Attractors, Input-to-State Stability, and Control Sets

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Abstract

The relations between attractors, input-to-state-stability, and controllability properties are discussed. In particular it is shown that loss of the attractor property under perturbations is connected with a qualitative change in the controllability properties due to a merger with a control set.

1 Introduction

The fundamental notion of input-to-state-stability relates the admissible inputs and initial values to the amplitudes of the corresponding trajectories, cp. Sontag [5], Grüne [3]. For a smooth control system

$$\dot{x}(t) = f(x(t), u(t))$$

with inputs u taking values in $U \subset \mathbb{R}^m$, we define Input-to-State-Stability in the following way.

Definition 1.1. *A positively invariant compact set A is called Input-to-State-Stable (ISS) on a neighborhood B of A with attraction rate β of class $\mathcal{KL}$ and robustness gain γ of class $\mathcal{K}_\infty$ if the following inequality holds for every $x \in B$ and $t \geq 0$:*

$$\|\varphi(t, x, u)\|_A \leq \max\{\beta(\|x\|_A, t), \gamma(\|u\|_\infty)\}.$$

Here $\|\cdot\|_A$ denotes the distance to the set A. Since $\gamma(0) = 0$, it follows that A is attracting for the unperturbed system with $u = 0$; and near A there are attracting sets for small perturbation ranges. The ingredients of this definition are attracting sets and their behavior under perturbations with varying maximal amplitudes. In order to get more insight into the behavior of systems under perturbations and the ISS-property, we will start with an attractor for the unperturbed system; then we discuss the behavior under perturbations with $\|u\|_\infty \leq \rho$, for varying $\rho > 0$. In particular we will relate the loss of the attractivity property to a change in the controllability behavior; see Corollary 3.1.

There are a huge variety of different notions for attractors in control theory and dynamical systems theory. We will use notions going back to the work of C. Conley that are well established in dynamical systems theory. They will also be used for the perturbed system via the notion of control flows.

We consider the following class of systems

$$\dot{x}(t) = f(x(t), u(t)),\ u \in \mathcal{U}, \tag{1.1}$$

where f is C^∞ and $\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}$. We assume that unique global solutions $\varphi(t, x, u)$ exist for $t \in \mathbb{R}$. We also assume that the vector space V spanned by these smooth vector fields

$$V = \operatorname{span}\{f(\cdot, u),\ u \in U\}$$

is finite dimensional and that

$$F = \{f(\cdot, u),\ u \in U\}$$

is compact and convex. Let

$$\mathcal{F} = \{v \in L_\infty(\mathbb{R}, V),\ v(t) \in F \text{ for } t \in \mathbb{R}\}.$$

System (1.1) defines a continuous flow on $\mathcal{F} \times \mathbb{R}^d$ (with weak*-topology on $\mathcal{F}$)

$$\Phi_t(v, x) = (v(t + \cdot), \varphi(t, x, v)),\ t \in \mathbb{R}.$$

We call this the associated non-parametric control flow. It is closely related to control flows as considered in [1] with the shift on the space $\mathcal{U}$ of control functions; hence the time dependent vector fields are parametrized by the control functions and it has to be assumed that the system is control-affine and the control range U is compact and convex. Nonparametric control flows inherit all properties of control flows, mainly due to the fact that the shift on $\mathcal{F}$ is chain transitive. Details will appear elsewhere.

For simplicity we suppose that everything is contained in the interior of a compact invariant set $K \subset \mathbb{R}^d$. Thus we consider the control flow on the compact metric space $\mathcal{F} \times K$.

Remark 1.1. *This apparently very restrictive assumption can often be achieved if the involved vector fields are smoothly changed outside a large ball. Then one has to add in an appropriate way invariant sets "near infinity". The technical details are somewhat involved and hence will not be presented here (some constructions in this direction are included in [2]).*

2 Attractors and Chain Control Sets

First we discuss the behavior on the level of chain control sets or, equivalently, of chain transitivity for the control flow. In the next section we combine this with control sets.

