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Exponential Growth Behavior of Bilinear Control Systems

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Abstract

For a general class of bilinear control systems on vector bundles the exponential growth rates and corresponding decompositions into subbundles are described.

1. Introduction*

In this paper, we describe the exponential growth behavior, i.e., the Lyapunov exponents of bilinear control systems on vector bundles. Roughly, these are linear differential equations, where the coefficients are determined by a nonlinear control system. Section 1 describes the main examples of these systems, in particular: linearized control systems, and introduces the relevant notions. Section 2 describes the Lyapunov spectrum under an inner pair condition and presents a result on local stable manifolds.

2. Problem Formulation

We consider the following class of bilinear control systems on vector bundles:

$$\begin{array}{c}\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^m u_i(t)X_i(x(t)) \\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m,\ u(t) \in U,\ t \in \mathbb{R}\}\end{array} \tag{1}$$

where $X_i : F \to \mathbf{T}F$, $i = 0, 1, ..., m$, are C^∞ vector fields and $\pi : F \to M$ is a vector bundle with base space M, which is a connected paracompact Riemannian C^∞ manifold; the set $U \subset \mathbb{R}^m$ is compact and convex. Denote by $\varphi(t, x, u)$ the solutions of (1) corresponding to $u \in \mathcal{U}$ with initial condition $\varphi(0, x, u) = x \in F$. We assume existence and uniqueness of solutions for all $t \in \mathbb{R}$. We require that the fibers F_y, $y \in M$, of the vector bundle F are preserved under φ, i.e.,

$$x_1 \in F_{\pi x_2} \text{ implies } \varphi(t, x_1, u) \in F_{\pi\varphi(t,x_2,u)}$$

for all $t \in \mathbb{R}, u \in \mathcal{U}$, and that the solution map is linear on the fibers:

$$\varphi(t, \alpha x_1 + x_2, u) = \alpha\varphi(t, x_1, u) + \varphi(t, x_2, u)$$

for all α, $t \in \mathbb{R}$ and $x_1, x_2 \in F$ with $\pi x_1 = \pi x_2$.

For $(u, x) \in \mathcal{U} \times F$, $x \notin Z$, the zero section in F, the exponential growth rate or Lyapunov exponent of the corresponding trajectory is given by

$$\lambda(u, x) = \limsup_{t\to\infty} \frac{1}{t} \log \mid \varphi(t, x, u) \mid \tag{2}$$

and the Lyapunov spectrum Σ_{Ly} of the system (1) is the set of all Lyapunov exponents

$$\Sigma_{Ly} = \{\lambda(u, x),\ (u, x) \in \mathcal{U} \times (F \setminus Z)\}. \tag{3}$$

The aim of this paper is to describe the Lyapunov spectrum and the pairs (u, x) for which the Lyapunov exponents are attained.

The control system on F induces a control system on M which can be described in the following form:

$$\dot{y}(t) = Y_0(y(t)) + \sum_{i=1}^m u_i(t)Y_i(y(t)). \tag{4}$$

Here $Y_i : M \to \mathbf{T}M$, $i = 0, 1, ..., m$, are C^∞ vector fields given by

$$Y_i(y) = \mathbf{T}\pi(X_i(\pi^{-1}(y))),$$

where $\mathbf{T}\pi : \mathbf{T}F \to \mathbf{T}M$ is the linearization of π. Locally, the bilinear control system (1) is described by

$$\begin{array}{c}\dot{y}(t) = f_0(y(t)) + \sum_{i=1}^m u_i(t)f_i(y(t)) \\ \dot{x}(t) = [A_0(y(t)) + \sum_{i=1}^m u_i(t)A_i(y(t))]\, x(t).\end{array} \tag{5}$$

Hence one has a coupled system of differential equations, where the nonlinear part of the control system describing the motion $y(t)$ on M drives the linear part $x(t)$ via the coefficients $A_i(y(t))$; the controls $u(t)$ may be applied to both parts.

