Pluriharmonic maps, twisted loops and twistors

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Abstract

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A minimal surface in euclidean space has two very special properties: (A) It allows a twisted circle of isometric deformations preserving the tangent plane (Associated family), and (B) it is just the real part of a holomorphic map (Weierstrass representation). In fact, these two properties hold more generally for a pluriharmonic map f of a simply connected complex manifold into euclidean space. If instead the target space is a Riemannian symmetric space P, Property (A) essentially remains true, however by lack of global parallel displacements a parallel isomorphism between the tangent spaces of the associated family is needed. Consequently Property (B) gets more complicated: f arises by projecting a "superhorizontal" holomorphic map $\hat f$ into a certain infinite dimensional flag manifold (adjoint orbit) fibering over P. the "universal twistor space". The map $\hat f$ takes values in a finite dimensional sub-twistor space iff the associated family is trivial.

Introduction

Among the most beautiful objects in Geometry are the minimal surfaces in 3-space. One of their spectacular properties is the existence of a (so called *associated*) family of deformations preserving the interior distances and the surface normal while rotating the principal directions. The best known example is the deformation of the catenoid into the helicoid, cf. `http://www.ag.jku.at/verbieg_en.html`. It starts and ends with the catenoid which however is turned inside-out during the deformation. This is an example of a *twisted loop* of surfaces: It comes back to its original shape, but only after applying a *point reflection* on the ambient space. The same phenomenon occurs when euclidean 3-space is replaced with a *symmetric space* P, a Riemannian manifold with isometric point reflections at every point. Moreover, the minimal surface can be replaced by a harmonic map of a surface into P. In fact, a surface is only the easiest example of a complex manifold, and we may equally well consider a *pluriharmonic* map of an arbitrary complex manifold. Also these maps can be deformed by twisted loops, described as mappings into some space of loops. These are holomorphic mappings of a certain kind, and therefore pluriharmonic maps can be obtained from holomorphic data; the Weierstrass representation for minimal surfaces is the best known example.

Sometimes the situation is rigid and the twisted loop of deformations arises only by isometries of the ambient space; such pluriharmonic maps are called *isotropic*. Of course they can exist only if the point reflections in P can be deformed to the identity; symmetric spaces with this property are called *inner*. This case is much simpler and has been studied for a long time, starting with the work of Calabi [4]. It is closely connected with the concept of a *twistor* (cf. [3]) which was first investigated by R. Penrose [13] in connection to Relativity. Roughly speaking, a twistor is a complex structure on a tangent space of an inner symmetric space P, and the set of all twistors forms a fibre bundle over P. The original example studied by Penrose was the "classical" twistor fibration $\mathbb{C}P^3 \to S^4$; any point of complex projective 3-space can be viewed as a complex structure on some tangent space of S^4. Each isotropic pluriharmonic map into a symmetric space can be lifted to some twistor space over P. The most classical twistor fibration was also the most successfull: With its help R. Bryant [1] gave an explicit description of all minimal spheres in S^4.

In the present survey article we want to explain, following work of K. Uhlenbeck, J. Rawnsley and others, how the general (non-isotropic) case can also be understood in terms of a generalized twistor theory. This more general twistor space is infinite dimensional, a space of twisted loops, and since it contains all other twistor spaces, we would like to call it "universal". This is mainly a re-interpretation of well known facts which however could give a unified view point to the theory. Most of the details missing in this survey can be found in [7]. We thank J. Dorfmeister for many useful hints and discussion.

0 Harmonic maps

A smooth manifold M is called *Riemannian* if there is an inner product on each of its tangent spaces, depending smoothly on the base point. For any smooth curve $c : [a, b] \to M$ the *length* $\mathcal{L}(c) = \int_a^b |c'(t)|\, dt$ is defined, giving M the structure of a metric space which locally is approximated by euclidean space. All euclidean notions are applicable but have different properties. The main difference arises for the *parallel displacement* of tangent vectors and the corresponding differentiation of tangent vector fields, the so called *Levi-Civita derivative*: The parallel displacement becomes path dependent, and the Levi-Civita derivatives with respect to two coordinate directions do not commute; in fact their commutator is the basic invariant of Riemannian geometry, the *curvature tensor*.