A reference for the facts stated here is [1], Appendix B; see Robinson [4] for the relation to gradient-like systems and Lyapunov functions.

Definition 2.1. *For a flow Φ on a compact metric space X a compact invariant set A is an attractor if it admits a neighborhood N such that $A = \omega(N) = \{y \in X,$ there are $t_k \to \infty$ and $x_k \in N$ with $\Phi(t_k, y_k) \to y\}$.*

We also allow the empty set as an attractor. A neighborhood N as in Definition 2.1 is called an *attractor neighborhood.* Every attractor is compact and invariant, and a repeller is an attractor for the time reversed flow (with limit sets denoted by $\omega^*(N)$). Every attractor comes with an associated complementary repeller $A^* = \{x \in X,\ \omega(x) \cap A\} = \varnothing$. Then for every $x \notin A \cup A^*$

$$\omega^*(x) \subset A^* \text{ and } \omega(x) \subset A.$$

For all considered flows we assume that there are only finitely many connected components of the chain recurrent set $\mathcal{R}$, which consists of all $x \in X$ such that for all $\varepsilon,\ T > 0$ there is an (ε, T)-chain from x to x consisting of $x_0 = x,\ x_1, ..., x_n = x$ and $T_i > T$ with $\mathrm{d}(\Phi(T_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, ..., n-1$. The connected components of the chain recurrent set are chain transitive, i.e., consist of points which can be connected by (ε, T)-chains for all $\varepsilon,\ T > 0$. Observe that an attractor A consists of chain recurrent components $\mathcal{M}_{i_1}, ..., \mathcal{M}_{i_k}$ together with the connecting trajectories. To be more precise define

$$[\mathcal{M}_j, \mathcal{M}_k] = \{x \in X,\ \omega^*(x) \subset \mathcal{M}_j \text{ and } \omega(x) \subset \mathcal{M}_k\}.$$

Then

$$A = \bigcup\nolimits_j \mathcal{M}_{i_j} \cup \bigcup\nolimits_{j,\ k} \left[\mathcal{M}_{i_j}, \mathcal{M}_{i_k}\right].$$

A Morse decomposition consists of finitely many subsets $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ of X (called Morse sets) such that there is a strictly increasing sequence of attractors

$$\emptyset = A_0 \subset A_1 \subset A_2 \subset ... \subset A_n = X,$$

with

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*,\ 0 \le i \le n-1.$$

A finest Morse decomposition exists if and only if the chain recurrent set has finitely many connected components; its Morse sets are these connected components.

We denote the connected components of the chain recurrent set of the unperturbed system $\dot{x} = f(x, 0)$ by $E_1^0, ...E_n^0$, and consider a compact set $A^0 \subset \mathrm{int}K$ which is an attractor. We number the E_i^0 such that

$$A^0 = \bigcup\nolimits_{i=1}^l E_i^0 \cup \bigcup\nolimits_{j,\ k=1}^l \left[E_j^0, E_k^0\right]. \tag{2.2}$$

In order to allow for different maximal amplitudes of the inputs, we consider the ranges $U^\rho = \rho \cdot U,\ \rho \ge 0$. It is easily seen that the corresponding trajectories coincide with the trajectories $\varphi^\rho(t, x, u)$ of

$$\dot{x}(t) = f^\rho(x(t), u(t)) = f(x(t), \rho u(t)),\ u \in \mathcal{U}.$$

The maximal chain transitive sets E_i^0 of the unperturbed system are contained in chain control sets E_i^ρ of the ρ-system for every $\rho > 0$. These are the maximal controlled invariant subsets of $\mathbb{R}^d$: for every two elements $x,\ y$ and all $\varepsilon,\ T > 0$ there are $x_0 = x,\ x_1, ..., x_n = y,\ u_0, ..., u_n \in \mathcal{U}$ and $T_0, ..., T_{n-1} > T$ with $\mathrm{d}(\varphi^\rho(T_i, x_i, u_i), x_{i+1}) < \varepsilon$. Their lifts

