Average and Asymptotic Properties of Apportionment Methods for Proportional Representation

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Chapter 1

Introduction

Because in democratic systems the electoral outcome decides on the line of future policy, the process of voting is of great importance for society. In general, an election consists of two parts, both influencing its result. In the first step, each voter gives his vote to one of the parties participating in the election. The numbers of votes in favor of the competing parties then give rise to vote proportions, which specify the share of voters supporting a party. In the second step, almost continuous vote proportions have to be translated into integer numbers of seats in the parliament. Translating electoral votes into specific seat allocations, the process of apportionment, unavoidably influences the final distribution of power, because in general it involves some kind of adjustment of the fractional seats that would arise if literal calculation were possible. As a consequence, it is an important issue in proportional representation systems to measure the effects of this adjustment process, in order to judge which apportionment method is most suitable for application.

The following chapters will concentrate on some of the most popular apportionment methods: The quota method of greatest remainders and the stationary divisor methods. We will investigate whether these apportionment methods, on average, treat smaller and larger parties equally or allow a systematic advantage in either direction. For measuring the effect of the adjustment process, the concept of seat biases has been introduced. Seat biases are defined as averages of the difference between the seats actually apportioned to the competing parties and their ideal shares of seats. Of course, apportionment methods should result in vanishing seat biases for legal reasons. Assuming repeated application of these methods, we will be able to determine the seat biases affecting the various parties. A geometric-combinatorial approach to the calculation of seat biases will be introduced, which turns out to be highly useful in order to evaluate this expectation. It is based on a combination of knowledge about the geometry of sets of vote proportions leading to a specific seat allocation and of a combinatorial method of accounting for all possible seat allocations. The political character of the problem of a violated proportionality calls for quantitative seat bias results, which become accessible in a rigorous fashion by means of the geometric-combinatorial approach.

The geometry of rounding polytopes, which are the sets of vote proportions rounded to a specific seat allocation, is discussed in chapter 2. For the mentioned apportionment methods, the vertices and surface volumes of all rounding polytopes will be calculated. These results are necessary in order to determine and compare average properties of the methods, such as the seat biases. To deal with average behaviour, we assume uniformly distributed vote proportions in the following, implying that the probability of a specific rounding polytope is proportional to its surface volume. Chapter 2 begins with a short survey of rounding methods, fixing several notations. Then we turn to the calculation of vertices and volumes of rounding polytopes, seperately for the quota method of greatest remainders and the divisor methods. To do that, we decompose the rounding polytopes into simplices, for which the volume can be evaluated by means of their vertices.

By the assumption of uniformly distributed vote proportions, we derive the distribution of the seat allocations for the quota method of greatest remainders and stationary divisor methods in chapter 3. Then an appropriate combinatorial method to account for all seat allocations is presented and a formula for the calculation of seat biases is given, based on polynomials reflecting the combinatorial structure of the problem. Via explicit expressions for these apportionment polynomials we finally can calculate seat biases.

As an example, seat bias formulas for systems of four parties are given in chapter 4. Furthermore, we will realize that the concept of seat biases can be extended to electoral systems with thresholds, that is, with minimum numbers of votes a party must reach in order to be eligible to participate in the apportionment process. All the seat biases are found to decrease from their maxima to zero, when the threshold increases from zero to its maximum, and that this decrease is linear. The final section of chapter 4 deals with further criteria to decide which apportionment method is most suitable for application. For several methods and numbers of parties, the probability of violating two important criteria is calculated by means of the geometric-combinatorial approach.

In chapter 5 we prove a previous conjecture on asymptotic seat biases of the quota method of greatest remainders and of the stationary divisor methods, when the size of parliament tends to infinity. The proof relies on the general formula for calculating seat biases from chapter 3, and on knowledge about the leading terms of the apportionment polynomials in the size of parliament, which we derive in the second section.

Our analysis of the seat bias, which is the conditional expectation of the seat excess, is complemented in chapter 6 by a study of the seat excess variance. For two and three parties, the variance is determined for the quota method of greatest remainders and for stationary divisor methods, where the derivation relies on a calculation of barycenters of rounding polytopes, for which we use generalized apportionment polynomials. Moreover, numerical simulations and a study of Bavarian electoral data are discussed.

In chapter 7 an alternative seat bias model is addressed and compared to the model used for the previous studies. Instead of taking the voter's point of view by assuming a uniform distribution of the vote proportions, the apportionment-oriented model stresses the importance of the rounding process for the allocation of seats. We will prove for the stationary divisor methods that these two models reveal the same asymptotic behaviour when the number of seats in parliament grows. For that purpose, we proceed analogous to the calculation of seat biases in chapters 2 and 3. However, the geometrical part now is much simpler, whereas the combinatorial part needs additional considerations.

Chapter 2

Surface Volumes of Rounding Polytopes

Consider a vector $\mathbf{w} = (w_1, \ldots, w_\ell)^t$ of $\ell \geq 2$ non-negative continuous weights that sum to one. In the following, these weights are a set of probabilities. The rounding problem consists of rounding each weight w_i to a non-negative integer m_i such that the rounding result $\mathbf{m} = (m_1, \ldots, m_\ell)^t$ sums to a given integer accuracy M, i.e. the continuous weight w_i is approximated by the rational proportion m_i/M. It is well-known that rounding the weights w_i individually may leave a discrepancy between the sum of the rounding results m_i and the desired accuracy M, see [16, Section 1]. However, such a discrepancy is often infeasible, and rounding methods are needed that yield rounding results summing to the predetermined accuracy. An example is the apportionment of seats in a parliament with the fixed house size M, by rounding proportions of votes. Other examples can be found in statistics [30, 31].

In this chapter new mathematical insight into traditional rounding methods is developed by characterizing the sets of weight vectors $\mathbf{w}$ which get rounded to a fixed integer vector $\mathbf{m}$. These sets are polytopes, for all methods considered here. As the weights are constrained to sum to one, the rounding polytopes are of dimension $\ell - 1$, where ℓ is the number of weights to be rounded. For a given rounding method, the vertices and surface volumes of all rounding polytopes are determined. The derivation of the results is based on monographs by Balinski and Young [5] as well as Kopfermann [22], and original work by Pólya [29]. The findings have been published in [11] and [12]. They are important for the comparison of different methods in terms of their average behaviour. For such an average behaviour it is common to assume uniformly distributed weights [1–4, 6, 7, 16, 32, 41], so that the probability of a rounding polytope is proportional to its surface volume.

2.1 Rounding Methods and Rounding Polytopes

Let the *probability simplex* S^ℓ be the set of non-negative vectors summing to one

$$S^\ell := \left\{ \mathbf{w} \in [0,1]^\ell : \sum_{i=1}^{\ell} w_i = 1 \right\},$$

and let the weight vector $\mathbf{w} = (w_1, \dots, w_\ell)^t$ be uniformly distributed on S^ℓ. Rounding $\mathbf{w}$ to the given integer accuracy M means to map it to a vector of non-negative integers $\mathbf{m}$ with components summing to M. A *rounding method* therefore is a mapping

$$A : S^\ell \to G^\ell(M),$$

where

$$G^\ell(M) := \left\{ \mathbf{m} \in \mathbb{N}_0^\ell : \sum_{i=1}^{\ell} m_i = M \right\}.$$

Subsequently, we will only consider accuracies $M > \ell$. Details on the more pathological case $M \leq \ell$ can be found in [11].