Next we discuss a number of examples defining a bilinear control system on a vector bundle. The simplest examples are bilinear control systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^m u_i(t)A_i\right] x(t), \tag{6}$$

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where $A_0, A_1, ..., A_m$ are (constant) $d \times d$ matrices. They fit into the general class of systems (1) by defining $F = \mathbb{R}^d$. Here the nonlinear part is trivial. Note that for $U = \{0\}$, one obtains a time invariant linear differential equation $\dot{x}(t) = A_0 x(t)$, and the Lyapunov spectrum reduces to the set of the real parts of the eigenvalues of the matrix A_0.

The next class of examples arises, when the coefficients of a linear equation on $\mathbb{R}^d$ are determined by the solutions of a nonlinear system on some manifold M. More specifically, consider the system

$$\begin{aligned} \dot{y}(t) &= Y_0(y(t)) + \sum_{i=1}^{m} u_i(t) Y_i(y(t)) \quad (7) \\ \dot{x}(t) &= A_0(y(t))x(t). \end{aligned}$$

We may interpret y as a background disturbance, whose dynamics are modelled by a differential equation with unknown time varying parameters $(u_i(t))$ with bounded amplitudes. These disturbances act on the differential equation with state $x(t) \in \mathbb{R}^d$, but are not influenced by $x(t)$. This system fits into the general model (1), if we define the state space as $F = \mathbb{R}^d \times M \to M$. A simple example of this type on $F = \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{S}^1$ is the linear oscillator, where the period of the restoring force is perturbed

$$\begin{aligned} \dot{y} &= \omega + u(t), \ \ddot{x} + x \sin y = 0 \\ u(t) &\in [-\rho, \rho] \text{ with } \rho > 0. \end{aligned}$$

We may combine (6) and (7) to obtain systems where the u_i also appear in the x-equation.

Another generalization of the bilinear system (6) are control systems obtained by linearizing along trajectories: Consider a control system on a manifold M described by

$$\dot{y}(t) = Y_0(y(t)) + \sum_{i=1}^{m} u_i(t) Y_i(y(t)) \quad (8)$$

Then linearization along the trajectories yields a bilinear control system on the tangent bundle $F = \mathbf{T}M \to M$ described by

$$\frac{d}{dt}\mathbf{T}y(t) = \mathbf{T}Y_0(\mathbf{T}y(t)) + \sum_{i=1}^{m} u_i(t)\mathbf{T}Y_i(\mathbf{T}y(t)) \quad (9)$$

where for a vector field Y on M its linearization is denoted by $\mathbf{T}Y = (Y, DY)$. Locally, this means: If $Y_j = \sum_{k=1}^{d} \alpha_{kj}(y)\frac{\partial}{\partial y_k}$, denote the Jacobian of the coefficient functions by

$$A_j(y) = \left(\frac{\partial \alpha_{kj}(y)}{\partial y_l}\right)_{l,k}$$

Then $\mathbf{T}Y_j(y, v) = (\alpha_j(y), A_j(y)v)$, and the system is described by a pair of coupled differential equations given by

$$\begin{aligned} \dot{y}(t) &= \alpha_0(y(t)) + \textstyle\sum_{i=1}^{m} u_i(t)\alpha_i(y(t)) \\ \dot{v}(t) &= A_0(y(t))v(t) + \textstyle\sum_{i=1}^{m} u_i(t)A_i(y(t))v(t) \end{aligned} \quad (10)$$

Note that (6) occurs as a special case of this system, if $y^0 \in M$ is a common fixed point of the vector fields Y_i, $i = 0, ..., m$, and the system is linearized in y^0 (take $A_i := A_i(y^0)$, $i = 0, ..., m$).