Classical euclidean geometry is investigated by using substructures: lines, planes etc. In Riemannian geometry, the rôle of lines is taken by *geodesic lines* which have parallel tangent vectors and which locally minimize the length and also the *energy* $\mathcal{E}(c) = \int_a^b |c'(t)|^2\, dt$ among all curves connecting two given points. If metric completeness is assumed, any two points are joined by a geodesic line, like in euclidean geometry. But what are the substitutes for planes and higher dimensional subspaces? A plane in space contains the line passing through any two of its points. In Riemannian geometry, a submanifold with this property is called *totally geodesic*. However such submanifolds are very rare unless we restrict attention to spaces of constant curvature which are locally just spherical, euclidean or hyperbolic spaces.

Therefore we consider another generalization of geodesics to higher dimensions, using the energy minimizing property. If M and P are Riemannian manifolds and $f : M \to P$ a smooth map, the derivative of f at a point $x \in M$ is a linear map $df_x : T_xM \to T_{f(x)}P$ between the corresponding tangent spaces of M and P. The vector space $\mathrm{Hom}(T_xM, T_{f(x)}P)$ inherits an inner product and hence a norm from the inner products on T_xM and $T_{f(x)}P$ given by the Riemannian metrics; in fact

$$|df_x|^2 = |df_x.e_1|^2 + ... + |df_x.e_m|^2 \tag{1}$$

for an orthonormal basis $(e_1, ..., e_m)$ of T_xM. For any compact subset $M_o \subset M$ the *energy* or *Dirichlet integral* of $f|M_o$ is

$$\mathcal{E}(f|M_o) = \int_{M_o} |df_x|^2 \, dv_x \tag{2}$$

where dv_x denotes the volume element of M at x determined by the Riemannian metric. A map $f : M \to P$ is called *harmonic* if the variation of its energy vanishes,

$$\delta\mathcal{E}(f|M_o) := \frac{d}{ds}\mathcal{E}(f_s|M_o)|_{s=0} = 0 \tag{3}$$

for any compact subset $M_o \subset M$, where $f_s : M \to P$ is any smooth variation of f with $f = f_s$ outside M_o. As always, this variational principle is equivalent to its Euler differential equation:

$$\Delta f := \text{trace } Ddf = 0 \tag{4}$$

involving the Levi-Civita derivative D on $\text{Hom}(TM, TP)$ which is induced by the Riemannian metrics on M and P.

Harmonic maps exist for all dimensions of M. A harmonic map of the real line $M = \mathbb{R}$ is just a geodesic. The next case when M is a surface is most interesting since then the energy (2) is invariant under conformal changes of the metric g on M. In fact, if g is replaced with $\mu^2 g$ for some smooth positive scaling function μ on M, then $|df_x|^2$ takes up a factor $\mu(x)^2$ while the 2-dimensional volume element dv_x is divided by $\mu(x)^2$, hence the energy remains unchanged. But an oriented surface with a conformal class of metrics is nothing else than a 1-dimensional complex manifold where the complex structure J on the tangent space is the rotation by the angle $\pi/2$. Hence harmonic maps $f : M \to P$ are already defined when P is Riemannian but M is only a complex 1-dimensional manifold (a *Riemann surface*) without specified metric. We will see that locally all harmonic maps of Riemann surfaces into symmetric spaces can be obtained in terms of meromorphic functions on M as has been shown in [8]. These maps became interesting to physicists under the name σ-models (cf. [6]).

If $\dim M > 2$, harmonic maps in general do not have such nice properties. However there is an interesting special case where the methods of complex analysis still apply. Let M be a *complex* manifold of any dimension. A map $f : M \to P$ is called *pluriharmonic* if $f|C$ is harmonic for any complex 1-dimensional submanifold (*complex curve*) $C \subset M$. If we compare harmonic maps of surfaces to geodesics, then pluriharmonic maps play the rôle of totally geodesic submanifolds, and they do not always exist as we will see. However there are many interesting examples. Under certain conditions, a harmonic map $f : M \to P$ of a Kähler manifold M is automatically pluriharmonic, in particular this holds if P has nonpositive curvature operator (cf. [15], [11]).