The *quota method of greatest remainders* (Hamilton, Hare) is a rounding method that operates in two stages. In the first stage, the proportions $w_i M$ are rounded down to their integer parts $\bar{m}_i$. In the unlikely case that all $w_i M$ are integers the discrepancy

$$\delta := M - \sum_{i=1}^{\ell} \bar{m}_i \in \mathbb{N}_0$$

vanishes and we have $\mathbf{m} := (\bar{m}_i, \dots, \bar{m}_\ell)$. If there is a positive discrepancy, the fractional parts $w_i M - \bar{m}_i$ are ranked in the second stage (where ties are broken arbitrarily). Then the vector $\mathbf{m}$ is obtained by setting $m_i = \bar{m}_i + 1$ for the δ largest remainders and $m_i = \bar{m}_i$ for the $\ell - \delta$ smallest remainders.

A popular family of rounding methods is given by the *divisor methods*. Following the definition of Balinski and Rachev [4, p. 3], a divisor method is based on a strictly isotonic sequence $s(k)$ of reals such that $k \leq s(k) \leq k+1$ for all $k \in \mathbb{N}_0$ and there exists no pair of integers k_1, k_2 with $d(k_1) = k_1 + 1$ and $d(k_2) = k_2$. This sign-post sequence defines a *rounding function*

$$r : [0, \infty) \to \mathbb{N}_0, \quad x \mapsto r(x) := \begin{cases} k & \text{for } x \in [k, s(k)), \\ k+1 & \text{for } x \in [s(k), k+1). \end{cases}$$

Ties $x = s(k)$ may be broken in a different way than setting $r(x) = k+1$. However, since we consider weights for which a tie appears with probability zero, this ambiguity does not affect our results. The divisor method with sign-post sequence $s(k)$ maps a weight vector $\mathbf{w}$ into the integer vector $A(\mathbf{w}) \in G^\ell(M)$ such that there exists a divisor $D \in (0, \infty)$ with $m_i := A(w_i) = r(w_i/D)$ for all i.

Important sub-classes of the family are the q*-stationary divisor methods* with parameter $q \in [0,1]$ based on the sequence $s(k) = k + q$, and the p*-power mean divisor methods* with parameter $p \in \mathbb{R}$ based on the sequence $s(k) = [(k^p + (k+1)^p)/2]^{1/p}$. They give rise to five popular divisor methods, see [5, p. 61]:

1.) Adams: $s(k) = k$ (rounding up, $q = 0$, $p = -\infty$),

2.) Webster, Sainte-Laguë: $s(k) = k + 0.5$ (standard rounding, $q = 0.5$, $p = 1$),

3.) Jefferson, D'Hondt: $s(k) = k + 1$ (rounding down, $q = 1$, $p = \infty$),

4.) Hill, Huntington: $s(k) = \sqrt{k(k+1)}$ (geometric rounding, $p = 0$),

5.) Dean: $s(k) = k(k+1)/(k+0.5)$ (harmonic rounding, $p = -1$).

Marshall, Olkin, and Pukelsheim [23] give a comparison of these five methods in terms of majorization. An implementation of divisor methods following Dorfleitner and Klein [10] is provided by the computer program BAZI, see `http://www.uni-augsburg.de/bazi`.

The stationary divisor methods with parameter $q \in [0,1]$ are defined by the rounding function r_q that rounds down if the fractional part of a nonnegative number is less than q, and up if it is greater than q. More formally, denote the integer and fractional part of a number $x \geq 0$ by $\lfloor x \rfloor = \text{IntegerPart}(x)$ and $x - \lfloor x \rfloor = \text{FractionalPart}(x)$, respectively. Then
$$r_q(x) = \begin{cases} \lceil x \rceil = \text{IntegerPart}(x) + 1 & \text{for FractionalPart}(x) > q, \\ \lfloor x \rfloor = \text{IntegerPart}(x) & \text{for FractionalPart}(x) < q. \end{cases}$$
Ties occur if $\text{FractionalPart}(x) = q$; in this case the definition can stipulate $r_q(x) = \lfloor x \rfloor$ or $r_q(x) = \lceil x \rceil$.

All rounding methods presented above map a weight vector with permuted entries to the permuted integer vector

$$A((w_{\sigma(1)}, \ldots, w_{\sigma(\ell)})^t) = (m_{\sigma(1)}, \ldots, m_{\sigma(\ell)})^t$$

for any permutation σ. This property will be tacitly used in the subsequent proofs.

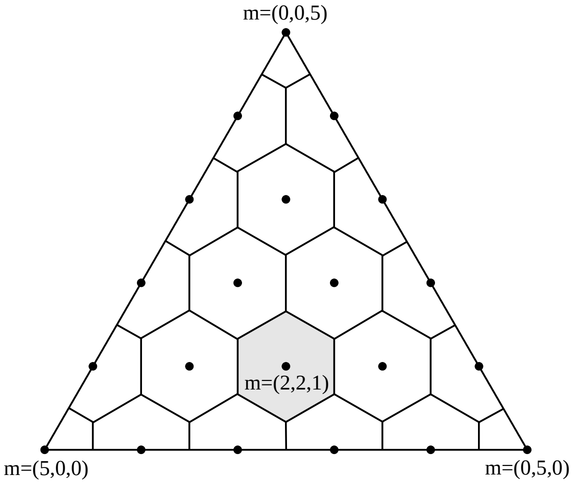
In the sequel we study the sets of weight vectors $\mathbf{w}$ rounded to a given integer vector $\mathbf{m} \in G^\ell(M)$. For both the quota method of greatest remainders and the divisor methods ties are broken arbitrarily. For example, let $\mathbf{w} = (0.5, 0.5)$ and $M = 3$; then the rounding results $\mathbf{m}_1 = (2,1)$ and $\mathbf{m}_2 = (1,2)$ are possible. We define the sets, for $\mathbf{m} \in G^\ell(M)$,

$$P_A(\mathbf{m}) := \text{cl}\left\{\mathbf{w} \in S^\ell : A(\mathbf{w}) = \mathbf{m}\right\}$$

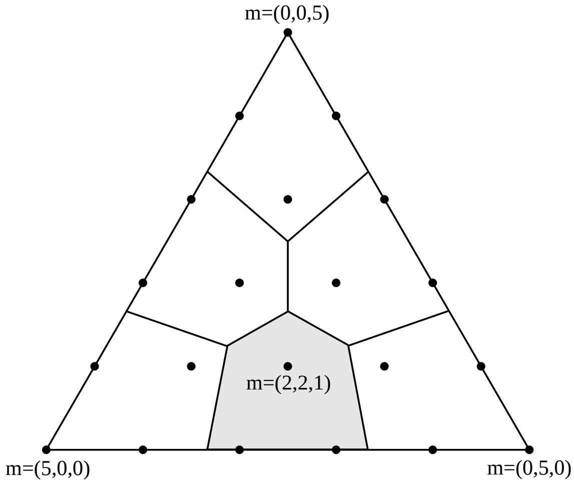
of weight vectors that can be rounded to $\mathbf{m}$ under A when the ties are broken arbitrarily, where set closure is denoted by cl. We obtain

$$\mathbf{w} \in P_A(\mathbf{m}) \Longleftrightarrow \mathbf{m} \in A(\mathbf{w}).$$

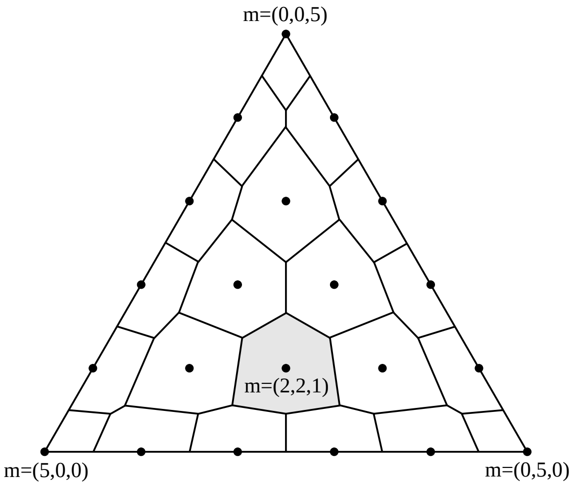
With these preparations we now are able to characterize the above rounding methods via linear inequalities.