$$\mathcal{E}_i^\rho = \left\{ \begin{array}{c} (f(\cdot, u), x) \in \mathcal{F} \times \mathbb{R}^d, \\ u \in \mathcal{U} \text{ and} \\ \varphi^\rho(t, x, u) \in E_i^\rho \text{ for } t \in \mathbb{R} \end{array} \right\} \tag{2.3}$$

are the maximal chain transitive sets of the corresponding control flows Φ^ρ. Here, for convenience, $f(\cdot, u)$ denotes the vector field-valued function

$$t \mapsto f(\cdot, u(t)) \in \mathcal{F} \subset L_\infty(\mathbb{R}, V).$$

Every chain transitive set for *small* positive $\rho > 0$ is of this form with a unique $E_i^0,\ i = 1, ...n$. Sadly, for larger ρ-values, there may exist further maximal chain transitive sets $\mathcal{E}^\rho$ containing no chain transitive set of the unperturbed system. An easy example is obtained by looking at systems where for some $\rho_0 > 0$ a saddle node bifurcation occurs in $\dot{x} = f(x, \rho)$. A more intricate example is [1], Example 4.7.8. We will ignore this fact here, mainly, because we cannot contribute much to its analysis. Instead we concentrate on the maximal chain transitive sets $\mathcal{E}_i^\rho,\ i = 1, ..., n$. Observe that for larger ρ-values they may intersect and hence coincide and change attraction properties; it is this process that we will analyze. Upper semicontinuity of chain transitive components on parameters immediately yields the following first result.

Proposition 2.1. *For all $\rho > 0$ and all $i = 1, ..., n$, there are maximal chain transitive sets $\mathcal{E}_i^\rho$ depending upper semicontinuously on ρ.*

Next we state the situation for small $\rho > 0$.

Proposition 2.2. *Assume that for every $\rho > 0$ every maximal chain transitive set contains a chain transitive set E_i^0 of the unperturbed system. Then there is $\rho_0 > 0$ such that for all ρ with $\rho_0 > \rho \geq 0$ there is an attractor A^ρ of the ρ-system of the form*

$$A^\rho = \bigcup\nolimits_{j=1}^l \mathcal{E}_j^\rho \cup \bigcup\nolimits_{j,k=1}^l \left[\mathcal{E}_j^\rho, \mathcal{E}_k^\rho\right],$$

It depends upper semicontinuously on ρ and all $\mathcal{E}_j^\rho$ are different.

Proof. Every attractor for ρ is a union of chain transitive components and the corresponding intervals. Since chain transitive components depend upper semicontinuously on parameters, this also follows for the intervals. Furthermore note that $\mathcal{E}_j^\rho \subset \mathcal{E}_j^{\rho'}$ for $\rho' > \rho \geq 0$. Hence the chain transitive components $\mathcal{E}_j^\rho$ contained in A^ρ must satisfy $l \in \{1, ..., l\}$ and for ρ small enough, they are different. □

Looking at the notion of Input-to-State Stability, we observe that in this context only those attractors are of interest which are input-global in the following sense: The attraction property should hold for *arbitrary* inputs $u \in \mathcal{U}^\rho$. This is not part of the definition of attractors. Luckily, this is automatically satisfied for control flows as shown by the following proposition.

Proposition 2.3. *Consider an attractor A for the nonparametric control flow Φ associated to system (1.1). Then it has an attractor neighborhood of the form $\mathcal{F} \times B$ with $B \subset \mathbb{R}^d$, i.e.,*

$$A = \omega(\mathcal{F} \times B).$$

Proof. If A^* is the complementary repeller for A, then the distance between the projections $\pi_2 A$ and $\pi_2 A^*$ of A and A^*, respectively, to $\mathbb{R}^d$ is greater than some positive number δ. In fact: Otherwise, there are $x \in \pi_2 A \cap \pi_2 A^*$ and $u,\ u^* \in \mathcal{U}$ with $(u, x) \in A$ and $(u^*, x) \in A^*$. Define $w \in \mathcal{U}$ by

$$w(t) = \begin{cases} u^*(t) & \text{for} \quad t \leq 0 \\ u(t) & \text{for} \quad t > 0 \end{cases}.$$

Then $(w, x) \in A$, since invariance of A and A^* imply $\omega(w, x) = \omega(u, x) \subset A$ and $\omega^*(w, x) = \omega^*(u^*, x) \subset A^*$. This contradicts $A \cap A^* = \varnothing$.

