Lyapunov exponents and robust stabilization

F. Colonius*, D. Hinrichsen** and F. Wirth**

*Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY
colonius@math.uni-augsburg.de

**Institut für Dynamische Systeme
Universität Bremen
28334 Bremen
GERMANY
dh@mathematik.uni-bremen.de
fabian@math.uni-bremen.de

1 Description of the problems

In recent years there has been considerable interest in the analysis and synthesis of time-varying control systems. Here we present some open problems in this area. Let $\mathbf{K} = \mathbb{R}, \mathbb{C}$ and consider a family of bilinear systems

$$\begin{aligned} \dot{x}(t) &= \left(A_0 + \sum_{i=1}^{m} v_i(t) A_i\right) x(t) \qquad (18.1) \\ x(0) &= x^0 \in \mathbf{K}^n \\ v(t) &= (v_1(t), \dots, v_m(t))' \in V \text{ a.a. } t \geq 0 \end{aligned}$$

where $A_0, \dots, A_m \in \mathbf{K}^{n \times n}$ and $V \subset \mathbf{K}^m$ is a convex, compact set with $0 \in \operatorname{int} V$. The set of admissible control functions $\mathcal{V}$ is the set of measurable functions $v : \mathbf{R} \to V$. The solution determined by an initial condition x^0 and a control $v \in \mathcal{V}$ is denoted by $\varphi(\cdot; x^0, v)$.
The exponential growth rate or *Lyapunov exponent* of a trajectory with initial state $0 \neq x^0 \in \mathbf{K}^n$ and control $v \in \mathcal{V}$ is defined by

$$\lambda(x^0, v) := \limsup_{t \to \infty} \frac{1}{t} \log \|\varphi(t; x^0, v)\| .$$

It is well known that for fixed $v \in \mathcal{V}$ there are at most n different Lyapunov exponents of (18.1). For constant controls $v(\cdot) \equiv v$ the Lyapunov exponents are simply the real parts of the eigenvalues of $A_0 + \sum_{i=1}^{m} v_i A_i$. If $v \in \mathcal{V}$ is periodic the Lyapunov exponents given by v are the *Floquet exponents* from classical Floquet theory. The Lyapunov and the Floquet spectrum are defined by

$$\Sigma_{Ly} := \{\lambda(x^0, v) \; ; \; 0 \neq x^0 \in \mathbf{K}^n, \, v \in \mathcal{V}\},$$

$$\Sigma_{Fl} := \{\lambda(x^0, v) \; ; \; 0 \neq x^0 \in \mathbf{K}^n, \, v \in \mathcal{V}, v \text{ periodic}\}.$$

Under suitable assumptions the closure of the Floquet spectrum $\mathrm{cl}\,\Sigma_{Fl}$ is given by intervals whose number does not exceed the dimension of the state space n, see [1] for the continuous and [8] for the discrete-time case. The investigation of this property is closely related to controllability properties of the system obtained by projection onto the sphere. In general these spectral intervals are only accessible through involved numerical computations except for easy cases like the following example taken from [1].
Consider the system

$$\dot{x}(t) = \left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + v(t) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) x(t).$$

For a control range $V = [-\rho, \rho]$ it may be shown that the Floquet intervals are given by

$$\left(-\frac{\rho}{2} - \sqrt{2 + \frac{\rho^2}{4}}, \frac{\rho}{2} - \sqrt{2 + \frac{\rho^2}{4}} \right), \quad \left(-\frac{\rho}{2} + \sqrt{2 + \frac{\rho^2}{4}}, \frac{\rho}{2} + \sqrt{2 + \frac{\rho^2}{4}} \right).$$

Furthermore, in this example the Lyapunov spectrum is equal to the closure of the Floquet spectrum. It is still not known whether this is always the case.

Problem 1: Floquet versus Lyapunov spectrum. Prove or disprove that for all systems of the form (18.1)

$$\Sigma_{Ly} = \mathrm{cl}\,\Sigma_{Fl}.$$

In the formulation of (18.1) the v_i may be interpreted as time-varying perturbations or as open-loop controls. We will now concentrate on the interpretation of v as a time-varying disturbance counteracting a linear control input into the system. Thus we consider the following system

$$\dot{x}(t) = \left(A_0 + \sum_{i=1}^{m} v_i(t) A_i \right) x(t) + Bu(t), \tag{18.2}$$

where the assumptions on the disturbance v and the set V are as before and $B \in \mathbf{K}^{n \times p}, u(t) \in \mathbf{K}^p$.

We propose to consider the following robust stabilization problem: Construct a (possibly time-varying) feedback $F : \mathbb{R}_+ \to \mathbf{K}^{p\times n}$ such that

$$\dot{x}(t) \quad = \quad \left(A_0 + \sum_{i=1}^{m} v_i(t) A_i \right) x(t) + BF(t)x(t)\,, \qquad (18.3)$$

is exponentially stable for all disturbances $v \in \mathcal{V}$. Such a map F is called *robustly stabilizing state-feedback.* A further question of interest is to restrict the set of admissible values for F, namely to require that $F(t) \in \Gamma \subset \mathbf{K}^{p\times n}$, where Γ satisfies a set of conditions similar to those on V.

Problem 2: Robust stabilization. Develop a (game theoretic?) robust stabilization approach for system (18.2). In particular, solve the following problems.

