An Invariance Radius for Nonlinear Systems

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ABSTRACT *The stability radius of linear differential equations gives a measure for the robustness of stability with respect to (real, complex, or dynamic) perturbations. In this chapter a generalization for asymptotically stable equilibria of nonlinear systems is proposed and analyzed. It specifies the maximal perturbation range, for which the control set surrounding the equilibrium retains its invariance. It is shown that this value is attained when the invariant control set touches the boundary of its invariant domain of attraction. Then it merges with another (variant) control set and itself becomes variant.*

5.1 Introduction

The notion of a stability radius for linear systems which was introduced by D. Hinrichsen and A.J. Pritchard in [5], [6] has proved to be a very efficient and fruitful tool for the analysis of perturbed linear systems. In this chapter we propose a certain nonlinear analogue of this notion.

Consider a nominal nonlinear system in $\mathbb{R}^d$ given by a smooth vector field f_0,

$$\dot{x} = f_0(x)$$

with an asymptotically stable equilibrium x_0; hence $f_0(x_0) = 0$. We are interested in the stability behavior of this equilibrium under persistent, time-varying, and bounded perturbations. Thus we consider a system of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) =: f(x, u(t)), \qquad u(t) = (u_i(t)) \in U,$$

with smooth vector fields $f_1, \ldots, f_m$; the perturbations $u(t)$ take values in a given convex and compact subset U of $\mathbb{R}^m$ with $0 \in \operatorname{int} U$. Also in

[1]F. Colonius was supported by the Nonlinear Control Network, TMR-Project.

this nonlinear situation it is convenient to introduce a parameter $\rho \geq 0$, which indicates the size of the perturbations and to replace the pointwise constraint $u(t) \in U$ by

$$u(t) \in U^\rho := \rho \cdot U.$$

In particular, for $\rho = 0$, one recovers the nominal unperturbed system. We are interested in the supremal ρ-value, such that the stability behavior of the nominal system is preserved. In general, however, it is not obvious how to make this precise.

If the equilibrium is singular (i.e., it remains fixed under all perturbations) $f(x_0, u) = 0$ for all $u \in U$, then the stability radius is simply the supremal ρ-value, such that x_0 is asymptotically stable for all $u \in \mathcal{U}^\rho := \{u : \mathbb{R} \to \mathbb{R}^m,\ u(t) \in \rho \cdot U \text{ for almost all } t\}$. This can be compared to the stability radius of the linearized system

$$\dot{y} = \frac{\partial}{\partial x} f(x_0, u(t))y, \qquad u(t) \in \mathcal{U}^\rho.$$

Here Lyapunov or Bohl exponents can be used; see Chapter 3 by A. Paice and F. Wirth in this book.

If the equilibrium does not remain fixed under the disturbances, we have to specify the relevant notion of stability. Considering the perturbation as a control, one finds that for positive $\rho > 0$ there is a control set $D(\rho)$ containing x_0; for ρ small enough, this control set is invariant. We define the corresponding invariance radius as the maximal ρ-value for which the control set $D(\rho)$ containing x_0 remains invariant. As our analysis shows, this ρ-value, where the control set becomes variant, is connected with a merging of control sets and hence with a discontinuity (in the Hausdorff metric) of the map $\rho \longmapsto D(\rho)$. Furthermore, we briefly discuss how the invariance radius is connected with the occurrence of hyperbolic equilibria on the boundary of the invariant control set.

Invariant control sets determine the generic stability behavior of the system (see [3]). They are also of interest in the analysis of Markovian systems, where they are associated with the supports of the invariant measures; see, for example, [2].

In Section 5.2 we give some background material on the behavior of control systems. In Section 5.3, we first present a motivating one-dimensional example. Then the invariance radius is defined and its properties are described. Furthermore, a two-dimensional model for a continuous flow stirred tank reactor is discussed.

5.2 Background on Invariant Control Sets and Chain Control Sets

In this section, we cite the pertinent results from [3] on the global controllability structure of control systems.

We consider the following control system

$$\begin{array}{c} \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),\ t \in \mathbb{R}, \\ u \in \mathcal{U}^\rho = \{u : \mathbb{R} \to \mathbb{R}^m,\ u(t) \in U^\rho := \rho \cdot U \text{ for all } t \in \mathbb{R}, \text{ measurable}\}, \end{array} \tag{5.1}$$

where for $i = 0, \ldots, m$ the vector fields f_i on $\mathbb{R}^d$ are C^∞. We assume that the control range $U \subset \mathbb{R}^m$ is nonvoid, convex, and compact with $0 \in \operatorname{int} U$, that $\rho \geq 0$, and that for every initial state $x \in \mathbb{R}^d$ and every control function $u \in \mathcal{U}^\rho$ there exists a unique trajectory $\varphi(t, x, u)$, $t \in \mathbb{R}$, satisfying $\varphi(0, x, u) = x$. If the dependence on ρ does not play a role, we simply write $\mathcal{U}$. Throughout we assume that the Lie algebra generated by the vector fields $f_0, f_1, \ldots, f_m$ satisfies for every $x \in \mathbb{R}^d$ the following rank condition