Now let $v \in \mathcal{F}$ be arbitrary and take x in the $\delta/2$-neighborhood of $\pi_2 A$. If $\omega(v, x)$ is not contained in A, then (v, x) is in the complementary repeller A^*. But then the distance between $\pi_2 A^*$ and $\pi_2 A$ must be smaller than $\delta/2$. This contradiction shows that (v, x) is in an attractor neighborhood of A. □

The discussion up to now completely describes the attractor properties for small positive ρ. Here the attractors and the chain control sets reflect the properties of the unperturbed system. For larger ρ-values this need not be the case, because chain control sets contained in the attractor A^ρ may merge with other chain control sets. We cannot describe the changes in the attractor properties on the level of chain control sets. Instead we have to go to control sets, which is possible under additional assumptions.

3 Loss of Attractivity and Control Sets

The purpose of this section is to describe loss of the input-global attraction property. In general, attractors are sets of the form

$$A^\rho = \bigcup\nolimits_{j=1}^{l} \mathcal{E}_j^\rho \cup \bigcup\nolimits_{j,\ k=1}^{l} \left[\mathcal{E}_j^\rho, \mathcal{E}_k^\rho\right]. \tag{3.4}$$

We assume that for all ρ with $0 < \rho < \rho_1$ the set A^ρ is an attractor. Recall that a control set D is a maximal controlled invariant set such that

$$D \subset \mathrm{cl}\mathcal{O}^+(x) \text{ for all } x \in D. \tag{3.5}$$

Here $\mathcal{O}^+(x)$ denotes the reachable set from x. A control set C is an invariant control set if equality holds in (3.5). Throughout we assume that local accessibility holds, i.e., that the small time reachable sets in forward and backward time $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$, respectively, have nonvoid interiors for all x and all $T > 0$. Recall that there are only finitely many, say $1 \leq m < n$, invariant control sets in K, and that control sets depend lower semicontinuously on parameters. If a chain control set is the closure of a control set, then it depends continuously on ρ. This equality holds for all up to at most countably many ρ-values under an inner pair condition guaranteeing that the reachable sets for $\rho \geq 0$ are contained in the reachable sets for ρ' with $\rho' > \rho$; see [1], Theorem 4.7.5..

Remark 3.1. *The inner-pair condition may appear unduly strong. However it is easily verified for small $\rho > 0$ if the unperturbed system has a controllable linearization (more information is given in [1], Chapter 4.) For general $\rho > 0$ the inner pair condition holds, e.g., for coupled oscillators if the number of perturbations is equal to the degrees of freedom; for this result and more general conditions see the forthcoming Ph.D. Thesis of Tobias Gayer).*

We assume that for all ρ with $\rho_1 > \rho > 0$ the chain control sets E_i^ρ are the closures of control sets with nonvoid interior. Then some of these chain control sets in the attractor must be invariant. Thus we can write

$$E_i^\rho = \mathrm{cl} C_i^\rho \text{ for } i = 1, ..., l_1$$
$$E_i^\rho = \mathrm{cl} D_i^\rho \text{ for } i = l_1, ..., l,$$

for invariant control sets C_i^ρ and variant control sets D_i^ρ. It follows that

$$\mathcal{E}_i^\rho = \mathrm{cl}\mathcal{C}_i^\rho \text{ for } i = 1, ..., l_1,$$
$$\mathcal{E}_i^\rho = \mathrm{cl}\mathcal{D}_i^\rho \text{ for } i = l_1, ..., l,$$

and hence for $\rho < \rho_1$ the attractors are

$$A^\rho = \bigcup_{i=1}^{l_1} \mathcal{C}_i^\rho \cup \bigcup_{i=l_1}^{l} \mathcal{D}_i^\rho \cup \bigcup_{i=1}^{l_1} \bigcup_{j=l_1}^{l'} \left[\mathcal{D}_j^\rho, \mathcal{C}_i^\rho\right]. \tag{3.6}$$

We analyze the case where for $\rho = \rho_1$ the set A^{ρ_1} has lost the attractor property. The following example illustrates some of the issues involved.