Returning to the general bilinear control system (1) on a vector bundle F over a manifold M we note that there is an associated control flow

$$\begin{aligned} \Phi_t &: \mathcal{U} \times F \to \mathcal{U} \times F, \ t \in \mathbb{R} \\ \Phi_t(u, x) &\longmapsto (u(t + \cdot), \varphi(t, x, u)) \end{aligned}$$

Then Φ is a linear flow on the vector bundle $\pi : \mathcal{U} \times F \to \mathcal{U} \times M$. Hence the theory of linear flows on vector bundles, cp. [1] can be applied to bilinear control systems.

3. Results

We construct an 'outer approximation' of the Lyapunov spectrum (3) given by the Morse spectrum and an 'inner' approximation, the Floquet spectrum. Then the Lyapunov spectrum is sandwiched in between. The Morse spectrum is associated to the chain control sets in the projective bundle, the Floquet spectrum to the control sets. An inner pair assumption will imply that the chain control sets coincide with the closures of the control sets. Then also the closure of the Floquet spectrum and the Morse spectrum, and hence the Lyapunov spectrum all coincide.

We start by defining the Floquet spectrum, which is based on periodic coefficient functions for the linear part. In order to motivate the construction, we go back to the special case of constant coefficients. Roughly, the chain control sets generalize the sums of generalized eigenspaces corresponding to eigenvalues with equal real part, while the control sets generalize the eigenspaces. Consider a time invariant linear equation in $\mathbb{R}^d$ without the origin of the form

$$\dot{x}(t) = A_0 x(t),\, x(0) = x_0 \neq 0 \qquad (11)$$

with $A_0 \in \mathbb{R}^{d\times d}$. Project this equation down to the unit sphere $\mathbb{S}^{d-1}$ by defining $s(t) = \frac{x(t)}{|x(t)|}$. Then one obtains by the chain rule a differential equation for $s(t)$,

$$\dot{s}(t) = h_0(s(t)) \text{ with } h_0(s) := [A_0 - s^T A_0 s]s.$$

For an eigenvector $x \in \mathbb{R}^d$ corresponding to a real eigenvalue the projection s of x on the unit sphere is an equilibrium point of this differential equation. Conversely, every equilibrium point on the sphere corresponds to an eigenvector for a real eigenvalue λ.

Now consider the bilinear control system (6) in $\mathbb{R}^d$. The same procedure as above yields an induced control system on the unit sphere $\mathbb{S}^{d-1}$ given by

$$\dot{s}(t) = h(u(t), s(t)),\ u \in \mathcal{U}$$

with analogous notation. For the Lyapunov exponents one obtains an integral expression of the form

$$\lambda(u,x) = \lim \sup_{t\to\infty} \frac{1}{t}\int_0^t q(u(\tau), s(\tau))\, d\tau$$

A typical effect of control actions in the system on the unit sphere can be seen in the system

$$\dot{x}(t) = [A_0 + u(t)A_1]\,x(t)$$

with $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $U = [-\rho, \rho]$, $\rho > 0$ small. For constant u, the eigenspaces corresponding to $u \equiv 0$ are rotated. Then the unstable eigendirections (i.e., the corresponding equilibria on the unit sphere) for $u \in U$ form invariant, closed control sets, while the stable eigendirections form variant, open control sets. Note that by identifying points on opposite sides of the sphere one obtains two control sets (now in projective space $\mathbb{P}^1$) and one of them is invariant. In higher dimension, we consider periodic trajectories in the control sets; they will correspond to eigenspaces of fundamental solutions corresponding to periodic controls. The corresponding Floquet exponents, naturally, are Lyapunov exponents, and hence the Floquet spectrum will provide an 'inner' approximation to the Lyapunov spectrum.