$$\dim \mathcal{LA}\{f_0, f_1, \ldots, f_m\}(x) = d.$$

In particular, this implies local accessibility for all $\rho > 0$; that is, for all $x \in \mathbb{R}^d$ and all $T > 0$ one has

$$\operatorname{int} \mathcal{O}^{\rho,+}_{\leq T}(x) \neq \emptyset \quad \text{and} \quad \operatorname{int} \mathcal{O}^{\rho,-}_{\leq T}(x) \neq \emptyset,$$

where $\mathcal{O}^{\rho,+}_{\leq T}(x) = \{y \in \mathbb{R}^d,\ y = \varphi(t, x, u) \text{ with } 0 \leq t \leq T \text{ and } u \in \mathcal{U}^\rho\}$ and $\mathcal{O}^{\rho,-}_{\leq T}(x) = \{y \in \mathbb{R}^d,\ x = \varphi(t, y, u) \text{ with } 0 \leq t \leq T \text{ and } u \in \mathcal{U}^\rho\}$ are the positive and negative reachable sets (or orbits) from x. We discuss the behavior of these control systems in relation to the behavior of the unperturbed system

$$\dot{x} = f_0(x). \tag{5.2}$$

Fix a compact positively invariant set $K \subset M$ for the unperturbed system and suppose that it has an asymptotically stable equilibrium $x_0 \in \operatorname{int} K$.

In order to describe the global controllability behavior of the control system (5.1), we assume that K is positively invariant for all controls $u \in \mathcal{U}^\rho$, $\rho \geq 0$, and recall the following concepts and results.

Definition 5.1. *A subset D with nonvoid interior of the state space $\mathbb{R}^d$ is a* control set *if it is a maximal subset with the property that for all $x \in D$ the inclusion*

$$D \subset \operatorname{cl}\{\varphi(t, x, u),\ u \in \mathcal{U},\ \textit{and}\ t \geq 0\}$$

holds. A control set C is an invariant control set if

$$\mathrm{cl}\, C = \mathrm{cl}\{\varphi(t,x,u),\ u \in \mathcal{U},\ \textit{and}\ t \geq 0\}.$$

If the dependence of D on ρ plays a role, we indicate this by the argument ρ in $D(\rho)$. The same remark applies also to all other notions introduced below. The local accessibility assumption guarantees that the invariant control sets are the closed control sets and that they have nonvoid interior. Recall that every point in K can be steered into an invariant control set.

Definition 5.2. *The* domain of attraction *of a control set D is*

$$\mathbf{A}(D) = \left\{x \in \mathbb{R}^d,\ \mathrm{cl}\, \mathcal{O}^+(x) \cap D \neq \emptyset\right\},$$

and for an invariant control set C the invariant domain of attraction *is*

$$\mathbf{A}^{inv}(C) = \left\{x \in \mathbb{R}^d,\ \begin{array}{c} \textit{if}\ \mathrm{cl}\, \mathcal{O}^+(x) \cap C' \neq \emptyset \textit{ for an invariant} \\ \textit{control set } C',\ \textit{then } C' = C \end{array}\right\}.$$

Note that the domain of attraction of a control set D is open with $D \subset \mathbf{A}(D)$. For an invariant control set C the invariant domain of attraction is the largest invariant subset in the domain of attraction, and clearly, $C \subset \mathbf{A}^{inv}(C)$. However, $\mathbf{A}^{inv}(C)$ is not necessarily open, and trajectories starting in $\mathbf{A}^{inv}(C)$ do not necessarily converge to C. In particular, $\mathbf{A}^{inv}(C)$ may contain variant control sets.

Invariant control sets may be considered as a generalization of asymptotically stable equilibria for the perturbed system. We also need the following weaker concept, chain control sets, which allows for arbitrarily small jumps in the trajectories.

For ε, $T > 0$, a controlled (ε, T)-chain from x to y is given by $n \in \mathbb{N}$, times $T_0, \ldots, T_{n-1} \geq T$, points $x_0 = x, \ldots, x_{n-1}$, $x_n = y$, and controls $u_0, \ldots, u_{n-1} \in \mathcal{U}$ such that

$$d(\varphi(T_i, x_i, u_i), x_{i+1}) < \varepsilon \qquad \text{for all } i = 0, \ldots, n-1.$$

Definition 5.3. *A* chain control set *$E \subset \mathbb{R}^d$ is a maximal subset such that (i) for all $x, y \in E$ and all ε, $T > 0$ there is a controlled (ε, T)-chain from x to y; and (ii) for all $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(t,x,u) \in E$ for all $t \in \mathbb{R}$.*

Chain control sets are closed and every control set is contained in a chain control set. Furthermore, if two control sets D_1 and D_2 satisfy $\mathrm{cl}\, D_1 \cap \mathrm{cl}\, D_2 \neq \emptyset$, then there is a chain control set E with $D_1 \cup D_2 \subset E$.

