Morse Decompositions and Spectra on Flag Bundles

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For linear flows on vector bundles, the chain recurrent components of the induced flows on flag bundles are described and a corresponding Morse spectrum is constructed.

KEY WORDS: Morse spectrum; linear flows; chain recurrence; flags.

1. INTRODUCTION

Smooth ergodic theory and the theory of random dynamical systems were very successful in relating Lyapunov exponents to other local and global characteristics of dynamical systems. An example is the Pesin formula relating positive Lyapunov exponents to entropy. For the topological theory the situation is less satisfactory (in spite of considerable progress). This is, among other things, due to the fact that the "linear algebra" provided by the Oseledets Theorem is more efficient than the known topological concepts of spectra. In the present paper (continuous) linear flows on vector bundles are considered. They encompass, in particular, linear differential equations with almost periodic coefficients, linearized autonomous differential equations, and bilinear and linearized control systems; compare Sacker–Sell [4, 8]. We show that the constructions used for the chain recurrent components of linear flows in projective bundles (Salamon–Zehnder [10]) and for the Morse spectrum [3] can be generalized so that they provide precise results in higher dimensions. This is based on a classification of the chain recurrent components for the induced flows on

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flag and Grassmann bundles providing the finest Morse decomposition. The importance of flags in this context is also emphasized by related results on the classification of control sets and chain control sets in the theory of Lie semigroups (San Martin and Tonelli [11], Braga Barros and San Martin [2]). Applications to other characteristics are left for further work.

In Section 2 we recall some concepts from topological dynamics and their application to linear flows on projective vector bundles together with the Morse spectrum. In Section 3, the chain recurrent components in flag bundles are constructed via a finest Morse decomposition. In Section 4, this is used for the construction of a Morse spectrum.

2. CHAIN TRANSITIVE COMPONENTS AND THE MORSE SPECTRUM IN PROJECTIVE SPACE

In this section we collect some definitions and results related to the spectral theory of linear flows on vector bundles.

We first recall the following notions and facts from the theory of flows on compact metric spaces (going back to the work of Conley [5]; proofs can, e.g., be found in [4, Appendix B]). A set $K \subset X$ is called invariant if $x \cdot \mathbb{R} \subset K$ for all $x \in K$; a compact subset $K \subset X$ is called isolated invariant, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int} N$, such that $x \cdot \mathbb{R} \subset N$ implies $x \in K$. Thus an invariant set K is isolated if every trajectory that remains close to K actually belongs to K. The ω-limit set of a subset $Y \subset X$ is defined as

$$\omega(Y) = \left\{ y \in X, \quad \begin{matrix} \text{there are } t_k \to \infty \text{ and } y_k \in Y \\ \text{such that } y_k \cdot t_k \to y \end{matrix} \right\} = \bigcap_{t>0} \operatorname{cl}(Y \cdot [t, \infty)).$$

Analogously, $\omega^*(Y)$ is defined for t tending to $-\infty$.

Definition 1. A Morse decomposition of a flow on a compact metric space is a finite collection $\{\mathcal{M}_i, i = 1,..., n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that:

(i) For all $x \in X$ one has $\omega(x), \omega^*(x) \subset \bigcup_{i=1}^n \mathcal{M}_i$.

(ii) Suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1},..., \mathcal{M}_{j_l}$ and $x_1,..., x_l \in X \setminus \bigcup_{i=1}^n \mathcal{M}_i$ with $\omega^*(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1,..., l$; then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$.

The elements of a Morse decomposition are called Morse sets.

A Morse decomposition $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ is called finer than a Morse decomposition $\{\mathcal{M}'_1,..., \mathcal{M}'_{n'}\}$, if for all $j \in \{1,..., n'\}$ there is $i \in \{1,..., n\}$

with $\mathcal{M}_i \subset \mathcal{M}'_j$. The intersection of two Morse decompositions $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ and $\{\mathcal{M}'_1,..., \mathcal{M}'_{n'}\}$ defines a Morse decomposition $\{\mathcal{M}_i \cap \mathcal{M}'_j, i, j\}$, where only those indices i, j with $\mathcal{M}_i \cap \mathcal{M}'_j \neq \emptyset$ are allowed. Note that, in general, intersections of infinitely many Morse decompositions do not define a Morse decomposition. Morse sets are ordered via

$$\begin{aligned}\mathcal{M}_i \preccurlyeq \mathcal{M}_k \text{ if there are } \mathcal{M}_{j_0} = \mathcal{M}_i, \mathcal{M}_{j_1},..., \mathcal{M}_{j_l} = \mathcal{M}_k \text{ and } x_1,..., x_l \in X \\ \text{with } \omega^*(x_k) \subset \mathcal{M}_{j_{k-1}} \text{ and } \omega(x_k) \subset \mathcal{M}_{j_k} \text{ for } k = 1,..., l. \qquad (2.1)\end{aligned}$$

We enumerate the Morse sets in such a way that $\mathcal{M}_i \preccurlyeq \mathcal{M}_j$ implies $i \leqslant j$.

Morse decompositions can be constructed from attractors and their complementary repellers.

Definition 2. For a flow on a compact metric space X a compact invariant set A is an attractor if it admits a neighborhood N such that $\omega(N) = A$. A repeller is a compact invariant set R that has a neighborhood N^* with $\omega^*(N^*) = R$.

We also allow the empty set as an attractor. A neighborhood N as in Definition 2 is called an attractor neighborhood. Every attractor is compact and invariant, and a repeller is an attractor for the time reversed flow. Furthermore, if A is an attractor in X and $Y \subset X$ is a compact invariant set, then $A \cap Y$ is an attractor for the flow restricted to Y. For an attractor A, the set $A^* = \{x \in X, \omega(x) \cap A = \emptyset\}$ is a repeller, called. the complementary repeller. Then (A, A^*) is called an attractor-repeller pair. Note that A and A^* are disjoint. There is always the trivial attractor-repeller pair $A = X$, $A^* = \emptyset$.

The following result characterizes Morse decompositions via attractor-repeller sequences (it is often taken as a definition; cp. Rybakowski [7, Definition III.1.5 and Theorem III.1.8, Salamon [9], or Salamon and Zehnder [10]).

Theorem 1. *For a flow on a compact metric space X a finite collection of subsets $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors*

$$\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = X,$$

such that

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \qquad \textit{for} \quad 0 \leqslant i \leqslant n-1.$$

For $x, y \in X$ and $\varepsilon, T > 0$ an (ε, T)-chain from x to y is given by a natural number $n \in \mathbb{N}$, together with points

$$x_0 = x, \qquad x_1, \ldots, x_n = y \in X \qquad \text{and times} \quad T_0, \ldots, T_{n-1} \geqslant T,$$

such that $d(x_i \cdot T_i, x_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, n-1$. A subset $Y \subset X$ is chain transitive if for all $x, y \in Y$ and all $\varepsilon, T > 0$ there exists a (ε, T)-chain from x to y. A point $x \in X$ is chain recurrent if for all $\varepsilon, T > 0$ there exists a (ε, T)-chain from x to x. The chain recurrent set $\mathscr{R}$ is the set of all chain recurrent points. The connected components of the chain recurrent set $\mathscr{R}$ coincide with the maximal chain transitive subsets of $\mathscr{R}$. Furthermore, the flow restricted to a connected component of $\mathscr{R}$ is chain transitive. The connected components of $\mathscr{R}$ are called the chain recurrent components. The chain recurrent set and attractors are related in the following way.

Theorem 2. *The chain recurrent set $\mathscr{R}$ satisfies*

$$\mathscr{R} = \bigcap \{A \cup A^*, A \text{ is an attractor}\}.$$

In particular, there exists a finest Morse decomposition $\{\mathscr{M}_1, \ldots, \mathscr{M}_n\}$ if and only if the chain recurrent set $\mathscr{R}$ has only finitely many connected components. In this case, the Morse sets coincide with the chain recurrent components of $\mathscr{R}$ and the flow restricted to a Morse set is chain transitive and chain recurrent.