Example 3.1. *Consider a locally accessible system contained in a compact set $K \subset \mathbb{R}^d$ with five control sets*

$$C_1,\ C_2,\ D_1,\ D_2,\ D_3$$

where C_1 and C_2 are invariant control sets, D_1 is open, and

$$D_2 \subset \mathcal{O}^+(D_1), C_1 \subset \mathcal{O}^+(D_2),$$
$$D_3 \subset \mathcal{O}^+(D_2),\ C_2 \subset \mathcal{O}^+(D_3).$$

and it is not possible to steer the system from $\mathrm{cl}D_3$ *to* C_1. *Assume furthermore that the closures of these control sets are the chain control sets. Then there is the following increasing sequence of attractors:*

$$A_0 = \varnothing,\ \ A_1 = \mathcal{C}_1,\ \ A_2 = \mathcal{C}_1 \cup \mathcal{C}_2,$$
$$A_3 = A_2 \cup \mathcal{D}_3 \cup [\mathcal{D}_3, \mathcal{C}_2],$$
$$A_4 = A_3 \cup [\mathcal{D}_2, \mathcal{C}_1] \cup [\mathcal{D}_2, \mathcal{C}_2] \cup [\mathcal{D}_2, \mathcal{D}_3],$$
$$A_5 = A_4 \cup \mathcal{D}_1 \cup [\mathcal{D}_1, \mathcal{D}_2] \cup [\mathcal{D}_1, \mathcal{D}_3] \cup [\mathcal{D}_1, \mathcal{C}_1] \cup [\mathcal{D}_1, \mathcal{C}_2] = \mathcal{F} \times K,$$

with corresponding repellers

$$A_0^* = \mathcal{F} \times K,$$
$$A_1^* = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{C}_2 \cup [\mathcal{D}_1, \mathcal{D}_2] \cup [\mathcal{D}_1, \mathcal{D}_3] \cup [\mathcal{D}_1, \mathcal{C}_2] \cup [\mathcal{D}_2, \mathcal{D}_3] \cup [\mathcal{D}_2, \mathcal{C}_2] \cup [\mathcal{D}_3, \mathcal{C}_2],$$
$$A_2^* = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup [\mathcal{D}_1, \mathcal{D}_2] \cup [\mathcal{D}_1, \mathcal{D}_3] \cup [\mathcal{D}_2, \mathcal{D}_3],$$
$$A_3^* = \mathcal{D}_1 \cup \mathcal{D}_2 \cup [\mathcal{D}_1, \mathcal{D}_2],$$
$$A_4^* = \mathcal{D}_1,$$
$$A_5^* = \varnothing.$$

This sequence yields the finest Morse decomposition

$$\mathcal{M}_{5-i} = A_{i+1} \cap A_i^*,\ \ i = 0,\ 1, ..., 4,$$

which consists of the lifted (chain) control sets:

$$\mathcal{M}_5 = A_1 \cap A_0^* = \mathcal{C}_1,$$
$$\mathcal{M}_4 = A_2 \cap A_1^* = \mathcal{C}_2,$$
$$\mathcal{M}_3 = A_3 \cap A_2^* = \mathcal{D}_3,$$
$$\mathcal{M}_2 = A_4 \cap A_3^* = \mathcal{D}_2,$$
$$\mathcal{M}_1 = A_5 \cap A_4^* = \mathcal{D}_1.$$

