomorphims $\mathrm{d}f \in \mathrm{Hom}(TM, f^*TP)$ where f^*TP is the pull back bundle over M with fibre $(f^*TP)_z = T_{f(z)}P$ at any point $z \in M$. Hence $\mathrm{d}f$ is a 1-form on M with values in the vector bundle $E = f^*TP$, and in fact it is *closed*, i.e. for any vector fields X, Y on M we have

$$d^D \mathrm{d}f(X, Y) := D_X(\mathrm{d}f(Y)) - D_Y(\mathrm{d}f(X)) - \mathrm{d}f([X, Y]) = 0. \tag{1}$$

If M also carries a Riemannian metric, we can define the *Hessian* $D\,\mathrm{d}f\colon TM \otimes TM \to f^*TP$,

$$D\,\mathrm{d}f(X, Y) = D_X(\mathrm{d}f(Y)) - \mathrm{d}f(D_X Y), \tag{2}$$

and (1) says that this bilinear map is symmetric.

Now we ask the converse question: Given manifolds M and P and a metric vector bundle E over M with a compatible covariant derivative D and an E-valued 1-form $\delta \in \mathrm{Hom}(TM, E)$, when does there exist a smooth map $f\colon M \to P$ such that $E \cong f^*TP$ and $\delta \cong \mathrm{d}f$? More precisely

$$\Phi \circ \delta = \mathrm{d}f \tag{3}$$

for some bundle isometry $\Phi\colon E \to f^*TP$ which is *parallel*, i.e. invariant under parallel displacements. Such a δ will be called *integrable*. By (1), a necessary condition for integrability is $d^D\delta = 0$. Clearly, if M is simply connected and $P = \mathbb{R}^n$ with its Euclidean inner product, this condition is also sufficient.

But instead of $\mathbb{R}^n$ we want to consider a *symmetric space*. This is a Riemannian manifold P such that for every $p \in P$ there is an isometry s_p fixing p with $(\mathrm{d}s_p)_p = -I$, called *geodesic symmetry* or *point reflection* at p; clearly $s_p^{-1} = s_p$. In particular, P is *homogeneous*, i.e. its (full) isometry group G acts transitively, and hence P may be viewed as a coset space G/K where K is the isotropy group (stabilizer subgroup) of some chosen point $o \in P$. Clearly, the point reflection s_o commutes with K, hence K is fixed by the *Cartan involution* $\sigma \in \mathrm{Aut}(G)$,

$$\sigma(g) = s_o g s_o^{-1}. \tag{4}$$

In fact $\mathrm{Fix}(\sigma) = \{g \in G;\ \sigma(g) = g\}$ contains K as a normal subgroup of finite index. A local characterization of symmetric spaces is the *parallelity of its Riemannian curvature tensor* $R^P(X, Y, Z) = [D_X, D_Y]Z - D_{[X,Y]}Z$. The trilinear map on the tangent spaces defined by R^P is called a *Lie triple product*.

Now $d^D\delta = 0$ is no longer sufficient for the integrability of δ, but a further condition is needed: We assume that on the vector bundle E there is also a parallel Lie triple product $R^P\colon E \otimes E \otimes E \to E$ whose restriction to any fibre is isometrically isomorphic to the curvature tensor on any tangent space of P. We call (E, D, R^P) a vector bundle *of type* P. We have (cf. [10]):

THEOREM 1. *Let M be simply connected and (E, D, R^P) a vector bundle of type P over M and let $R^E\colon TM \otimes TM \otimes E \to E$ be its curvature tensor. Given $\delta \in \mathrm{Hom}(TM, E)$, there exists $f\colon M \to P = G/K$ and a parallel bundle isometry $\Phi\colon E \to f^*TP$ such that $\Phi \circ \delta = \mathrm{d}f$ if and only if*

$$d^D\delta = 0, \tag{5}$$

$$R^E(X,Y)\xi = R^P(\delta X, \delta Y)\xi \tag{6}$$

for all sections X, Y of TM and ξ of E. The map f is unique up to isometries of P, i.e. any other solution is of the form $g \circ f$ for some $g \in G$.

2. Complex Rotations and the Levi Form

From now on let M be always a simply connected *complex* manifold. The almost complex structure on TM will be called J. Consider the eigenspace decomposition of J on the complexified tangent bundle

$$T^cM = T'M + T''M, \tag{7}$$

where $J = i\,I$ on $T'M$ and $J = -i\,I$ on $T''M$. Let $\theta \in [0, 2\pi]$. Put $\lambda = e^{-i\theta}$ and

$$R_\lambda := (\cos\theta)I + (\sin\theta)J. \tag{8}$$

Then $R_\lambda = \lambda^{-1} I$ on $T'M$ and $R_\lambda = \lambda\, I$ on $T''M$.

Let $f\colon M \to P = G/K$ be a smooth map and put $\delta_\lambda = \mathrm{d}f \circ R_\lambda \in \mathrm{Hom}(TM, f^*TP)$. Under which conditions is δ_λ integrable for any $\lambda \in S^1$? The answer was given in [11]: if and only if f is *pluriharmonic*, i.e. if $f|_C$ is harmonic for any complex one-dimensional submanifold $C \subset M$ or equivalently if the *Levi form* of f vanishes, $Lf = 0$.* The Levi form can be explained best by using a local *Kähler metric* on M, i.e. a Riemannian metric such that J is orthogonal and parallel. In particular, the eigenbundles $T'M$ and $T''M$ are parallel. Locally, such a metric always exists on a complex manifold, e.g. the flat metric in a complex coordinate chart has this property. Then the Hessian $h = D\,\mathrm{d}f$ is defined and its complex linear extension to T^cM splits as $h = h^{(2,0)} + h^{(1,1)} + h^{(0,2)}$: For any $X, Y \in T^cM$ we put $h^{(2,0)}(X,Y) = h(X', Y')$, $h^{(0,2)}(X,Y) = h(X'', Y'')$, and

$$h^{(1,1)}(X,Y) = h(X', Y'') + h(X'', Y') = \tfrac{1}{2}(h(X,Y) + h(JX, JY)), \tag{9}$$

where $X' = (1/2)(X - iJX)$ and $X'' = (1/2)(X + iJX)$ are the components of X with respect to the splitting (7). The $(1,1)$-part given by (9) is called *Levi form* $Lf = h^{(1,1)}$. Surprisingly, this is *independent of the Kähler metric*. In fact, $h(X,Y) = D_X(\mathrm{d}f(Y)) - \mathrm{d}f(D_XY)$, and only the second term depends on the metric of M. If X and Y are complex vector fields with $X \in T'M$ and $Y \in T''M$, then by parallelity of these bundles we have $D_XY \in T''M$ and $D_YX \in T'M$. Hence $D_XY = D_YX \in T' \cap T'' = 0$ if $[X,Y] = 0$. Using a local basis of commuting vector fields in T' and T'', e.g. $\partial/\partial z_i$ and $\partial/\partial \bar{z}_j$ for a complex coordinate system $z = (z_1, \ldots, z_m)$ on M, we see the independence of $Lf = D\,\mathrm{d}f^{(1,1)}$ from the chosen Kähler metric.

* A slightly different definition was given by Ohnita [17]: He calls f pluriharmonic if $\mathrm{d}f \circ J$ is integrable. This is easily seen to be equivalent to our definition.

THEOREM 2. *Let M be a simply connected complex manifold, $P = G/K$ a symmetric space with either nonnegative or nonpositive curvature and $f: M \to P$ be a smooth map. Then $\delta_\lambda = \mathrm{d}f \circ R_\lambda$ is integrable for all $\lambda \in S^1$ if and only if f is pluriharmonic.*

Remark. The proof of Theorem 2 is contained in [11]. As a matter of fact, the conclusion '$\Rightarrow$' is easy : the pluriharmonicity follows from (5) alone. To see this observe that $\delta_\lambda = \lambda\,\mathrm{d}f$ on $T''M$ and $\delta_\lambda = \lambda^{-1}\,\mathrm{d}f$ on $T'M$, hence $(D_{X'}\delta)Y'' = \lambda h(X', Y'')$ while $(D_{Y''}\delta)X' = \lambda^{-1}h(Y'', X')$. Due to the symmetry of $h = D\,\mathrm{d}f$, these expressions are equal for $\lambda \neq \pm 1$ only if $h(X', Y'') = 0$; this is the pluriharmonicity of f. The converse conclusion '$\Leftarrow$', in particular the proof of (5) for δ_λ is more complicated and needs the semi-definiteness of the curvature operator of P.