For the definition of vector bundles $\tau: \mathscr{V} \to B$ we refer to Karoubi [6] (or [4, Appendix B]): Locally, they are the product of an open subset of the metric space B with a finite dimensional Hilbert space. We always assume that the base space B is compact and connected. A linear flow Φ on a vector bundle $\pi: \mathscr{V} \to B$ is a flow Φ on $\mathscr{V}$ such that for all $\alpha \in \mathbb{R}$ and $v_1, v_2 \in \mathscr{V}$ with $\pi(v_1) = \pi(v_2)$ and $t \in \mathbb{R}$ one has

$$\pi(\Phi(t, v_1)) = \pi(\Phi(t, v_2)), \qquad \Phi(t, \alpha(v_1 + v_2)) = \alpha\Phi(t, v_1) + \alpha\Phi(t, v_2).$$

Where notationally convenient, we write instead of $\Phi(t, v)$ either $\Phi_t(v)$ or $\Phi(t)\, v$. The flow Φ induces flows on the base space B (corresponding to transport of the fibers) and on the projective bundle $\mathbb{P}\mathscr{V}$. The following theorem goes back to Selgrade [12].

Theorem 3 (Selgrade). *Let Φ be a linear flow on a vector bundle $\pi: \mathscr{V} \to B$ with chain recurrent flow on the base space B. Then the chain recurrent set of the induced flow $\mathbb{P}\Phi$ on the projective bundle $\mathbb{P}\mathscr{V}$ has finitely many, linearly ordered, components $\{\mathscr{M}_1, \ldots, \mathscr{M}_l\}$, and $1 \leqslant l \leqslant d := \dim \mathscr{V}_b$,*

$b \in B$. Every chain recurrent component $\mathcal{M}_i$ defines an invariant subbundle of $\mathcal{V}$ via

$$\mathcal{V}_i = \mathbb{P}^{-1}(\mathcal{M}_i) = \{v \in \mathcal{V}, v \notin Z \textit{ implies } \mathbb{P}v \in \mathcal{M}_i\}$$

and the following decomposition into a Whitney sum holds:

$$\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_l. \tag{2.2}$$

For points $v \in \mathcal{V}$ not in the zero section Z in $\mathcal{V}$ the Lyapunov exponent or exponential growth rate of the corresponding trajectory is given by

$$\lambda(v) = \limsup_{t \to \infty} \frac{1}{t} \log |\Phi_t v| \tag{2.3}$$

and the Lyapunov spectrum Σ_{Ly} of the linear flow Φ is the set of all Lyapunov exponents

$$\Sigma_{\mathrm{Ly}} = \{\lambda(v), v \in \mathcal{V} \backslash Z\}. \tag{2.4}$$

The concept of Morse spectrum is defined via (ε, T)-chains in the projective bundle. Recall that for $\varepsilon, T > 0$ an (ε, T)-chain ζ in $\mathbb{P}\mathcal{V}$ of Φ is given by $n \in \mathbb{N}$, $T_0, \ldots, T_{n-1} \geqslant T$, and $p_0, \ldots, p_n$ in $\mathbb{P}\mathcal{V}$ with $d(\Phi(T_i, p_i), p_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. Define the finite time exponential growth rate of such a chain (or "chain exponent") by

$$\lambda(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} (\log |\Phi(T_i, v_i)| - \log |v_i|),$$

where $v_i \in \mathbb{P}^{-1}(p_i)$.

Definition 3. Let Φ be a linear flow on a vector bundle $\pi: \mathcal{V} \to B$ and let $\mathcal{L} \subset \mathbb{P}\mathcal{V}$ be a compact invariant set for the induced flow $\mathbb{P}\Phi$ on $\mathbb{P}\mathcal{V}$ such that $\mathbb{P}\Phi \mid \mathcal{L}$ is chain transitive. Then the Morse spectrum over $\mathcal{L}$ is

$$\Sigma_{\mathrm{Mo}}(\mathcal{L}, \Phi) = \left\{ \begin{matrix} \lambda \in \mathbb{R}, \text{ there are } \varepsilon^k \to 0, T^k \to \infty \text{ and } (\varepsilon^k, T^k)\text{-chains} \\ \zeta^k \text{ in } \mathcal{L} \text{ with } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty \end{matrix} \right\}.$$

The Morse spectrum $\Sigma_{\mathrm{Mo}}(\Phi)$ of Φ is defined as the union of the Morse spectra on the chain recurrent components in $\mathbb{P}\mathcal{V}$.

The main results on the Morse spectrum are collected in the following theorem.

Theorem 4. *Let Φ be a linear flow on a vector bundle $\pi: \mathcal{V} \to B$ with chain recurrent flow on the base space B. Then*

$$\Sigma_{\mathrm{Ly}}(\Phi) \subset \Sigma_{\mathrm{Mo}}(\Phi) = \bigcup_{i=1}^{l} \Sigma_{\mathrm{Mo}}(\mathcal{M}_i, \Phi),$$

where $\mathcal{M}_i \subset \mathbb{P}\mathcal{V}$, $i = 1,\ldots, l$, are the chain recurrent components of the projective flow $\mathbb{P}\Phi$. Furthermore, for every $i = 1,\ldots, l$,

$$\Sigma_{\mathrm{Mo}}(\mathcal{M}_i, \Phi) = [\kappa^*(\mathcal{M}_i), \kappa(\mathcal{M}_i)]$$

with $\kappa^*(\mathcal{M}_i) < \kappa^*(\mathcal{M}_j)$ and $\kappa(\mathcal{M}_i) < \kappa(\mathcal{M}_j)$ for $i < j$; the boundary points $\kappa^*(\mathcal{M}_i)$, $\kappa(\mathcal{M}_i)$ are Lyapunov exponents of Φ.

We also note that, for a chain recurrent base space, the Morse spectrum coincides with the Sacker–Sell spectrum; one obtains the exponentially dichotomous subbundles provided by Sacker–Sell theory by taking in the decomposition (2.2) the sum of all subbundles with intersecting Morse spectral intervals.

3. MORSE DECOMPOSITIONS ON FLAG BUNDLES

In this section we describe the chain recurrent components in the flag bundles. The existence of the corresponding finest Morse decomposition follows from the linear structure of the attractors. The proof proceeds via induction on the length of the flags; in the induction step, the argument is reduced to the one-dimensional Selgrade theorem, Theorem 3, by constructing appropriate vector bundles.

Throughout the rest of this paper, we consider a linear flow on a d-dimensional vector bundle $\pi: \mathcal{V} \to B$ with chain transitive compact base space B. We find associated Grassmann bundles $\mathbb{G}_i\mathcal{V}$ by repeating the construction for vector bundles (cp., e.g., Appendix B in [4]): Grassmann bundles are locally trivial fiber bundles where the fibers are Grassmannians. Recall that Grassmannians can be considered as elements of the projective space of exterior products; a subspace is identified with the line spanned by a simple element whose entries span the subspace. Analogously, a Grassmann bundle can be identified with a subset of the projective bundle of an exterior product bundle $\Lambda^k\mathcal{V}$. We consider the distance on the Grassmann bundle that is induced by the metric on the corresponding projective bundle. We will also consider flag bundles whose elements are sequences of subspaces V_i in a fiber $\mathcal{V}_b$

$$V_1 \subset V_2 \subset \cdots \subset V_k$$

with $\dim V_i = i$ for $i \in \bar{k} = \{1,\dots,k\}$ and $k \leqslant d$. We denote the corresponding flag bundle by

$$\mathbb{F}_k\mathscr{V} = \left\{ F_k = (b, V_1, V_2,\dots, V_k), \quad \begin{array}{c} b \in B \text{ and } V_i \subset \mathscr{V}_b \text{ with } V_i \subset V_{i+1} \\ \text{and } \dim V_i = i \text{ for } i \in \bar{k} \end{array} \right\}.$$

Thus for $k = d$ we have the complete flag bundle $\mathbb{F}\mathscr{V} := \mathbb{F}_d\mathscr{V}$. Furthermore, observe that $\pi\colon \mathbb{F}_k\mathscr{V} \to \mathbb{F}_{k-1}\mathscr{V}$ has a natural, locally trivial, fiber bundle structure. Where convenient, we denote the corresponding fibers by $(\mathbb{F}_k\mathscr{V})_{F_{k-1}}$ for $F_{k-1} \in \mathbb{F}_{k-1}\mathscr{V}$. With a slight abuse of notation we also write $v \in F_k$ if $v \in V_k$. We supply the flag bundles with the metric induced by the Grassmann bundles $\mathbb{G}_i\mathscr{V}$

$$d(F_k, F'_k) = \max_{i = 1,\dots,k} d_{\mathbb{G}_i}((b, V_i), (b', V'_i)).$$

In the following we denote the flows induced by the linear flow Φ on the Grassmann, the flag and the exterior bundles again by Φ; by the context it will always be clear which flow is meant.

First we will discuss how Morse decompositions in $\mathbb{F}_k$ and $\mathbb{F}_j$, $k > j$, are related.