Returning to the general system (1) or the local version (5), we see that periodicity of the control functions is not enough to obtain linear differential equations with periodic coefficients. We also need that the corresponding trajectory in the base space M is periodic. Furthermore, we have to consider not just trajectories and control sets in projective space $\mathbb{P}^{d-1}$, but in the projective bundle $\mathbb{P}F$: The bilinear control system (1) on the vector bundle F induces the following projective control system on the projective bundle $\mathbb{P}\pi : \mathbb{P}F \to M$:

$$\frac{d}{dt}\mathbb{P}x = \mathbb{P}X_0(\mathbb{P}x) + \sum_{i=1}^{m} u_i(t)\mathbb{P}X_i(\mathbb{P}x), \quad (12)$$

where $\mathbb{P}X$ is the projection of a vector field X on F onto $\mathbb{P}F$, i.e., the $\mathbb{P}X_j$ read locally

$$\mathbb{P}X_j(y,s) = (f_j(y), h(A_j(y), s)),$$
$$h(A_j(y), s) = \left[A_j(y) - s^T A_j(y)s \cdot Id\right] s.$$

Here ^{T} denotes transposition and Id is the $d\times d$ identity matrix. Then (12) is locally described by

$$\begin{aligned} \dot{y}(t) &= f_0(y(t)) + \textstyle\sum_{i=1}^m u_i(t) f_i(y(t)) \\ \dot{s}(t) &= h_0(y(t), s(t)) + \textstyle\sum_{i=1}^m u_i(t) h_i(y(t), s(t)). \end{aligned} \qquad (13)$$

The trajectories of (12) are denoted by $\mathbb{P}\varphi(t, \mathbb{P}x, u)$, $t \in \mathbb{R}$. We formally define the Floquet spectrum of the system (1) as follows.

Definition 1 *Let ${}_{\mathbb{P}}D$ be a control set with nonvoid interior of the system (12) on the projective bundle $\mathbb{P}F$ induced by the bilinear system (1) on the vector bundle F. The Floquet spectrum of the system (1) over ${}_{\mathbb{P}}D$ is defined as*

$$\Sigma_{Fl}({}_{\mathbb{P}}D) = \left\{ \begin{array}{c} \lambda(u,x),\ (u, \mathbb{P}x) \in \mathcal{U} \times int\, {}_{\mathbb{P}}D \\ u \text{ is piecewise constant and } \tau- \\ \text{periodic with } \mathbb{P}\varphi(\tau, \mathbb{P}x, u) = \mathbb{P}x \end{array} \right\}$$

The Floquet spectrum over a control set D in the base space M of the system (1) is

$$\Sigma_{Fl}(D) = \bigcup \Sigma_{Fl}({}_{\mathbb{P}}D),$$

where the union is taken over all contol sets ${}_{\mathbb{P}}D$ with nonvoid interior and $\mathbb{P}\pi({}_{\mathbb{P}}D) \subset D$.

Obviously, one has for every control set ${}_{\mathbb{P}}D \subset \mathbb{P}F$ the inclusion

$$\mathbf{\Sigma}_{Fl}({}_{\mathbb{P}}D) \subset \mathbf{\Sigma}_{Ly}({}_{\mathbb{P}}D),$$

where

$$\mathbf{\Sigma}_{Ly}({}_{\mathbb{P}}D) = \left\{ \begin{array}{c} \lambda(u,x),\ (u,x) \in \mathcal{U} \times F \,\text{with} \\ \varphi(t, \mathbb{P}x, u) \in cl\,{}_{\mathbb{P}}D \,\text{for all}\, t \in \mathbb{R} \end{array} \right\}$$

In this sense, the Floquet spectrum furnishes an 'inner' approximation to the Lyapunov spectrum.

Next we construct an 'outer' approximation of the Lyapunov spectrum given by the Morse spectrum. This concept is based on topological considerations, see [2]. One considers chain control sets in the projective bundle $\mathbb{P}F$ or, equivalently, the chain recurrent components of the corresponding control flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}F$ as an appropriate generalization of sums of generalized eigenspaces.

