HYPERBOLIC CONTROL SETS AND CHAIN CONTROL SETS

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ABSTRACT. A shadowing lemma for hyperbolic control flows is proved. A consequence is that hyperbolic chain control sets are closures of control sets.

1. INTRODUCTION

In this paper we discuss controllability problems for control affine systems. Via a shadowing lemma, we show that chain control sets which are hyperbolic (in a sense made precise below) are closures of control sets. This is motivated, in particular, by parameter dependent problems. Here it is known that control sets whose closures are chain control sets depend continuously on parameters. Hence our results show that hyperbolicity guarantees this continuity property.

More precisely, we consider control-affine systems of the form

$$\begin{aligned} \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), \quad t \in \mathbb{R}, \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m,\ (u_i(t)) \in U \text{ for almost all } t \in \mathbb{R}\}; \end{aligned} \tag{1.1}$$

here X_0, $X_1, \ldots, X_m$ are vector fields on a d-dimensional Riemannian manifold N and $U \subset \mathbb{R}^m$. We assume throughout that for all $(u,x) \in \mathcal{U} \times N$ there is a unique solution $\varphi(t,x,u)$, $t \in \mathbb{R}$, of this differential equation with the initial condition $\varphi(0,x,u) = x$ and control u. Furthermore, we assume that the vector fields are sufficiently smooth. If the control range U is a convex and compact set, one can associate with system (1.1) the control flow given by

$$\Phi : \mathbb{R} \times \mathcal{U} \times N \to \mathcal{U} \times N, \quad \Phi_t(u,x) = (u(t+\cdot), \varphi(t,x,u)).$$

1991 *Mathematics Subject Classification.* 93B05, 37D05.

Key words and phrases. Control sets, hyperbolicity, shadowing lemma.

Then (see, e.g., Colonius and Kliemann [2]) Φ becomes a continuous dynamical system if we endow $\mathcal{U}$ with the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^m)$, and $\mathcal{U} \times N$ is a metrizable space with $\mathcal{U}$ compact. We fix a metric on this space.

We will use hyperbolicity assumptions for this skew product flow to show a shadowing lemma. Another version of a shadowing lemma for general discrete time skew product flows has been proved by Meyer and Sell [6]. Their version, however, does not allow for jumps in the base space. The version presented here closes jumps in the smooth component, naturally, without eliminating jumps in the base space where no smooth structure is present.

In Sec. 2, we collect basic definitions and cite some relevant material. In Sec. 3, a shadowing lemma is proved and used to show that a chain control set with hyperbolic lift (i.e., there is a hyperbolic decomposition of the control flow restricted to the lift of the chain control set) coincides with the closure of a control set.

This paper addresses the regular situation, where local accessibility holds. Grünvogel [5] studied the controllability behavior near a singular point (which remains fixed under every control). He constructed control sets using properties of Lyapunov exponents via stable and unstable manifold theory.

2. Problem formulation and background

In this section we cite some notions and results from nonlinear control theory (cf. [2]).

Throughout we assume that system (1.1) is locally accessible, i.e., for all $x \in N$ and all $T > 0$ one has

$$\operatorname{int} \mathcal{O}^+_{\leq T}(x) \neq \emptyset \quad \text{and} \quad \operatorname{int} \mathcal{O}^-_{\leq T}(x) \neq \emptyset,$$

where $\mathcal{O}^+_{\leq T}(x) = \{y \in N,\ y = \varphi(t, x, u) \text{ with } 0 \leq t \leq T \text{ and } u \in \mathcal{U}\}$ and $\mathcal{O}^-_{\leq T}(x) = \{y \in N,\ x = \varphi(t, y, u) \text{ with } 0 \leq t \leq T \text{ and } u \in \mathcal{U}\}$ are the positive and negative reachable sets (or orbits) from x. This is guaranteed by the accessibility rank condition

$$\dim \Delta_{\mathcal{LA}}(x) = \dim N = d \quad \text{for all} \quad x \in N. \tag{2.1}$$

where $\mathcal{LA} = \mathcal{LA}\{X(\cdot, u),\ u \in U\}$ denotes the Lie algebra generated by the system vector fields and $\Delta_{\mathcal{LA}}(x)$ is for $x \in N$ the subspace of the tangent space $\mathbb{T}_x N$ generated by the vector fields in $\mathcal{LA}$.