Proposition 1. *Let $\{{}_j\mathscr{M}_1,\dots, {}_j\mathscr{M}_n\}$ be a Morse decomposition in $\mathbb{F}_j\mathscr{V}$ for the attractor sequence $\varnothing = {}_jA_0 \subset {}_jA_1 \subset \cdots \subset {}_jA_n = \mathbb{F}_j\mathscr{V}$. Define for $k > j$*

$${}_kA_i := \{(F_j, V_{j+1},\dots, V_k) \in \mathbb{F}_k\mathscr{V}, F_j \in {}_jA_i\}.$$

Then $\{{}_k\mathscr{M}_1,\dots, {}_k\mathscr{M}_n\}$ is a Morse decomposition in $\mathbb{F}_k\mathscr{V}$. Conversely, consider a Morse decomposition $\{{}_k\mathscr{M}_1,\dots, {}_k\mathscr{M}_n\}$ in $\mathbb{F}_k\mathscr{V}$ with attractor sequence $\varnothing = {}_kA_0 \subset {}_kA_1 \subset \cdots \subset {}_kA_n$. Then

$${}_jA_i := \{F_j \in \mathbb{F}_j\mathscr{V}, \text{ there are } V_{j+1} \subset \cdots \subset V_k \text{ with } (F_j, V_{j+1},\dots, V_k) \in {}_kA_i\}$$

is an attractor sequence in $\mathbb{F}_j\mathscr{V}$ with Morse sets

$${}_j\mathscr{M}_i := \{F_j \in \mathbb{F}_j\mathscr{V}, \text{ there are } V_{j+1} \subset \cdots \subset V_k \text{ with } (F_j, V_{j+1},\dots, V_k) \in {}_k\mathscr{M}_i\}$$

In particular, every projection of a Morse set to $\mathbb{F}_j\mathscr{V}$ contains a chain recurrent component of $\mathbb{F}_j\mathscr{V}$.

Proof. Let ${}_jN_i$ be attractor neighborhoods of ${}_jA_i$, i.e., one has $\omega({}_jN_i) = {}_jA_i$. Then

$${}_kN_i := \{(F_j, V_{j+1},\dots, V_k) \in \mathbb{F}_k\mathscr{V}, F_j \in {}_jN_i\}$$

are attractor neighborhoods of ${}_kA_i$. The analogous construction for the complementary repellers yields the assertion.

For the converse assertion observe that for attractor neighborhoods ${}_kN_i$ one obtains attractor neighborhoods of ${}_jA_i$ as

$$ {}_jN_i := \{F_j \in \mathbb{F}_j\mathscr{V}, \text{ there are } V_{j+1} \subset \cdots \subset V_k \text{ with } (F_j, V_{j+1},\ldots, V_k) \in {}_kN_i\}. $$

□

Remark 1. Observe that analogous results hold for arbitrary flag bundles corresponding to dimensions $i_1 < i_2 < \cdots < i_j$ and $k_1 < k_2 < \cdots < k_{j'}$, provided that $\{i_1,\ldots, i_j\} \subset \{k_1,\ldots, k_{j'}\}$.

The following first main result is a flag version of Selgrade's Theorem.

Theorem 5. *Consider a linear flow Φ on a vector bundle $\pi: \mathscr{V} \to B$ and suppose that the induced flow on the compact base space B is chain transitive.*

(i) Then for every $1 \leqslant k \leqslant d$ there exists a unique finest Morse decomposition $\{{}_k\mathscr{M}_{i_j}\}$ of the induced flow Φ on the flag bundle $\mathbb{F}_k\mathscr{V}$; here $i_j \in \{1,\ldots, d\}^k$ is a multiindex; and the number of chain transitive components in the flag bundle $\mathbb{F}_k\mathscr{V}$ is bounded by $d!/(d-k)!$.

(ii) Let $\mathscr{M}_i$, $i \in \{1,\ldots, d\}^{k-1}$ be a chain recurrent component in the flag bundle $\mathbb{F}_{k-1}\mathscr{V}$ and consider the $d-k+1$-dimensional vector bundle $\pi: \mathscr{W}(\mathscr{M}_i) \to \mathscr{M}_i$ with fibers $\mathscr{W}(\mathscr{M}_i)_{F_{k-1}} = \mathscr{V}_b/V_{k-1}$ for $F_{k-1} = (b, V_1,\ldots, V_{k-1}) \in \mathscr{M}_i$. Then every chain recurrent component ${}_{\mathbb{P}}\mathscr{M}_{i_j}$, $j = 1,\ldots, k_i \leqslant k-d+1$, of the projective bundle $\mathbb{P}\mathscr{W}(\mathscr{M}_i)$ determines a chain recurrent component ${}_k\mathscr{M}_{i_j}$ of $\mathbb{F}_k\mathscr{V}$ via

$$ {}_k\mathscr{M}_{i_j} = \{F_k = (F_{k-1}, V_k) \in \mathbb{F}_k\mathscr{V}, F_{k-1} \in \mathscr{M}_i \text{ and } \mathbb{P}(V_k/V_{k-1}) \subset {}_{\mathbb{P}}\mathscr{M}_{i_j}\}. $$

and every chain recurrent component in $\mathbb{F}_k\mathscr{V}$ is of this form. This inductively determines the multiindex i_j.

Proof. We proceed by induction over the dimension k. By Selgrade's Theorem, Theorem 3, the assertion holds for $k = 1$. Now suppose that the assertion holds for $k-1$. Note first that by Proposition 1 there is a Morse decomposition in $\mathbb{F}_k\mathscr{V}$ which projects down to the finest Morse decomposition in $\mathbb{F}_{k-1}\mathscr{V}$. Thus we may restrict our attention to attractor sequences

$$ \varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathbb{F}_k\mathscr{V} $$

with the following property: For the corresponding attractors and repellers in $\mathbb{F}_{k-1}\mathscr{V}$ one has that $\mathbb{F}_{k-1}A_j \cap \mathbb{F}_{k-1}A_j$ is a chain recurrent component of $\mathbb{F}_{k-1}\mathscr{V}$. Next observe that for a fixed chain recurrent component $\mathscr{M}$ in $\mathbb{F}_{k-1}\mathscr{V}$, every Morse decomposition $\{\mathscr{M}_j\}$ in $\mathbb{F}_k\mathscr{V}$ induces a Morse decomposition $\{\mathscr{M}_j \cap \pi^{-1}\mathscr{M}\}$ of $\pi^{-1}\mathscr{M}$; this follows at once from the definition of a Morse decomposition and compactness and invariance of $\pi^{-1}\mathscr{M}$. For every Morse set in $\mathbb{F}_k\mathscr{V}$ the projection to $\mathbb{F}_{k-1}\mathscr{V}$ contains at least one chain recurrent component of $\mathbb{F}_{k-1}\mathscr{V}$. There exists a Morse decomposition in $\mathbb{F}_k\mathscr{V}$ projecting down to the finest Morse decomposition in $\mathbb{F}_{k-1}$ and a refinement of such a Morse decomposition in $\mathbb{F}_k\mathscr{V}$ yields a refinement of at least one Morse decomposition over a chain recurrent component in $\mathbb{F}_{k-1}\mathscr{V}$. Hence we may restrict our attention to Morse decompositions $\{\mathscr{M}_i\}$ of $\pi^{-1}\mathscr{M}$ over a fixed chain recurrent component $\mathscr{M}$ in $\mathbb{F}_{k-1}\mathscr{V}$. Such a Morse decomposition corresponds to an attractor sequence A_j, $j = 1, \ldots, n$, in the fiber bundle

$$\pi^{-1}\mathscr{M} = \{F_k = (F_{k-1}, V_k) \in \mathbb{F}_k\mathscr{V}, F_{k-1} \in \mathscr{M}\}.$$

Now consider the vector bundle π: $\mathscr{W}(\mathscr{M}) \to \mathscr{M}$ with fibers

$$(\mathscr{W}(\mathscr{M}))_{F_{k-1}} = \mathscr{V}_b / V_{k-1} \qquad \text{for} \quad F_{k-1} = (b, V_1, \ldots, V_{k-1}) \in \mathscr{M}. \tag{3.1}$$