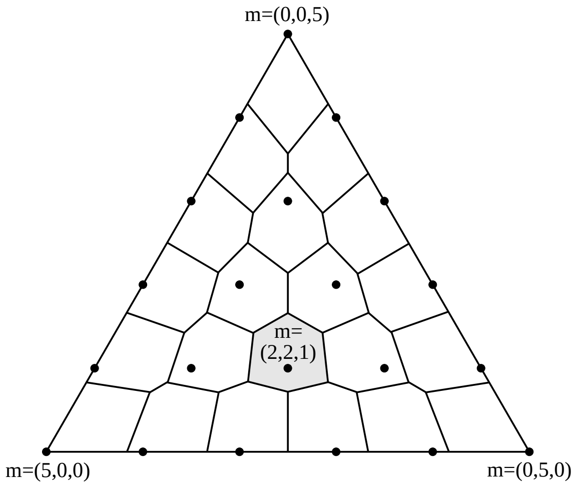
(a) Quota method of greatest remainders.



(b) Divisor method: rounding up.



(c) Divisor method: standard rounding.



(d) Divisor method: rounding down.

Figure 2.1: Rounding polytopes for $\ell = 3$ weights and accuracy $M = 5$.

Lemma 2.1 (Characterization via linear inequalities)
Let $\mathbf{m} \in G^\ell(M)$ be a rounding result and let $\mathbf{w} \in S^\ell$ be a weight vector.

*(a) Let A be the quota method of greatest remainders.
Then $\mathbf{w} \in P_A(\mathbf{m})$ if and only if*

$$Mw_i - m_i \leq Mw_j - m_j + 1 \quad \textit{for all } i, j = 1, \ldots, \ell \textit{ with } i \neq j. \tag{2.1}$$

*(b) Let A be the divisor method with sign-post sequence s.
Then $\mathbf{w} \in P_A(\mathbf{m})$ if and only if*

$$w_i s(m_j - 1) \leq w_j s(m_i) \quad \textit{for all } i, j = 1, \ldots, \ell \textit{ with } i \neq j. \tag{2.2}$$

Proof. See [22, pp. 196, 202] and [5, p. 100]. **qed**

The inequalities of Lemma 2.1 describe $P_A(\mathbf{m})$ as a polyhedron. As $P_A(\mathbf{m}) \subseteq S^\ell$ and S^ℓ is bounded, $P_A(\mathbf{m})$ is a polytope, which we call the *rounding polytope* of the rounding result $\mathbf{m}$ under the method A. Figure 2.1 shows rounding polytopes for four methods in barycentric coordinates, i.e. a point $\mathbf{w}$ in one of the triangles represents the vector of the shortest distances from this point to the triangle edges. Note that a divisor method with $s(0) = 0$ rounds exclusively to interior lattice points; compare the case of rounding up in Figure 2.1.

By characterizing $P_A(\mathbf{m})$ in terms of its vertices, the surface volume of $P_A(\mathbf{m})$ can be computed. In the special case that A is a q-stationary divisor method and that $m_i \geq 1$ for all i, these results were already obtained by Kopfermann [22, Section 6.2]. The boundary cases with $m_i = 0$ for some i need particular attention, see Figure 2.1. Going beyond the work of Kopfermann the following considerations comprise all boundary cases for divisor methods as well as a full treatment of the quota method of greatest remainders.

We decompose the polytope $P_A(\mathbf{m})$ into simplices whose surface volumes can be calculated by determinant formulas [40, p. 278]. A d-dimensional simplex is a d-dimensional polytope with $d+1$ vertices $v_0, \ldots, v_d$, and d-dimensional volume equal to $1/d!$ times the modulus of the determinant of a $d \times d$-matrix with columns $v_i - v_0$, $i = 1, \ldots, d$.

A surface volume is defined by means of a full-dimensional volume after a projection, see [15, Section V.4]. Here, when

$$\pi : S^\ell \to \left\{ \mathbf{w} \in [0,1]^{\ell-1} : \sum_{i=1}^{\ell-1} w_i \leq 1 \right\}, \quad \mathbf{w} \mapsto (w_1, \ldots, w_{\ell-1}),$$

is the projection on the first $\ell - 1$ components, we get for any measurable set $A \subseteq S^\ell$

$$\mathrm{Vol}(A) = \sqrt{\ell} \times \mathrm{Vol}(\pi(A)).$$

Note that the volume on the left hand side of this equation is a surface volume, whereas the volume on the right hand side is full-dimensional. In particular, the entire simplex S^ℓ has volume

$$\mathrm{Vol}(S^\ell) = \frac{\sqrt{\ell}}{(\ell-1)!}.$$

Because in the following no confusion is possible, we will refer to surface volumes simply as volumes.

2.2 Quota Method of Greatest Remainders

Consider rounding via the quota method of greatest remainders and let $\mathbf{m} \in G^\ell(M)$ be a possible rounding result. Moreover, let

$$R(\mathbf{m}) := \{i : m_i \neq 0\}$$

denote the set of indices of non-zero components of $\mathbf{m}$, with cardinality $r(\mathbf{m}) := |R(\mathbf{m})|$. In this section, we will first study the vertices of $P(\mathbf{m}) := P_A(\mathbf{m})$, and then calculate the volume of the polytope.

By Lemma 2.1, the translation $T : \mathbf{w} \mapsto \mathbf{x} = \mathbf{w} - \mathbf{m}/M$ maps any rounding polytope $P(\mathbf{m})$ that lies in the interior of S^ℓ, i.e. $r(\mathbf{m}) = \ell$, into the *standard polytope*

$$P_0 := \left\{ \mathbf{x} \in \mathbb{R}^\ell : \sum_{i=1}^{\ell} x_i = 0,\ x_i \leq x_j + \frac{1}{M} \quad \text{for all } i \neq j \right\}.$$

If $m_i = 0$, then the constraint $w_i \geq 0$ remains invariant under the translation T, i.e. it is translated into the constraint $x_i \geq 0$. Hence, a rounding polytope $P(\mathbf{m})$ with $r(\mathbf{m}) < \ell$ is translated into the restricted standard polytope

$$P_0 \ \cap \bigcap_{i \notin R(\mathbf{m})} \left\{ \mathbf{x} \in \mathbb{R}^\ell : x_i \geq 0 \right\}.$$

In particular, $P(\mathbf{m})$ and $P(\bar{\mathbf{m}})$ are congruent whenever $r(\mathbf{m}) = r(\bar{\mathbf{m}})$. Theorem 2.1 gives the vertices of the restricted standard polytope, and adding $\mathbf{m}/M$ results in the vertices of $P(\mathbf{m})$. We denote the row vectors in $\mathbb{R}^\ell$ with all components equal to 1 or 0 by $\mathbf{1}_\ell$ and $\mathbf{0}_\ell$, respectively.