COROLLARY 1. *Pluriharmonic maps come in families (so called associated families) f_λ, $\lambda \in S^1$ where $\mathrm{d}f \circ R_\lambda \cong \mathrm{d}f_\lambda$, more precisely*

$$\Phi_\lambda \circ \mathrm{d}f \circ R_\lambda = \mathrm{d}f_\lambda \tag{10}$$

for some parallel bundle isometry Φ_λ: $f^*TP \to f_\lambda^*TP$ preserving the curvature tensor R^P.

The fact that Φ_λ is isometric and preserves R^P is equivalent to saying that for any $z \in M$ we have $\Phi_\lambda(z) = dg_{f(z)}$ for some isometry $g \in G$, more precisely, for every $z \in M$ there exists $g \in G$ (depending on z) with $g(f(z)) = f_\lambda(z)$ and $\mathrm{d}g_{f(z)} = \Phi_\lambda(z) \in \mathrm{Hom}(T_{f(z)}P, T_{f_\lambda(z)}P)$ where $\mathrm{d}g$ is the differential of g considered as an isometry g: $P \to P$. Viewing G as a group acting on TP we may simply say $\Phi_\lambda(z) = g \in G$. From now on we will call f_λ, or more precisely $(f_\lambda, \Phi_\lambda)$, the *associated family* of f.

Let us consider the special case $\lambda = -1$. Then $R_{-1} = -I$ and we may easily write down a solution of (10): $f_{-1} = f$ and $\Phi_{-1} = -I$. More generally, if $(f_\lambda, \Phi_\lambda)$ is a solution for some λ, we have $R_{-\lambda} = -R_\lambda$ and hence $(f_\lambda, -\Phi_\lambda)$ is a solution for $-\lambda$. Considering $\Phi_\lambda(z)$ as an element of G we may express this by saying $\Phi_{-\lambda}(z) = \Phi_\lambda(z)s_{f(z)}$ or shortly

$$\Phi_{-\lambda} = \Phi_\lambda s_f, \tag{11}$$

where $s_p \in G$ is the geodesic symmetry at $p \in P$.

Remark. We would like to point out that the construction of an associated family as explained above requires M to be simply connected.

3. Loop Group Formulation

Let $f: M \to P = G/K$ be any smooth map. We will always assume that $f(z_o)$ is the base point $o = eK \in G/K$ for some fixed $z_o \in M$. If M is *contractible*, f

can be lifted to a smooth mapping $F: M \to G$ such that $f(z) = F(z)K \in G/K$; this will be called a *framing* for f. Any other framing is obtained as $\tilde{F} = Fk$ for some smooth mapping (*gauge transformation*) $k: M \to K$. Passing to Fk with $k = F(z_o)^{-1} \in K$, we may assume $F(z_o) = e$ (= unit element of G). If M is arbitrary, such framings are possibly no longer defined globally, but on any contractible open subset $M_o \subset M$. From now on M_o will aways denote such a subset of M.

If f is pluriharmonic, we obtain the associated family $(f_\lambda, \Phi_\lambda)$ to which we assign the framing

$$F_\lambda = g_\lambda \Phi_\lambda F, \tag{12}$$

where $g_\lambda = \Phi_\lambda(z_o)^{-1} \in G$. In fact F_λ is a framing of $g_\lambda \circ f_\lambda$ since $\Phi_\lambda(z)$ (viewed as an element of G and hence as an isometry of P) maps $f(z)$ onto $f_\lambda(z)$. Moreover, by (11) we have

$$F_{-\lambda} = (g_\lambda s_o)^{-1} \Phi_\lambda s_f F = s_o^{-1} g_\lambda \Phi_\lambda F s_o = \sigma(F_\lambda) \tag{13}$$

where $\sigma(g) = s_o g s_o^{-1}$ is the Cartan involution of G corresponding to the symmetric space P. Thus F_λ can be considered as an element of the *twisted loop group*

$$\Lambda := \Lambda_\sigma G := \{\gamma: S^1 \to G;\ \gamma(-\lambda) = \sigma(\gamma(\lambda))\}, \tag{14}$$

where the loops $\gamma \in \Lambda$ need to satisfy certain regularity assumptions (e.g. H^1 or C^∞). Thus starting from the pluriharmonic map $f: M \to G/K$ we have defined a map

$$\mathcal{F}: M_o \to \Lambda,\ \ \mathcal{F}(z)(\lambda) = F_\lambda(z). \tag{15}$$

We can make $\mathcal{F}$ unique and hence globally defined on M by composing it with the canonical projection

$$\pi: \Lambda \to \mathcal{Z} := \Lambda/K, \tag{16}$$

where $K \subset \Lambda$ denotes the subgroup of constant loops in Λ with values in K. Note that the twisting condition $\gamma(-\lambda) = \sigma(\gamma(\lambda))$ implies that constant loops automatically lie in the fixed group of σ in which K is normal and of finite index. Summing up the preceding discussion we obtain the following theorem:

THEOREM 3. *Let M be a simply connected complex manifold and $z_o \in M$. Then for any pluriharmonic map $f: M \to P$ with $f(z_o) = o$ there exists precisely one map $\bar{\mathcal{F}}: M \to \mathcal{Z} = \Lambda/K$ with $ev_1 \circ \bar{\mathcal{F}} = f$ and with local lifts $\mathcal{F} = (F_\lambda): M_o \to \Lambda$ given by (12) such that $F_\lambda F_1^{-1}$ is parallel along f.*

4. The Maurer–Cartan Form

Now we assume that the isometry group G of the symmetric space P is a (closed) matrix group: $G \subset GL(\mathbb{R}^N) \subset \mathbb{R}^{N\times N}$ for some N. Let $\mathfrak{g}$ denote its Lie algebra. To any smooth map $F: M \to G$ we assign a $\mathfrak{g}$-valued 1-form

$$\alpha = F^{-1}\,\mathrm{d}F \in \Omega^1(M;\mathfrak{g}), \tag{17}$$

which satisfies $\mathrm{d}\alpha = -F^{-1}\,\mathrm{d}F\,F^{-1} \wedge \mathrm{d}F = -\alpha \wedge \alpha$. Vice versa, given any $\alpha \in \alpha^1(M,\mathfrak{g})$ on a simply connected manifold M, it is an easy consequence of the Frobenius theorem that there exists a map $F\colon M \to G$ with $\alpha = F^{-1}\,\mathrm{d}F$ if and only if

$$\mathrm{d}\alpha = -\alpha \wedge \alpha, \tag{18}$$

and this F is unique up to left translations with (constant) elements of G. Using the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to the symmetric space $P = G/K$, the Maurer–Cartan form splits accordingly as

$$\alpha = \alpha_k + \alpha_p. \tag{19}$$

A second splitting compatible to the first one is obtained from the complex structure:

$$\alpha = \alpha' + \alpha'', \tag{20}$$

where $\alpha'(X) = \alpha(X')$ and $\alpha''(X) = \alpha(X'')$.

Now let $f\colon M \to G/K$ be pluriharmonic. As explained in the previous sections, we have the associated family $(f_\lambda, \Phi_\lambda)$ with framing $F_\lambda = g_\lambda \Phi_\lambda F$ as given in (12). The corresponding Maurer–Cartan form is

$$\alpha_\lambda = F_\lambda^{-1}\,\mathrm{d}F_\lambda = F^{-1}\Phi_\lambda^{-1}\,\mathrm{d}\Phi_\lambda F + \alpha. \tag{21}$$

Due to the parallelity of Φ_λ, the first term on the right-hand side of (21) takes values in $\mathfrak{p}$ as the subsequent lemma shows:

LEMMA 1. *Let $f, \tilde{f}\colon M \to P = G/K$ be smooth maps such that $\Phi \circ \mathrm{d}\tilde{f} = \mathrm{d}f$ for some linear isometry $\Phi\colon f^*TP \to \tilde{f}^*TP$ preserving the curvature tensor of P. Thus Φ can be considered also as a smooth map $\Phi\colon M \to G$. Let $F\colon M \to G$ be a framing of f. Then Φ is parallel as a section of* $\mathrm{Hom}(f^*TP, \tilde{f}^*TP)$ *if and only if $F^{-1}\Phi^{-1}\,\mathrm{d}\Phi F \in \mathfrak{p}$.*