This is a vector bundle of dimension $d-k+1$, since it is obtained as a quotient bundle of $\mathscr{V} \to \mathscr{M}$ modulo the $k-1$-dimensional sub-vector bundle with fibers V_{k-1} over $F_{k-1} = (b, V_1, \ldots, V_{k-1})$. Note that this is a subbundle, since its fibers have constant dimension and it is closed. Next we show that an attractor A in $\pi^{-1}\mathscr{M}$ yields an attractor $\hat{A}$ in the projective bundle $\mathbb{P}\mathscr{W}(\mathscr{M})$. In fact, let N be an attractor neighborhood of A with $A = \omega(N)$ where

$$\omega(N) = \{F_k \in \pi^{-1}\mathscr{M}, \text{there are } t^j \to \infty \text{ and } F_k^j \in N \text{ with } \Phi(t^j, F_k^j) \to F_k\}.$$

Consider the subsets $\hat{A}$ and $\hat{N}$ in $\mathbb{P}\mathscr{W}(\mathscr{M})$ with fibers over $F_{k-1} = (b, V_1, \ldots, V_{k-1})$ given by

$$\hat{A}_{F_{k-1}} = \mathbb{P}\{v + V_{k-1}, \text{there is } F_k = (F_{k-1}, V_k) \in A \text{ with } v \in V_k\}$$

and

$$\hat{N}_{F_{k-1}} = \mathbb{P}\{v + V_{k-1}, \text{there is } F_k = (F_{k-1}, V_k) \in N \text{ with } v \in V_k\}.$$

Then $\hat{N}$ is a neighborhood of $\hat{A}$ and $\omega(\hat{N}) = \hat{A}$ and hence $\hat{N}$ is an attractor neighborhood of $\hat{A}$. In fact, the neighborhood property follows easily in the projective bundle with fibers $\mathbb{P}\mathscr{V}_b$ over $F_{k-1} = (b, V_1, \ldots, V_{k-1}) \in \mathscr{M}$ and remains true in $\mathscr{W}(\mathscr{M})$, since the projection is open. To see the limit property, consider $\mathbb{P}(v+V_{k-1}) \in \hat{A}_{F_{k-1}}$ with $F_k = (F_{k-1}, V_k) \in A$ and $v \in V_k$. Since $\omega(N) = A$, there are $t^j \to \infty$ and $F_k^j = (F_{k-1}^j, V_k^j) \in N$ with $\Phi(t^j, F_k^j) \to F_k$. Then also $\Phi(t^j, V_{k-1}^j) \to V_{k-1}$ and $\Phi(t^j, V_k^j) \to V_k$. This implies that there are $v^j \in V_k^j$ with

$$\mathbb{P}(v^j + V_{k-1}^j) \to \mathbb{P}(v + V_{k-1}).$$

Thus $\hat{A} \subset \omega(\hat{N})$. Conversely, consider a point $\mathbb{P}(v+V_{k-1}) \in \omega(\hat{N})$. Thus there are $t^j \to \infty$ and v^j, V_{k-1}^j with $(F_{k-1}^j, V_k^j) \in N$ and $\Phi(t^j, \mathbb{P}(v^j + V_{k-1}^j)) \to \mathbb{P}(v+V_{k-1})$. Since $\omega(N) = A$, it follows by compactness that there is V_k such that for a subsequence $\Phi(t^j, F_k^j) \to F_k = (F_{k-1}, V_k) \in A$ with $v \in V_k$. Hence $\mathbb{P}(v+V_{k-1}) \in \hat{A}$.

For the vector bundle $\mathscr{W}(\mathscr{M})$, Theorem 3 implies that there exists a finest Morse decomposition in the projective bundle, since the base space $\mathscr{M}$ is chain transitive. By the construction it follows that a refinement of a Morse decomposition in $\pi^{-1}\mathscr{M}$ yields a refinement of the corresponding Morse decomposition in $\mathbb{P}\mathscr{W}(\mathscr{M})$; hence there exists a finest Morse decomposition in $\pi^{-1}\mathscr{M}$ as claimed. □

Remark 2. The proof of this theorem shows, that the projections of the chain recurrent components in flags $\mathbb{F}_k\mathscr{V}$ to flags $\mathbb{F}_j\mathscr{V}$ with $j < k$ yield the chain recurrent components in $\mathbb{F}_j\mathscr{V}$.

The result above also allows us to describe the chain recurrent components on the Grassmann bundles.

Proposition 2. *There exists a finest Morse decomposition on every Grassmann bundle $\mathbb{G}_k\mathscr{V}$. Its Morse sets which are the chain recurrent components are given by the projection of the chain recurrent components from the complete flag bundle.*

Proof. Let $\mathscr{M} \subset \mathbb{F}\mathscr{V}$ be a chain recurrent component. Then

$$\{V_k \in \mathbb{G}_k\mathscr{V}, \text{there is } (V_1, \ldots, V_k, V_{k+1}, \ldots, V_d) \in \mathscr{M}\}$$

is obviously chain transitive in $\mathbb{G}_k\mathscr{V}$. Now consider a Morse decomposition $\{{}_{\mathbb{F}}\mathscr{M}_1, \ldots, {}_{\mathbb{F}}\mathscr{M}_{l_d}\}$ in $\mathbb{F}\mathscr{V}$ with attractor sequence $\emptyset = {}_{\mathbb{F}}A_0 \subset {}_{\mathbb{F}}A_1 \subset \cdots \subset {}_{\mathbb{F}}A_n$. We claim that

$${}_{\mathbb{G}_k}A_i := \{V_k \in \mathbb{G}_k\mathscr{V}, \text{there are } V_j \text{ such that } (V_1, \ldots, V_k, V_{k+1}, \ldots, V_d) \in {}_{\mathbb{F}}A_i\}$$

is an attractor sequence in $\mathbb{G}_k\mathscr{V}$ with Morse sets

$${}_{\mathbb{G}_k}\mathscr{M}_i := \{V_k \in \mathbb{G}_k\mathscr{V}, \text{there are } V_j \text{ such that } (V_1, \ldots, V_k, V_{k+1}, \ldots, V_d) \in {}_{\mathbb{F}}\mathscr{M}_i\}.$$

In fact, for attractor neighborhoods ${}_{\mathbb{F}}N_i$ of ${}_{\mathbb{F}}A_i$ one obtains attractor neighborhoods of ${}_{\mathbb{G}_k}A_i$ as

$$N_i := \{V_k \in \mathbb{G}_k\mathscr{V}, \text{there are } V_j \text{ such that } (V_1,\ldots, V_k, V_{k+1},\ldots, V_d) \in {}_{\mathbb{F}}N_i\}.$$

The same arguments applied to the complementary repellers proves the claim. This shows that the projections of the chain recurrent components of the complete flag to a Grassmann bundle yield chain recurrent sets which belong to a Morse decomposition. Hence the finest Morse decomposition on the Grassmann bundle is given by the projection from the complete flag bundle. □

Remark 3. For every linear flow on a chain recurrent compact base space one obtains a tree in the following way: Above the root B, the nodes at level 1 are the chain recurrent components in $\mathbb{G}_1\mathscr{V}$. The nodes at level k are the chain recurrent components $\mathscr{M}_{i_j}$ in the flag bundles $\mathbb{F}_k\mathscr{V}$ and there is an edge from the node $\mathscr{M}_{i_j} \subset \mathbb{F}_{k-1}$ to $\mathscr{M}_{i_j} \subset \mathbb{F}_k\mathscr{V}$ if $\mathscr{M}_{i_l}$ projects down to $\mathscr{M}_{i_j}$. This is equivalent to $i_l = (i_j, m)$ for some $m \in \{1,\ldots, k_{i_j}\}$.

The chain recurrent components on Grassmann or flag bundles need not be linearly ordered. However, one obtains a unique maximal (and, via time reversal, minimal) element.

Corollary 1. On every flag bundle there is a unique maximal chain recurrent component.

Proof. This is seen inductively: For $k = 1$, this is part of Selgrade's Theorem. Suppose that $\mathscr{M}$ is a maximal chain recurrent component in $\mathbb{F}_{k-1}\mathscr{V}$. Then there exists, again by Selgrade's Theorem, a unique maximal chain recurrent component in the projective vector bundle $\mathbb{P}\mathscr{W}(\mathscr{M})$. One can easily see that this corresponds to a maximal chain recurrent component in $\mathbb{F}_k\mathscr{V}$. □

Remark 4. It also follows that the chain recurrent components on the complete flag bundle project down to the chain recurrent components on arbitrary flag bundles.

Remark 5. In Salamon–Zehnder [10], the proof of Selgrade's theorem is based on the fact that attractors in the projective bundle define subbundles, i.e., the intersection with a fiber is linear and has constant dimension. An analogous construction for flag bundles shows that also the intersection of an attractor with a fiber $\mathcal{W}(\mathcal{M})_{F_{k-1}}$ is linear and has constant dimension. This may be used to give a direct proof of Theorem 5, without taking recourse to the one-dimensional theorem.