Theorem 2.1 (Vertices of the restricted standard polytope)
The restricted standard polytope

$$P_0 \ \cap \bigcap_{i \notin R(\mathbf{m})} \left\{ \mathbf{x} \in \mathbb{R}^\ell : x_i \geq 0 \right\}$$

has $2^\ell - 2^{\ell - r(\mathbf{m})} - 1$ *vertices* $\mathbf{v}(\lambda)$*, which are induced by vectors* $\lambda \in \{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\}$ *with* $\lambda_j = 0$ *for some index* $j \in R(\mathbf{m})$*. The components of* $\mathbf{v}(\lambda)$ *are, for* $i = 1, \dots, \ell$*,*

$$v_i(\lambda) = \begin{cases} \frac{1}{M}\left(1 - \frac{e(\lambda)}{\ell - z(\lambda)}\right) & \text{if } \lambda_i = 1, \\ -\frac{1}{M}\frac{e(\lambda)}{\ell - z(\lambda)} & \text{if } \lambda_i = 0 \text{ and } i \in R(\mathbf{m}), \\ 0 & \text{if } \lambda_i = 0 \text{ and } i \notin R(\mathbf{m}), \end{cases} \tag{2.3}$$

where $z(\lambda) := |\{i \notin R(\mathbf{m}) : \lambda_i = 0\}|$ *and* $e(\lambda) := |\{1 \leq i \leq \ell : \lambda_i = 1\}|$.
If $r(\mathbf{m}) = 1$*, then* $\mathbf{v}(\mathbf{0}_\ell) = \mathbf{0}_\ell$ *is an additional vertex and the restricted standard polytope has* $2^\ell - 2^{\ell - r(\mathbf{m})} = 2^{\ell-1}$ *vertices.*
There are no other vertices than the indicated $\mathbf{v}(\lambda)$*.*

In order to prove Theorem 2.1 we investigate the standard polytope P_0, and later the restricted standard polytope. By the parallelotope decomposition of P_0 in Lemma 2.2 we are able to determine the vertices of the standard polytope in Lemma 2.3. Thus, let the vector $\mathbf{u}(k) \in \mathbb{R}^\ell$ have component k equal to $(\ell-1)/M\ell$ and all other components equal to $-1/M\ell$.

Lemma 2.2 (Parallelotope decomposition of the standard polytope)
Define the parallelotopes

$$L_i := \left\{ \sum_{k\neq i} \mu_k \mathbf{u}(k) : \mu_k \in [0,1] \right\} \subseteq \mathbb{R}^\ell, \quad i = 1, \ldots, \ell.$$

Then $\operatorname{int}(L_i)$ *and* $\operatorname{int}(L_j)$ *are disjoint if* $i \neq j$*, and* $P_0 = \bigcup_{i=1}^\ell L_i$*.*

Proof. A vector $\mathbf{x} \in \operatorname{int}(L_i) \cap \operatorname{int}(L_j)$ can be expressed as

$$\mathbf{x} = \sum_{k\neq i} \mu_k \mathbf{u}(k) = \sum_{k\neq j} \delta_k \mathbf{u}(k)$$

with all μ_k and δ_k positive. It follows that

$$\mathbf{x} = \mathbf{x} - \delta_i \sum_{k=1}^{\ell} \mathbf{u}(k) = \sum_{k\neq i,j} (\delta_k - \delta_i)\, \mathbf{u}(k) - \delta_i \mathbf{u}(j).$$

Since $\mathbf{u}(1), \ldots, \mathbf{u}(i-1), \mathbf{u}(i+1), \ldots, \mathbf{u}(\ell)$ form a basis of $\{\mathbf{x} : \sum_i x_i = 0\}$, it follows that $\mu_j = -\delta_i$, which contradicts the fact that μ_j and δ_i are positive.

To see $\bigcup_{i=1}^\ell L_i \subseteq P_0$, let $\mathbf{x} = \sum_{k=1}^\ell \mu_k \mathbf{u}(k)$ with $\mu_k \in [0,1]$. For $p \neq q$,

$$\begin{aligned} x_p - x_q &= \sum_{k=1}^{\ell} \mu_k \left(u_p(k) - u_q(k)\right) = \mu_p \left(\frac{\ell-1}{M\ell} + \frac{1}{M\ell} \right) + \mu_q \left(-\frac{1}{M\ell} - \frac{\ell-1}{M\ell} \right) \\ &= \frac{1}{M} (\mu_p - \mu_q) \leq \frac{1}{M}. \end{aligned}$$

Hence, $\mathbf{x} \in P_0$.

Conversely, let $\mathbf{x} \in P_0$. We need to show $\mathbf{x} \in L_i$, for some i. As a set of $\ell - 1$ vectors among $\mathbf{u}(1), \ldots, \mathbf{u}(\ell)$ forms a basis of $\{\mathbf{x} : \sum_i x_i = 0\}$, we have $\mathbf{x} = \sum_{k=1}^{\ell-1} \mu_k \mathbf{u}(k)$. Since $\sum_{k=1}^\ell \mathbf{u}(k) = \mathbf{0}_\ell$, we write $\mathbf{x} = \sum_{k=1}^\ell \mu_k \mathbf{u}(k)$ with $\mu_k \geq 0$ for all k and $\mu_i = 0$ for some i. We obtain, for $p \neq q$,

$$\frac{1}{M} \geq \frac{x_p - x_q}{M} = \frac{\mu_p - \mu_q}{M} \geq \frac{\mu_p}{M}.$$

Thus $\mu_p \leq 1$ for all p, which implies $\mathbf{x} \in L_i$. **qed**

Lemma 2.3 (Vertices of the standard polytope)
Every vector $\lambda \in \{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\}$ *induces a vertex* $\mathbf{u}(\lambda)$ *of the standard polytope* P_0 *via*

$$u_i(\lambda) = \begin{cases} \frac{1}{M}\frac{\ell - e(\lambda)}{\ell} & \text{if } \lambda_i = 1, \\ -\frac{1}{M}\frac{e(\lambda)}{\ell} & \text{if } \lambda_i = 0, \end{cases} \tag{2.4}$$

for $i = 1, \dots, \ell$, *with* $e(\lambda) := |\{1 \le i \le \ell : \lambda_i = 1\}|$. *There are no other vertices.*