1 Associated families and symmetric spaces

Another peculiarity for $\dim M = 2$ is the existence of so called *associated families* of harmonic maps. Consider the case $P = \mathbb{R}^n$. Then (4) becomes $f_{22} = -f_{11}$ (where the indices mean partial derivatives). This is the integrability condition for the differential form $df \circ J = f_2 dx_1 - f_1 dx_2$; in other words, $df \circ J$ is closed iff f is harmonic. Hence locally $df \circ J = d\tilde{f}$ for another (so called *conjugate*) harmonic function $\tilde{f}$, and taking linear combinations, we obtain a circle of such maps $f_\theta = f\cos\theta + \tilde f \sin\theta$ with

$$df_\theta = df \circ R_\theta \tag{5}$$

where R_θ denotes the rotation by the angle θ on the tangent space of M. This is called the *associated family* of f. This result can be extended in two ways. We may replace the surface by an arbitrary complex manifold M and put $R_\theta = I\cos\theta + J\sin\theta$ where I is the identity and J the complex structure on TM. Moreover we replace $\mathbb{R}^n$ by any *symmetric space* P.

A Riemannian manifold P is called *symmetric* if for every $p \in P$ there is an isometry $s_p : P \to P$ fixing p with derivative $(ds_p)_p = -I$; this is called *point reflection* or *symmetry at* p. Thus s_p reverses any geodesic line passing through p. As a consequence, the group of isometries $I(P)$ acts transitively on P since any two points $o, q \in P$ can be joined by a geodesic segment $c : [0,1] \to P$, and we can map o to q by the point reflection s_p where $p = c(\frac{1}{2})$ is the midpoint of c.

Locally, symmetric spaces are characterized by the fact that the curvature tensor

$$R^P(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z$$

(acting on tangent vector fields X, Y, Z on P) is *parallel* on P, i.e. it commutes with the parallel displacements on P. On any tangent space T_pP it defines a trilinear map (a so called *Lie triple product*) R^P which completely encodes the local structure of the symmetric space P. In particular, any isometric linear map $\phi : T_pP \to T_qP$ preserving R^P (i.e. $R^P(\phi X, \phi Y)\phi Z = \phi R^P(X,Y)Z$) "extends" to an isometry $g \in I(P)$ with $g(p) = q$ and $dg_p = \phi$. For $p = q$ any such ϕ will be called an *automorphism* of T_pP; it extends to an isometry of P fixing the point p.

Equation (5) as it stands can hold only for $P = \mathbb{R}^n$ where all tangent spaces are identified by global parallel displacement. But otherwise df_x and $d(f_\theta)_x$ take values in different tangent spaces, $T_{f(x)}P$ and $T_{f_\theta(x)}P$. Thus we replace (5) by

$$df_\theta = \Phi_\theta \circ df \circ R_\theta \tag{6}$$

where $\Phi_\theta(x)$ is an isomorphism between $T_{f(x)}P$ and $T_{f_\theta(x)}P$ for any $x \in M$ which is as nice as possible:

- $\Phi_\theta(x)$ is a linear isometry preserving the curvature tensor R^P,
- $\Phi_\theta(x)$ is parallel with respect to x.

A family of smooth maps $f_\theta : M \to P$ satisfying (6) will be called an *associated family* of $f = f_0$. A main result of [10] characterizes pluriharmonic maps by associated families:

Theorem 1. *Let M be a simply connected complex manifold and P a symmetric space of nonpositive or nonnegative curvature. Then a smooth map $f : M \to P$ is pluriharmonic if and only if it has an associated family f_θ. This is uniquely determined up to isometries of P.*

From the uniqueness we can derive more properties of the associated family. Since $R_{\theta+\pi} = -R_\theta$, we obtain a solution $(f_{\theta+\pi}, \Phi_{\theta+\pi})$ of (6) for the rotation angle $\theta + \pi$ from a solution (f_θ, Φ_θ) for θ, namely

$$f_{\theta+\pi} = f_\theta, \quad \Phi_{\theta+\pi} = -\Phi_\theta. \tag{7}$$

In the second equation we may replace $-\Phi_\theta(x)$ by $\Phi_\theta(x)s_{f(x)}$ obtaining

$$\Phi_{\theta+\pi}(x) = \Phi_\theta(x)s_{f(x)}, \tag{8}$$

since $\Phi_\theta(x)$ can be considered as an isometry of P sending $f(x)$ onto $f_\theta(x)$ and $s_{f(x)}$ acts as $-I$ on $T_{f(x)}P$.