Next we describe the relation of the chain recurrent components in the flag bundles to the chain recurrent components in projective space by constructing an appropriate Morse decomposition.

Theorem 6. *Let Φ be a linear flow on a vector bundle $\pi: \mathcal{V} \to B$ with chain transitive compact base space B and dimension d. Let $\mathcal{V}_i$ with dimension d_i, $i = 1,\dots, l$, be the subbundles which project down to the finest Morse decomposition of the projective flow $\mathbb{P}\Phi$: $\mathbb{P}\mathcal{V} \to B$, according to Selgrade's Theorem. Define for $1 \leqslant k \leqslant d$ the index set*

$$I(k) := \{(k_1, k_2,\dots, k_l), k_1 + k_2 + \cdots + k_l = k \text{ and } 0 \leqslant k_i \leqslant d_i\}.$$

Then a Morse decomposition in the Grassmann bundle $\mathbb{G}_k\mathcal{V} \to B$ is given by the sets

$$\mathcal{N}^k_{k_1,\dots,k_l} = \mathbb{G}_{k_1}\mathcal{V}_1 \oplus \cdots \oplus \mathbb{G}_{k_l}\mathcal{V}_l, \qquad (k_1,\dots, k_l) \in I(k),$$

with the interpretation that, on the right hand side we have in every fiber $\mathcal{V}_b$ over $b \in B$ the sum of arbitrary k_i-dimensional subspaces of $\mathcal{V}_{i,b}$. In particular, every chain recurrent component of $\mathbb{G}_k\mathcal{V}$ is contained in one of these Morse sets.

Remark 6. We can also write

$$\begin{aligned}\mathcal{N}^k_{k_1,\dots,k_l} &= \{(b, V) \in \mathbb{G}_k\mathcal{V}, b \in B \text{ and } \dim((b, V) \cap \mathcal{V}_i) = k_i \text{ for } i = 1,\dots, l\}\\ &= \left\{(b, V) \in \mathbb{G}_k\mathcal{V}, V = \bigoplus_i V_i \text{ with } V_i \subset \mathcal{V}_{i,b} \text{ and } \dim V_i = k_i\right\}.\end{aligned}$$

Proof. It is clear that the $\mathcal{N}^k_{k_1,\dots,k_l}$ are nonvoid, pairwise disjoint and isolated compact invariant sets. First we show that every ω-limit set is contained in a set of the type $\mathcal{N}^k_{k_1,\dots,k_l}$ with $(k_1,\dots, k_l) \in I(k)$. The assertion is clear for $k = 1$, by Selgrade's Theorem. Suppose that it holds for all dimensions less than k and consider for an element $(b, W) \in \mathbb{G}_k\mathcal{V}$ the ω-limit set $\omega(b, W)$. Let the k-dimensional subspace W be spanned by $w^1,\dots, w^k$, i.e.,

$$W = \operatorname{span}\{w_1,\dots, w_k\},$$

and consider a sequence $t_n \to \infty$ such that $\Phi_{t_n} W$ converges to some element $W^+ \in \mathbb{G}_k \mathscr{V}_{b^+}$. Define

$$U = \operatorname{span}\{w_1, \ldots, w_{k-1}\} \in \mathbb{G}_{k-1} \mathscr{V}.$$

Passing, if necessary, to a subsequence, we may assume that $\Phi_{t_n} U$ converges to an element $U^+ \in \mathbb{G}_{k-1} \mathscr{V}_{b^+}$. Using the induction hypothesis, we find

$$U^+ \in \mathscr{N}^{k-1}_{k_1, \ldots, k_l} = \mathbb{G}_{k_1} \mathscr{V}_1 \oplus \cdots \oplus \mathbb{G}_{k_l} \mathscr{V}_l;$$

here $(k_1, \ldots, k_l) \in I(k-1)$. Furthermore, we may assume that the sequences

$$\frac{\Phi_{t_n} w^k}{|\Phi_{t_n} w^k|}, \qquad n \in \mathbb{N},$$

converge. It is clear, that its distance to one of the subbundles $\mathscr{V}_r$, $r \in \{1, \ldots, l)$ converges to 0. Hence $\Phi_{t_n} W$ converges to the set

$$\mathscr{N}^k_{k_1, \ldots, k_r + 1, \ldots, k_l}.$$

Since this is an isolated compact invariant set, it contains the ω-limit set $\omega(b, W)$. Similarly, one sees that every ω^*-limit set is contained in a set of this type.

It remains to prove the no-cycle condition. First we introduce a (lexicographic) order on $I(k)$ and hence on $\{\mathscr{N}^k_{k_1, \ldots, k_l}, (k_1, \ldots, k_l) \in I(k)\}$. For elements of $I(k)$ define $(k'_1, \ldots, k'_l) \prec (k_1, \ldots, k_l)$ if there exists $t \in \{1, \ldots, l\}$ such that

$$k'_l \leqslant k_l, k'_{l-1} \leqslant k_{l-1}, \ldots, k'_{t-1} \leqslant k_{t-1}, k'_t < k_t.$$

We claim that for all $(b, W) \in \mathbb{G}_k \mathscr{V}$ the implication

$$\text{if } \omega^*(b, W) \subset \mathscr{N}^k_{k'_1, \ldots, k'_l} \text{ and } \omega(b, W) \subset \mathscr{N}^k_{k_1, \ldots, k_l}, \text{ then } \mathscr{N}^k_{k'_1, \ldots, k'_l} \prec \mathscr{N}^k_{k_1, \ldots, k_l}$$

holds. Then the no-cycle property (ii) in the definition of a Morse decomposition is verified and the theorem is proven.

First observe that the assertion holds for $k = 1$, again by Selgrade's Theorem. Now suppose that it holds for all dimensions less than k and consider $(b, W) \in \mathbb{G}_k \mathscr{V}$ with

$$\omega^*(b, W) \subset \mathscr{N}^k_{k'_1, \ldots, k'_l} \qquad \text{and} \qquad \omega(b, W) \subset \mathscr{N}^k_{k_1, \ldots, k_l}.$$

As before, let the k-dimensional subspace W be spanned by $w^1,\dots, w^k$, and consider sequences $t_n^\pm \to \infty$ such that $\Phi_{t_n^\pm} W$ converges to some element $W^\pm \in \mathbb{G}_k \mathscr{V}_{b^\pm}$. Define

$$U = \operatorname{span}\{w_1,\dots, w_{k-1}\} \in \mathbb{G}_{k-1}\mathscr{V}.$$

Passing, if necessary, to a subsequence, we may assume that $\Phi_{t_n^\pm} U$ converges to an element $U^\pm \in \mathbb{G}_{k-1}\mathscr{V}_{b^\pm}$. As earlier, we find

$$U^- \in \mathscr{N}^{k-1}_{k'_1,\dots,k'_l} = \mathbb{G}_{k'_1}\mathscr{V}_1 \oplus \cdots \oplus \mathbb{G}_{k'_l}\mathscr{V}_l \qquad \text{and}$$
$$U^+ \in \mathscr{N}^{k-1}_{k_1,\dots,k_l} = \mathbb{G}_{k_1}\mathscr{V}_1 \oplus \cdots \oplus \mathbb{G}_{k_l}\mathscr{V}_l,$$

where $(k_1,\dots, k_l)$, $(k'_1,\dots, k'_l) \in I(k-1)$. The induction hypothesis implies that $(k'_1,\dots, k'_l) \preccurlyeq (k_1,\dots, k_l)$. Furthermore, we may assume that the sequences

$$\frac{\Phi_{t_n^\pm} w^k}{|\Phi_{t_n^\pm} w^k|}, \qquad n \in \mathbb{N},$$

converge. It is clear, that for $n \to \pm\infty$ its distance to one of the subbundles $\mathscr{V}_{r^\pm}$, $r^\pm \in \{1,\dots, l)$ tends to 0 and $r^- \leqslant r^+$. Thus $W^- \in \mathscr{N}^k_{k'_1,\dots,k'_{r^-}+1,\dots,k'_l}$ and $W^+ \in \mathscr{N}^k_{k_1,\dots,k_{r^+}+1,\dots,k_l}$. Clearly, $(k'_1,\dots, k'_{r^-}+1,\dots, k'_l) \preccurlyeq (k_1,\dots, k_{r^+}+1,\dots, k_l)$. Furthermore, if here equality holds, then (b, W) is in the corresponding Morse set. Thus the assertion holds. □

Remark 7. For autonomous differential equations $\dot{x} = Ax$ in $\mathbb{R}^d$ one can show that the Morse decomposition constructed above is the finest one. In general, this may not be the case.