To describe the global behavior of system (1.1) we consider the following concepts.

Definition 1. A *control set* D is a subset of N with the following properties: (i) for all $x \in D$ the inclusion $D \subset \operatorname{cl}\mathcal{O}(x)$ holds, (ii) for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$, and (iii) the set D is maximal (with respect to set inclusion) with these properties. A control set C is an invariant control set if $\operatorname{cl} C = \operatorname{cl}\mathcal{O}^+(x)$ for all $x \in C$.

Note that the local accessibility assumption guarantees that the invariant control sets are the closed control sets and that they have nonvoid interior; and in the interior of control sets complete controllability holds. A set with nonvoid interior is a control set if it is a maximal set where approximate controllability holds. We also introduce a variant allowing for (small) jumps between pieces of trajectories.

Definition 2. Fix $x, y \in N$ and let $\varepsilon, T > 0$. A *controlled* (ε, T)*-chain* ς from x to y is given by $n \in \mathbb{N}$, $x_0, \ldots, x_n \in N$, $u_0, \ldots, u_{n-1} \in \mathcal{U}$ and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) \leq \varepsilon \quad \text{for all} \quad j = 0, \ldots, n-1.$$

If for every $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y, then the point x is chain controllable to y.

In analogy with control sets, chain control sets are defined as maximal regions of chain controllability.

Definition 3. A set $E \subset N$ is called a *chain control set* of system (1.1) if (i) for all $x \in E$ there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$, (ii) for all $x, y \in E$ and $\varepsilon,\ T > 0$ there is a controlled (ε, T)-chain from x to y, and (iii) E is maximal with these properties.

Thus chain control sets E have the properties that through every point there is a trajectory that remains in E for all times $t \in \mathbb{R}$ (not just for $t \geq 0$, as we require for control sets) and one can reach each point from any other point by trajectories with arbitrarily small jumps. Chain control sets are closed and every control set is contained in a chain control set. However, there may exist points in a chain control set, which do not belong to a control set. The maximality property guarantees that chain control sets are disjoint. The following result from [2], Theorem 4.3.11 shows that the chain control sets coincide with the projections on the state space of the chain recurrent components of the control flow Φ. Recall that for a continuous flow $\Psi : \mathbb{R} \times X \to X$ on a metric space X and $\varepsilon,\ T > 0$ an (ε, T)-chain from $x \in X$ to $y \in X$ is given by $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\Psi(t_j, x_j), x_{j+1}) \leq \varepsilon \quad \text{for all} \quad j = 0, ..., n-1.$$

A subset $Y \subset X$ is chain transitive, if for all $x,\ y \in Y$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y.

The set $\mathcal{U}$ is compact and metrizable in the weak* topology of $L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}^m))^*$; a metric is given by

$$d(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left| \int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \, dt \right|}{1 + \left| \int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \, dt \right|}, \tag{2.2}$$

where $\{x_n,\ n \in \mathbb{N}\}$ is a countable, dense subset of $L_1(\mathbb{R}, \mathbb{R}^m)$, and $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathbb{R}^m$. With this metric, $\mathcal{U}$ is a compact, complete, separable metric space. On $\mathcal{U} \times N$ we take the metric $d((u,x),(v,y)) = \max\{d(u,v), d(x,y)\}$.

For a chain control set E, we consider its lift to $\mathcal{U} \times N$ defined by

$$\mathcal{E} = \{(u,x) \in \mathcal{U} \times N,\ \varphi(t,x,u) \in E \ \text{ for all } \ t \in \mathbb{R}\}. \tag{2.3}$$

Observe that $\mathcal{E}$ is compact iff E is compact.