Observe that one can obtain this finest Morse decomposition also by other increasing attractor sequences, e.g.,

$$A_0 = \varnothing,\ \ A_1 = \mathcal{C}_2,$$
$$A_2 = \mathcal{C}_2 \cup \mathcal{D}_3 \cup [\mathcal{D}_3, \mathcal{C}_2],$$
$$A_3 = A_2 \cup \mathcal{C}_1,$$
$$A_4 = A_3 \cup [\mathcal{D}_2, \mathcal{C}_1] \cup [\mathcal{D}_2, \mathcal{C}_2] \cup [\mathcal{D}_2, \mathcal{D}_3],$$
$$A_5 = A_4 \cup \mathcal{D}_1 \cup [\mathcal{D}_1, \mathcal{D}_2] \cup [\mathcal{D}_1, \mathcal{D}_3] \cup [\mathcal{D}_1, \mathcal{C}_1] \cup [\mathcal{D}_1, \mathcal{C}_2] = \mathcal{U} \times K.$$

Now consider what may happen when the input range is increased with ρ*: The corresponding (chain) control sets increase. If* C_2 *intersects the closure of* D_3*, there is no essential change in the attractor structure: Just the attractors* $\mathcal{C}_1$ *and* $\mathcal{C}_1 \cup \mathcal{C}_2$ *merge into one attractor containing these two lifted control sets. Things are different, if* C_1 *and the closure of* D_2 *intersect. Then the attractor* $\mathcal{C}_1$ *has vanished. It is only recovered as part of an attractor containing also* $\mathcal{C}_2$*.*

The situation alluded to in the previous example is characterized by the facts that the closure of the control set is strictly contained in the chain control set; and arbitrarily close to C_1 there are points which can be steered into another invariant control set. Then one finds the control set D_2 which then merges with C_1. Thus the loss of the attraction property of the attractor $\mathcal{C}_1$ is accompanied by this phenomenon. We formalize this intuition in the following way.

Definition 3.1. *For a set $I \subset \mathbb{R}^d$, the domain of attraction is*

$$\mathbf{A}(I) = \left\{ x \in \mathbb{R}^d,\ \mathrm{cl}\mathcal{O}^+(x) \cap \mathrm{int} I \neq \varnothing \right\},$$

and the invariant domain of attraction is

$$\mathbf{A}^{inv}(I) = \left\{ x \in \mathbb{R}^d,\ \begin{array}{c} \textit{if } C \subset \mathrm{cl}\mathcal{O}^+(x) \textit{ is} \\ \textit{an invariant control} \\ \textit{set, then } C \subset I \end{array} \right\}. \tag{3.7}$$

Recall that for every $x \in K$ there is an invariant control set in $\mathrm{cl}\mathcal{O}^+(x)$ and $\mathrm{cl}\mathcal{O}^+(x) = \mathrm{cl}\ \mathrm{int}\mathcal{O}^+(x)$. Furthermore observe that for an invariant control set C the invariant domain of attraction satisfies

$$\mathbf{A}^{inv}(C) = \left\{ x \in \mathbb{R}^d,\ \begin{array}{c} \text{if } \mathrm{cl}\mathcal{O}^+(x) \cap C' \neq \varnothing \text{ for} \\ \text{an invariant control set } C', \\ \text{then } C = C' \end{array} \right\}.$$

We will analyze the case where $I = \pi_2 \mathcal{I}$ is the projection to $\mathbb{R}^d$ of an invariant set $\mathcal{I} \subset \mathcal{F} \times \mathbb{R}^d$ of the form

$$\mathcal{I} = \bigcup\nolimits_{i=1}^{l_1} \mathcal{C}_i \cup \bigcup\nolimits_{i=l_1}^{l} \mathcal{D}_i \cup \bigcup\nolimits_{i=1}^{l_1} \bigcup\nolimits_{j=l_1+1}^{l} [\mathcal{D}_j, \mathcal{C}_i].$$

with invariant control sets C_i and variant control sets D_j. We suppose that $\mathcal{I}$ is not an attractor due to the fact that the projection I intersects the boundary of its invariant domain of attraction,

$$I \cap \partial \mathbf{A}^{inv}(I) \neq \varnothing.$$

Thus arbitrarily close to I one finds points x_n such that for some control $u_n \in \mathcal{U}$ one can steer the system away from I into an invariant control set (and then stay there). Hence it is clear that in this case $\mathcal{I}$ is not an attractor. If for increasing input range the attractors are strictly increasing, it will also follow that they must contain other invariant control sets. We will show that this occurs if the attractor $\mathcal{I}$ merges with some variant control set as observed in the example above.