4. THE SPECTRUM

In this section we define a Morse spectrum on flag bundles. We will show that for every chain recurrent component in the complete flag $\mathbb{F}\mathscr{V}$ one obtains exactly d intervals of Morse exponents; the jth interval corresponds to the chain exponents of j-dimensional subspaces in the considered flags. Thus the total spectrum consists of at most $d!$ times d intervals. Furthermore the intervals are contained in the sums of one-dimensional growth rates.

Let Φ: $\mathbb{R} \times \mathscr{V} \to \mathscr{V}$ be a linear flow on a vector bundle π: $\mathscr{V} \to B$. For the induced flow on the complete flag bundle $\mathbb{F}\mathscr{V}$ let the chain recurrent components be given by $\mathscr{M}_i \subset \mathbb{F}\mathscr{V}$, $i \in I$. For $\varepsilon, T > 0$ an (ε, T)-chain ζ of Φ in $\mathbb{F}\mathscr{V}$ is given by

$$n \in \mathscr{N}, \quad T_0,\dots, T_{n-1} \geqslant T, \quad F^0 = (V^0_1,\dots, V^0_d),\dots, F^n = (V^n_1,\dots, V^n_d) \in \mathbb{F}\mathscr{V}$$

with $d(\Phi(T_i, F^i), F^{i+1}) < \varepsilon$ for $i = 0,\ldots, n-1$. Define the flag of exponential growth rates of ζ by

$$\Lambda_k(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \left(\sum_{i=0}^{n-1} \log |\Phi(T^i, V_k^i)| - \log |V_k^i|\right), \qquad k = 1,\ldots, d,$$

where norms are taken for elements in the exterior product spaces $\Lambda^k \mathscr{V}$ which project down to V_k^i identified with the corresponding element in $\mathbb{P}\Lambda^k\mathscr{V}$.

Definition 4. Let $\mathscr{L} \subset \mathbb{F}\mathscr{V}$ be a compact invariant set for the induced flow Φ on $\mathbb{F}\mathscr{V}$ and assume that $\Phi_{|\mathscr{L}}$ is chain transitive. Then the Morse (flag) spectrum over $\mathscr{L}$ is defined as

$$\Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi) = \left\{ \begin{array}{c} (\Lambda_1,\ldots, \Lambda_d) \in \mathbb{R}^d, \text{there are } \varepsilon^k \to 0, T^k \to \infty \text{ and } (\varepsilon^k, T^k)\text{-chains} \\ \zeta^k \text{ in } \mathscr{L} \text{ with } (\Lambda_1(\zeta^k),\ldots, \Lambda_d(\zeta^k)) \to_{k\to\infty} (\Lambda_1,\ldots, \Lambda_d) \end{array} \right\}.$$

Furthermore, the Morse spectrum of the linear flow on the flag bundle $\mathbb{F}\mathscr{V}$ is defined as

$$\Sigma_{\mathrm{Mo}}(\Phi) = \bigcup \Sigma_{\mathrm{Mo}}(\mathscr{M}, \Phi),$$

where the union is taken over all chain recurrent components $\mathscr{M}$ in $\mathbb{F}\mathscr{V}$.

We also abbreviate

$$\Sigma_{\mathrm{Mo}}^k(\mathscr{L}, \Phi) = \{\Lambda_k, \text{there is } (\Lambda_1,\ldots, \Lambda_k,\ldots, \Lambda_d) \in \Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi)\}$$

First we show that it is actually sufficient to consider periodic chains in the definition of the Morse spectrum, i.e., the exponents of chains from a point to itself. This property will be used frequently. The proof is based on [4], Lemma B.2.23, which gives a uniform upper bound for the time needed to connect two points in a chain recurrent component.

Proposition 3. *For a linear flow Φ on a vector bundle let $\mathscr{L} \subset \mathbb{F}\mathscr{V}$ be a compact invariant set of the induced flow Φ on $\mathbb{F}\mathscr{V}$ such that $\Phi_{|\mathscr{L}}$ is chain transitive. Then the Morse spectrum of Φ over $\mathscr{L}$ satisfies*

$$\Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi) = \left\{ \begin{array}{l} \Lambda = (\Lambda_1,\ldots, \Lambda_d) \in \mathbb{R}^d, \textit{there are } \varepsilon^k \to 0, T^k \to \infty \textit{ and periodic} \\ (\varepsilon^k, T^k)\textit{-chains } \zeta^k \textit{ in } \mathscr{L} \textit{ with } (\Lambda_1(\zeta^k),\ldots, \Lambda_d(\zeta^k)) \to_{k\to\infty} (\Lambda_1,\ldots, \Lambda_d) \end{array} \right\}.$$

Proof. Let $\Lambda \in \Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi)$ and fix $\varepsilon, T > 0$. It suffices to prove that for every $\delta > 0$ there exists a periodic (ε, T)- chain ζ with $|\Lambda - \Lambda(\zeta)| < \delta$ (here we take the Euclidean norm in $\mathbb{R}^d$). By [4], Lemma B.2.23, there exists $\bar{T}(\varepsilon, T) > 0$ such that for all $p, p' \in \mathscr{L}$ there is a (ε, T)-chain from p to p' with total time $\leqslant \bar{T}(\varepsilon, T)$. For $T' > T$ choose an (ε, T')-chain ζ' with $|\Lambda - \lambda(\zeta')| < \frac{\delta}{2}$ given by, say, $(V_1^0, \dots, V_d^0), \dots, (V_1^n, \dots, V_d^n) \in \mathbb{F}\mathscr{V}$ with $|V_i| = 1$ and times $T^0, \dots, T^{n-1} > T'$. Then for all $j = 1, \dots, d$

$$\Lambda_j(\zeta') = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} \log |\Phi(T^i, V_j^i)|.$$

Concatenate this with an (ε, T)-chain ξ from $(V_1^n, \dots, V_d^n)$ to $(V_1^0, \dots, V_d^0)$ with points $(W_1^0, \dots, W_d^0) = (V_1^n, \dots, V_d^n), \dots, (W_1^m, \dots, W_d^m) = (V_1^0, \dots, V_d^0)$ with $|W_j^i| = 1$, and times $S^0, \dots, S^{m-1} > T$, and total time $\sum_{i=0}^{m-1} S^i \leqslant \bar{T}(\varepsilon, T)$. This yields a periodic (ε, T)-chain $\xi \circ \zeta'$ with chain exponents

$$\Lambda_j(\xi \circ \zeta') = \left(\sum_{i=0}^{n-1} T_i + \sum_{i=0}^{m-1} S_i\right)^{-1} \left[\sum_{i=0}^{n-1} \log |\Phi(T^i, V_j^i)| + \sum_{i=0}^{m-1} \log |\Phi(S^i, W_j^i)|\right].$$

Choosing T' large enough one obtains for all j

$$|\Lambda_j(\zeta) - \Lambda_j(\xi \circ \zeta')| < \frac{\delta}{2},$$

which yields the assertion. □

The following result describes the behavior of the spectrum under time reversal. We omit the proof, since it is completely analogous to the one-dimensional case; compare [4, Proposition 5.3.4].

Proposition 4. *For a linear flow Φ on a vector bundle $\pi: \mathscr{V} \to B$ let the corresponding time reversed flow Φ^* be defined by*

$$\Phi^*(t, v) = \Phi(-t, v), \qquad t \in \mathbb{R}, \quad v \in \mathscr{V}.$$

Then $\mathscr{R}(\Phi) = \mathscr{R}(\mathbb{P}\Phi^*)$ in $\mathbb{F}\mathscr{V}$ and $\Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi^*) = -\Sigma_{\mathrm{Mo}}(\mathscr{L}, \Phi)$ for every compact invariant set $\mathscr{L} \subset \mathbb{F}\mathscr{V}$ such that $\Phi_{|\mathscr{L}}$ is chain transitive.

Next we will show that the Morse spectrum over a chain transitive set in $\mathbb{F}\mathscr{V}$ is an interval. The proof is based on a "mixing" of exponents near the extremal values of the spectrum.