Theorem 1. *L et $E \subset N$ be a chain control set. Then $\mathcal{E} \subset \mathcal{U} \times N$ as defined by* (2.3) *is a maximal invariant chain transitive set for the control flow $(\mathcal{U} \times N, \Phi)$. Conversely, let $\mathcal{E} \subset \mathcal{U} \times N$ be a maximal invariant chain transitive set for $(\mathcal{U} \times N, \Phi)$. Then the projection $\pi_N \mathcal{E}$ on N is a chain control set.*

We note that, analogously, the lifts of control sets are the topologically transitive sets of the control flows.

In order to state a result on continuous dependence of control sets, we consider parameter-dependent control-affine systems of the form

$$\dot{x} = X_0(\alpha, x) + \sum_{i=1}^{m} u_i(t) X_i(\alpha, x), \quad u \in \mathcal{U}, \tag{2.4}$$

where $X_0, \ldots, X_m$ are C^∞-vector fields on a Riemannian manifold N depending continuously on their arguments (α, x) and $X_i(\cdot, \alpha)$ and the set $\mathcal{U}$ of admissible controls are as above; the parameters α are chosen in a subset A of $\mathbb{R}^k$. We indicate the parameter dependence by a superscript α. Then the following continuity result holds (see [3]). We assume in this theorem that the state space N of the control system is compact. If N is not compact one has to appropriately restrict the attention to a compact invariant subset.

Theorem 2. *Let D^{α_0} be a control set of $(2.4)^{\alpha_0}$ with $\alpha_0 \in A$, and assume that system $(2.4)^{\alpha_0}$ satisfies the accessibility rank condition* (2.1) *on the closure $\operatorname{cl} D^{\alpha_0}$. Assume that $\operatorname{cl} D^{\alpha_0}$ coincides with the chain control* *set E^{α_0} containing D^{α_0}. Then for all α in a neighborhood of α_0 there are unique control sets D^α depending continuously in the Hausdorff metric on α at $\alpha = \alpha^0$ with $D^\alpha \cap \operatorname{int} D^{\alpha_0} \neq \emptyset$.*

3. Chain control sets and a shadowing lemma

In this section, we prove a shadowing lemma for control flows following the arguments of Franke and Selgrade [4]. Then this is applied to chain control sets.

First we introduce some notions. For the control flow Φ on $\mathcal{U} \times N$ we write, where convenient, $\Phi_t(u,x) = (u,x)\cdot t$. The set $(u,x)\cdot\mathbb{R}$ is called the orbit of (u,x). A reparametrization of an orbit is an orientation-preserving homeomorphism on $\mathbb{R}$ fixing the origin. An (ε, T)-chain given by $n \in \mathbb{N}$, $T_0, \dots, T_{n-1} \geq T$, $(u_0,x_0), \dots, (u_n,x_n) \in \mathcal{U} \times N$, will be written in the form

$$(u_0,x_0)*t = (u_i,x_i)\cdot\left(t - \sum_{j=0}^{i-1} T_j\right) \text{ for } \sum_{j=0}^{i-1} T_j \leq t \leq \sum_{j=0}^{i} T_j,\ i = 0,\dots,n-1;$$

analogously for (ε, T)-chains with infinitely many jumps. An orbit $(u,x)\cdot\mathbb{R}$ is said to δ-shadow an infinite (ε, T)-chain $(u_0,x_0)\cdot\mathbb{R}$ if there exists a reparametrization g such that for the projections on N one has $d(\pi_N((u_0,x_0)*t), \pi_N((v,y)\cdot g(t))) < \delta$ for all $t \in \mathbb{R}$, and analogously on subintervals of $\mathbb{R}$ by restricting g. Observe that we require the shadowing property only for the smooth component.

Let $\mathcal{N}$ be a compact isolating α-neighborhood of the lift $\mathcal{E}$ of a compact chain control set E and $\theta = d(\mathcal{E}, \mathcal{U} \times N \setminus \mathcal{N})$. Define for some $\lambda \in (0,1)$

$$W^+(u,x) = \{y \in N,\ (u,y) \in \mathcal{N} \text{ and } d(\varphi(t,x,u), \varphi(t,y,u)) < \alpha\lambda^t \text{ for all } t > 0\},$$

$$W^-(u,x) = \{y \in N,\ (u,y) \in \mathcal{N} \text{ and } d(\varphi(t,x,u), \varphi(t,y,u)) < \alpha\lambda^{-t} \text{ for all } t > 0\},$$

$$W^+((u,x)\cdot(a,b)) = \bigcup_{t\in(a,b)} W^+((u,x)\cdot t).$$

The following shadowing property holds (see Franke and Selgrade [4], Lemma 3.1).