While the invariant domain of attraction need not be open, this is true for the domain of attraction.

Lemma 3.1. *The domain of attraction $\mathbf{A}(I)$ is open.*

Proof. Consider $x \in \mathbf{A}(I)$. Then there are $u \in \mathcal{U}$ and $T > 0$ such that $\varphi(T, x, u) \in \mathrm{int} I$. This remains true for all initial points in a neighborhood of x. $\square$

The analysis below will be based on constructing control sets in the open set

$$L = \bigcup_{i=l+1}^{m} [\mathbf{A}(I) \cap \mathbf{A}(C_i)] . \tag{3.8}$$

Observe that the union is taken over all invariant control set C_i with $C_i \cap I = \varnothing$. A control set $D \subset L$ is called an L-invariant control set if $x \in D$ and $\varphi(t, x, u) \notin D$ for some $t > 0$ and $u \in \mathcal{U}$ implies $\varphi(t, x, u) \notin L$. The following technical lemma is needed.

Lemma 3.2. *For every $x \in L$ there are $J \subset \{l+1, ..., m\}$ and $y \in \mathcal{O}^+(x)$ such that $y \in \mathbf{A}(I) \cap \bigcap_{j \in J} \mathbf{A}(C_j)$ and J is a minimal index set in the following sense:*
If $\varphi(t, y, u) \in L$ for some $t > 0$ and $u \in \mathcal{U}$, then $\varphi(t, y, u) \in \bigcap_{j \in J} \mathbf{A}(C_j)$.

Proof. Since $x \in L$, there exists $J_1 \subset \{l+1, ..., m\}$ with $x \in \bigcap_{j \in J_1} \mathbf{A}(C_j)$. If there are $t_1 > 0$ and $v_1 \in \mathcal{U}$ with $y_1 := \varphi(t_1, x, v_1) \in L \setminus \bigcap_{j \in J_1} \mathbf{A}(C_j)$, then there exists a proper subset $\emptyset \neq J_2 \subset J_1$ with $y_1 \in \bigcap_{j \in J_2} \mathbf{A}(C_j)$. Proceeding recursively, one ends up, after finitely many steps, at a point $y \in \mathcal{O}^+(x)$ with a minimal index set J. $\square$

Note that a minimal index set has at least one element. Furthermore, the lemma implies that for each L-invariant control set D there is $J \subset \{l+1, ..., m\}$ such that for each $x \in \mathrm{int}\, D$ the index set J is minimal.

Proposition 3.1. *Assume that $\mathcal{I}$ is an invariant set of the form (3) and consider the set L in (3.8). Then there exists at least one and at most finitely many L-invariant control sets D and every point in L can be steered into an L-invariant control set.*

Proof. This is follows from [1], Theorem 3.3.10 (as in [2]). Consider $x \in \mathbf{A}(I) \cap \bigcap_{j \in J} \mathbf{A}(C_j)$, where $J \subset \{l+1, ..., m\}$ is some minimal index set for x. Then one constructs an L-invariant control set $D \subset \mathrm{cl}\mathcal{O}^+(x)$ with

$$\partial D \cap \partial L \cap \mathbf{A}(C_j) \neq \varnothing \text{ for } j \in J.$$

Then the cited theorem implies the assertions. $\square$

We use this result in order to show that loss of the attractor property is connected with the merger with some control set.