Theorem 7. *For a linear flow Φ on a vector bundle $\pi: \mathscr{V} \to B$ let $\mathscr{L} \subset \mathbb{F}\mathscr{V}$ be closed and invariant such that $\Phi_{|\mathscr{L}}$ is chain transitive. Define for $k \in \{1,\dots, d\}$*

$$\kappa_k^*(\mathscr{L}) = \inf \Sigma_{\mathrm{Mo}}^k(\mathscr{L}, \Phi), \qquad \kappa_k(\mathscr{L}) = \sup \Sigma_{\mathrm{Mo}}^k(\mathscr{L}, \Phi).$$

Then for every k one has

$$\Sigma_{\mathrm{Mo}}^k(\mathscr{L}, \Phi) = [\kappa_k^*(\mathscr{L}), \kappa_k(\mathscr{L})].$$

Proof. It suffices to show that for all $\lambda \in [\kappa_k^*(\mathscr{L}), \kappa_k(\mathscr{L})]$, all $\delta > 0$, and all $\varepsilon, T > 0$ there is a periodic (ε, T)-chain ζ in $\mathscr{L}$ with

$$|\Lambda_k(\zeta) - \lambda| < \delta. \tag{4.1}$$

Then closedness of the Morse spectrum will yield the result. For fixed $\delta > 0$ and $\varepsilon, T > 0$, there are periodic (ε, T)-chains in $\mathscr{L}$ with

$$\Lambda_k(\zeta^*) < \kappa_k^*(\mathscr{L}) + \delta \quad \text{and} \quad \Lambda_k(\zeta) > \kappa_k(\mathscr{L}) - \delta.$$

Denote the initial points of ζ^* and ζ by F^{0^*} and F^0, respectively. By chain transitivity there are (ε, T)-chains ζ_1 from F^{0^*} to F^0 and ζ_2 from F^0 to F^{0^*}, both in $\mathscr{L}$. For $m \in \mathbb{N}$ let ζ^{*m} and ζ^m be the m- fold concatenation of ζ^*, and of ζ, respectively. Then for $m, n \in \mathbb{N}$ the concatenation $\zeta^{m,n} = \zeta_2 \circ \zeta^m \circ \zeta_1 \circ \zeta^n$ is a periodic (ε, T)-chain in $\mathscr{L}$. Note that the exponents of concatenated chains are convex combinations of the corresponding exponents. Hence for every $\lambda \in [\Lambda_k(\zeta^*), \Lambda_k(\zeta)]$ one finds numbers $m, n \in \mathbb{N}$ such that $|\Lambda_j(\zeta^{m,n}) - \lambda| < \delta$. This proves (4.1). □

Next we will discuss the relation of the Morse spectrum to the growth rates on Grassmann bundles. We already know by Proposition 2 that the chain recurrent components in the Grassmann bundles are the projections of the chain recurrent components in the flag bundle. It is also clear that every kth-interval of the flag spectrum is contained in the corresponding interval for the Grassmannian bundle. We prove the following theorem supplying an ergodicity result for the extremal growth rates.

Theorem 8. *For every chain recurrent component $\mathscr{M} \subset \mathbb{F}\mathscr{V}$ and every $k = 1,\dots, d$ there exist ergodic measures $\mu(\mathscr{M})$ and $\mu^*(\mathscr{M})$ on $\mathscr{M}$ such that*

$$\kappa_k(\mathscr{M}) := \sup \Sigma_{\mathrm{Mo}}^k(\mathscr{M}, \Phi) = \lim_{t \to \pm\infty} \frac{1}{t} \log |\Phi(t, V)|$$

for $\mathbb{G}_k\mu(\mathcal{M})$-almost all $V \in \mathbb{G}_k\mathcal{V}$, and

$$\kappa_k^*(\mathcal{M}) := \inf \Sigma_{\mathrm{Mo}}^k(\mathcal{M}, \Phi) = \lim_{t \to \pm\infty} \frac{1}{t} \log |\Phi(t, V)|$$

for $\mathbb{G}_k\mu^*(\mathcal{M})$-almost all $V \in \mathbb{G}_k\mathcal{V}$. Furthermore, for all $j < k$ one has that

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\Phi(t, V)|$$

exist and are constant for $\mathbb{G}_j\mu^*(\mathcal{M})$- almost all and $\mathbb{G}_j\mu^*(\mathcal{M})$-almost all $V \in \mathbb{G}_j\mathcal{V}$, respectively.

Proof. This is proven using the corresponding result for the one-dimensional spectrum: The linear flow Φ is cohomologous to a subflow of a smooth linear flow Ψ. Then for $j = 1, \ldots, d$, the flows $\Lambda^j\Phi$ are cohomologous to subflows of $\Lambda^j\Psi$ and these are also smooth linear flows. Hence we may assume that all the flows induced by Φ are smooth. Thus for the flow on $\mathbb{F}\mathcal{V}$ the jth exponential growth rates in the chains are actually integrals. Now one finds ergodic invariant measures with support in $\mathcal{M}$ for which the boundary points of the kth spectrum are attained as limits of growth rates. Then the first assertion holds. By ergodicity of these measures all jth growth rates are also limits and constant over the support of $\mathcal{M}$. The ergodic measures $\mu^*(\mathcal{M})$, $\mu(\mathcal{M})$ induce ergodic measures $\mu^*(\mathcal{M}, B)$, $\mu(\mathcal{M}, B)$ on the base space B. The Multiplicative Ergodic Theorem of Oseledets (compare Arnold [1]), implies that the Lyapunov exponents

$$\lambda_1, \ldots, \lambda_d \qquad \text{and} \qquad \lambda_1^*, \ldots, \lambda_d^*$$

corresponding to these measures are constant almost everywhere. Their sums give the growth rates of the volume elements. Hence one has for all k and $\mathbb{G}_k\mu(\mathcal{M})$-almost all $V \in \mathbb{G}_k\mathcal{V}$ a representation

$$\lim_{t \to \pm\infty} \frac{1}{t} \log |\Phi(t, V)| = \lambda_{i_1} + \cdots + \lambda_{i_k};$$

similarly for $\mu^*(\mathcal{M})$. □

The following theorem collects the previous results on the structure of the Morse spectrum.

Theorem 9. *Let Φ be a linear flow on a vector bundle $\pi\colon \mathscr{V} \to B$ with chain transitive base space B and dimension d. Then the Morse spectrum on the complete flag $\mathbb{F}\mathscr{V}$ is given by*

$$\Sigma_{\mathrm{Mo}}(\Phi) = \bigcup \Sigma_{\mathrm{Mo}}(\mathscr{M}_i),$$

where $\mathscr{M}_i \subset \mathbb{F}\mathscr{V}$, $i \in \{1,..., d\}^{d!}$, are the chain recurrent components of the induced flow Φ on $\mathbb{F}\mathscr{V}$. Furthermore, for every i the k-spectrum is an interval,

$$\Sigma^k_{\mathrm{Mo}}(\mathscr{M}_i) = \bigcup\, [\kappa^*_k(\mathscr{M}_i), \kappa(\mathscr{M}_i)]$$

The boundary points $\kappa^*_k(\mathscr{M}_i)$, $\kappa_k(\mathscr{M}_i)$ are sums of regular Lyapunov exponents of Φ.

Next we relate the Morse spectrum on flags to the one-dimensional spectrum on the projective bundle. One will expect that the spectral values of k-dimensional subspaces can be represented as the sum of k one-dimensional spectral values. This is, in particular, provided by the following theorem which relates the spectrum of a chain recurrent component to the Morse set $\mathscr{N}_{k_1,..., k_d}$ as defined in Theorem 6.

Theorem 10. *Let $\mathscr{M}$ be a chain recurrent component in the complete flag $\mathbb{F}\mathscr{V}$ and consider the Morse set*

$$\mathscr{N}^k_{k_1,..., k_l} = \mathbb{G}_{k_1}\mathscr{V}_1 \oplus \cdots \oplus \mathbb{G}_{k_l}\mathscr{V}_l, \qquad (k_1,..., k_l) \in I(k),$$

containing the projection to the Grassmann bundle $\mathbb{G}_k\mathscr{M}$ according to Theorem 6. Then the kth interval of the Morse spectrum of $\mathscr{M}$ satisfies

$$\Sigma^k_{\mathrm{Mo}}(\mathscr{M}) \subset \sum_{l=i_1}^{i_k} \Sigma_{\mathrm{Mo}}(\mathscr{V}_{i_l}).$$

Proof. For an ergodic measure μ on $\mathscr{N}^k_{k_1,..., k_l}$, the projection to B is ergodic and hence the bundle $\mathscr{V}$ can be written as the direct sum of measurable subbundles $V_j(\mu)$ consisting of points where the Lyapunov exponents $\lambda_j(\mu)$ are attained as limits; furthermore for every k-dimensional subspace V the volume growth rate is the sum of k Lyapunov exponents: On the other hand, every Oseledets bundle $V_j(\mu)$ is contained in a bundle $\mathscr{V}_i$; see [4, Corollary 5.5.17]. Hence the (Oseledets) Lyapunov exponents are elements of the Morse spectrum of a corresponding bundle $\mathscr{V}_i$. For the Lyapunov exponents in $\mathbb{G}_k\mathscr{M}$, these bundles must be the $\mathscr{V}_{i_l}$, $i_l = i_1,..., i_k$, and they occur with the multiplicity of the Lyapunov exponents. Applying this to the ergodic measures μ and μ^* where the supremum and infimum, respectively, of $\Sigma^k_{\mathrm{Mo}}(\mathcal{M})$, are attained, yields the assertion. □

Another consequence of the ergodic presentation result in Theorem 8 is the following result on the spectrum over chain recurrent components in Grassmann bundles. It shows that the Morse spectrum of chain recurrent components in Grassmann bundles is well defined.