A class of particular interest is formed by the so called *isotropic* pluriharmonic maps whose associated family is constant: $f_\theta = f$ for all θ (up to isometries of P). Then (6) becomes

$$df = \Phi_\theta \circ df \circ R_\theta, \tag{9}$$

and this time each $\Phi_\theta(x)$ is an isometry preserving R^P and mapping $T_{f(x)}P$ onto itself. We can choose $\theta \mapsto \Phi_\theta(x)$ to be a one-parameter group, a homomorphism of the unit circle: If (f, Φ_θ) and $(f, \Phi_{\theta'})$ are solutions of (9) for θ and θ', then $(f, \Phi_\theta\Phi_{\theta'})$ is a solution for $\theta + \theta'$ (since $R_{\theta+\theta'} = R_\theta R_{\theta'}$) and hence we can assume

$$\Phi_{\theta+\theta'} = \Phi_\theta\Phi_{\theta'}, \quad \Phi_\pi = -I \tag{10}$$

2 Twistor lifts

Twistors (cf. [13], [3]) have been introduced in order to apply complex analysis to non-complex symmetric spaces. On a complex (so called *hermitian*) symmetric space there is a *complex structure* on any tangent space T_pP which by definition is a Lie triple automorphism j with $j^2 = -I$. In complex coordinates, j is just the multiplication by $i = \sqrt{-1}$, and it is invariant under parallel displacements. But if no such complex structure is given, the idea is to consider the set of *all possible* complex structures on T_pP. Any of these belongs to a one-parameter group of automorphims of T_pP called *twistors at p*. More precisely, if $G = I(P)^o$ denotes the identity component of the isometry group $I(P)$, a *twistor at p* is by definition a smooth homomorphism $\tau : S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \to G$ fixing p and passing through the point reflection $s_p = \tau_\pi$; the corresponding complex structure on T_pP is given by $j = \tau_{\pi/2}$. Of course this is possible only if the point reflections belong to the identity component of the isometry group; symmetric spaces with this property are called *inner*. E.g. the even dimensional spheres are inner, but not so the odd dimensional ones.

A *twistor space* Z over P is the conjugacy class of some twistor τ_o within G. If τ is a twistor at p, then $g\tau g^{-1}$ is a twistor at gp, and since G acts transitively, Z contains twistors at all points of P. Thus Z fibres over P with fibre Z_p being the set of all twistors at p; let $\mathrm{p} : Z \to P$ be the projection. Further Z is a complex manifold: Every $\tau \in Z$ defines a complex structure $\tau_{\pi/2}$ on $T_{\mathrm{p}(\tau)}P$ which extends canonically to a complex structure on $T_\tau Z$. In fact Z can be viewed as an adjoint orbit (an orbit of the adjoint representation of G on its Lie algebra $\mathfrak{g}$) since the one-parameter group is determined by its infinitesimal generator in $\mathfrak{g}$, and it is well known that all adjoint orbits are complex manifolds (quotients of complex Lie groups by closed complex subgroups).

The easiest example is the 4-dimensional sphere $P = S^4 \subset \mathbb{R}^5$ where the twistors $\tau \in Z_p$ are one-parameter groups of orthogonal matrices fixing p and acting by oriented planar rotations on two orthogonal planes in $p^\perp$. Any such τ is conjugate under $SO(5)$ to $(\tau_o)_\theta = \mathrm{diag}\,(1, \rho_\theta, \rho_\theta)$ with $\rho_\theta = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$; more precisely, Z is the conjugacy class of τ_o in $SO(5)$. The stabilizer of τ_o is the subgroup $U(2) \subset SO(4) \subset SO(5)$ and hence $Z \cong SO(5)/U(2)$ is complex projective 3-space $\mathbb{C}P^3$ (recall that $SO(5) = PSp(2) \subset PU(4)$ acts transitively on $\mathbb{C}P^3 = PU(4)/U(3)$ and $PSp(2) \cap U(3) = U(2)$).