Proof. The parallel displacements on P are given by curves $g(t)$ in G which are horizontal with respect to the principal fibration $G \to G/K$. Hence Φ is parallel iff it maps horizontal curves in G again onto horizontal curves. More precisely, if $z(t)$ is a curve in M and $t \mapsto g(t) \in G$ a horizontal lift of $t \mapsto f(z(t)) \in G/K$, then $t \mapsto \Phi(z(t))g(t) \in G$ is a horizontal lift of $t \mapsto \tilde{f}(z(t))$. In other words, $(\Phi(z)g)' = \Phi(z)'g + \Phi(z)g'$ is horizontal. But g' and hence $\Phi(z)g'$ are horizontal vectors anyway since left translation by the element $\Phi(z(t)) \in G$ preserves horizontality. Thus $\Phi(z)'g$ is horizontal for any curve $z(t)$ in M and every $g \in G$. Therefore, the 1-forms $\mathrm{d}\Phi F$ and $F^{-1}\Phi^{-1}\,\mathrm{d}\Phi F$ take horizontal values iff Φ is parallel. This proves the claim since the horizontal space at $e \in G$ is $\mathfrak{p} \subset \mathfrak{g} = T_eG$. □

As a consequence, we obtain from (21):

$$(\alpha_\lambda)_k = \alpha_k \tag{22}$$

for any $\lambda \in S^1$. The other component $(\alpha_\lambda)_p$ is just the horizontal lift of $\mathrm{d}f_\lambda$, up to isometries of P, and from (10) we obtain

$$(\alpha_\lambda)_p = \alpha_p \circ R_\lambda = \lambda^{-1}\alpha'_p + \lambda\,\alpha''_p. \tag{23}$$

A smooth map $\mathcal{F}\colon M_o \to \Lambda$ with

$$\mathcal{F}^{-1}\,\mathrm{d}\mathcal{F} = \alpha_k + \lambda^{-1}\alpha'_p + \lambda\,\alpha''_p \tag{24}$$

for some $\alpha \in \alpha^1(M, \mathfrak{g})$ will be called an *extended framing*.

THEOREM 4. *Let M be a simply connected complex manifold. A smooth map $f\colon M \to G/K$ is pluriharmonic if and only if there is locally an extended framing $\mathcal{F}$ such that $f = \pi \circ \mathcal{F}(1)$ where $\pi\colon G \to G/K$ is the canonical projection. The map $\bar{\mathcal{F}} = \pi \circ \mathcal{F}\colon M \to \Lambda/K$ is globally defined and uniquely determined by f.*

Proof. It only remains to show that $f = \pi \circ \mathcal{F}(1)$ is pluriharmonic if $\mathcal{F}\colon M \to \Lambda$ satisfies (24). This follows from Theorem 2: Putting $F_\lambda = \mathcal{F}(\lambda)$ and $f_\lambda = \pi \circ F_\lambda$, we obtain

$$\mathrm{d}f_\lambda = \mathrm{d}\pi.\mathrm{d}F_\lambda = F_\lambda.\mathrm{d}\pi.(F_\lambda^{-1}\,\mathrm{d}F_\lambda) = F_\lambda.\mathrm{d}\pi.(\alpha_k + \alpha_p \circ R_\lambda) = F_\lambda.(\alpha_p \circ R_\lambda).$$

In particular we have $\mathrm{d}f_1 = F_1\,\alpha_p$ and hence $\mathrm{d}f_\lambda = \tilde{\Phi}_\lambda\,\mathrm{d}f_1 \circ R_\lambda$ with $\tilde{\Phi}_\lambda = F_\lambda F_1^{-1}$. From (24) and the previous lemma we see that $\tilde{\Phi}_\lambda$ is parallel which finishes the proof. □

5. Holomorphic Framings

Let us assume further that $G \subset GL(\mathbb{R}^N)$ can be complexified in the following sense: We complexify the Lie algebra $\mathfrak{g} \subset \mathbb{R}^{N\times N}$ by putting $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g} \subset \mathbb{C}^{N\times N}$ and we assume that $\mathfrak{g}^c$ is the Lie algebra of a closed subgroup $G^c \subset GL(\mathbb{C}^N)$ such that $G = \{g \in G^c;\ \overline{g} = g\}$.* Moreover, we require that the Cartan involution σ on G extends holomorphically to G^c and commutes with the complex conjugation. Now we consider the complex extension of the twisted loop group

$$\Lambda^c := \Lambda_\sigma G^c = \{\gamma\colon S^1 \to G^c\}. \tag{25}$$

Any $\gamma \in \Lambda^c$ can be written as a matrix Fourier series

$$\gamma(\lambda) = \sum_{j=-\infty}^{\infty} a_j \lambda^j. \tag{26}$$

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* Obviously, this holds if G is the zero set of a polynomial on $\mathbb{R}^{N\times N}$ which is true for all classical groups and for all adjoint representations of semisimple groups; note that the automorphism group of a real Lie algebra is defined by such polynomials.

Let Σ denote the space of all such Fourier series (26) and consider the subspaces

$$\Sigma^+ = \left\{ \sum_{j\geq 0} a_j \lambda^j \right\}, \quad \Sigma^- = \left\{ \sum_{j<0} a_j \lambda^j \right\}.$$

We define the subgroups*

$$\begin{aligned} \Lambda^+ &= \{\gamma \in \Lambda^c;\ \gamma, \gamma^{-1} \in \Sigma^+\}, \\ \Lambda^- &= \{\gamma \in \Lambda^c;\ \gamma, \gamma^{-1} \in I + \Sigma^-\}. \end{aligned} \tag{27}$$

LEMMA 2 (cf. [8, 17]). *If $\mathcal{F}: M_o \to \Lambda$ is an extended framing, there exists some $\mathcal{V}_+: M_o \to \Lambda^+$ such that $\mathcal{H} = \mathcal{F}\mathcal{V}_+: M_o \to \Lambda^c$ is holomorphic on M_o. In other words, $\mathcal{F}$ is holomorphic modulo Λ^+.*

Proof. Consider $\alpha = \mathcal{F}^{-1}\,\mathrm{d}\mathcal{F}$ and

$$\eta = \mathcal{H}^{-1}\,\mathrm{d}\mathcal{H} = \mathcal{V}_+ \alpha \mathcal{V}_+^{-1} + \mathcal{V}_+^{-1}\,\mathrm{d}\mathcal{V}_+. \tag{28}$$

To establish the holomorphicity of $\mathcal{H}$ we need to find $\mathcal{V}_+$ such that the antilinear part η'' of η vanishes. But $\eta'' = \mathcal{V}_+^{-1}(\alpha''\mathcal{V}_+ + d''\mathcal{V}_+)$ vanishes if and only if

$$d''\mathcal{V}_+ = -\alpha''\mathcal{V}_+. \tag{29}$$

Since $\alpha'' = \mathcal{F}^{-1}d''\mathcal{F}$, the integrability condition for (29) is satisfied. Moreover, $\alpha'' = \alpha_k'' + \lambda\alpha_p''$ takes values in the Lie algebra of Λ^+, hence we can find a solution $\mathcal{V}_+$ with values in Λ^+. □

Remark. From (28) we see that the Fourier series of η like that of α starts with λ^{-1}:

$$\eta = \sum_{j>-1} \eta_j \lambda^j. \tag{30}$$

The choice of $\mathcal{H}$ is not unique; in fact, each solution (29) takes precisely the form $\tilde{\mathcal{V}}_+ = \mathcal{V}_+\mathcal{W}_+$ for some holomorphic map $\mathcal{W}_+: M_o \to \Lambda^+$. In the next section we will use this freedom to choose a very special η where all but the first coefficient η_{-1} vanish.

Now consider the group $\hat{K} = \mathrm{Fix}(\sigma) = \{g \in G;\ \sigma(g) = g\}$ which contains the isotropy group K as a normal subgroup of finite index (cf. Section 1). Let $\Gamma = \hat{K}/K$.