For $\varepsilon, T > 0$ an (ε, T)-chain ζ of $\mathbb{P}\Phi$ is given by $n \in \mathbb{N}$, $T_0, ..., T_n \geq T$, and $(u_0, p_0), ..., (u_n, p_n)$ in $\mathcal{U} \times \mathbb{P}F$ with $d(\mathbb{P}\Phi(T_i, u_i, p_i), (u_{i+1}, p_{i+1})) < \varepsilon$ for $i = 0, ..., n-1$. Define the finite time exponential growth rate of such a chain ζ (or 'chain exponent') by

$$\lambda(\zeta) = \frac{\sum_{i=0}^{n-1} (\log \mid \varphi(T_i, x_i, u_i) \mid - \log \mid x_i \mid)}{\sum_{i=0}^{n-1} T_i}$$

where $x_i \in \mathbb{P}^{-1}(p_i)$.

Definition 2 *Let ${}_{\mathbb{P}}\mathcal{L} \subset \mathcal{U} \times \mathbb{P}F$ be a compact invariant set for the induced flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}F$ and assume that $\mathbb{P}\Phi \mid {}_{\mathbb{P}}\mathcal{L}$ is chain transitive. Then the Morse spectrum over ${}_{\mathbb{P}}\mathcal{L}$ is*

$$\mathbf{\Sigma}_{Mo}({}_{\mathbb{P}}\mathcal{L}) = \left\{ \begin{array}{c} \lambda \in \mathbb{R},\ \text{there are}\, \varepsilon^k \to 0 \\ T^k \to \infty \,\text{and}\, (\varepsilon^k, T^k)\text{−chains}\, \zeta^k \\ \text{in}\, {}_{\mathbb{P}}\mathcal{L} \,\text{with}\, \lambda(\zeta^k) \to \lambda \,\text{as}\, k \to \infty \end{array} \right\}$$

For a compact invariant set $\mathcal{L} \subset \mathcal{U} \times M$ define the Morse spectrum over $\mathcal{L}$ as

$$\mathbf{\Sigma}_{Mo}(\mathcal{L}) = \bigcup \mathbf{\Sigma}_{Mo}({}_{\mathbb{P}}\mathcal{E}),$$

where the union is taken over all chain recurrent components ${}_{\mathbb{P}}\mathcal{E}$ of $\mathbb{P}\Phi \mid (\mathbb{P}\pi)^{-1}\mathcal{L}$.

Of particular interest is the Morse spectrum over a compact chain control set $E \subset M$. Then the lift $\mathcal{E} = \{(u,x),\ \varphi(t,x,u) \in E \,for\, t \in \mathbb{R}\}$ of E is a compact invariant set in $\mathcal{U} \times M$ and we write

$$\mathbf{\Sigma}_{Mo}(E) := \mathbf{\Sigma}_{Mo}(\mathcal{E}).$$

Define the Lyapunov spectrum over a control set D in the base space M as

$$\begin{aligned} \mathbf{\Sigma}_{Ly}(D) &= \{\lambda(u,x),\ (u,x) \in \mathcal{U} \times (F \setminus Z), \\ \pi\varphi(t, \pi x, u) &\in D \text{ for all } t \geq 0\}. \end{aligned}$$

There exists a chain control set E with $D \subset E$ and provided E is compact the following inclusions hold

$$\mathbf{\Sigma}_{Fl}(D) \subset \mathbf{\Sigma}_{Ly}(D) \subset \mathbf{\Sigma}_{Mo}(E).$$

Hence the Lyapunov spectrum lies in between the Floquet and the Morse spectrum. We are mainly interested in the Lyapunov spectrum, hence we present conditions which imply that the Floquet and the Morse spectrum, and hence the Lyapunov spectrum all coincide.

We denote for $U \subset \mathbb{R}^m$ compact and convex with $0 \in int\, U$ and $\rho \geq 0$

$$U^\rho = \rho U = \{\rho u,\ u \in U\}.$$

To complete the picture we also consider the case of unbounded perturbations. Let $\mathcal{U}^\infty = \bigcup_{\rho > 0} \mathcal{U}^\rho = L_\infty(\mathbb{R}, \mathbb{R}^m)$ and denote the corresponding control system by $(1)^\infty$. All quantities defined above will be written with a superscript ρ to indicate their dependence on the control range U^ρ for $0 \leq \rho \leq \infty$. Note that for $\rho = 0$ we obtain just a single differential equation.