Proof. Obviously $\mathbf{u}(\lambda) = \sum_{k=1}^{\ell} \lambda_k \mathbf{u}(k)$, which yields that

$$\mathbf{u}(\lambda) \in \bigcap_{i:\lambda_i = 0} L_i \subset P_0.$$

The $\mathbf{u}(\lambda)$ with $e(\lambda) = 1$ are in fact the $\mathbf{u}(k)$ in the definition of the parallelotopes L_i. Due to symmetry with respect to permutations it suffices to concentrate on vectors $\mathbf{u}(\lambda)$ with the first $e(\lambda)$ components of λ equal to 1. Such a $\mathbf{u}(\lambda)$ solves

$$B_\ell \mathbf{u}(\lambda) := \begin{array}{c} \begin{array}{ccccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle \dots & \scriptstyle e(\lambda) & \scriptstyle e(\lambda)+1 & \scriptstyle \dots & \scriptstyle \ell \end{array} \\ \begin{pmatrix} 1 & & & & -1 & & \\ \vdots & & & & & \ddots & \\ 1 & & & & & & -1 \\ & 1 & & & -1 & & \\ & & \ddots & & \vdots & & \\ & & & 1 & -1 & & \\ 1 & 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix} \end{array} \mathbf{u}(\lambda) = \frac{1}{M} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}.$$

Since B_ℓ is a non-singular matrix, $\mathbf{u}(\lambda)$ is a vertex. By Lemma 2.2, P_0 is the convex hull of all $2^\ell - 2$ vertices $\mathbf{u}(\lambda)$. Therefore, no other vertices exist. **qed**

Proof of Theorem 2.1. Let $\mathbf{x}$ be a vertex of

$$P_0 \cap \bigcap_{i \notin R(\mathbf{m})} \left\{ \mathbf{x} \in \mathbb{R}^\ell : x_i \ge 0 \right\}.$$

Define $K := \{i \notin R(\mathbf{m}) : x_i = 0\}$ and let $k := |K|$. If $K = \emptyset$, i.e. $x_i > 0$ for all $i \notin R(\mathbf{m})$, then $\mathbf{x}$ must already be a vertex of P_0 and, consequently, one of the $\mathbf{v}(\lambda)$ with $z(\lambda) = 0$. Otherwise, the vector consisting of the components x_i of $\mathbf{x}$ with index $i \notin K$ must be a vertex of the $(\ell - k)$-dimensional standard polytope $P_0 \subset \mathbb{R}^{\ell - k}$ and thus $\mathbf{x}$ equals one of the $\mathbf{v}(\lambda)$ with $z(\lambda) = k$. Conversely, every $\mathbf{v}(\lambda)$ is a vertex since it fulfills

$$\begin{pmatrix} B_{\ell - z(\lambda)} & \\ & I_{z(\lambda)} \end{pmatrix} \mathbf{v}(\lambda) = \frac{1}{M} \begin{pmatrix} \mathbf{1}_{\ell - z(\lambda) - 1} \\ \mathbf{0}_{z(\lambda)+1} \end{pmatrix}.$$

Finally, we obtain $\mathbf{v}(\lambda) \ne \mathbf{v}(\bar\lambda)$ if λ and $\bar\lambda$ are two distinct vectors in $\{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\}$ such that there exist indices $i, j \in R(\mathbf{m})$ with $\lambda_i = \bar\lambda_j = 0$. The number of vertices hence equals $|\{\lambda \in \{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\} : \lambda_i = 0 \text{ for some } i \in R(\mathbf{m})\}| = 2^\ell - 2^{\ell - r(\mathbf{m})} - 1$. **qed**

Turning to the volumes of the rounding polytopes, a decomposition similar to Lemma 2.2 permits us to compute in Theorem 2.2 the volume of an arbitrary restricted standard polytope, which equals the volume of the associated rounding polytope. To this end it is convenient to introduce the set of 0-1 vectors in $\mathbb{R}^{\ell-1}$ having component sum k

$$\left\{ \begin{matrix} \ell-1 \\ k \end{matrix} \right\} := \left\{ \mathbf{t} \in \{0,1\}^{\ell-1} : \sum_{i=1}^{\ell-1} t_i = k \right\}.$$

Its cardinality is given by

$$\left| \left\{ \begin{matrix} \ell-1 \\ k \end{matrix} \right\} \right| = \binom{\ell-1}{k}.$$

In addition, for $\mathbf{t} = (t_1, \ldots, t_{\ell-1})^t \in \{0,1\}^{\ell-1}$ and $j \leq \ell - 1$ we define

$$t_{(j)} := \sum_{i=1}^{j} t_i.$$

Theorem 2.2 (Volumes of rounding polytopes for the Hamilton method)
The volume of $P(\mathbf{m})$ depends only on $r := r(\mathbf{m})$ and is given by

$$\operatorname{Vol}(P(\mathbf{m})) = \frac{\sqrt{\ell}}{\binom{\ell}{r} M^{\ell-1}} \sum_{\mathbf{t} \in \left\{ \begin{smallmatrix} \ell-1 \\ \ell-r \end{smallmatrix} \right\}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{r + t_{(j)}} \right)^{t_{j+1}}. \qquad (2.5)$$

Corollary 2.1 (Volumes of special rounding polytopes)
If $r(\mathbf{m}) \in \{1, \ell\}$, then the volume of $P(\mathbf{m})$ is given by

$$\operatorname{Vol}(P(\mathbf{m})) = \frac{\sqrt{\ell}}{M^{\ell-1}} \times \left\{ \begin{array}{ll} 1/\ell! & \text{for } r(\mathbf{m}) = 1, \\ 1 & \text{for } r(\mathbf{m}) = \ell. \end{array} \right. \qquad (2.6)$$

In order to prove Theorem 2.2, we establish in Lemma 2.4 a volume formula based on determinants. Simplifying this formula subsequently yields the Theorem.

Lemma 2.4 (Volume of the restricted standard polytope)
Let $1 \in R(\mathbf{m})$; otherwise permute the indices without changing volumes. Then

$$\operatorname{Vol}\left(P_0 \cap \bigcap_{i \notin R(\mathbf{m})} \{ x \in \mathbb{R}^\ell \mid x_i \geq 0 \} \right) = r \times \operatorname{Vol}(U_1), \qquad (2.7)$$

where U_1 is the convex hull of $\{\mathbf{v}(\lambda) : \lambda \in \{0,1\}^\ell, \lambda_1 = 0\}$.
Let π_1 be the projection onto the components with index different from 1. Then

$$\operatorname{Vol}(U_1) = \frac{\sqrt{\ell}}{\binom{\ell-1}{\ell-r}} \sum_{\mathbf{t} \in \left\{ \begin{smallmatrix} \ell-1 \\ \ell-r \end{smallmatrix} \right\}} \left| \det \left(\pi_1[\mathbf{v}(\lambda^1(\mathbf{t}))], \ldots, \pi_1[\mathbf{v}(\lambda^{\ell-1}(\mathbf{t}))] \right) \right| \qquad (2.8)$$

with

$$\lambda^j(\mathbf{t}) := \sum_{i=1}^{j-t_{(j)}} \epsilon_{i+1} + \sum_{i=1}^{t_{(j)}} \epsilon_{\ell-n+i}$$

and ϵ_i denoting the vector of the canonical basis in $\mathbb{R}^\ell$ having component i equal to 1 and all other components 0.