Lemma 1. *Suppose that system* (1.1) *satisfies the following condition (S):*
There are $\alpha, \beta, a, \lambda > 0$ such that for all $(u,x), (v,y) \in \mathcal{E}$ with $d((u,x),(v,y)) < \beta$ the intersection $W^+((u,x)\cdot(-a,a)) \cap W^-(v,y)$ is a single point.

Then for each $\delta > 0$ *there is* $\varepsilon > 0$ *such that each finite* $(\varepsilon, 1)$*-chain in* $\mathcal{E}$ *is* δ*-shadowed by some* $q \in \mathcal{N}$*. Also, if the* $(\varepsilon, 1)$*-chain is* $(u_0, x_0) * [a, b]$ *with* $0 \in [a, b]$*, then the parametrization* g *can be chosen such that*

$$\frac{3}{4}t + \frac{a}{2} - 1 < g(t) < \frac{5}{4}t - \frac{a}{2} + 1.$$

Proof. Given $\delta > 0$, choose α so that $\alpha < \theta/2$ and so that

$$\alpha + \alpha + \frac{\alpha}{\lambda(1-\lambda)} < \frac{\delta}{3},$$

where λ is the hyperbolic constant for $\mathcal{E}$. Let $0 < \gamma < 1/2$ be so small that $d((u, x) \cdot t, (u, x)) < \min\{\delta/3, \alpha/3\}$ for all $|t| < \gamma$ and $(u, x) \in \mathcal{N}$. By the hypothesis, there are positive numbers γ (possibly smaller than γ above) and $\beta < \theta/2$ such that $W^+((u, x)) \cap W^-((v, y) \cdot (-\gamma, \gamma))$ is a single point for all (u, x), $(v, y) \in \mathcal{N}$ with $d((u, x), (v, y)) < \beta$. Choose an integer $M > 2$ large enough so that $\lambda^M \alpha < \beta/2$.

Choose $K \in \mathbb{N}$ such that $\sum\limits_{n=K}^{\infty} \frac{1}{2^n} < \frac{\beta}{4}$ and choose M large enough such that for all $n = 1, \ldots, K$

$$\left| \int\limits_{\mathbb{R}\setminus[-M,M]} \langle u(t) - v(t), x_n(t) \rangle \, dt \right| < \beta/4.$$

Thus for all u, $v \in \mathcal{U}$ one has

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left| \int\limits_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \, dt \right|}{1 + \left| \int\limits_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \, dt \right|} <$$

$$< \sum_{n=1}^{\infty} \frac{1}{2^n} \int\limits_{-M}^{M} \left| \langle u(t) - v(t), x_n(t) \rangle \right| dt + \beta/2.$$

Choose $\varepsilon < \min(\beta, 1/M)$ small enough such that every $(\varepsilon, 1)$-chain $(u_0, x_0) * [a, b]$ with $b - a \leq M$ and at most $M+1$ jumps satisfies $d((u_0, x_0) \cdot t, (u_0, x_0) * t) < \beta/2$ for all $t \in [0, M]$. This is possible by the uniform continuity of φ.

Take an integer $r > 0$ and consider an $(\varepsilon, 1)$-chain $(u_0, x_0) * [0, rM]$. Note that the number of jumps in any segment of $(u_0, x_0) * [0, rM]$ of time length M is at most $M + 1$. This chain can be chosen in $\mathcal{E}$, since $\mathcal{E}$ is a chain recurrent component (see [2], Theorem B.2.20).