Also in the general case twistors are composed by planar rotations which however may have different velocities: The tangent space $\mathfrak{p} = T_pP$ is an orthogonal sum of subspaces $\mathfrak{p}_k$ on which τ acts by planar rotations $\rho_{k\theta}$. But since $\tau_\pi = -I$, the weights k must be odd integers. Following [3], we call a twistor τ *canonical* if the lowest weight space $\mathfrak{p}_1$ generates $\mathfrak{p}$ as a Lie triple algebra, i.e. $\mathfrak{p} = \mathfrak{p}_1 + R^P(\mathfrak{p}_1, \mathfrak{p}_1, \mathfrak{p}_1) + ...$ The space $\mathfrak{p} = T_pP$ can be embedded naturally into $T_\tau Z$ as the *horizontal subspace* for the fibration $\mathrm{p} : Z \to P$, and the subspace $\mathfrak{p}_1 \subset \mathfrak{p} \subset T_\tau Z$ will be called *superhorizontal*.

Now let $f : M \to P$ be an isotropic pluriharmonic map which is *full*, i.e. $f(M)$ does not belong to a proper totally geodesic subspace of P. Then by the results of the previous chapter $\Phi(x) = (\theta \mapsto \Phi_\theta(x))$ is a twistor at $f(x)$ for any $x \in M$ and thus defines a map $\Phi : M \to Z$ with $\mathrm{p} \circ \Phi = f$, the so called *twistor lift*. From the parallelity of $x \mapsto \Phi(x)$ we see that $d\Phi$ takes values in the horizontal bundle of the fibration $\mathrm{p} : Z \to P$ ("horizontal" and "parallel" are just the same notions for the principal bundle $G \to P$ and its associated bundles.). More precisely, using (9) we see that $d\Phi$ takes values in the superhorizontal subbundle on which the twistor has weight one and hence agrees with the complex structure j (more precisely, with the rotation group generated by j); this shows that Φ is also holomorphic. Vice versa it is easy to see that the projection of a superhorizontal holomorphic map is an isotropic pluriharmonic map. Thus we obtain:

Theorem 2. *Isotropic pluriharmonic maps $f : M \to P$ are precisely the projections of holomorphic superhorizontal maps into twistor spaces over P.*

3 Loop space lifts

Now let us consider an arbitrary pluriharmonic map $f : M \to P$. Fixing a base point $p \in P$ we have a canonical projection $\pi : G \to P$, $\pi(g) = gp$ where G is the identity component of the isometry group of P. If we also fix a suitable basis B of the vector space T_pP, then $g(B)$ is a basis for $T_{gp}P$ and hence G can be considered as a certain set of bases (*frames*) of the tangent spaces of P. On a contractible open subset $M' \subset M$, the map f can be lifted to G, yielding a smooth map $F : M' \to G$ which projects onto $f|M'$, i.e. $f(x) = F(x)p$. This is called a *local framing* of f since it provides each tangent space $T_{f(x)}P$ with a frame $F(x)$. Obviously, two such framings $F, \tilde{F}$ differ by a map into the isotropy group $K = \{g \in G;\ gp = p\}$, more precisely, $\tilde{F} = FF_K$ for some smooth map $F_K : M' \to K$.

We have already seen that pluriharmonic maps come in associated one-parameter families f_θ satisfying (6). A framing $F : M' \to G$ of f defines also a framing $F_\theta = \Phi_\theta F$ for f_θ. More generally we may put

$$F_\theta = g_\theta \Phi_\theta F \tag{11}$$

for an arbitrary isometry $g_\theta \in G$, replacing f_θ with $g_\theta f_\theta$. We will use this freedom as follows: We fix base points $x_o \in M$ and $p = f(x_o) \in P$. We may assume $F(x_o) = I$, and we choose $g_\theta = \Phi_\theta(x_o)^{-1}$ whence $F_\theta(x_o) = I$ for all θ. Recall that $F(x) \in G$ maps p to $f(x)$ and hence conjugates the point reflections s_p and $s_{f(x)}$. From (8) we obtain $g_{\theta+\pi} = s_p g_\theta$ and hence $F_{\theta+\pi}(x) = g_{\theta+\pi}\Phi_{\theta+\pi}(x)F(x) = s_p g_\theta \Phi_\theta(x) s_{f(x)} F(x) = s_p F_\theta(x) s_p$, thus

$$F_{\theta+\pi}(x) = \sigma(F_\theta(x)) \tag{12}$$

where $\sigma \in \mathrm{Aut}(G)$ denotes the conjugation by s_p. Therefore each map $\theta \mapsto F_\theta(x)$ belongs to the *twisted loop group*

$$\hat{G} = \Lambda_\sigma(G) = \{\gamma : S^1 \to G;\ \gamma_{\theta+\pi} = \sigma(\gamma_\theta)\} \tag{13}$$