Proof. Let U_i be the convex hull of $\{\mathbf{v}(\lambda) : \lambda \in \{0,1\}^\ell, \lambda_i = 0\}$. Since we can express every $\mathbf{v}(\lambda)$ with $\lambda_i = 0$ as a linear combination $\sum_{j\neq i}^{\ell} \mu_j \mathbf{u}(j)$ by setting

$$\mu_j = \begin{cases} 1 & \text{for } \lambda_j = 1, \\ 0 & \text{for } \lambda_j = 0 \text{ and } j \in R(\mathbf{m}), \\ \frac{e(\lambda)}{\ell - z(\lambda)} & \text{for } \lambda_j = 0 \text{ and } j \notin R(\mathbf{m}), \end{cases}$$

we know that $U_i \subseteq L_i$. Thus, the interior of $U_i \cap U_j$ is empty if $i \neq j$ and for all $i \notin R(\mathbf{m})$ it follows that

$$\mathrm{Vol}(U_i) \leq \mathrm{Vol}\left(L_i \ \cap \bigcap_{i \notin R(\mathbf{m})} \{\mathbf{x} \in \mathbb{R}^\ell : x_i \geq 0\} \right) = 0.$$

The definition of U_i as convex hull of vertices leads to

$$P_0 \ \cap \bigcap_{i \notin R(\mathbf{m})} \{\mathbf{x} \in \mathbb{R}^\ell : x_i \geq 0\} = \bigcup_{i \in R(\mathbf{m})} U_i.$$

Since permuting the components i and j maps U_i in U_j and leaves the volume invariant, we have

$$\mathrm{Vol}\left(P_0 \cap \bigcap_{i \notin R(\mathbf{m})} \{\mathbf{x} \in \mathbb{R}^\ell : x_i \geq 0\} \right) = r \times \mathrm{Vol}(U_1).$$

To calculate the volume of U_1, we decompose it into simplices. Let therefore

$$\lambda^j := (0, \underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{\ell-1-j})^t$$

and let Σ_1 be the group of permutations of $\{1, \ldots, \ell\}$ leaving 1 fix. Then U_1 is the union of the simplices Δ_σ, $\sigma \in \Sigma_1$, that are defined as convex hull of $\mathbf{v}(\sigma(\lambda^j))$, $j = 0, \ldots, \ell - 1$. Note that $\mathrm{int}(\Delta_\sigma) \cap \mathrm{int}(\Delta_\tau) = \emptyset$ if $\sigma \neq \tau$. The volume of a simplex Δ_σ is $\sqrt{\ell}$ times the full-dimensional volume of the projected simplex $\pi_1(\Delta_\sigma)$, which can be calculated by the determinant formula.

Let $\sigma, \tau \in \Sigma_1$, and define the equivalence relation

$$\sigma \sim \tau \ :\Longleftrightarrow \ [\sigma(i) \notin R(\mathbf{m}) \iff \tau(i) \notin R(\mathbf{m}) \quad \text{for all } i] \, .$$

Then $\sigma \sim \tau$ implies that Δ_σ and Δ_τ have the same volume because they can be mapped into each other by a permutation. Since each equivalence class consists of $(\ell - n - 1)!n!$

permutations, we obtain the formula for $\mathrm{Vol}(U_1)$ stated in the theorem by summing over representatives of the equivalence classes. This is done by indexing the sum with vectors $\mathbf{t} \in \{0,1\}^{\ell-1}$ where $t_i = 1$ means that all permutations in the corresponding equivalence class fulfill $\sigma(i+1) \notin R(\mathbf{m})$ and $t_i = 0$ signifies $\sigma(i+1) \in R(\mathbf{m})$. **qed**

Proof of Theorem 2.2. By definition, the vectors $\lambda^j(\mathbf{t}) \in \{0,1\}^\ell$ have the form

$$\lambda^j(\mathbf{t}) = (0, \underbrace{1, ..., 1}_{j-t_{(j)}}, 0, ..., 0, \underbrace{1, ..., 1}_{t_{(j)}}, 0, ..., 0)^t$$

with exactly j components equal to one. Let Λ be the square matrix with columns equal to the last $\ell-1$ components of the vectors $\lambda^j(\mathbf{t})$, $j = 1, ..., \ell-1$. Since $\lambda_i^j(t) = 1$ implies $\lambda_i^{j+1}(\mathbf{t}) = 1$, we may transform Λ into an upper triangular matrix by permuting its rows. This transformation leaves the absolute value of the determinant of Λ unchanged. Now, the same permutation shall be applied to

$$\mathbf{v}(\Lambda) := \left(\pi_1[\mathbf{v}(\lambda^1(\mathbf{t}))], ..., \pi_1[\mathbf{v}(\lambda^{\ell-1}(\mathbf{t}))]\right).$$

By (2.3) and since $e(\lambda^j) = j$ and $z(\lambda^j) = \ell - r - t_{(j)}$, an appropriate permutation of rows leads to

$$\det(\mathbf{v}(\Lambda)) = \det \begin{pmatrix} \frac{1}{M}\left(1 - \frac{1}{r+t_{(1)}}\right) & \cdots & \frac{1}{M}\left(1 - \frac{\ell-1}{r+t_{(\ell-1)}}\right) \\ & \ddots & \vdots \\ \star & & \frac{1}{M}\left(1 - \frac{\ell-1}{r+t_{(\ell-1)}}\right) \end{pmatrix}.$$

Here, and in the remainder of the evaluation of the determinant, we ignore possible sign changes due to the absolute value in (2.8). The lower triangular part $(\star)$ of the permuted matrix $\mathbf{v}(\Lambda)$ corresponds to zeros in the vectors $\lambda^j(\mathbf{t})$, $j = 1, ..., \ell-1$; subsequently only the first sub-diagonal is of interest. By (2.3) and the definition of $\lambda^j(\mathbf{t})$, the sub-diagonal entry in the j-th column equals 0 if $t_{j+1} = 1$ and

$$-\frac{1}{M}\left(\frac{j}{r+t_{(j)}}\right) \quad \text{if } t_{j+1} = 0.$$

To simplify the determinant, we subtract the first row from all other rows. This gives

$$\det(\mathbf{v}(\Lambda)) = \det \begin{pmatrix} \frac{1}{M}\left(1 - \frac{1}{r+t_{(1)}}\right) & \cdots & \cdots & \frac{1}{M}\left(1 - \frac{\ell-1}{r+t_{(\ell-1)}}\right) \\ & 0 & \cdots & 0 \\ & & \ddots & \vdots \\ \star & & & 0 \end{pmatrix},$$

where the sub-diagonal entry in column j is

$$a_j = \begin{cases} -\frac{1}{M}\left(1 - \frac{j}{r+t_{(j)}}\right) & \text{if } t_{j+1} = 1, \\ -\frac{1}{M} & \text{if } t_{j+1} = 0. \end{cases}$$

λ	$e(\lambda)$	$\mathbf{v}(\lambda)$
$(0,0,1)^t$	1	$\frac{1}{5}\left(-\frac{1}{3},-\frac{1}{3},+\frac{2}{3}\right)^t$
$(0,1,1)^t$	3	$\frac{1}{5}\left(-\frac{2}{3},+\frac{1}{3},+\frac{1}{3}\right)^t$
$(0,1,0)^t$	1	$\frac{1}{5}\left(-\frac{1}{3},+\frac{2}{3},-\frac{1}{3}\right)^t$
$(1,1,0)^t$	2	$\frac{1}{5}\left(+\frac{1}{3},+\frac{1}{3},-\frac{2}{3}\right)^t$
$(1,0,0)^t$	1	$\frac{1}{5}\left(+\frac{2}{3},-\frac{1}{3},-\frac{1}{3}\right)^t$
$(1,0,1)^t$	2	$\frac{1}{5}\left(+\frac{1}{3},-\frac{2}{3},+\frac{1}{3}\right)^t$

Table 2.2: Vertices $\mathbf{v}(\lambda)$ of the standard rounding polytope P_0 for the quota method of greatest remainders, as determined by Theorem 2.1 ($M = 5$, $\ell = 3$).