The following inner pair condition guarantees that an equilibrium x_0 of the nominal system is in the interior of a control set for $\rho > 0$.

$$\text{For every } \rho > 0 \text{ one has } x_0 \in \mathrm{int}\mathcal{O}^{\rho,+}(x_0). \tag{5.3}$$

In this case we call the pair $(0, x_0) \in \mathcal{U}^\rho \times \mathbb{R}^d$ an inner pair. Consider the following maps defined on $[0, \infty)$ with values in the set of compact subsets of $\mathbb{R}^d$,

$$\rho \mapsto \operatorname{cl} D(\rho) \qquad \text{and} \qquad \rho \mapsto E(\rho), \tag{5.4}$$

where $D(\rho)$ and $E(\rho)$ are the control sets and chain control sets containing x_0. Here the existence of $E(\rho)$ is clear; the existence of the control sets $D(\rho)$ follows from the next theorem (see [3, Chapter 4]).

Theorem 5.1. *Assume that the inner pair condition* (5.3) *is satisfied. Then for every $\rho > 0$ there is a control set $D(\rho)$ with $x_0 \in \operatorname{int} D(\rho)$ and*

$$\{x_0\} = \bigcap_{\rho>0} D(\rho) = \bigcap_{\rho>0} E(\rho).$$

It is clear that $\operatorname{cl} D(\rho) \subset E(\rho)$. Under a strengthened inner pair condition, the relation between the control sets and chain control sets above is much closer. Let

$$\mathcal{D}(\rho) := \operatorname{cl}\{(u, x) \in \mathcal{U}^\rho \times \mathbb{R}^d,\ \varphi(t, x, u) \in \operatorname{int} D(\rho) \text{ for all } t \in \mathbb{R}\}.$$

We require that for $\rho' > \rho$ every $(u, x) \in \mathcal{D}(\rho)$ is an inner pair for the ρ'-system.

Theorem 5.2. *Assume that for all $\rho' > \rho > 0$ and all $(u, x) \in \mathcal{D}(\rho)$ there is $T > 0$ with $\varphi(T, x, u) \in \operatorname{int} \mathcal{O}^{+,\rho'}(x)$. Then the map $\rho \mapsto \operatorname{cl} D(\rho)$ is lower semicontinuous and the map $\rho \mapsto E(\rho)$ is upper semicontinuous (with respect to the Hausdorff metric). The sets of continuity points for both maps coincide, and $\rho^* \in (0, \infty)$ is a continuity point if and only if $\operatorname{cl} D(\rho^*) = E(\rho^*)$. There are at most countably many points of discontinuity.*

Thus the strengthened inner pair condition guarantees that "almost always" the chain control sets coincide with the closures of control sets. The exceptional points are of particular interest.

In what follows we need an existence result for control sets that are invariant relative to a given subset L of the state space; see [3, Section 3.3].

Definition 5.4. *For a subset $L \subset M$ a control set $D \subset L$ is called L-*invariant, *if $x \in D$ and $\varphi(t, x, u) \notin D$ for some $t > 0$ and $u \in \mathcal{U}$ implies $\varphi(t, x, u) \notin L$.*

Hence a trajectory can leave an L-invariant control set only if it also leaves the set L. Control sets with nonvoid interior are closed if and only if they are invariant. For L-invariant control sets one obtains an analogous result, provided that–roughly speaking – a trajectory which starts in L and leaves L cannot return to L and local accessibility in L holds.

Theorem 5.3. *Let L be a subset of the state space M satisfying the no-return condition*

$$\text{if } z \in \operatorname{cl} \mathcal{O}^+(x) \text{ for some } x \in L \text{ and } \mathcal{O}^+(z) \cap L \neq \emptyset, \text{ then } z \in L \tag{5.5}$$

and the L-accessibility condition

$$\text{for all } y \in L \text{ and all } T > 0 \text{ one has } \operatorname{int} \left[\mathcal{O}^+_{\leq T}(y) \cap L\right] \neq \emptyset. \tag{5.6}$$

Assume that there exists a compact set $Q \subset L$ such that for all $y \in L$ one has

$$\operatorname{cl} \mathcal{O}^+(y) \cap Q \neq \emptyset. \tag{5.7}$$

Then there exists an L-invariant control set D.

Remark 5.1. *For an L-invariant control set D the exit boundary*

$$\Gamma^*(D) := \{x \in \partial D, \text{ there are } u \in \mathcal{U} \text{ and } t > 0 \text{ with } \varphi(t, x, u) \notin D\}$$

coincides with $L \cap \partial D$. Also one sees immediately that the no-return condition (5.5) *implies that every control set D with* $\operatorname{int} D \cap L \neq \emptyset$ *is contained in L.*

Next we discuss the generic (in a topological sense) limit behavior of trajectories. For this purpose we endow the set of control functions $\mathcal{U}$ which is a subset of $L_\infty(\mathbb{R}, \mathbb{R}^m)$ with the corresponding weak* topology. Then $\mathcal{U}$ becomes a compact metric space (cp. [3]) and the domains of attraction in the product space have the following properties.