LEMMA 3. *The inclusion $\Lambda \subset \Lambda^c$ induces an equivariant isomorphism of the coset spaces $\Lambda/\hat{K} \xrightarrow{\cong} \Lambda^c/\Lambda^+$. Consequently, $\mathcal{Z}$ is a finite covering of Λ^c/Λ^+:*

$$\bar{\mathcal{Z}} := \Lambda^c/\Lambda^+ \cong \mathcal{Z}/\Gamma. \tag{31}$$

* Without using coordinates, Λ^+ contains all $\gamma: S^1 \to G^c$ which are the boundary values of a holomorphic map $\hat{\gamma}: D \to G^c$, where $D \subset \mathbb{C}$ denotes the open unit disk, while $\gamma \in \Lambda^-$ gives the boundary values of a holomorphic map $\hat{\gamma}: \hat{\mathbb{C}} \setminus \bar{D} \to G^c$ with $\gamma(\infty) = I$ (where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

Proof. We show first that $\Lambda \subset \Lambda^c$ acts transitively on the coset space Λ^c/Λ^+. This is the so called *Iwasawa decomposition* (cf. [8, 16, 21]): Any $\gamma^c \in \Lambda^c$ can be decomposed as $\gamma^c = \gamma\gamma^+$ with $\gamma \in \Lambda$ and $\gamma^+ \in \Lambda^+$. Thus

$$\Lambda^c/\Lambda^+ = \Lambda/(\Lambda^+ \cap \Lambda).$$

But $\Lambda \cap \Lambda^+$ contains only the constant loops since $\overline{\sum_j a_j \lambda^j} = \sum_j \overline{a_j} \lambda^{-j}$. Since the constant loops in $\Lambda = \Lambda_\sigma(G)$ are fixed by σ they belong to $\hat{K}$. □

Let L, L^c, L^+, L^- be the Lie algebras of the loop groups Λ, Λ^c, Λ^+, Λ^-, respectively. The tangent space $\mathfrak{z} = L^c/L^+ = (L^- \oplus L^+)/L^+$ of $\mathcal{Z}$ at the base point $e\Lambda^+$ can be identified with the Lie algebra L^- and has a filtration by finite-dimensional subspaces $\mathfrak{z}_1 \subset \mathfrak{z}_2 \subset \cdots \subset \mathfrak{z}$ with

$$\mathfrak{z}_r = \left\{ \sum_{j=1}^{r} a_{-j}\lambda^{-j} \in L^c \right\}. \tag{32}$$

Viewed as subspaces of $\mathfrak{z} = L^c/L^+$, the $\mathfrak{z}_r$ are invariant under $\mathrm{Ad}(\Lambda^+)$ and hence they determine distributions $\hat{\mathfrak{z}}_r$ on $\bar{\mathcal{Z}}$ and on $\mathcal{Z}$. The first of these spaces, $\mathfrak{z}_1$, generates all the others under the Lie product; it is called the *superhorizontal subspace*. A mapping into $\mathcal{Z}$ is called *superhorizontal* if its differential takes values in $\hat{\mathfrak{z}}_1$.

THEOREM 5. *Let M be simply connected. The pluriharmonic maps $f\colon M \to P = G/K$ are in 1:1 correspondence to the superhorizontal and holomorphic maps $\bar{\mathcal{F}}\colon M \to \mathcal{Z} = \Lambda/K$, and we have $f = ev_1 \circ \bar{\mathcal{F}}$ where $ev_1\colon \Lambda/K \to G/K$, $\gamma K \mapsto \gamma(1)K$ is the evaluation map at $\lambda = 1$.*

Proof. If a pluriharmonic map $f\colon M \to P$ is given and $\mathcal{F}\colon M_o \to \Lambda$ is an extended frame, then $\bar{\mathcal{F}} = \pi \circ \mathcal{F}\colon M \to \mathcal{Z}$ has the desired properties by the preceding lemmas and (24). It remains to show that $f = ev_1 \circ \bar{\mathcal{F}}$ is pluriharmonic provided that $\bar{\mathcal{F}}\colon M \to \mathcal{Z}$ is holomorphic and superhorizontal. Let $\mathcal{F}\colon M \to \Lambda$ locally be a lift of $\bar{\mathcal{F}}$ with respect to $\pi\colon \Lambda \to \Lambda/K = \mathcal{Z}$. Since $\bar{\mathcal{F}}$ is superhorizontal, $\mathrm{d}\pi(\mathcal{F}^{-1}\,\mathrm{d}\mathcal{F})$ takes values in $\mathfrak{z}_1$, in other words, $\tilde{\omega} = \mathcal{F}^{-1}\,\mathrm{d}\mathcal{F}$ takes values in $\mathfrak{z}_1 + L^+$ which shows that $\alpha = \sum_{j \geq -1} \alpha_j \lambda^j$. But since α is real, i.e. $\overline{\alpha} = \alpha$, all α_j with $j \geq 2$ vanish and we obtain

$$\alpha = \alpha_{-1}\lambda^{-1} + \alpha_0 + \alpha_1\lambda. \tag{33}$$

From the twisting condition $\sigma\alpha(-\lambda) = \alpha(\lambda)$ we see that $\alpha_j \in \mathfrak{k}^c$ for even j and $\alpha_j \in \mathfrak{p}^c$ for odd j which shows that $\alpha_0 \in \mathfrak{k}$ and $\alpha_{\pm 1} \in \mathfrak{p}^c$. On the other hand, from the holomorphicity of $\bar{\mathcal{F}}$ we obtain $\alpha'' \equiv 0 \bmod L^+$, hence

$$\alpha'' = \alpha_0'' + \alpha_1''\lambda + \cdots. \tag{34}$$

Comparing the last two equations we infer that $\alpha_{-1}'' = 0$ and hence by reality we have $\alpha_1' = 0$. This implies that α is of the form (24),

$$\alpha = \lambda^{-1}\alpha_p' + \alpha_k + \lambda\,\alpha_p'' \tag{35}$$

which finishes the proof. □

Remarks. (1) We may as well replace $\mathcal{Z} = \Lambda/K$ by $\bar{\mathcal{Z}} = \Lambda/\hat{K} = \Lambda^c/\Lambda^+$ since $\mathcal{Z}$ is a covering of $\bar{\mathcal{Z}}$ and M is simply connected: Maps from M into $\mathcal{Z}$ can be projected to $\bar{\mathcal{Z}}$, and those into $\bar{\mathcal{Z}}$ can be lifted to $\mathcal{Z}$, and holomorphicity and superhorizontality are preserved.

(2) Theorem 5 shows that the rôle of $\mathcal{Z}$ is very similar to that of the *twistor spaces* for the special case of isotropic (pluri)harmonic maps (cf. [5] and Section 8 below). Therefore, $\mathcal{Z}$ will be called *general twistor space*⋆ and $\bar{\mathcal{F}}$ the *general twistor lift* of f.

6. The Normalized Potential

The last theorem has shown that any pluriharmonic map $f\colon M \to P = G/K$ can be obtained from holomorphic data, the map $\bar{\mathcal{F}}\colon M \to \mathcal{Z}$. Though the target space $\mathcal{Z}$ is infinite dimensional, the differential $\mathrm{d}\bar{\mathcal{F}}$ takes values in the pull back of the finite-dimensional superhorizontal subbundle. In the present section we will replace this bundle-valued $(1, 0)$-form by a vector valued but meromorphic one, called η_{-1}, which still determines f. For harmonic maps on surfaces this construction was carried out in [8], reducing harmonic maps of a surface into a symmetric space to certain meromorphic differentials on the surface. A similar construction was known since long time for minimal surfaces in Euclidean space, the Weierstrass representation, and therefore the representation of f in terms of η_{-1} was also called a *generalized Weierstrass representation*.

In fact, let $\mathcal{F}\colon M_o \to \Lambda$ be an extended framing of f. In Section 5 we constructed a holomorphic framing $\mathcal{H}\colon M_o \to \Lambda^c$ with $\mathcal{H} = \mathcal{F}\mathcal{V}_+$ for some $\mathcal{V}_+\colon M_o \to \Lambda_+$. At all points $z \in M_o$ where $\mathcal{H}(z)$ is not too far from the unit element we may split $\mathcal{H}$ uniquely as

$$\mathcal{H} = \mathcal{H}_-\mathcal{H}_+, \tag{36}$$

where $\mathcal{H}_\pm$ takes values in $\Lambda^\pm$ (*Birkhoff decomposition*) In fact as in [8] it follows that $\mathcal{H}_\pm$ are meromorphic on all of M_o. Accordingly both factors $\mathcal{H}_\pm$ are meromorphic (i.e. all coefficients of the Fourier series (26) of $\mathcal{H}$ depend meromorphically on $z \in M_o$). Now we claim

$$\mathcal{H}_- = \mathcal{F}_-, \tag{37}$$

where $\mathcal{F} = \mathcal{F}_-\mathcal{F}_+$ is the Birkhoff decomposition of $\mathcal{F}$. In fact,

$$\mathcal{F}_-\mathcal{F}_+ = \mathcal{F} = \mathcal{H}\mathcal{V}_+^{-1} = \mathcal{H}_-\mathcal{H}_+\mathcal{V}_+^{-1},$$

⋆ Like the finite-dimensional twistor spaces, $\mathcal{Z}$ can be viewed formally as an adjoint orbit, using the embedding $\varphi\colon \mathcal{Z} \to L$, $\varphi(\gamma K) = \gamma'\gamma^{-1}$. For trivial K this embedding was discussed in [21, ch. 8.9]. In fact, since $\phi(g\gamma K) = g\phi(\gamma K)g^{-1} + g'g^{-1}$ for any $g \in \Lambda$, the image $\varphi(\mathcal{Z})$ is an adjoint orbit of the affine Kac–Moody group extending Λ.

and by the uniqueness of the Birkhoff decomposition we obtain $\mathcal{F}_- = \mathcal{H}_-$.