Given the matrix A_0, the perturbation structure $A_1, \ldots, A_m$, the perturbation set $\mathcal{V}$ and the gain restriction Γ.

a) Find necessary and sufficient conditions for the existence of an admissible robustly stabilizing time-invariant state-feedback.

b) Find necessary and sufficient conditions for the existence of an admissible robustly stabilizing periodic state-feedback.

c) Determine conditions under which the existence of a time-varying admissible robustly stabilizing feedback is equivalent to the existence of an admissible time-invariant (periodic) one.

d) Develop algorithms for the computation of a robustly stabilizing feedback controllers for (18.2).

2 History of the problem

Exponential growth rates of time-varying linear systems have already been studied by Floquet and Lyapunov. A further breakthrough was achieved in 1968 when Oseledets proved the famous multiplicative ergodic theorem which showed under ergodicity assumptions the existence of well behaved Lyapunov exponents in the stochastic framework. This result motivated a flurry of further research on Lyapunov exponents in smooth ergodic theory and for stochastic systems, which in turn motivated a similar analysis in the deterministic case over the last decade.

3 Motivations

For a large class of systems where Floquet and Lyapunov spectra coincide approximations to the spectra can be computed by optimal control methods (via numerical solutions of Hamilton-Jacobi-Bellman equations, see [3]), thus a positive solution of Problem 1 also shows computability of the Lyapunov spectrum. This is of particular interest as it has been shown in [6] that there is no algorithm to compute Lyapunov exponents in the stochastic framework in the general case. The complete Lyapunov spectrum can be used for different problem areas arising in control:

1. Robustness analysis: Interpreting v as a disturbance, the time-varying stability radius of a stable matrix A_0 corresponding to perturbations of the form (18.1) can be defined as the infimum of those values $\rho \in \mathbb{R}_+$ for which the maximal Lyapunov exponent of (18.1) with control range $V_\rho := \rho V$ is nonnegative.

2. Stabilization: Interpreting v as a control input, the existence of an appropriately defined feedback that stabilizes (18.1) can be characterized in terms of properties of the Lyapunov spectrum. In contrast to the first item knowledge of the complete spectrum is necessary.

Problem 2 is similar to the robust stabilization problems that have been at the center of control theory in recent years. The following singular H_∞ problem is a special case (with *constant unconstrained* feedbacks F): Given $A \in \mathbf{K}^{n\times n}$, $D \in \mathbf{K}^{n\times \ell}$, $E \in \mathbf{K}^{q\times n}$ and $\rho > 0$, find a feedback matrix $F \in \mathbf{K}^{p\times n}$ such that all the systems

$$\dot{x}(t) = (A + D\Delta(t)E + BF)x, \quad \Delta(\cdot) \in L_\infty(0,\infty;\mathbf{K}^{\ell\times q}), \ \|\Delta(\cdot)\|_{L_\infty} \leq \rho \tag{18.4}$$

are stable. This problem may be formulated as in Problem 2 by choosing a basis $A_1, \ldots, A_m$ of the subspace $\{D\Delta E \in \mathbf{K}^{n\times n} \ ; \ \Delta \in \mathbf{K}^{\ell\times q}\}$ and considering the system

$$\dot{x}(t) = (A + \sum_{i=1}^{m} v_i(t)A_i + BF)x\,,$$

where $V = \{v \in \mathbf{K}^m \ ; \ \exists\Delta : \|\Delta\| \leq \rho \text{ and } \sum_{i=1}^m v_i A_i = D\Delta E\}$.
Note, however, that Problem 2 encompasses a far larger problem class. For instance, feedback constraints are directly taken into account. Furthermore, arbitrary time-varying affine parameter perturbations can be represented as in (18.2) so that Problem 2 has features of a time-varying μ-synthesis problem.

4 Available results

The Lyapunov spectrum of families of bilinear systems of the form (18.1) has been studied intensively, e.g. in [1], [2]. In these publications the bilinear system is projected onto the projective space $\mathbf{PS}^{n-1}$ and controllability properties of the projected nonlinear system are used to characterize the Floquet spectrum. The main assumption used in this approach is that the nonlinear system is locally accessible. Then by a result due to E. Joseph, the equality in Problem 1 holds in dimension n=2. Furthermore, it has been shown in the discrete-time case that $\mathrm{cl}\,\Sigma_{Fl} \subset \Sigma_{Ly}$ using a method that naturally extends to the continuous-time case, [8]. This result is based on the standard accessibility assumptions which allow in particular to construct Lyapunov exponents from sequences of Floquet exponents by connecting periodic orbits.

Considering scaled ranges $V_\rho := \rho V$ it is shown in [1] that, under an additional controllability condition, $\mathrm{cl}\,\Sigma_{Fl}(\rho) = \Sigma_{Ly}(\rho)$ for all but countably many $\rho > 0$. Here $\Sigma_{Ly}(\rho)$ and $\Sigma_{Fl}(\rho)$ denote the Lyapunov and Floquet spectra of (18.1) with perturbation range restricted to V_ρ.

So far these results have been applied to the stabilization problem [3], [7]. A numerical stabilization scheme based on the theory of Lyapunov exponents is described in [3]. Robustness analysis has been treated in [9].

Problem 2 has hardly been dealt with. In the complex case $\mathbf{K} = \mathbb{C}$ the singular H_∞ problem (18.4) is solvable if and only if there exists $F \in \mathbb{C}^{m\times n}$ such that the complex stability radius $r_{\mathbb{C}}(A+BF, D, E) > \rho$, see [4]. Necessary and sufficient conditions for the existence of such a feedback are given in [5]. In the real case $\mathbf{K} = \mathbb{R}$ the problem is still unsolved.

5 References

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