Theorem 5.4. *Let $C_1, \dots, C_k$ be the invariant control sets in K. Then for all i,*

$$\mathcal{A}(C_i) := \{(u, x) \in \mathcal{U} \times K, \text{ there is } T > 0 \text{ with } \varphi(T, x, u) \in C_i\}$$

is open in $\mathcal{U} \times K$ and $\bigcup_{i=1}^k \mathcal{A}(C_i)$ is open and dense in $\mathcal{U} \times K$.

5.3 An Invariance Radius for Nonlinear Systems

In order to motivate our definition of an invariance radius for nonlinear systems, we discuss the following simple one-dimensional system.

Example 5.1. Consider the scalar system

$$\dot{x}(t) + V'(x(t)) + u(t) = 0 \qquad \text{in } \mathbb{R},\ u(t) \in [-\rho, \rho],$$

where $\rho \geq 0$, and $V : \mathbb{R} \to \mathbb{R}$ is a smooth function with derivative $V'(x)$. Let the graph of V be as sketched in Figure 5.1. Then for $\rho = 0$ the equilibrium

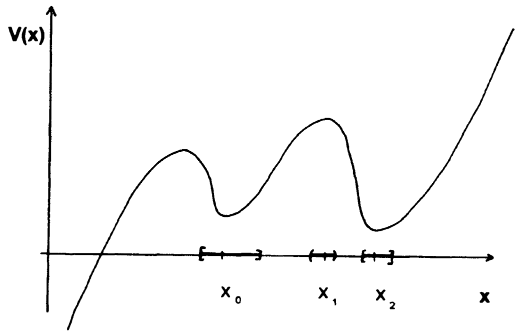

FIGURE 5.1. Equilibria and control sets for the one-dimensional example.

x_0 is asymptotically stable, and x_2 is another asymptotically stable equilibrium, while the equilibrium x_1 is unstable. For small positive ρ there is an invariant control set $D(\rho)$ containing x_0. For increasing ρ the control set $D(\rho)$ loses its invariance: Depending on $V'(x)$, either it is possible to leave it at the left side, and there are unbounded trajectories starting in x_0; or, there are trajectories leaving $D(\rho)$ and entering the invariant control set containing x_2. In general, it is not possible to return to $D(\rho)$ unless ρ is further increased. The only exception is the situation where the invariant control sets around x_0 and x_2 merge directly, which apparently is a nongeneric situation. Observe, furthermore, that there is a (variant) control set containing the unstable equilibrium x_1. The control set $D(\rho)$ loses its invariance when it merges with this control set.

We propose to define the invariance radius r of x_0 as the ρ-value where $D(\rho)$ for the first time loses its invariance.

In the following we show how to generalize the situation in this very special example to general nonlinear systems.

Formally, we define the invariance radius as follows.

Definition 5.5. *Let x_0 be an asymptotically stable equilibrium of the nominal system* (5.2)*. Then the* invariance radius *of x_0 for the perturbed system* (5.1) *is*

$$r = \sup\{\rho > 0, \textit{ the control set } D(\rho') \ni x_0 \textit{ is invariant for all } 0 < \rho' < \rho\}.$$

According to Theorem 5.1, this is well defined provided that the inner pair condition (5.3) holds. It is the main purpose of this chapter to describe what happens at $\rho = r$.

First we note that also for $\rho = r$ the control set $D(r)$ is invariant.

Proposition 5.1. *Under the conditions above, the control set $D(r)$ is invariant and*

$$D(r) = \mathrm{cl}\left[\bigcup_{0<\rho<r} D(\rho)\right].$$

Proof: Abbreviate

$$B := \bigcup_{0<\rho<r} D(\rho).$$

For every $x, y \in B$ one has $y \in \mathrm{cl}\,\mathcal{O}^{+,r}(x)$. This follows, since there is a ρ with $0 < \rho < r$ such that $x, y \in D(\rho)$. Hence the assertion follows from controllability in $D(\rho)$. This implies that B is contained in a control set D with nonvoid interior.

We claim that D is contained in $\mathrm{cl}\,B$. Otherwise, note that $\mathrm{cl}\,D = \mathrm{cl}\,\mathrm{int}\,D$. Thus one finds a compact subset A of $\mathrm{int}\,D$ with $x_0 \in A$ and $A \cap (D \setminus \mathrm{cl}\,B) \neq \emptyset$. Then by lower semicontinuity of control sets (see [3, Theorem 3.2.28]) for some $\rho < r$ there is a control set containing A. This control set must be of the form $D(\rho)$ and has nonvoid intersection with $D \setminus \mathrm{cl}\,B$. This is impossible by definition of B. This also proves the equality above with $D(r)$ replaced by $\mathrm{cl}\,D(r)$.