Hence we may use $\mathcal{F}_-$ in place of $\mathcal{H}$ as a holomorphic framing in the sense of Lemma 2 (with $\mathcal{V}_+ = \mathcal{F}_+^{-1}$); we will call it the *normalized framing*. Note that $\mathcal{F}_-$ is independent of the chosen extended framing $\mathcal{F}$ since any other extended framing is of the form $\mathcal{F}' = \mathcal{F}k$ for some mapping $k: M_o \to K$. Since $K \subset \Lambda^+$, the (Λ^-)-components of $\mathcal{F}$ and $\mathcal{F}'$ with respect to the Birkhoff decomposition agree. Hence $\mathcal{F}_-$ is a meromorphic mapping defined globally on M.

From the remark subsequent to Lemma 2 we see that the Fourier series of $\eta_- := \mathcal{F}_-^{-1}\,\mathrm{d}\mathcal{F}_-$ starts with $\eta_{-1}\lambda^{-1}$, but on the other hand it takes values in L^-, hence there are no other terms. Thus

$$\eta_- = \mathcal{F}_-^{-1}\,\mathrm{d}\mathcal{F}_- = \eta_{-1}\lambda^{-1} \tag{38}$$

for some meromorphic $\eta_{-1} \in \Omega^{(1,0)}(M_o, \mathfrak{g}^c)$. From the integrability condition $\mathrm{d}\eta_- = -\eta_- \wedge \eta_-$ which follows from (38) we obtain by comparing coefficients for the powers of λ:

$$\mathrm{d}\eta_{-1} = 0, \tag{39}$$

$$\eta_- \wedge \eta_- = 0. \tag{40}$$

THEOREM 6. *The pluriharmonic maps $f: M \to P = G/K$ are locally in* 1:1 *correspondence with meromorphic* $(1,0)$*-forms $\eta_{-1} \in \Omega^{(1,0)}(M, \mathfrak{p}^c)$ satisfying* (39) *and* (40).

Proof. It remains to construct a pluriharmonic map $f: M_o \to P$ from a given holomorphic $\eta_{-1} \in \Omega^{(1,0)}(M_o, \mathfrak{g}^c)$. We put $\eta_- = \eta_{-1}\lambda^{-1}$. This is a holomorphic $(1,0)$-form with values in Λ^- which is integrable by (39) und (40). Thus there exists a holomorphic map $F_-: M_o \to \Lambda^- \subset \Lambda^c$, This descends to a holomorphic superhorizontal map $\bar{\mathcal{F}} = \pi \circ \mathcal{F}_-: M_o \to \Lambda^c/\Lambda^+ = \mathcal{Z}$, and by Theorem 5, $f = ev_1 \circ \bar{\mathcal{F}}$ is pluriharmonic. More precisely, we can get a (real) extended framing $\mathcal{F}$ of f by the Iwasawa decomposition of $\mathcal{F}_-$:

$$\mathcal{F}_- = \mathcal{F}\mathcal{V}_+ \tag{41}$$

with $\mathcal{F}(z) \in \Lambda$ and $V_+(z) \in \Lambda^+$ for all $z \in M_o$ (in fact $\mathcal{V}_+ = \mathcal{F}_+^{-1}$). □

These computations can be carried out explicitly in the case of *finite uniton number*. A pluriharmonic map $f: M \to G/K$ is called *of finite uniton number* if the corresponding extended frames $\mathcal{F}$ take values in the *algebraic loop group* $\Lambda^{\mathrm{alg}} \subset \Lambda$ which consists of those loops $\gamma \in \Lambda$ such that γ and γ^{-1} are *finite* Laurent series.⋆ The finite uniton number property holds if and only if η_- takes values in some nilpotent subalgebra of $\mathfrak{g}^c$, cf. [4, 14]. We will discuss a subcase in Section 8.

⋆ This holds iff γ extends to an *algebraic* map $\hat{\gamma}: \mathbb{C}^* \to G^c$.

7. The Dressing Action

For a simply connected complex manifold M the pluriharmonic maps $f: M \to G/K$ correspond bijectively to the superhorizontal holomorphic maps $\bar{\mathcal{F}}: M \to \bar{\mathcal{Z}} = \Lambda^c/\Lambda^+$ (cf. Section 5). Moreover we have seen that the action of Λ^c on $\hat{\mathcal{Z}} = \Lambda^c/\Lambda^+$ by left translations is holomorphic and preserves the superhorizontal distribution $\hat{\mathfrak{z}}_1$. Thus for every $\gamma \in \Lambda^c$ and any superhorizontal holomorphic map $\bar{\mathcal{F}}: M \to \bar{\mathcal{Z}}$ we get another such map $\gamma\bar{\mathcal{F}}: M \to \bar{\mathcal{Z}}$ with $(\gamma\bar{\mathcal{F}})(z) = \gamma\bar{\mathcal{F}}(z)$ for all $z \in M$. By Theorem 5 this induces an action of Λ^c on the set of pluriharmonic maps $f: M \to G/K$ called the *dressing action*:

$$\gamma * f := ev_1(\gamma\bar{\mathcal{F}}).$$

Note that the dressing action of the *real* loop group Λ does not give anything new: For $\gamma \in \Lambda$ we have $\gamma * f = \gamma_1 F \bmod K = \gamma_1 f$ which is congruent to f since $\gamma_1 \in G$. Thus due to the Iwasawa splitting $\Lambda^c = \Lambda\Lambda^+$ the dressing action essentially can be restricted to Λ^+.

THEOREM 7. *For any pluriharmonic map $f: M \to G/K$ and any $\gamma \in \Lambda^c$ there is another pluriharmonic map $f' = \gamma * f: M \to G/K$ such that the corresponding general twistor lifts satisfy $\bar{\mathcal{F}}' = \gamma\bar{\mathcal{F}}$. The normalized holomorphic framing $\mathcal{F}_-'$ is the (Λ_-)-part of $\gamma\mathcal{F}_-$ with respect to the Birkhoff decomposition, in other words there is a map $\mathcal{V}_+ = V_0 + V_1\lambda + V_2\lambda^2 + \cdots: M \to \Lambda^+$ with*

$$\mathcal{F}_-'\mathcal{V}_+ = \gamma\mathcal{F}_-. \tag{42}$$

For the normalized potentials η_- and η_-' we obtain

$$\eta_-' = \mathrm{Ad}(V_0)\eta_- \tag{43}$$

where $V_0 = \mathcal{V}|_{\lambda=0}$ takes values in $\hat{K}^c = \mathrm{Fix}^c(\sigma)$.

Proof. Equation (42) follows since $\mathcal{F}_-' \equiv \mathcal{F}' \equiv \gamma\mathcal{F} \equiv \gamma\mathcal{F}_- \bmod \Lambda_+$. Further, from $F_-' = \gamma\mathcal{F}_-\mathcal{V}_+^{-1}$ we obtain $\eta_-' = \mathcal{V}_+\eta_-\mathcal{V}_+^{-1} - \mathrm{d}\mathcal{V}_+\mathcal{V}_+^{-1}$. We have $\eta_- = \eta_{-1}\lambda^{-1}$ and $\mathcal{V}^{-1} = V_o^{-1} + W_1\lambda + W_2\lambda^2 + \cdots$ and hence $\eta' \equiv V_0\eta V_o^{-1} \bmod L^+$. But both sides have no component in L^+ whence we obtain (43). Moreover, V_0 takes values in $\hat{K}^c$ since $\sigma V_0 = V_0$ by the twisting condition (14). □

8. The Isotropic Case

A pluriharmonic map $f: M \to P = G/K$ is called *isotropic* if its associated family is trivial, $f_\lambda = f$ up to congruence for all $\lambda \in S^1$, i.e. $f_\lambda = g_\lambda f$ for some $g_\lambda \in G$ independent of z. This is the realm of the 'classical' twistor theory (cf. [5, 11]) saying that such maps arise as projections from holomorphic superhorizontal maps into certain flag manifolds fibering over G/K. One of the most spectacular applications was Bryant's explicit formula for all minimal 2-spheres in the 4-sphere using the twistor fibration $\mathbb{C}P^3 \to S^4$, see [1]. We will show that twistor theory is generalized by our theory in Section 5.