Since by definition $t_{(\ell-1)} = \ell - r$ it follows that

$$\det(\mathbf{v}(\Lambda)) = \frac{\prod_{j=1}^{\ell-2} a_j}{M\ell},$$

which implies the result stated in Theorem 2.2. **qed**

In order to illustrate the previous results we consider the rounding polytope for

$$\mathbf{m} := (2,2,1)^t,$$

which is highlighted in Figure 2.1(a). By Theorem 2.1 with $r(\mathbf{m}) = 3$, the $2^3 - 2^0 - 1 = 6$ vertices of the polytope are determined via the $\mathbf{v}(\lambda)$ given in Table 2.2. Adding $\mathbf{m}/M = (2/5, 2/5, 1/5)^t$ to the $\mathbf{v}(\lambda)$ leads to the vertices of $P(\mathbf{m})$, which we state in the order of appearance on a clockwise tour on the edges of the rounding polytope:

$$\frac{1}{15}(5,5,5)^t \to \frac{1}{15}(4,7,4)^t \to \frac{1}{15}(5,8,2)^t \to \frac{1}{15}(7,7,1)^t \to \frac{1}{15}(8,5,2)^t \to \frac{1}{15}(7,4,4)^t.$$

It follows from Corollary 2.1 that $\mathrm{Vol}(P(\mathbf{m})) = \sqrt{3}/25 \approx 0.069$.

2.3 Divisor Methods

Consider rounding via the divisor method with sign-post sequence $s(k)$. Set $s(-1) := 0$. Again, we first address the vertices of $P(\mathbf{m}) := P_A(\mathbf{m})$, and then calculate the volume of the polytope. For q-stationary divisor methods we can simplify the volume formula.

Note that the cases when $m_i = 0$ for some i need no separate treatment. The set

$$\bar{R}(\mathbf{m}) := \{i : s(m_i - 1) \neq 0\}$$

takes over the role of $R(\mathbf{m})$, and we set $\bar{r}(\mathbf{m}) := |\bar{R}(\mathbf{m})|$. The assumption $M > \ell$ implies $\bar{r}(\mathbf{m}) \geq 1$, and we have $R(\mathbf{m}) = \bar{R}(\mathbf{m})$ for $s(0) > 0$. Theorem 2.3 gives the vertices of the rounding polytopes for divisor methods.

Theorem 2.3 (Vertices of rounding polytopes for divisor methods)

For divisor methods, a rounding polytope $P(\mathbf{m})$ comprises $2^\ell - 2^{\ell-\bar{r}(\mathbf{m})} - 1$ vertices $\mathbf{v}(\lambda)$, which are induced by vectors $\lambda \in \{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\}$ with $\lambda_j = 0$ for some index $j \in \bar{R}(\mathbf{m})$. The components of $\mathbf{v}(\lambda)$ are, for $i = 1, \ldots, \ell$,

$$v_i(\lambda) = \begin{cases} \frac{s(m_i)}{c(\lambda)} & \text{if } \lambda_i = 1, \\ \frac{s(m_i-1)}{c(\lambda)} & \text{if } \lambda_i = 0, \end{cases} \tag{2.9}$$

with the normalization $c(\lambda) := \sum_{i:\lambda_i=1} s(m_i) + \sum_{i:\lambda_i=0} s(m_i - 1)$.
If $\bar{r}(\mathbf{m}) = 1$, then $\mathbf{v}(\mathbf{0}_\ell)$ is also a vertex and $P(\mathbf{m})$ has $2^\ell - 2^{\ell-\bar{r}(\mathbf{m})} = 2^{\ell-1}$ vertices.
There are no other vertices than the indicated $\mathbf{v}(\lambda)$.

Remark 2.1 (Degenerate polytopes)

If $s(0) = 0$ and there exists $m_i = 0$, then $s(m_i) = s(m_i - 1) = 0$. This implies that $P(\mathbf{m})$ is degenerate in the sense that $w_i = 0$ for all $\mathbf{w} \in P(\mathbf{m})$. Hence, $\dim(P(\mathbf{m})) \leq \ell - 2$ and $P(\mathbf{m}) \subset P(\bar{\mathbf{m}})$ for some $\bar{\mathbf{m}}$ with $\bar{m}_j \geq 1$ for all j.

Proof of Theorem 2.3. Without loss of generality, assume that the components of $\mathbf{m}$ are ordered from largest to smallest, i.e. $m_1 \geq m_2 \geq \ldots \geq m_\ell$ (otherwise permute the components appropriately). Since $M > \ell$ it holds that $m_1 \geq 2$ and $s(m_1 - 1) > 0$.

Fulfilling the inequalities (2.2) every $\mathbf{v}(\lambda)$ lies in $P(\mathbf{m})$. To see that $\mathbf{v}(\lambda)$ is indeed a vertex, we concentrate first on $\bar{r}(\mathbf{m}) = \ell$, and without loss of generality on λ having the first k components equal to 0 and all other components equal to 1. Now, $\mathbf{v}(\lambda)$ solves the system $B_\ell \mathbf{v}(\lambda) = 0$ with the matrix

$$B_\ell := \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & k & k+1 & \cdots & \ell \end{array} \\ \begin{pmatrix} -s(m_{k+1}) & & & & s(m_1-1) & & \\ \vdots & & & & & \ddots & \\ -s(m_\ell) & & & & & & s(m_1-1) \\ & -s(m_{k+1}) & & & s(m_2-1) & & \\ & & \ddots & & \vdots & & \\ & & & -s(m_{k+1}) & s(m_k-1) & & \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix} \end{array},$$

for which only non-zero entries are shown. Because $s(m_1-1)>0$ and $s(m_{k+1})>0$, B_ℓ is of full rank ℓ; thus $\mathbf{v}(\lambda)$ is a vertex. If $\bar{r}(\mathbf{m})<\ell$, then every component $v_i(\lambda)$ with value zero fulfills the constraint $w_i \geq 0$ with equality, and we can argue in analogy to the case $\bar{r}(\mathbf{m})=\ell$ by replacing the dimension ℓ by the number of non-zero components of $\mathbf{v}(\lambda)$.

No other vertices exist, because the convex hull of all $\mathbf{v}(\lambda)$ of form (2.9) is the whole polytope $P(\mathbf{m})$. This will be shown by establishing that

$$P(\mathbf{m}) = \bigcup_{i\in\bar{R}(\mathbf{m})} Q_i,$$

where Q_i is the convex hull of all $\mathbf{v}(\lambda)$ with $\lambda_i=0$ and $\operatorname{int}(Q_i)\cap\operatorname{int}(Q_j)=\emptyset$ if $i\neq j$.