It remains to show that the control set $D(r)$ is invariant. Otherwise, there exist $T > 0$ and $u \in \mathcal{U}^r$ with $\varphi(T, x_0, u) \notin D(r)$. Then continuous dependence on the control and the initial value imply that there are $\rho < r$, and a control $v \in \mathcal{U}^\rho$ with $\varphi(T, x_0, v) \notin D(\rho) \subset D(r)$. This contradicts invariance of $D(\rho)$. □

Let $K = \mathrm{cl}\,\mathrm{int}\,K \subset M$ be a compact, connected, positively invariant (for all $u \in \mathcal{U}$) set containing at least two invariant control sets. Then K contains finitely many invariant control sets $C_1, \ldots, C_k$, $k \geq 2$. In order to study the behavior of a (distinguished) invariant control set, say of C_1, define

$$L = \bigcup_{i=2}^{k} [\mathbf{A}(C_1) \cap \mathbf{A}(C_i)] \cap \mathrm{int}\,K. \tag{5.8}$$

Thus L is a nonvoid open set consisting of the points in $\mathbf{A}(C_1) \cap \mathrm{int}\,K$ that can be steered to at least one other invariant control set C_i. In order to avoid certain degeneracies, we have to be careful about the behavior at the boundary ∂K of K. We require that trajectories in L remain bounded away from ∂K. More precisely, assume that the following strong invariance condition is satisfied.

For all $x \in L$ there is $\varepsilon_x > 0$ with $d(\varphi(t, x, u), \partial K) \geq \varepsilon_x$, $\forall u \in \mathcal{U}$, $t \geq 0$, and there is $\varepsilon_0 > 0$ such that for all $x \in \mathrm{cl}\,L$ and $u \in \mathcal{U}$ we have that $y = \lim_{k\to\infty} \varphi(t_k, x, u) \in L$ for $t_k \to \infty$ implies $d(y, \partial K) \geq \varepsilon_0$.

We need some topological properties of the boundary of L. Let ∂L and $\partial^K L$ denote the boundaries of L in $\mathbb{R}^d$ and K, respectively, and define

$$\partial_j L = \partial^K L \cap \mathbf{A}(C_j), \qquad j = 1, \ldots, l, \quad \text{and} \quad \partial_0 L = \partial L \cap \partial K. \quad (5.9)$$

Note that for each $y \in \partial^K L$ the following alternative holds: if $y \in \mathbf{A}(C_1)$ then $y \notin \mathbf{A}(C_i)$ for every $2 \le i \le k$; or $y \notin \mathbf{A}(C_1)$ and there is (at least one) $i \in \{2, \ldots, k\}$ with $y \in \mathbf{A}(C_i)$. Then

$$\partial^K L = \bigcup_{i=1}^{l} \partial_j L \quad \text{and} \quad \partial L = \bigcup_{i=0}^{l} \partial_j L.$$

Using that the domains of attraction are open, one sees that the first union is disjoint; furthermore, the sets $\partial_j L$, $j = 1, \ldots, l$, are open and isolated (i.e., their closures are disjoint) in $\partial^K L$. Here we again use that every point in K can be steered into an invariant control set C_i. In the topology of $\mathbb{R}^d$

$$y \in \operatorname{cl} \partial_1 L \ \cap \operatorname{cl} \bigcup_{i=2}^{l} \partial_j L \ \text{ implies } y \in \partial_0 L \subset \partial K. \quad (5.10)$$

Note further that $\partial^K L$ consists of at least two (different) $\partial_j L$. Analogous definitions can be given for every connected component of L and all properties stated above remain valid.

The following technical lemma is needed.

Lemma 5.1. *For every $x \in L$ there are $J \subset \{2, \ldots, l\}$ and $y \in \mathcal{O}^+(x)$ such that $y \in \mathbf{A}(C_1) \cap \bigcap_{j \in J} \mathbf{A}(C_j)$ and J is a minimal index set in the following sense.*

If $\varphi(t, y, u) \in L$ for some $t > 0$ and $u \in \mathcal{U}$, then $\varphi(t, y, u) \in \bigcap_{j \in J} \mathbf{A}(C_j)$.

Proof: Since $x \in L$, there exists $J_1 \subset \{2, \ldots, l\}$ with $x \in \bigcap_{j \in J_1} \mathbf{A}(C_j)$. If there are $t_1 > 0$ and $v_1 \in \mathcal{U}$ with $y_1 := \varphi(t_1, x, v_1) \in L \setminus \bigcap_{j \in J_1} \mathbf{A}(C_j)$, then there exists a proper subset $\emptyset \neq J_2 \subset J_1$ with $y_1 \in \bigcap_{j \in J_2} \mathbf{A}(C_j)$. Proceeding recursively, one ends up, after finitely many steps, at a point $y \in \mathcal{O}^+(x)$ with a minimal index set J. □

Note that a minimal index set has at least one element. Furthermore, the lemma implies that for each L-invariant control set D there is $J \subset \{2, \ldots, l\}$ such that for each $x \in \operatorname{int} D$ the index set J is minimal.

Next we construct a set Q satisfying condition (5.7).