Let us briefly recall the classical twistor theory as represented in [11]. We will always assume that f is *full*, i.e. it does not take values in a proper totally geodesic subspace of P. From (10) we obtain a family of parallel maps Φ_λ with

$$df = \Phi_\lambda \circ df \circ R_\lambda \tag{44}$$

but this time each $\Phi_\lambda(z)$ is an automorphism of $T_{f(z)}P$, in other words, $\Phi_\lambda(z)$ lies in the isotropy subgroup of the point $f(z) \in P$ which is $F(z)K\ F(z)^{-1}$ (where $F\colon M_o \to G$ is a frame along f). From (44) and $R_\lambda R_\mu = R_{\lambda\mu}$ and $R_{-1} = -I$ we see

$$\Phi_\lambda \Phi_\mu = \Phi_{\lambda\mu}, \quad \Phi_{-1} = -I, \tag{45}$$

i.e. the map $\Phi(z)\colon \lambda \mapsto \Phi_\lambda(z)\colon S^1 \to G$ is an homomorphism with $\Phi_{-1}(z) = s_{f(z)}$. Moreover, by the parallelity of Φ_λ the value set $\{\Phi_\lambda(z);\ z \in M\} \subset G$ is contained in a conjugacy class for every $\lambda \in S^1$; recall that parallel translations on P are restrictions of isometries, i.e. elements of G. Thus there is a circle subgroup $q = (q_\lambda)\colon S^1 \to G$ with

$$q_{-1} = s_o, \tag{46}$$

such that $\Phi\colon z \mapsto \Phi(z)$ takes values in the conjugacy class of q, more precisely

$$\Phi_\lambda = F q_\lambda F^{-1}. \tag{47}$$

The conjugacy class of the circle q is the *twistor space* $Z_q = G/C_q$ where

$$C_q = \{g \in G;\ g q_\lambda g^{-1} = q_\lambda\ \forall_\lambda\}. \tag{48}$$

This is a complex manifold: If we let $G_+, G_- \subset G^c$ be the Lie subgroups corresponding to the Lie subalgebras

$$\mathfrak{g}_+ = \sum_{j \geq 0} \mathfrak{g}_j, \quad \mathfrak{g}_- = \sum_{j<0} \mathfrak{g}_j, \tag{49}$$

where $\mathfrak{g}^c = \sum_j \mathfrak{g}_j$ denotes the eigenspace decomposition of $\mathrm{Ad}(q_\lambda)$:

$$\mathfrak{g}_j = \ker(\mathrm{Ad}(q_\lambda) - \lambda^{-j} I) \subset \mathfrak{g}^c. \tag{50}$$

Then we have

$$Z_q = G/C_q = G^c/G_+, \tag{51}$$

and G^c/G_+ is a coset space of a complex Lie subgroup and hence a complex manifold.

From (46) we see that C_q is contained in $\hat{K} = \mathrm{Fix}(\sigma)$ and hence in K since the centralizer of any torus is connected (cf. [15]). Therefore the map $\pi\colon gqg^{-1} \cong$

$gC_q \mapsto gK\colon Z_q \to G/K$ is a well defined fibration. Hence we have $f = \pi \circ \Phi$ where $\Phi\colon M \to Z_q$ is called the *twistor lift*. It follows from (44) and the parallelity of Φ_λ (cf. [11]) that this map is holomorphic and *superhorizontal*, i.e. its differential takes values in the *superhorizontal subbundle* of TZ_q obtained from $\mathfrak{g}_1^r := (\mathfrak{g}_1 + \mathfrak{g}_{-1}) \cap \mathfrak{g}$ by left translations. From (44) and the fullness of f it follows that $\mathfrak{g}_1^r$ generates $\mathfrak{g}$ as a Lie algebra (cf. [11]) and hence the circle q is *canonical* in the sense of [5]. Thus all isotropic pluriharmonic maps $f\colon M \to G/K$ arise as projections of superhorizontal holomorphic maps into a twistor space Z_q for some canonical circle $q \subset K$.

We want to derive these statements within our general twistor theory. There we have chosen our extended frame $\mathcal{F} = (F_\lambda)$ as follows (cf. (12):

$$F_\lambda = g_\lambda \Phi_\lambda F = g_\lambda F\, k_\lambda \tag{52}$$

with $k_\lambda(z) = F^{-1}\Phi_\lambda F \in K$. Now this is a homomorphism with respect to λ which is conjugate to q_λ for all z,

$$k_\lambda(z) = k(z) q_\lambda k(z)^{-1}. \tag{53}$$

Hence $F_\lambda = g_\lambda F\, k\, q_\lambda k^{-1}$ or equivalently $F_\lambda k = g_\lambda F k\, q_\lambda$. Using the gauge freedom, we may replace the framing F by Fk which we now call F. In the new notation we have $F_\lambda = g_\lambda F\, q_\lambda$. Since we require also the normalization $F(z_o) = e$, we obtain $g_\lambda = q_\lambda^{-1}$ and hence

$$F_\lambda = q_\lambda^{-1} F\, q_\lambda. \tag{54}$$

This equation has been observed in the surface case, cf. [7], [14]. Since q is a circle subgroup, there are only finitely many powers of λ in the matrix q_λ and hence the isotropic case belongs to the case of *finite uniton number* (cf. Section 6). Moreover we see from (54) that $\mathcal{F} = (F_\lambda)$ takes values in a finite-dimensional subgroup $\Lambda^q \cong G$ of Λ:

$$\Lambda^q = \{q^{-1} g q;\ g \in G\} \subset \Lambda, \tag{55}$$

where $q^{-1}gq(\lambda) := q_\lambda^{-1} g q_\lambda$. Such a curve $q^{-1}gq$ is constant if g lies in the centralizer C_q of the circle group q. Thus the projection $\pi\colon \Lambda \to \mathcal{Z} = \Lambda/K$ maps Λ^q onto Λ^q/C_q which is isomorphic to the twistor space $Z_q = G/C_q$. More precisely, we have the embedding

$$j\colon G/C_q \to \mathcal{Z} = \Lambda/K, \quad gC_q \mapsto q_\lambda^{-1} g\, q_\lambda K, \tag{56}$$

which is (1) holomorphic, (2) equivariant under G via the embedding $G \to \Lambda^q \subset \Lambda$, $g \mapsto q_\lambda^{-1} g q_\lambda$, (3) sends the superhorizontal distribution of Z_q into that of $\mathcal{Z}$ and (4) satisfies

$$ev_1 \circ j = \pi\colon G/C_q \to G/K\colon\ gC_q \mapsto gK. \tag{57}$$

THEOREM 8. *Let $f\colon M \to G/K$ be a pluriharmonic map. Then the following properties are equivalent:*

(1) *f is isotropic,*
(2) *$\bar{\mathcal{F}}$ takes values in a finite-dimensional twistor space $j(Z_q) \subset \mathcal{Z}$,*
(3) *The normalized potential η_{-1} takes values in $\mathfrak{g}_{-1} = \ker(\mathrm{Ad}(q_\lambda) - \lambda I)$ for some canonical circle subgroup $q\colon S^1 \to G$.*

Proof. We have already seen (1) ⇒ (2). Next assume (2). Let $\mathcal{F} = (F_\lambda)\colon M_o \to \Lambda$ be a local lift of $\bar{\mathcal{F}}$. Then $F_\lambda = q_\lambda^{-1} F q_\lambda k$ for some $k\colon M_o \to K$ where $F = F_1$. Putting $\lambda = 1$ we obtain $k = e$ and $F_\lambda = q_\lambda^{-1} F q_\lambda$. Now the Birkhoff decomposition restricts to the finite-dimensional Bruhat splitting $G^c = G_- G_+$ corresponding to q (cf. (49)). In fact, if $F = F_- F_+$ is the Bruhat decomposition of F (for $F(z)$ close enough to e), both factors are invariant under conjugation with q_λ^{-1} and $\mathcal{F}_\pm = \mathrm{Ad}(q_\lambda^{-1}) F_\pm$. Thus $\eta_- = \mathrm{Ad}(q^{-1})\alpha_-$ where $\alpha_- = F_-^{-1}\,\mathrm{d}F_-$. But on the other hand $\eta_- = \lambda^{-1}\eta_{-1}$ and hence α_- takes values in $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ (where $\mathrm{Ad}(q_{-1}) = \lambda^{-1}$) and $\eta_{-1} = \alpha_-$.