We define inductively a set of r points (u'_k, x'_k) in $\mathcal{E}$ as follows:

$$(u'_0, x'_0) = (u_0, x_0) \in \mathcal{E}$$

and for $k = 0, 1, \dots, r-1$

$$x'_{k+1} = W^-((u'_k, x'_k) \cdot M) \cdot \tau_{k+1}) \cap W^+((u_0, x_0) * (k+1)M),$$

where $|\tau_{k+1}| < \gamma$; the corresponding controls u'_k are defined by the relation

$$u'_{k+1} = \begin{cases} u_0 * (k+1)M & \text{for} \quad t > 0, \\ u'_k(M + \tau_{k+1} + t) & \text{for} \quad t \leq 0. \end{cases}$$

This definition is correct: assume that $(u'_k, x'_k) \in \mathcal{E}$ for $k \geq 0$. The segment of $(u_0, x_0) * [0, rM]$ from $(u_0, x_0) * kM$ to $(u_0, x_0) * (k+1)M$ is an $(\varepsilon, 1)$-chain with at most $M + 1$ jumps, so by our choice of ε,

$$d(((u_0, x_0) * kM) \cdot M, (u_0, x_0) * (k+1)M) < \frac{\beta}{2}.$$

Since $x'_k \in W^+((u_0, x_0) * kM)$, one has

$$d(\varphi(M, x'_k, u_0 * kM), \varphi(M, (u_0, x_0) * kM) \leq \lambda^M \alpha < \frac{\beta}{2}.$$

Together, these two estimates give

$$\begin{aligned} &d(\pi_N((u_0 * kM, x'_k) \cdot M), \pi_N((u_0, x_0) * (k+1)M)) = \\ &= d(\varphi(M, x'_k, u_0 * kM), \pi_N((u_0, x_0) * (k+1)M) < \\ &< \beta. \end{aligned}$$

Observe that by definition the control $u_0 * kM$ coincides with u'_k on the interval $[0, \infty)$. Hence

$$\varphi(M, x'_k, u_0 * kM) = \varphi(M, x'_k, u'_k)$$

and we find

$$d(\varphi(M, x'_k, u'_k), \pi_N((u_0, x_0) * (k+1)M) < \beta.$$

We already know that

$$d((u_0 * kM) \cdot M, u_0 * (k+1)M) < \beta/2.$$

Since the control $(u_0 * kM) \cdot M$ coincides with $u'_k \cdot M$ on the interval $(-M, M)$, it follows that

$$d(u'_k \cdot M, (u_0, x_0) * (k+1)M) < \beta.$$

We conclude that the distance of $(u'_k, x'_k) \cdot M$, $(u_0, x_0) * (k+1)M \in \mathcal{E}$ is less than β. Hence there is a unique point

$$x'_{k+1} = W^-((u'_k, x'_k) \cdot (M + \tau_{k+1})) \cap W^+((u_0, x_0) * (k+1)M))$$

for some τ_{k+1} with $|\tau_{k+1}| < \gamma$. Since $\alpha < \theta/2$ and $x'_{k+1} \in W^+((u_0, x_0) * (k+1)M))$, it follows that $d(\varphi(t, x'_{k+1}, u_0 * (k+1)M), \mathcal{E}) < \alpha$ for all $t \geq 0$.

And since $x'_{k+1} \in W^-((u'_k, x'_k) \cdot (M + \tau_{k+1}))$, it follows that $d(\varphi(t, x'_{k+1}, u'_k \cdot (M + \tau_{k+1})), \mathcal{E}) < \alpha$ for all $t \leq 0$. Thus, $(u'_{k+1}, x'_{k+1}) \in \mathcal{E}$.

The orbit of (u'_r, x'_r) will be our δ-tracing orbit. The occurrence of τ_k causes some technical difficulties. They can be solved as in [4], proof of Lemma 3.1, pp. 30–31. We omit the details. □

As in [4], Lemma 3.4, one proves the following result.

Lemma 2. *Under the assumptions of Lemma* 1, *for each* $\delta > 0$ *there is a number* $\varepsilon > 0$ *such that each infinite* $(\varepsilon, 1)$*-chain in* $\mathcal{E}$ *is* δ*-traced by some* $(u', x') \in \mathcal{E}$.