Lemma 5.2. *Assume that $K \subset M$ with $K = \operatorname{cl} \operatorname{int} K$ is a compact, connected, positively invariant set for the control system* (5.1) *satisfying the strong positive invariance condition. Assume furthermore that the system is locally accessible from every point in K. Let $x \in \mathbf{A}(C_1) \cap \bigcap_{j \in J} \mathbf{A}(C_j) \cap K$, where $J \subset \{2, \ldots, l\}$*

is some minimal index set for x. *Then there exists an* L*-invariant control set* $D \subset \operatorname{cl} \mathcal{O}^+(x)$ *with*

$$\partial D \cap \partial_j L \neq \emptyset \quad \textit{for all } j \in J. \tag{5.11}$$

Proof: Let $x \in L$ and define

$$L_1 := L \cap \operatorname{cl} \mathcal{O}^+(x).$$

The proof is based on Theorem 5.3 applied to L_1 instead of L. It is easily seen that the set L_1 satisfies the no-return condition (5.5) and the L_1-accessibility condition (5.6). In order to construct a compact set Q with (5.7), we first introduce some notations. Denote the connected component of L that contains x by L_x, and let

$$Q_0 := \{y \in L_x,\ d(y, \operatorname{cl} \partial_0 L_x) \geq \varepsilon_x\},$$

where ε_x is chosen according to the strong invariance condition. For $\varepsilon > 0$ and $j \in \{1\} \cup J$ define

$$N_j(\varepsilon) := \{y \in L_x,\ d(y, \operatorname{cl} \partial_j L_x) \leq \varepsilon\},$$

$$Q_j(\varepsilon) := \{y \in Q_0,\ d(y, \operatorname{cl} \partial_j L_x) = \varepsilon\}.$$

Using (5.10), we can choose $\varepsilon > 0$ small enough such that

$$d(\partial_j L_x \cap Q_0, \partial_i L_x \cap Q_0) \geq 5\varepsilon \text{ for all } i,\ j \in J,\ i \neq j.$$

Hence the sets $Q_j(\varepsilon)$ are nonvoid, compact, and pairwise disjoint with distance at least 3ε. Decreasing, if necessary, the number ε further, we may assume that $x \in L_x \setminus \bigcup_{j \in J} N_j(2\varepsilon)$. Every trajectory $\{\varphi(t, y, u),\ t \geq 0\}$ with $y \in \operatorname{cl} \mathcal{O}^+(x) \cap (L_x \setminus N_j(2\varepsilon))$ that approaches C_j for $t \to \infty$ must exit through $\partial_j L_x \cap Q_0$ and must cross $Q_j(\varepsilon)$. For every y in this set there exists a control $u \in \mathcal{U}$ with this property. Furthermore we find that

$$\emptyset \neq Q_j(x, \varepsilon) := \operatorname{cl} \mathcal{O}^+(x) \cap Q_j(\varepsilon) \subset \mathbf{A}(C_1) \cap \bigcap_{j \in J} \mathbf{A}(C_j)$$

for all $j \in J$ and all $\varepsilon > 0$ small enough. Hence by Theorem 5.3 the existence of an L_1-invariant control set $D \subset \operatorname{cl} \mathcal{O}^+(x)$ follows. Obviously, an L_1-invariant control set is also L-invariant. Finally, condition (5.11) holds, because in the construction above ε can be made arbitrarily small and every L-invariant control set D has nonvoid intersection with every $Q_j(x, \varepsilon)$. □

We obtain the following result.

Proposition 5.2. *Assume that* $K \subset M$ *with* $K = \operatorname{cl} \operatorname{int} K$ *is a compact, connected, positively invariant set for the control system* (5.1) *satisfying the strong* *invariance condition. Furthermore, let $C_1, \ldots, C_l$ be the invariant control sets in K. Consider the set*

$$L = \bigcup_{j \in \{2,\ldots,k\}} [\mathbf{A}(C_1) \cap \mathbf{A}(C_j)] \cap \operatorname{int} K.$$

Then there exists at least one and at most finitely many L-invariant control sets D and every point in L can be steered into an L-invariant control set.

Proof: It only remains to show that the number of L-invariant control sets is finite. This follows as [1, Proposition 2.11]. □

The next theorem indicates the controllability situation when an invariant control set touches the boundary of its invariant domain of attraction. In this case, the chain control set containing the invariant control set must contain another control set.

Theorem 5.5. *Assume that $K \subset M$ with $K = \operatorname{cl} \operatorname{int} K$ is a compact, positively invariant set for the control system (5.1) satisfying the strong invariance condition. Let C_1 be an invariant control set with $C_1 \cap \partial \mathbf{A}^{inv}(C_1) \neq \emptyset$. Then there exists a control set D that is invariant relative to the set*

$$L = \bigcup_{j \in \{2,\ldots,k\}} [\mathbf{A}(C_1) \cap \mathbf{A}(C_j)] \cap \operatorname{int} K,$$

where $C_2, \ldots, C_k$ are the invariant control sets different from C_1 in K, such that the (unique) chain control set E containing C_1 satisfies

$$C_1 \cup D \subset E.$$

Proof: By our assumption, there exists a point $x \in C_1 \cap \partial \mathbf{A}^{inv}(C_1)$. Since the invariant control set C_1 is contained in the interior of its domain of attraction, it follows that there are $x_n \in \mathbf{A}(C_1) \setminus \mathbf{A}^{inv}(C_1)$ with $x_n \to x$. Thus there are invariant control sets different from C_1 that have nonvoid intersection with $\operatorname{cl} \mathcal{O}^+(x_n)$. Since the number of invariant control sets is finite, we may assume that there is a control set $C_j \neq C_1$ with $\operatorname{cl} \mathcal{O}^+(x_n) \cap C_j \neq \emptyset$ for all n.