Now let us assume (3). Then $\eta_- = \lambda^{-1}\eta_{-1} = \mathrm{Ad}(q_\lambda^{-1})\eta_{-1}$ and $\mathcal{F}_- = \mathrm{Ad}(q_\lambda^{-1}) F_-$ where F_- takes values in G_- with $\mathrm{d}F_- = F_-\eta_{-1}$. From the finite-dimensional Iwasawa decomposition $G^c = G G_+$ we obtain $F_- = F F_+$ with $F \in G$ and $F_+ \in G_+$. Applying $\mathrm{Ad}(q_\lambda^{-1})$ and comparing with the infinite-dimensional Iwasawa decomposition $\mathcal{F}_- = \mathcal{F}\mathcal{F}_+$ we see that $\mathcal{F}_\pm = q_\lambda^{-1} F_\pm q_\lambda$ and $\mathcal{F} = q_\lambda^{-1} F q_\lambda$. Since $q_\lambda \in K$ for all λ, the last equality $F_\lambda = q_\lambda^{-1} F q_\lambda$ projects onto $f_\lambda = q_\lambda^{-1} f$ and thus f_λ equals f up to the rigid motion q_λ. Hence f is isotropic. □

EXAMPLE 1 ([2, 9]). Let $P = G_p(\mathbb{C}^n) = U(n)/(U(p) \times U(q))$ with $p + q = n$ be the Grassmannian of complex p-dimensional subspaces of $\mathbb{C}^n$. Then the possible twistor spaces over P (cf. ([5], [3]) are certain (classical) flag manifolds. Recall that a *flag* over $\mathbb{C}^n$ can be viewed either as a chain of subspaces $W_1 \subset W_2 \subset \cdots \subset W_r = \mathbb{C}^n$ or as an orthogonal decomposition $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r$ where $W_1 = E_1$ and $W_{j+1} = W_j \oplus E_{j+1}$. Let Z be the space of all flags of a certain type (determined by the dimensions $d_j = \dim E_j$). Then there is a projection $\pi\colon Z \to P = G_p(\mathbb{C}^n)$ with $p = \sum_{j\ odd} d_j$ sending the flag $(E_1, \ldots, E_r)$ onto the subspace $\sum_{j\ odd} E_j$. This is a twistor fibration with $G = U(n)$, $G^c = GL(n, \mathbb{C})$, and the canonical circle q_λ is given by the matrix

$$q_\lambda = \mathrm{diag}(\lambda I_{d_1}, \lambda^2 I_{d_2}, \ldots, \lambda^r I_{d_r}). \tag{58}$$

The eigenspaces $\mathfrak{g}_k$ for q_λ consist of block matrices (blocks of size $d_1, \ldots, d_r$) where only the k-the block diagonal is nonzero.

A map $f\colon M \to Z$ is a chain of vector bundles (a ‘moving flag’) $z \mapsto (W_j(z))_{j=1}^r$ over M; it is holomorphic iff the bundles W_j are spanned locally by holomorphic sections, i.e. there is a holomorphic map $H = (h_1, \ldots, h_n)\colon M_o \to GL(n, \mathbb{C}) = U(n)^c$ such that $W_j = \mathrm{Span}(h_1, \ldots, h_{n_j})$ where $n_j = \dim W_j$.

Moreover, f is superhorizontal if W_j differentiates into W_{j+1}, i.e. $\mathrm{d}w_j$ takes values in W_{j+1} for any section w_j of W_j. Thus the moving flag $W_1 \subset W_2 \subset \cdots$ is obtained as follows. We start with $n_1 = d_1$ arbitrary holomorphic functions $h_1, \dots, h_{d_1} \colon M_o \to \mathbb{C}^n$ in 'general position', i.e. they are pointwise linear independent and all their partial derivatives are linearly independent up to the order when the full dimension n is exhausted. Then we let $W_1(z)$ be the linear span of the values of the functions h_i (for $i = 1, \dots, d_1$) at $z \in M_o$ and W_2 the span of the h_i and their first partial derivatives while W_3 is spanned by h_i and their first and second partial derivatives etc. Thus we obtain all holomorphic superhorizontal maps into Z and hence (by composing with the projection $\pi \colon Z \to P$) all isotropic pluriharmonic maps $f \colon M \to P = G_p(\mathbb{C}^n)$.

How is this elementary description related to our general theory? Let us consider the moving (nonorthogonal) decomposition $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$ where the first space V_1 is spanned at any point z by the values of $h_1, \dots, h_{d_1}$, the second one V_2 only by their first derivatives, V_3 by their second derivatives etc. Denoting $\mathbb{C}^n = \mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_r$ the canonical decomposition where $\mathbb{E}_i$ are orthogonal subspaces of dimension d_i spanned by consecutive parts of the canonical basis $(e_1, \dots, e_n)$, we have $H(\mathbb{E}_i) \subset V_i$. Moreover, since V_i is spanned by $(i-1)$-th derivatives of the h_i, $i = 1, \dots, d_1$, the 1-form $\mathrm{d}H|_{\mathbb{E}_i}$ takes values in $V_{i+1} = H(\mathbb{E}_{i+1})$. Thus $\eta = H^{-1}\,\mathrm{d}H$ maps each $\mathbb{E}_i$ into $\mathbb{E}_{i+1}$ for $i = 1, \dots, r-1$. Only the last space $\mathbb{E}_r$ can be mapped anywhere. Hence η takes values in $\mathfrak{g}_{-1} + \mathfrak{g}_+$ which consists of those block matrices where only the upper triangular part and the first lower diagonal is nonzero. In order to obtain the normalized potential, according to Chapter 6 we have to decompose $H = H_- H_+$ where H_- is strictly lower triangular and H_+ upper triangular (including the diagonal). Then $\eta_- = H_-^{-1}\,\mathrm{d}H_-$ is the normalized potential taking values in $\mathfrak{g}_{-1}$, i.e. it has only block entries on the first lower block diagonal.

9. Pluriharmonic Maps into Lie Groups

In this section we specialize to the case where P is a compact Lie group* G. In some sense, this is the general case: Any symmetric space $P = G/K$ can, up to coverings, be viewed as a connected component of the totally geodesic submanifold

$$\hat{P} = \{g \in G;\ \sigma(g) = g^{-1}\}, \tag{59}$$

using the *Cartan embedding*

$$P = G/K \to \hat{P} \subset G, \quad gK \mapsto g\sigma(g^{-1}). \tag{60}$$

Therefore, many authors (based on the fundamental article of Uhlenbeck [22]) restrict their attention to this case. We will see how our discussion specializes.

* What we need is a biinvariant semi-Riemannian metric on G; positive definiteness is not essential. All semisimple Lie groups and many others have this property.