These lemmas imply the following main result of the paper. For system (1.1) condition (S) implies that every chain control set is the closure of a control set.

Theorem 3. *Assume that system* (1.1) *is locally accessible and consider a chain control set* E *with nonvoid interior. If condition* (S) *holds for the lift* $\mathcal{E}$ *of* E, *then* E *coincides with the closure of a control set* D *with nonvoid interior.*

Proof. By Theorem 1, we know that $\mathcal{E}$ is a compact maximal invariant chain transitive set for the control flow $(\mathcal{U} \times N, \Phi)$ and $\Phi(t, \cdot)|\mathcal{E}$ is chain transitive. Consider $x \in \text{int } E$ and take an arbitrary point $z \in E$. By local accessibility there are $t > 0$ and points

$$x_0 \in \text{int}\,(\mathcal{O}^+_{\leq t}(x) \cap E) \quad \text{and} \quad x_1 \in \text{int}\,(\mathcal{O}^-_{\leq t}(x) \cap E).$$

There are u, u_0, $u_1 \in \mathcal{U}$ such that (u, z), (u_0, x_0), $(u_1, x_1) \in \mathcal{E}$. By Lemma 2, we find that for $\delta_n = 2^{-n}$ there are $\varepsilon_n > 0$ such that every $(\varepsilon_n, 1)$-chain is δ_n-shadowed by an orbit. Now take such $(\varepsilon_n, 1)$-chains from (u_0, x_0) to (u, z) and from (u, z) to (u_1, x_1). Their concatenation is a chain from (u_0, x_0) to (u_1, x_1), passing near (u, z). For $\delta_n \to 0$ one finds a trajectory of the system from $\mathcal{O}^+_{\leq t}(x)$ to $\mathcal{O}^-_{\leq t}(x)$. Hence x and, therefore, the whole trajectory belongs to a control set D, which, by continuous dependence on initial values, has nonvoid interior. Since the trajectories pass arbitrarily close to z, this point belongs to the closure of D. This completes the proof. □

The crucial assumption in the shadowing lemma above (and hence in the equality between chain control sets and control sets) is assumption (S). For a single differential equation it follows from the standard hyperbolicity condition; see [4]. In what follows we show that this hyperbolicity condition for the uncontrolled system also implies assumption (S) for systems with small control range.

More specifically, we consider the control system (1.1) with controls in

$$\mathcal{U}^\rho := \{u : \mathbb{R} \to \mathbb{R}^m, u(t) \in \rho \cdot\ U \text{ for almost all } \ t \in \mathbb{R}\},$$

where $U \subset \mathbb{R}^m$ is convex and compact with $0 \in U$, and $\rho \geq 0$ indicates the size of the control range. Observe that for every $\rho \geq 0$ these systems can be considered as given by the system of equations with fixed control range

$$\dot{x}(t) = X_0(x(t)) + \rho \cdot \sum_{i=1}^{m} u_i(t) X_i(x(t)), \quad t \in \mathbb{R},$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m,\ (u_i(t)) \in U \text{ for almost all } t \in \mathbb{R}\}.$$

Let $\mathcal{M}$ be a compact chain transitive set for the uncontrolled system $\dot{x} = X_0(x)$ and suppose that the following (uniform) hyperbolicity condition is satisfied: the tangent flow $T\Phi_t$ leaves a continuous splitting

$$T_{\mathcal{M}}\mathbb{R}^d = \mathcal{V}^{0,+} \oplus \mathcal{V}^{0,0} \oplus \mathcal{V}^{0,-}$$

invariant, where for some $0 < \lambda < 1$ and some adapted Riemannian metric
(i) if $(x, v) \in \mathcal{V}^{0,-}$ and $t < 0$, then $|T\Phi_t(x, v)| > \lambda^{-t}|v|$;
(ii) if $(x, v) \in \mathcal{V}^{0,+}$ and $t > 0$, then $|T\Phi_t(x, v)| < \lambda^{t}|v|$;
(iii) $\mathcal{V}^{0,0}$ is the span of the vector field X_0.
This implies condition (S) for this flow (see Franke and Selgrade [4], p. 29).