Let E^0 be a chain recurrent component for the uncontrolled system $(4)^0$ in the base space M. For $\rho > 0$ there are chain control sets of $(4)^\rho$ with $E^0 \subset E^\rho$. Under an inner pair assumption there are also control sets D^ρ with $E^0 \subset int\, D^\rho$. Similarly, consider the uncontrolled projective system on $\mathbb{P}F$

$$\frac{d}{dt}\mathbb{P}x = \mathbb{P}X_0(\mathbb{P}x),\ t \in \mathbb{R} \qquad (14)$$

and denote by ${}_{\mathbb{P}}E_i^0$ the corresponding chain recurrent components in the projective bundle $\mathbb{P}F$ with $\mathbb{P}\pi({}_{\mathbb{P}}E_i^0) \subset E^0$, where i is in some index set I. Fix $\rho^* > 0$. Under the following ρ-inner pair condition in the projective bundle $\mathbb{P}F$ there are also control sets around the ${}_{\mathbb{P}}E_i^0$ growing with ρ:

$$\begin{array}{c}\text{For all } \rho', \rho \in [0, \rho^*) \text{ with } \rho' > \rho \text{ and all} \\ \text{lifts } {}_{\mathbb{P}}\mathcal{E}_i^\rho \text{ of chain control sets each} \\ (u, x) \in {}_{\mathbb{P}}\mathcal{E}_i^\rho \text{ is an inner pair for } (12)^{\rho'}\end{array} \tag{15}$$

The following theorem, see [3], describes the control sets and chain control sets and the corresponding spectra and characterizes the Lyapunov spectrum.

Theorem 3 *Fix $0 < \rho^* \leq \infty$ and assume that for all $\rho \in [0, \rho^*)$ the systems (12)$^\rho$ are locally accessible. Let E^0 be a chain recurrent component for the uncontrolled system (4)^0 in the base space M, and for $\rho \in [0, \rho^*)$ let E^ρ denote the unique chain control set of (4)$^\rho$ with $E^0 \subset E^\rho$. Assume that there is a compact set $K \subset M$ such that $E^\rho \subset K$ and the set K is positively invariant under all controls $u \in \mathcal{U}^{\rho^*}$ and that the ρ-inner pair condition (15) is satisfied.*
Then the following assertions hold:
(i) For all $\rho \in [0, \rho^*)$ and for every chain recurrent component ${}_{\mathbb{P}}E_i^0$, $1 \leq i \leq l(0) \leq d$, of the equation (14) there are chain control sets ${}_{\mathbb{P}}E_i^\rho$ with ${}_{\mathbb{P}}E_i^0 \subset {}_{\mathbb{P}}E_i^\rho$ and $\mathbb{P}\pi({}_{\mathbb{P}}E_i^\rho) = E^\rho$. There are no further chain control sets with $\mathbb{P}\pi({}_{\mathbb{P}}E_i^\rho) \cap E^\rho \neq \emptyset$, their number $l(\rho)$ is decreasing in ρ and satisfies $1 \leq l(\rho) \leq d$.
(ii) For all $\rho \in (0, \rho^*)$ there are unique control sets D^ρ of (4)$^\rho$ with $E^0 \subset int\, D^\rho$ and control sets ${}_{\mathbb{P}}D_i^\rho$ of (12)$^\rho$ with ${}_{\mathbb{P}}E_i^0 \subset int\, {}_{\mathbb{P}}D_i^\rho$ and $\mathbb{P}\pi({}_{\mathbb{P}}D_i^\rho) = D^\rho$; for all but at most countably many ρ-values and all $1 \leq l(\rho) \leq d$

$$cl\, D^\rho = E^\rho \text{ and } cl\, {}_{\mathbb{P}}D_i^\rho = {}_{\mathbb{P}}E_i^\rho.$$