Corollary 2. *Let $\mathcal{M}_i$ and $\mathcal{M}_j$ be chain recurrent components in the flag bundle $\mathbb{F}\mathcal{V}$ such that their projections to the Grassmann bundle $\mathbb{G}_k\mathcal{V}$ coincide. Then*

$$[\kappa_k^*(\mathcal{M}_i), \kappa_k(\mathcal{M}_i)] = [\kappa_k^*(\mathcal{M}_j), \kappa_k(\mathcal{M}_j)].$$

Proof. By Theorem 8, $\kappa_k(\mathcal{M}_i)$ is attained in an ergodic invariant measure μ with Lyapunov exponents $\lambda_1(\mu),\dots, \lambda_d(\mu)$ and corresponding Oseledets subspaces $V_1,\dots, V_l$, $l \leqslant d$. Then for b in the support of the measure μ projected to B consider an element in the corresponding fiber of $\mathcal{M}_j$. Since $\mathbb{G}_k\mathcal{M}_j = \mathbb{G}_k\mathcal{M}_i$, the corresponding growth rates must coincide yielding $\kappa_k(\mathcal{M}_i) \leqslant \kappa_k(\mathcal{M}_j)$. Exchanging i and j and applying analogous arguments to the lower bounds of the spectral intervals one concludes the proof. □

Remark 8. In the same vein one sees that the Morse spectrum defined over subflag bundles coincides with the restriction of the Morse spectrum on the complete flag bundle.

Finally, we mention the relation of the Morse spectrum to singular values. Recall that for a linear map A on a Hilbert space H the singular values σ_k are given by the eigenvalues σ_k^2 of A^*A ordered such that $\sigma_1 \geqslant \cdots \geqslant \sigma_d$ and the singular value function is

$$\omega_k(A) = \sigma_1(A) \cdots \sigma_k(A).$$

Then (Temam [13, Chap. V, Proposition 1.4]) ω_k is the norm of the operator $\Lambda^k A$ induced by A on the exterior product $\Lambda^k H$ and

$$\omega_k(A) = \|\Lambda^k A\| = \sup\{\|Ax_1 \wedge \cdots \wedge Ax_k\|, \|x_1 \wedge \cdots \wedge x_k\| = 1\}.$$

Writing the cocycle maps on the fibers as

$$\Phi(t, b) := \Phi_t \mid \mathcal{V}_b : \mathcal{V}_b \to \mathcal{V}_{b \cdot t}.$$

we define the supremal uniform growth rate of the kth singular value function as

$$\Omega_k = \limsup_{t\to\infty} \frac{1}{t} \sup_{b\in B} \log \omega_k(\Phi(t, b)).$$

The relation to the Morse spectrum is described in the following corollary.

Corollary 3. For a linear flow Φ on a vector bundle $\pi\colon \mathscr{V} \to B$ the supremal growth rate in the Grassmann bundle $\mathbb{G}_k\mathscr{V}$ and the supremal uniform growth rate of the kth singular value function coincide, i.e.,

$$\sup \Sigma^k_{\mathrm{Mo}}(\mathscr{V}, \Phi) := \sup_{V\in\mathbb{G}_k\mathscr{V}} \limsup_{t\to\infty} \frac{1}{t} \log |\Phi_t V| = \Omega_k.$$

Proof. This follows, as in the one-dimensional case (see, e.g., [4; Proposition 5.4.15 and Lemma 5.2.7]), from Fenichel's Uniformity Lemma, now applied in the exterior product bundle. □

We conclude this paper with a simple example illustrating the chain recurrent components in the flag bundle and the Morse spectrum.

Example 1. Consider the autonomous differential equation

$$\dot{x} = Ax$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

In projective space $\mathbb{P}^3 = \mathbb{G}_1$ one obtains three (linearly ordered) chain recurrent components. In $\mathbb{G}_2$ there are the four chain recurrent components (here e_i denote the canonical base vectors of $\mathbb{R}^4$)

$$\mathscr{M}_{1,2} = \{\mathrm{span}\{e_1, x\}, x \in \mathrm{span}\{e_2, e_3\}\};$$
$$\mathscr{M}_{1,4} = \mathrm{span}\{e_1, e_4\} \qquad \text{(equilibrium)};$$
$$\mathscr{M}_{2,3} = \mathrm{span}\{e_2, e_3\} \qquad \text{(equilibrium)};$$
$$\mathscr{M}_{2,4} = \{\mathrm{span}\{x, e_4\}, x \in \mathrm{span}\{e_2, e_3\}\}.$$

In the order defined in (2.1) $\mathcal{M}_{2,3}$ and $\mathcal{M}_{1,4}$ are not comparable, and

$$\mathcal{M}_{2,4} \preccurlyeq \mathcal{M}_{2,3} \preccurlyeq \mathcal{M}_{1,2} \qquad \text{and} \qquad \mathcal{M}_{2,4} \preccurlyeq \mathcal{M}_{1,4} \preccurlyeq \mathcal{M}_{1,2}.$$

For the matrix

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

one also obtains three chain recurrent components in $\mathbb{G}_1$ and the four chain recurrent components in $\mathbb{G}_2$ given by

$$\begin{aligned} \mathcal{M}'_{1,2} &= \operatorname{span}\{e_1, e_2\} \qquad \text{(equilibrium)}; \\ \mathcal{M}'_{1,3} &= \{\operatorname{span}\{e_1, x\}, x \in \operatorname{span}\{e_3, e_4\}\}; \\ \mathcal{M}'_{2,3} &= \{\operatorname{span}\{e_2, x\}, x \in \operatorname{span}\{e_3, e_4\}\}; \\ \mathcal{M}'_{3,4} &= \operatorname{span}\{e_3, e_4\} \qquad \text{(equilibrium)}, \end{aligned}$$

and

$$\mathcal{M}'_{3,4} \preccurlyeq \mathcal{M}'_{2,3} \preccurlyeq \mathcal{M}'_{1,3} \preccurlyeq \mathcal{M}'_{1,2}.$$

One see that these two equations, which are topologically (but not C^1-) conjugate, can be distinguished topologically, if their extensions to the Grassmann manifold are considered. This example can modified to a linear flow with nontrivial base flow. For matrices $A_1,\dots, A_m \in \mathbb{R}^{d\times d}$, a set $U \subset \mathbb{R}^m$ with $0 \in U$ and a parameter $\rho > 0$ consider a bilinear control system given by

$$\dot{x} = Ax + \rho \sum_{i=1}^{m} u_i(t)\, A_i x, \tag{4.2}$$

with $u = (u_i) \in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}$. If $U \subset \mathbb{R}^m$ is compact and convex, the map

$$\Phi\colon \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathbb{R}^d, \qquad \Phi(t, u, x_0) = (u(t+\cdot), x(t, u, x_0))$$

is a linear flow (the control flow) over the compact metrizable and chain transitive base space $\mathcal{U}$ endowed with the weak* topology; here the flow on $\mathcal{U}$ is the shift, and $x(t, x_0, u)$ denotes the solution of (4.2) with initial value

x_0 at $t = 0$ and control function u. The chain recurrent set and the Morse spectrum depend upper semicontinuously on the parameter $\rho \geqslant 0$. Hence the chain recurrent components and the eigenvalues for $\dot{x} = Ax$ blow up to chain recurrent components and Morse spectral intervals, respectively, for the corresponding control flow over the base space $\mathcal{U}$; for small $\rho > 0$ the structures in the chain recurrent components and the Morse spectrum are retained. For larger ρ-values, some of the chain recurrent components may merge yielding less Morse spectral intervals.

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