where the loops γ are sufficiently regular (e.g. of class H^1). Thus we have obtained a map $\mathcal{F} : M' \to \hat{G}$ with $\mathcal{F}(x)_\theta := F_\theta(x)$. If we had chosen another framing $\tilde{F} = FF_K$, we would have got a corresponding map $\tilde{\mathcal{F}} = \mathcal{F}F_K$. By projecting $\mathcal{F}$ to the coset space $\hat{Z} = \hat{G}/K$ (where $K \subset \hat{G}$ denotes the subgroup of constant loops in the isotropy group $K \subset G$) we get a map $\bar{\mathcal{F}} = \mathcal{F}K$ which is independent of the choice of F and hence is globally defined on M. This space $\hat{Z}$ fibres over P where the projection $\hat{\mathrm{p}} : \hat{Z} \to P$ is the evaluation of the loop at the initial point: $\hat{\mathrm{p}}(\gamma K) = \gamma_0 p$. Now we have constructed a smooth map $\bar{\mathcal{F}} : M \to \hat{Z}$ which is a lift of $f : M \to P$, i.e. $\hat{\mathrm{p}} \circ \bar{\mathcal{F}} = f$.

This loop space $\hat{Z}$ is again a complex manifold, a quotient of two complex loop groups: $\hat{Z} = \hat{G}^c/\hat{G}^+$ where $\hat{G}^c$ is the set of twisted loops into the complexified group[1] G^c and $\hat{G}^+$ is the subgroup of those $\gamma : S^1 \to G^c$ such that both γ, γ^{-1} extend to analytic maps on the unit disk in $\mathbb{C}$. If G^c is a matrix group, we may write each $\gamma \in \hat{G}^c$ as a matrix Fourier series $\gamma_\theta = \sum_{k\in\mathbb{Z}} A_k e^{ik\theta}$, and $\gamma \in \hat{G}^+$ iff $A_k = 0$ for $k < 0$ and the same is true for γ^{-1}. It can be shown that $\bar{\mathcal{F}} : M \to \hat{Z}$ is holomorphic; in fact one constructs a holomorphic lift into $\hat{G}^c$ using the parallelity of Φ_θ. Moreover it follows from (6) that the differential of $\bar{\mathcal{F}}$ takes values in a finite dimensional homogeneous subbundle of the tangent bundle of $\hat{Z}$, which at the base point consists of the simplest possible (finite) Fourier series $Ae^{i\theta} + \bar{A}e^{-i\theta}$; this will be called the *superhorizontal subbundle.* Vice versa, we can characterize pluriharmonic maps by this property. Thus we arrive at a theorem which looks quite similar to the one in the isotropic case:

Theorem 3. *General pluriharmonic maps $f : M \to P$ are precisely the projections of holomorphic superhorizontal maps into the loop space $\hat{Z}$ over P.*

Remarks. 1. In fact, the differential $d\bar{\mathcal{F}}$ can be described in terms of a $\mathfrak{p}^c$-valued holomorphic differential form on M, called *normalized potential.* If M is a surface, this may be an arbitrary meromorphic 1-form, but in higher dimensions an integrability condition is needed (*curved flat condition*). In [8] it was shown how to obtain f back from the potential. This formula allows to compute explicit examples.

2. The loop group $\hat{G}^c$ acting on $\hat{Z}$ preserves the set of holomorphic superhorizontal maps $\bar{\mathcal{F}} : M \to \hat{Z}$. Hence it induces an action on the set of pluriharmonic maps which is a special case of the so called *dressing action.*

4 The "universal twistor"

Can the previous construction also be viewed as a sort of twistor lift? To answer this question we first try to understand the twistor construction as a special case of the loop space lift. This is not difficult since both times we have used Φ_θ. In the isotropic case, $\Phi_\theta(x)$ is a one-parameter group conjugate to a fixed twistor τ for each $x \in M$, and from $F_\theta = (\Phi_\theta(x_o))^{-1}\Phi_\theta F$ we obtain (after a slight modification of our frame F)

$$\mathcal{F} = \tau F \tau^{-1} \tag{14}$$

Recall that $\hat{Z} = \hat{G}/K$ and $Z = Ad(G)\tau = G/H$ where H is the centralizer of τ. Motivated by (14) we consider the group homomorphism

$$\rho_\tau : G \to \hat{G}, \quad g \mapsto \tau g \tau^{-1}. \tag{15}$$

[1]We can think of G as being a real matrix group defined by algebraic equations which may be complexified.