(iii) For all $\rho \in [0, \rho^*)$ and for each $i = 1, ..., l(\rho)$ the Morse spectrum has the form

$$\mathbf{\Sigma}_{Mo}({}_{\mathbb{P}}E_i^\rho) = [\kappa^*({}_{\mathbb{P}}E_i^\rho), \kappa({}_{\mathbb{P}}E_i^\rho)]$$

with $\kappa^*({}_{\mathbb{P}}E_i^\rho) < \kappa^*({}_{\mathbb{P}}E_j^\rho)$ and $\kappa({}_{\mathbb{P}}E_i^\rho) < \kappa({}_{\mathbb{P}}E_j^\rho)$ if ${}_{\mathbb{P}}E_i^\rho \neq {}_{\mathbb{P}}E_j^\rho$ and $i < j$.
(iv) For each $i = 1, ..., l$ the sets of continuity points in $(0, \rho^*)$ of the two set valued maps $\rho \mapsto cl\, \Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho)$ and $\rho \mapsto \Sigma_{Mo}({}_{\mathbb{P}}E_i^\rho)$ agree and $\rho = 0$ is a continuity point of the latter map. There are at most countably many points of discontinuity and at each continuity point ρ the equality $cl\, \Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho) = \Sigma_{Mo}({}_{\mathbb{P}}E_i^\rho)$ holds. In particular, if ρ is a continuity point for all $i = 1, ..., l$, then

$$\begin{aligned} cl\, \mathbf{\Sigma}_{Fl}(D^\rho) &= \bigcup_{i=1}^{l} cl\, \Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho) = \mathbf{\Sigma}_{Ly}^\rho(D^\rho) \\ &= \bigcup_{i=1}^{l} \mathbf{\Sigma}_{Mo}({}_{\mathbb{P}}E_i^\rho) = \mathbf{\Sigma}_{Mo}^\rho(E^\rho) \end{aligned}$$

This theorem gives - under the ρ-inner pair condition - a complete characterization of the Lyapunov spectrum and of the corresponding generalized 'eigenspaces' for bilinear control systems on vector bundles.

If the bilinear control system occurs as the linearization as in (9), a stable subbundle gives rise to stable manifolds of the nonlinear system. These manifolds will depend on the considered point y and on the applied control u. Suppose, for simplicity, that (8) describes a system in $\mathbb{R}^d$. The associated control flow is denoted by $\Phi_t : \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d$. The control flow associated with the linearized system (9) is $\mathbf{T}\Phi_t(u, y, v) : \mathcal{U} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d \times \mathbb{R}^d$, $t \in \mathbb{R}$, hence

$$\mathbf{T}\Phi_t(u, y, v) = (u(t + \cdot), \mathbf{T}\varphi(t, y, u)v).$$

We assume that a control set $D \subset \mathbb{R}^d$ of the nonlinear system (8) with compact closure and nonvoid interior is given and restrict the control flow Φ to the lift $\mathcal{D}$ of the control set D

$$\begin{aligned} \mathcal{D} &= cl\, \{(u, y) \in \mathcal{U} \times \mathbb{R}^d, \\ \varphi(t, y, u) &\in int\, D \text{ for all } t \in \mathbb{R}\}. \end{aligned}$$

Similarly, we restrict the linearized control flow to the vector bundle $\pi : \mathcal{V}_\mathcal{D} := \mathcal{D} \times \mathbb{R}^d \to \mathcal{D}$ with compact base $\mathcal{D}$. For simplicity of notation, we denote also these restrictions by Φ and $\mathbf{T}\Phi$, respectively.