Hence $x_n \in L$ and by the preceding proposition we find u_n and $t_n > 0$ such that $\varphi(t_n, x_n, u_n) \in \operatorname{int} D_n$ for some L- invariant control set D_n. Since the number of L-invariant control sets is finite and one can steer the system from every point of D_n into C_1, it follows that the chain control set E containing C_1 also contains some D_n. □

Remark 5.2. *The proof above uses arguments similar to those used for the existence of relatively invariant control sets in multistability regions, Colonius et al. [1], see also [3, Section 3.3].*

From the preceding theorem and Theorem 5.2 we immediately obtain the following consequence for systems with increasing control range. It shows that at the invariance radius r the invariant control set merges with another variant control set and itself becomes variant. This is the main result of this chapter.

Theorem 5.6. *Let the inner pair condition from Theorem 5.2 be satisfied and consider an asymptotically stable equilibrium x_0 of the nominal system. Suppose that for an increasing control set family $D(\rho)$ $\rho > 0$, containing x_0, which is invariant for $\rho \leq \rho_0$, one has*

$$D(\rho^0) \cap \partial \mathbf{A}^{inv}(D(\rho^0)) \neq \emptyset.$$

Assume that the chain control set $E(\rho^0)$ containing $D(\rho^0)$ does not contain another invariant control set. Then ρ^0 coincides with the invariance radius r, the map $\rho \mapsto D(\rho)$ is discontinuous at $\rho = r$ and $D(r) \neq E(r)$. Furthermore, there is a control set $D' \neq D(r)$ for the control range U^r such that for $\rho > r$ the control sets $D(\rho)$ satisfy

$$D(\rho) \supset D(r) \cup D'.$$

Finally, we treat the case of unbounded solutions. Instead of the existence of a compact, connected, positively invariant set K we consider a compact connected subset containing all invariant control sets and satisfying a no-return condition.

Theorem 5.7. *Consider an asymptotically stable equilibrium x_0 of the nominal system. Suppose that for an increasing control set family $D(\rho)$, $\rho > 0$, containing x_0, which is invariant for $\rho \leq \rho_0$, one has*

$$D(\rho^0) \cap \partial \mathbf{A}^{inv}(D(\rho^0)) \neq \emptyset.$$

Assume that there is a compact, connected subset $K = \operatorname{cl}\operatorname{int} K$ containing all invariant control sets $C_1, \ldots, C_l$ of the ρ^0-system, satisfying the strong invariance condition and the following no-return condition:

$$\begin{aligned} &\text{if } \varphi(T, z, u) \notin \operatorname{int} K \text{ for some } z \in K,\ T > 0, \text{ and } u \in \mathcal{U}^{\rho^0}, \\ &\text{then } \varphi(t, z, u) \notin K \text{ for all } t \geq T. \end{aligned} \tag{5.12}$$

Furthermore, suppose that the chain control set $E(\rho^0)$ containing $D(\rho^0)$ does not contain another invariant control set. Then ρ^0 coincides with the invariance radius r, the map $\rho \mapsto D(\rho)$ is discontinuous at $\rho = r$, and $D(r) \neq E(r)$. Furthermore, there is a control set $D' \neq D(r)$ for the control range U^r such that for $\rho > r$ the control set $D(\rho)$ satisfies

$$D(\rho) \supset D(r) \cup D'.$$

This theorem follows similarly to the preceding one; compare [3, Section 3.3] for analogous arguments in the analysis of multistability regions.

Next we discuss in more detail the question of when an invariant control set C touches the boundary of its invariant domain of attraction. In the one-dimensional system, Example 5.1, this happens when an unstable equilibrium corresponding to some constant control occurs at the boundary of C. A moment's reflection shows that the occurrence of an equilibrium at the boundary of C with nontrivial unstable manifold directed out of C does not necessarily lead to the loss of invariance. It does happen, if the unstable manifold approaches for $t \to \infty$ another invariant control set. If it returns to C, the invariance radius is not attained. If the unstable manifold approaches for $t \to \infty$ a variant control set, then, depending on its (global) properties, the invariance radius may or may not be attained. The following simple example of a chemical reactor shows some of these features (for this model compare Poore [7] or Golubitskii and Schaeffer [4] and also the analysis in [3, Chapter 9]).