Theorem 3.1. *Assume that $\mathcal{I} \subset \mathcal{F} \times K$ is an invariant set of the form (3) such that its projection I to $\mathbb{R}^d$ intersects the boundary of its invariant domain of attraction, $I \cap \partial \mathbf{A}^{inv}(I) \neq \emptyset$. Then there exists a variant control set D with $D \cap I = \varnothing$ such that $\mathrm{cl}\mathbf{A}(D) \cap I \neq \varnothing$.*

Proof. By our assumption there exists a point $x \in I \cap \partial \mathbf{A}^{inv}(I)$. Since I is contained in the interior of its domain of attraction, it follows that there are $x_n \in \mathbf{A}(I) \backslash \mathbf{A}^{inv}(I)$ with $x_n \to x$. Thus there are invariant control sets $C_i,\ i \in \{l+1, ..., m\}$ with $C_i \subset \mathrm{cl}\mathcal{O}^+(x_n)$. Since the number of invariant control sets is finite, we may assume that there is a single invariant control set, say C_{l+1}, with $C_{l+1} \cap I = \varnothing$ and $C_{l+1} \subset \mathrm{cl}\mathcal{O}^+(x_n)$ for all n. Hence x_n is in the set L defined in (3.8). By the preceding proposition we find $u_n \in \mathcal{U}$ and $t_n > 0$ such that $\varphi(t_n, x_n, u_n) \in \mathrm{int} D_n$ for some L-invariant control set D_n. Since the number of L-invariant control sets is finite we may assume that there is a single control set $D \subset L$ with these properties. One can steer the system from every point of D into I and into C_{l+1}. Hence $D \cap I = \varnothing$. □

As a corollary, we obtain a result showing that under the inner-pair condition loss of attractivity is connected with the merger of control sets. It is the main result of this paper. The inner-pair condition guarantees that the domains of attraction are strictly increasing, and thus one can steer from one of the control sets in I to the control set D constructed in the previous theorem; hence this control set is contained in every attractor containing $\mathcal{I}$. Recall that all attractors are of the form (3.4) and that generically they are of the form (3.6).

Corollary 3.1. *Assume the following inner-pair condition at ρ_1: For all $\rho > \rho_1$ and $(u, x) \in \mathcal{U}^{\rho_1} \times K$ there is $T > 0$ with $\varphi^{\rho_1}(T, x, u) \in \mathrm{int}\mathcal{O}^{\rho,+}(x)$. Consider the invariant sets in $\mathcal{F} \times K$*

$$\mathcal{I}^\rho = \bigcup_{i=1}^{l_1} \mathcal{C}_i^\rho \cup \bigcup_{i=l_1}^{l} \mathcal{D}_i^\rho \cup \bigcup_{i=1}^{l_1} \bigcup_{j=l_1}^{l} \left[\mathcal{D}_j^\rho, \mathcal{C}_i^\rho \right],$$

and assume that they are attractors for $\rho < \rho_1$ and that I^{ρ_1} intersects the boundary of its invariant domain of attraction defined in 3.7, i.e.,

$$I^{\rho_1} \cap \partial \mathbf{A}^{inv}(I^{\rho_1}) \neq \varnothing.$$

Then for all $\rho > \rho_1$ every attractor containing $\mathcal{I}^{\rho_1}$ contains a lifted variant control set $\mathcal{D}^{\rho_1}$ with $D^{\rho_1} \cap I^{\rho_1} = \varnothing$.

Proof. By the previous theorem, there is a control set D^{ρ_1} for the ρ_1-system such that $D^{\rho_1} \cap I^{\rho_1} = \varnothing$ and $\mathrm{cl}\mathbf{A}(D^{\rho_1}) \cap I^{\rho_1} \neq \varnothing$. Now consider for $\rho > \rho_1$ an attractor neighborhood of an attractor containing $\mathcal{I}^{\rho_1}$. By Proposition 2.3 we may assume that this neighborhood has the form $\mathcal{F}^\rho \times B$. Hence it contains a pair (u, x) with $\omega(u, x) \subset \mathcal{D}^{\rho_1}$. This implies that $\mathcal{D}^{\rho_1}$ is contained in this attractor. □

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