We assume that a decomposition into exponentially separated subbundles, one of them exponentially stable is given:

The linearized flow $\mathbf{T}\Phi$ on $\mathcal{V}_\mathcal{D}$ admits the following decomposition into invariant subbundles

$$\mathcal{V}_\mathcal{D} = \mathcal{V}^+ \bigoplus \mathcal{V}^- \tag{16}$$

such that there are constants $c_0 > 0$ and $\varepsilon_0 > 0$ with

$$| \mathbf{T}\Phi_t(u, y, v^+) | \leq c_0 \exp(-\varepsilon_0 t) \; | \mathbf{T}\Phi_t(u, y, v^-) | \tag{17}$$

for all $t \geq 0, (u,y,v^+) \in \mathcal{V}^+$, $(u,y,v^-) \in \mathcal{V}^-$ with $| v^+ | = | v^- | = 1$, and the subbundle $\mathcal{V}^+$ is stable,

$$\kappa^+ := \sup \mathbf{\Sigma}_{Ly}(\mathcal{V}^+) < 0 \qquad (18)$$

The condition (18) is equivalent to $\mathbf{\Sigma}_{Mo}(\mathcal{V}^+) \subset (-\infty, 0)$. Note that we do not assume hyperbolicity, that is, we allow $\mathbf{\Sigma}_{Ly}(\mathcal{V}^-) \cap (-\infty, 0) \neq \emptyset$.

If we assume, additionally, that the closure of the control set D is a chain control set, then the decomposition (16), (17) implies that the subbundles $\mathcal{V}^+$ and $\mathcal{V}^-$ are sums of subbundles $\mathcal{V}_i$, which project down to lifted chain control sets ${}_{\mathbb{P}}\mathcal{E}_i$ in the projective bundle $\mathbb{P}\mathbf{T}_{cl\,D}M$. This follows, since these $\mathcal{V}_i$ yield the finest decomposition into exponentially separated subbundles. Hence, in order to obtain such a decomposition, it suffices to check which chain control sets ${}_{\mathbb{P}}E_i$ over $cl\,D$ have negative spectrum.

The following theorem describes the corresponding stable manifold.

Theorem 4 *For the control system (8) consider a control set $D \subset \mathbb{R}^d$ with nonvoid interior and compact closure and lift $\mathcal{D}$. Suppose that for the linearized control system (9) the associated linearized control flow $\mathbf{T}\Phi$ on the vector bundle $\mathcal{V}_\mathcal{D} = \mathcal{D} \times \mathbb{R}^d \to \mathcal{D}$ admits the decomposition (16), (17), and suppose that the stable bundle $\mathcal{V}^+ \subset \mathcal{D} \times \mathbb{R}^d$ satisfies (18). Then there are $\delta > 0$ and a homeomorphism*

$$S^+ : \{(u,y,z) \in \mathcal{V}^+, \ | z | < \delta\} \to \mathcal{V}_\mathcal{D} = \mathcal{D} \times \mathbb{R}^d$$

of the form

$$S^+(u,y,z) = (u,y,s^+(u,y,z))$$

such that $\mathcal{W}^+ := \operatorname{Im} S^+$ called the local stable manifold corresponding to the stable subbundle $\mathcal{V}^+$ has the following properties:
(i) The set $\mathcal{W}^+$ is positively invariant under the control flow Φ, i.e., for $(u,y,\hat{y}) \in \mathcal{W}^+$ one has

$$(u(t+\cdot), \varphi(t,y,u), \varphi(t,\hat{y},u)) \in \mathcal{W}^+ \text{ for all } t \geq 0$$

(ii) for each $(u,y,\hat{y}) \in \mathcal{W}^+$ one has

$$\lim_{t\to\infty} e^{-\alpha t} [\varphi(t,\hat{y},u) - \varphi(t,y,u)] = 0$$

for every α with $\alpha > \kappa(\mathcal{V}^+)$.
(iii) for each $(u,y) \in \mathcal{D}$ the local stable manifolds at (u,y) defined by

$$\mathcal{W}^+_{(u,y)} = \{\hat{y} \in \mathbb{R}^d, \ (u,y,\hat{y}) \in \mathcal{W}^+\}$$

are topological manifolds and their dimension equals the dimension of $\mathcal{V}^+$.
(iv) The distance of the subbundle $\mathcal{W}^+$ to $\mathcal{V}^+$ can be made arbitrarily small in a Lipschitz sense.

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