Example 5.2. Consider the model of a continuous flow stirred tank reactor given by the equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1 - x_2)e^{x_1} \\ -x_2 + \alpha(1 - x_2)e^{x_1} \end{pmatrix} + u(t) \begin{pmatrix} x_c - x_1 \\ 0 \end{pmatrix}. \quad (5.13)$$

Here x_1 is the (dimensionless) temperature, x_2 is the product concentration, and a, α, B, x_c are positive constants. The parameter x_c is the coolant temperature, and hence the perturbation affects the heat transfer coefficient. In [7] Poore analyzes the bifurcation behavior of the nominal (i.e., $u(t) \equiv 0$) system. Here we choose parameter values such that for all constant controls $u(t) \equiv u \in [-\rho, \rho]$ the system (5.13) has exactly three fixed points as limit sets. Specifically, we take for our numerical analysis

$$a = 0.15, \qquad \alpha = 0.05, \qquad B = 7.0, \qquad x_c = 1.0. \qquad (5.14)$$

First let $U = [-0.15, 0.15]$. Because of the physical constraints we have to consider the system in the set $M = [0, \infty) \times [0, 1] \subset \mathbb{R}^2$. For each fixed $u \in U$ the equation (5.13) has three fixed points in M. Let $y^i = \alpha e^{z^i}/(1 + \alpha z^i)$, $i = 0, 1, 2$, and let $z^0 < z^1 < z^2$ be the zeros of the transcendental equation

$$-z - (a + u)(z - x_c) + B\alpha[1 - \frac{\alpha e^z}{1 + \alpha e^z}]e^z = 0.$$

Then these fixed points are given as two asymptotically stable ones, $x^0 = (z^0, y^0)$ and $x^2 = (z^2, y^2)$, and a hyperbolic one, $x^1 = (z^1, y^1)$; that is, the linearization about x^1 has one negative and one positive eigenvalue. The phase portrait of the nominal equation is indicated in Figure 5.2. There are exactly three control sets C_1, C_2, and D, containing the fixed points $x^i(u)$, $i = 0, 1, 2$, for $u \in \operatorname{int} U = (-0.15, 0.15)$ in their interior. The control sets C_1 and C_2 are invariant; the control set D is variant. The closures of these control sets are the three chain control sets of the system. Figure 5.2 shows the three control sets.

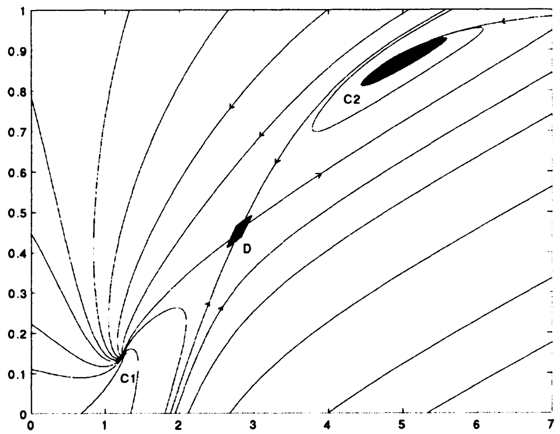

FIGURE 5.2. Phase portrait of the unperturbed ($u(t) \equiv 0$) continuous flow stirred tank reactor and the control sets.

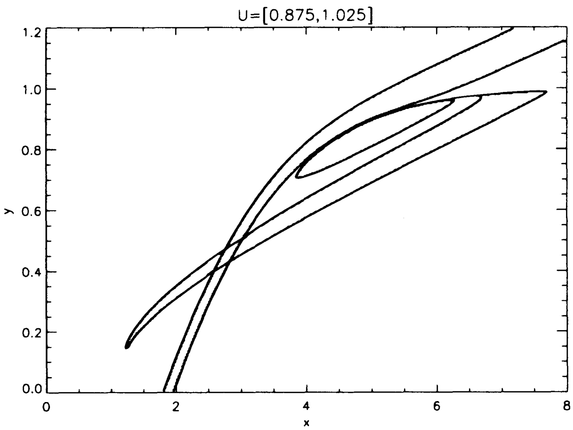

FIGURE 5.3. Control sets of the continuous flow stirred tank reactor and domains of attraction.

For different parameter values, Figure 5.3 shows the invariant control set C_2 as well as the positive and negative orbits from the hyperbolic equilibrium. Their intersection is the variant control set D. Although the numerics seem to indicate that for a slightly larger control range the invariant control set C_2 loses its invariance by intersecting the domain of attraction $\mathbf{A}(D)$ outside of $\operatorname{cl} D$ only, the situation is different: the relevant part of the boundary of $\mathbf{A}(D)$ is the stable manifold of a hyperbolic equilibrium in $\operatorname{cl} D$. Hence, if $\operatorname{cl} \mathbf{A}(D) \cap C_2 \neq \emptyset$, it follows from invariance of C_2 that this hyperbolic equilibrium also is in C_2 and thus $\operatorname{cl} D \cap C_2 \neq \emptyset$. Hence also in this example, the occurrence of an equilibrium on the boundary of the invariant control set with unstable manifold leading out of the invariant control set is responsible for the loss of invariance. If the perturbation range is further increased, the resulting variant control set merges with the remaining invariant control set C_1. This seems to happen when a hyperbolic equilibrium occurs on the boundary of C_1. Here this does not lead to the loss of invariance, since only one invariant control set is present.

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