The loop $\rho_\tau(g)$ is constant if and only if g commutes with τ, i.e. $g \in H$. Thus ρ_τ induces an equivariant embedding $\bar\rho_\tau : Z = G/H \to \hat Z = \hat G/K$. This is holomorphic and preserves superhorizontality. In fact, $d\rho_\tau$ maps a (complex) eigenvector X of $Ad(\tau)$ corresponding to an eigenvalue $e^{ik\theta}$ into the corresponding Fourier monomial: $(d\rho_\tau(X))_\theta = Ad(\tau_\theta)X = e^{ik\theta}X$.

The link between Z and $\hat Z$ becomes even more apparent if we consider $\hat Z$ like Z as an adjoint orbit. In fact, denoting the Lie algebra of $\hat G$ by $\hat{\mathfrak g}$ (consisting of the loops in $\mathfrak g$), we get an embedding $\hat Z \to \hat{\mathfrak g}$, $\gamma K \mapsto \gamma'\gamma^{-1}$ where $\gamma' = \frac{d}{d\theta}\gamma_\theta$. This can be considered as an adjoint orbit if we enlarge the Lie algebra $\hat{\mathfrak g}$ by an element δ with $ad(\delta)\xi := \xi'$ for any (sufficiently regular) $\xi \in \hat{\mathfrak g}$. Assuming G to be a matrix group we can represent each $\gamma \in \hat G$ as a multiplication operator and δ as a differential operator on matrix valued loops. Thus

$$Ad(\gamma)\delta = \gamma\delta\gamma^{-1} = \delta - \gamma'\gamma^{-1}, \tag{16}$$

and the mapping $\gamma K \mapsto \delta - \gamma'\gamma^{-1}$ is an embedding of $\hat Z$ as the adjoint orbit of δ in the enlarged Lie algebra.[2] Comparing with $Z = Ad(G)\tau$ we conclude that the one-parameter group generated by δ should be a "universal twistor" $\hat\tau$. This does not belong to $\hat G$ itself but to the automorphism group $\mathrm{Aut}(\hat{\mathfrak g})$; it is the shift of the loop parameter: For any $\xi \in \hat{\mathfrak g}$ we have

$$\hat\tau_\theta(\xi)_{\tilde\theta} = \xi_{\theta+\tilde\theta}. \tag{17}$$

Lemma. *All twistors τ acting on $\mathfrak g$ by the adjoint representation are restrictions of the "universal twistor" $\hat\tau$ on $\hat{\mathfrak g}$, more precisely,*

$$d\rho_\tau \circ Ad(\tau) = \hat\tau \circ d\rho_\tau \tag{18}$$

Proof. We have $\mathfrak g^c = \sum_k \mathfrak g_k$ with $Ad(\tau_\theta)X = e^{ik\theta}X$ for any $X \in \mathfrak g_k$. On the other hand, for any $X \in \mathfrak g_k$ we have $\hat\tau_\theta(d\rho_\tau(X)) = \hat\tau_\theta(\tilde\theta \mapsto e^{ik\tilde\theta}X) = e^{ik\theta}d\rho_\tau(X)$ which proves the claim. □

We sum up our discussion by the following

Theorem 4. *Any pluriharmonic map $f : M \to P$ is the projection of a holomorphic superhorizontal map $\bar{\mathcal F}$ into the universal twistor space $\hat Z$. The map f is isotropic iff $\bar{\mathcal F}(M)$ is contained in one of the finite dimensional twistor spaces $Z_\tau \subset \hat Z$.*

Remark. There is an important difference between the finite dimenisonal twistors τ and the universal one $\hat\tau$: The "universal twistor" does not act on P but on loop spaces, and therefore it is not a twistor in the sense of Section 2. But we may pass to the infinite dimensional symmetric space $\hat P = \hat G/\hat K$ which consists of the loops in P, and clearly $\hat\tau$ is a twistor on $\hat P$. This space also fibres over P via the evaluation at the initial point, and the loop space fibration factorizes over $\hat P$ as $\hat Z \to \hat P \to P$.

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