Theorem 4. *Suppose that the uncontrolled system $\dot{x} = X(x)$ is hyperbolic on a compact chain transitive component $\mathcal{M}$. Then for all $\rho > 0$ small enough, condition* (S) *holds on the lift $\mathcal{E}^\rho$ of the chain control set E^ρ of* $(1.1)^\rho$ *containing $\mathcal{M}$. If, in addition, local accessibility holds and E^ρ has nonvoid interior, then $E^\rho = \operatorname{cl} D$ for a control set D with nonvoid interior.*

Proof (sketch). Obviously, there exists an increasing family $\{E^\rho\}_{\rho \geq 0}$ of chain control sets containing the set $\mathcal{M}$. The hyperbolicity condition for the uncontrolled system is equivalent to condition (S) on $\{0\} \times \mathcal{M}$ for the skew product flow, again denoted by Φ_t, induced on the vector bundle

$$\mathbb{T}_{\mathcal{E}^\rho}(\mathcal{U} \times N) = \big\{(u, x, v) \in \mathcal{U} \times \mathbb{T}N,\ (u, x) \in \mathcal{E}^\rho\big\}.$$

Then, for every $\rho \geq 0$ the control flow Φ_t^ρ has a finest Morse decomposition in the projective bundle $\mathbb{P}_{\mathcal{E}^\rho}(\mathcal{U} \times N)$ which yields a decomposition into subbundles

$$\mathbb{T}_{\mathcal{E}^\rho}(\mathcal{U} \times N) = \mathcal{V}_1^\rho \oplus \cdots \oplus \mathcal{V}_k^\rho;$$

the Morse sets in the projective bundles are given by the projections of the subbundles $\mathcal{V}_i^\rho$; the Morse spectrum, and hence the Lyapunov spectrum is determined by these subbundles; and the Morse sets and the Morse spectra depend continuously on ρ for $\rho \to 0$, i.e.,

$$\lim_{\rho \searrow 0} \mathcal{V}_i^\rho = \mathcal{V}_i^0 \text{ and } \lim_{\rho \searrow 0} \Sigma_{Mo}(\mathcal{V}_i^\rho) = \Sigma_{Mo}(\mathcal{V}_i^0)$$

(cf. [2], Sec. 6.2, [1]; for the relation to the dichotomy or Sacker–Sell spectrum see also [2], Sec. 5.5). Hence hyperbolicity of the uncontrolled system implies hyperbolicity of the systems with $\rho > 0$ small enough. For these $\rho > 0$ the bundle $\mathbb{T}_{\mathcal{E}^\rho}(\mathcal{U} \times N)$ has a decomposition into a stable subbundle, an unstable subbundle and a one-dimensional subbundle which in the fibers for (u, x) with $u = 0$ is the subspace spanned by the vector field $X_0(x)$. According to the local invariant manifold theorem ([2], Theorem 6.4.3). the nonlinear system has local stable and unstable subbundles which are Lipschitz close to the linear subbundles. Since $\mathcal{E}^\rho$ is compact and the stable and that unstable manifolds as well as the flow are continuous with respect to (u, x), the transversality condition (S) holds with a uniform time interval $(-a, a)$ over $\mathcal{E}^\rho$, provided that $\rho > 0$ is small enough. The second assertion follows as Theorem 3.

Remark 1. The chain control sets E^ρ have nonvoid interiors provided that an inner pair condition holds (see [2], Corollary 4.5.11); this can be guaranteed, if $0 \in \operatorname{int} U$ and the linearized system satisfies a Lie algebraic condition, cf. [2], Proposition 4.5.19.

Remark 2. The considered system (1.1) is defined in continuous time. Nevertheless, there may exist nontrivial invariant sets with a trivial center bundle (this cannot occur for autonomous differential equations). In fact, for small control range, the chain control set around a hyperbolic equilibrium of the uncontrolled system will have trivial center bundle. This follows from lower semicontinuous dependence of the Morse spectrum on the control range; see [2], Corollary 13.1.5. In this situation, one can prove a shadowing lemma (with less technical difficulties) and obtain completely analogous results. We omit the details.

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