Let G be a Lie group with a biinvariant Riemannian metric. Then $G \times G$ acts isometrically on G by left and right translations L_g, $R_{g^{-1}}$, and also the inversion $\iota\colon g \mapsto g^{-1}$ is an isometry of G. Thus G is a symmetric space with symmetry $s_e = \iota$ at the unit element e, and the symmetry s_g at an arbitrary $g \in G$ is obtained by conjugating s_e with L_g or R_g which yields $s_g(x) = gx^{-1}g$. Representing G as a symmetric coset space we obtain

$$G = \tilde{G}/\tilde{K} := (G \times G)/\Delta, \quad \Delta := \{(g,g);\ g \in G\} \tag{61}$$

with the projection

$$\pi\colon G \times G \to G, \quad \pi(g,h) = gh^{-1} \tag{62}$$

and with the Cartan involution $\sigma \in \mathrm{Aut}(G \times G)$,

$$\sigma(g,h) = (h,g). \tag{63}$$

Now let $f\colon M \to G$ be pluriharmonic. Then there is globally a lift $\tilde{F}\colon M \to G \times G$ (a framing) given by

$$\tilde{F}(z) = (f(z), e). \tag{64}$$

For the corresponding Maurer–Cartan form we obtain

$$\tilde{\alpha} = \tilde{F}^{-1}\,\mathrm{d}\tilde{F} = (f^{-1}\mathrm{d}f, 0) =: (\alpha, 0). \tag{65}$$

We need to decompose $\tilde{\alpha} = \tilde{\alpha}_k + \tilde{\alpha}_p$ according to the Cartan decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}}$ of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ where $\tilde{\mathfrak{k}} = \{(A,A);\ A \in \mathfrak{g}\}$ is the Lie algebra of $\tilde{K} = \Delta = \mathrm{Fix}(\sigma)$ and $\tilde{\mathfrak{p}} = \{(A,-A);\ A \in \mathfrak{g}\}$. Thus,

$$\tilde{\alpha}_p = \tfrac{1}{2}(\alpha, -\alpha), \tag{66}$$

$$\tilde{\alpha}_k = \tfrac{1}{2}(\alpha, \alpha). \tag{67}$$

Introducing the λ-parameter and splitting further into the $(1,0)$ and $(0,1)$ components we obtain

$$\begin{aligned}\tilde{\alpha}_\lambda &= \lambda^{-1}\tilde{\alpha}'_p + \tilde{\alpha}_k + \lambda\tilde{\alpha}''_p \\ &= \tfrac{1}{2}\lambda^{-1}(\alpha', -\alpha') + \tfrac{1}{2}(\alpha,\alpha) + \tfrac{1}{2}\lambda(\alpha'', -\alpha'') \\ &= \left(\tfrac{1}{2}(1+\lambda^{-1})\alpha' + \tfrac{1}{2}(1+\lambda)\alpha'',\ \tfrac{1}{2}(1-\lambda^{-1})\alpha' + \tfrac{1}{2}(1-\lambda)\alpha''\right).\end{aligned} \tag{68}$$

THEOREM 9. *A smooth map $f\colon M \to G$ with Maurer–Cartan form $\alpha = f^{-1}\mathrm{d}f$ is pluriharmonic if and only if the 1-form*

$$\alpha_\lambda := \tfrac{1}{2}(1+\lambda^{-1})\alpha' + \tfrac{1}{2}(1+\lambda)\alpha'' \tag{69}$$

is integrable for all $\lambda \in S^1$.

Proof. It remains to show '$\Leftarrow$'. Denoting by $\mathcal{F} = (F_\lambda)$ a solution to

$$F_\lambda^{-1}\, \mathrm{d}F_\lambda = \alpha_\lambda, \tag{70}$$

we obtain a solution $\hat{\mathcal{F}} = (\hat{F}_\lambda)$ to $\hat{F}_\lambda^{-1}\, \mathrm{d}\hat{F}_\lambda = \hat{\alpha}_\lambda$ as

$$\hat{F}_\lambda = (F_\lambda, F_{-\lambda}). \tag{71}$$

Clearly, the normalization $\hat{F}_\lambda(z_o) = (e, e)$ at the base point z_o is equivalent with $F_\lambda(z_o) = e$. Thus, according to our general construction, an associated family (f_λ) of f is given by

$$f_\lambda = \pi \circ \hat{F}_\lambda = F_\lambda F_{-\lambda}^{-1}. \tag{72}$$

Hence f is pluriharmonic by Theorem 2. □

Remarks. (1) We would like to point out that the 'extended frame' in the sense of Uhlenbeck (cf. [13, 22]) is just the first component F_λ of the extended frame $\hat{\mathcal{F}} = (\hat{F}_\lambda)$ of the pluriharmonic map as defined in this paper (cf. (64)). It is therefore not surprising that the extended frame in the sense of this paper defines a pluriharmonic map f_λ for every $\lambda \in S^1$ while the extended frame in Uhlenbeck's sense defines a pluriharmonic map only for $\lambda = \pm 1$.

(2) We also note that the normalized potential $\hat{\eta}_- = \hat{\mathcal{F}}_-^{-1}\, \mathrm{d}\hat{\mathcal{F}}_-$ is of the form

$$\hat{\eta}_- = \left(\tfrac{1}{2}(1 + \lambda^{-1})\, \eta,\ \tfrac{1}{2}(1 - \lambda^{-1})\, \eta \right) \tag{73}$$

for some holomorphic $(1, 0)$-form $\eta \in \Omega^{(1,0)}(M, \mathfrak{g})$.

Acknowledgement

We would like to thank Martin Guest for comments and suggestions.

References

1. Bryant, R.: Conformal and minimal immersions of compact surfaces into the 4-sphere, *J. Differential Geom.* **17** (1982), 455–473.
2. Burstall, F. E.: Private communication.
3. Burstall, F. E., Eschenburg, J.-H., Ferreira, M. J. and Tribuzy, R.: Kähler submanifolds with parallel pluri-mean curvature, Preprint, math.DG/0111217, 2001.
4. Burstall, F. E. and Guest, M. A.: Harmonic two-spheres in compact symmetric spaces, *Math. Ann.* **309** (1997) 541–557.
5. Burstall, F. E. and Rawnsley, J. H.: *Twistor Theory for Riemannian Symmetric Spaces*, Lecture Notes in Math. 1424, Springer-Verlag, Berlin, 1990.
6. E. Calabi, E.: Minimal immersions of surfaces in Euclidean spheres, *J. Differential Geom.* **1** (1967), 111–125.
7. Chi, Q.-S., Fernandez, L. and Wu, H.: Normalized potentials of minimal surfaces in spheres, *Nagoya Math. J.* **156** (1999), 187–214.

8. Dorfmeister, J., Pedit, F. and Wu, H.: Weierstrass type representations of harmonic maps into symmetric spaces, *Comm. Anal. Geom.* **6** (1998), 633–668.
9. Erdem, S. and Wood, J. C.: On the construction of harmonic maps into a Grassmannian, *J. London Math. Soc. (2)* **28** (1983), 161–174.
10. Eschenburg, J.-H. and Tribuzy, R.: Existence and uniqueness of maps into affine homogeneous spaces, *Rend. Sem. Mat. Univ. Padova* **89**, 11–18.
11. Eschenburg, J.-H. and Tribuzy, R.: Associated families of pluriharmonic maps and isotropy, *Manuscripta Math.* **95** (1998), 295–310.
12. Ferus, D. and Pedit, F.: Curved flats in symmetric spaces, *Manuscripta Math.* **91** (1996), 445–454.
13. Guest, M.: *Harmonic Maps, Loop Groups, and Integrable Systems*, Cambridge Univ. Press, Cambridge, 1997.
14. Guest, M.: An update on harmonic maps of finite uniton number, via the zero curvature equation, in: M. Guest, R. Miyaoka and Y. Ohnita (eds), *Integrable Systems, Topology, and Physics*, Contemp. Math. 309, Amer. Math. Soc., Providence, RI, 2002.
15. Helgason, S.: *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
16. McIntosh, I.: Global solutions of the elliptic 2D periodic Toda lattice, *Nonlinearity* **7** (1994), 85–108.
17. Ohnita, Y.: Harmonic maps into symmetric spaces and integrable system theory, in: *Proceedings of the Conference 'Theory of Lie Groups and Manifolds'*, Sophia Kokyuroku in Mathematics 45, 2002, pp. 81–98.
18. Ohnita, Y. and Udagawa, S.: Harmonic maps of finite type into generalized flag manifolds and twistor fibrations, in: *Differential Geometry and Integrable Systems, Proceedings of the 9th MSJ-IRI Tokyo 2000*, Contemp. Math. 308, Amer. Math. Soc., Providence, NJ, 2002, pp. 245–270.
19. Ohnita, Y. and Valli, G.: Pluriharmonic maps into compact Lie groups and factorization into unitons, *Proc. London Math. Soc.* **61** (1990), 546–570.
20. Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through quadratic constraints, *Comm. Math. Phys.* **46** (1976), 207–221.
21. Pressley, A. and Segal, G.: *Loop Groups*, Oxford Science Publications, Clarendon Press, Oxford, 1986.
22. Uhlenbeck, K.: Harmonic maps into Lie groups (classical solutions of the chiral model), *J. Differential Geom.* **30** (1989), 1–50.