By definition, all Q_i are subsets of $P(\mathbf{m})$, which implies $\supseteq$ in the above equation. To see $\subseteq$, we first show that

$$\tilde{Q}_i := \{\mathbf{w}\in P(\mathbf{m}) : w_i s(m_k-1) \leq w_k s(m_i-1) \leq w_i s(m_k) \quad \text{for all } k\neq i\}$$

and Q_i coincide. Each $\mathbf{v}(\lambda)$ with $\lambda_i=0$ is an element of the polytope $\tilde{Q}_i$, thus $Q_i\subseteq\tilde{Q}_i$. Conversely, a vertex $\mathbf{w}$ of $\tilde{Q}_i$ has its component w_k, $k\neq i$, given as $s(m_k)w_i/s(m_i-1)$ or $s(m_k-1)w_i/s(m_i-1)$. The condition $\sum_{k=1}^{\ell} w_k = 1$ then implies that $\mathbf{w}=\mathbf{v}(\lambda)$ for a λ with $\lambda_i=0$, thus $\tilde{Q}_i\subseteq Q_i$. Next, let $\mathbf{w}\in P(\mathbf{m})$. Then we can choose an index $i\in\bar{R}(\mathbf{m})$ such that $s(m_i-1)>0$ and

$$\frac{w_i}{s(m_i-1)} \leq \frac{w_j}{s(m_j-1)} \quad \text{for all } j\in\bar{R}(\mathbf{m}).$$

As $\mathbf{w}$ fulfills the inequalities (2.2) it follows $\mathbf{w}\in Q_i$. Finally, according to the definition of $\tilde{Q}_i$, a point in $\operatorname{int}(Q_i)\cap\operatorname{int}(Q_j)$ fulfills

$$w_i s(m_j-1) < w_j s(m_i-1) < w_i s(m_j-1).$$

Hence, $\operatorname{int}(Q_i)\cap\operatorname{int}(Q_j)=\emptyset$ if $i\neq j$. **qed**

The knowledge about the vertices now allows us to decompose the projected cuboids Q_i into simplices whose volumes are computed by the determinant formula. This yields the volume of $P(\mathbf{m})$ given in Theorem 2.4.

Theorem 2.4 (Volumes of rounding polytopes for divisor methods)
If $s(m_i)>0$ for all i then

$$\operatorname{Vol}(P(\mathbf{m})) = \frac{\sqrt{\ell}}{c_0(\ell-1)!}\sum_{i\in\bar{R}(\mathbf{m})}\left(s(m_i-1)\prod_{j\neq i} d_j\right)\sum_{\sigma\in\Sigma_i}\frac{1}{\prod_{j\neq i} c_i^\sigma(j)}, \tag{2.10}$$

where Σ_i is the group of permutations of $\{1,\ldots,\ell\}$ leaving i fix, $d_j := s(m_j)-s(m_j-1)$, $c_0 := \sum_{j=1}^{\ell} s(m_j-1)$, and

$$c_i^\sigma(j) := \sum_{k=1,k\neq i}^{j} s(m_{\sigma(k)}) + \sum_{k=j+1,k\neq i}^{\ell} s(m_{\sigma(k)}-1) + s(m_i-1).$$

Remark 2.2 (Degenerate polytopes)

It follows from Remark 2.1 that if $s(m_i) = 0$ for some i, i.e. if $s(0) = 0$ and $m_i = 0$, then $\mathrm{Vol}(P(\mathbf{m})) = 0$.

Proof of Theorem 2.4. As permuting the components of a rounding result $\mathbf{m}$ does not change the volume of $P(\mathbf{m})$ we assume without loss of generality that $\mathbf{m}$ has ordered components, i.e. $m_1 \geq m_2 \geq \ldots \geq m_\ell$. This assumption implies $\bar{R}(\mathbf{m}) = \{1, \ldots, \bar{r}\}$.

The proof of Theorem 2.3 establishes

$$\mathrm{Vol}(P(\mathbf{m})) = \sum_{i \in \bar{R}(\mathbf{m})} \mathrm{Vol}(Q_i) = \sum_{i=1}^{\bar{r}(\mathbf{m})} \mathrm{Vol}(Q_i).$$

Since all Q_i can be treated analogously, only the calculation of $\mathrm{Vol}(Q_1)$ is demonstrated. The result for Q_i then is obtained by interchanging indices. Adopting the notation of the proof of Lemma 2.4 the arguments used there yield

$$\mathrm{Vol}(Q_1) = \frac{\sqrt{\ell}}{(\ell-1)!} \sum_{\sigma \in \Sigma_1} \left| \det\left(\pi_1[\mathbf{v}(\sigma(\lambda^1)) - \mathbf{v}(\mathbf{0}_\ell)], \ldots, \pi_1[\mathbf{v}(\sigma(\lambda^{\ell-1})) - \mathbf{v}(\mathbf{0}_\ell)]\right) \right|,$$

where

$$v_k(\mathbf{0}_\ell) = \frac{s(m_k - 1)}{c(\mathbf{0}_\ell)} = \frac{s(m_k - 1)}{c_0} \quad \text{for all } k = 1, \ldots, \ell.$$

In order to evaluate this determinant, we study the vertex $\mathbf{v}(\sigma(\lambda^j))$. Its first component is $s(m_1 - 1)/c(\sigma(\lambda^j))$, and its $\sigma(k)$-th component is $s(m_{\sigma(k)})/c(\sigma(\lambda^j))$ for $k = 2, \ldots, j+1$ and $s(m_{\sigma(k)} - 1)/c(\sigma(\lambda^j))$ for $k = j+2, \ldots, \ell$. Using $c_1^\sigma(j+1) = c(\sigma(\lambda^j))$ and setting

$$\mathbf{x}(\sigma(\lambda^j)) := c(\sigma(\lambda^j))\mathbf{v}(\sigma(\lambda^j))$$

we find that the determinant equals $D(\sigma)/c_0^{\ell-1} \prod_{j=2}^{\ell} c_1^\sigma(j)$ with

$$D(\sigma) := \det\left(\pi_1[c_0\mathbf{x}(\sigma(\lambda^1)) - c_1^\sigma(2)\mathbf{x}(\mathbf{0}_\ell)], \ldots, \pi_1[c_0\mathbf{x}(\sigma(\lambda^\ell)) - c_1^\sigma(\ell)\mathbf{x}(\mathbf{0}_\ell)]\right).$$

If $k \in \{\sigma(2), \ldots, \sigma(j+1)\}$, then

$$\begin{aligned} \left(\pi_1[c_0\mathbf{x}(\sigma(\lambda^j)) - c_1^\sigma(j+1)\mathbf{x}(\mathbf{0}_\ell)]\right)_k &= c_0 s(m_k) - c_1^\sigma(j+1)s(m_k - 1) \\ &= c_0 d_k - s(m_k - 1)\sum_{i=2}^{j+1} d_{\sigma(i)}, \end{aligned}$$

since $c_1^\sigma(j+1) = c_0 + \sum_{i=2}^{j+1} d_{\sigma(i)}$. If $k \in \{\sigma(j+2), \ldots, \sigma(\ell)\}$, then

$$\begin{aligned} \left(\pi_1[c_0\mathbf{x}(\sigma(\lambda^j)) - c_1^\sigma(j+1)\mathbf{x}(\mathbf{0}_\ell)]\right)_k &= c_0 s(m_k - 1) - c_1^\sigma(j+1)s(m_k - 1) \\ &= -s(m_k - 1)\sum_{i=2}^{j+1} d_{\sigma(i)}. \end{aligned}$$

In the evaluation of $D(\sigma)$ we can ignore possible sign changes. Switching row $\sigma(j)$ in row j results in

$$D(\sigma) = \det \begin{pmatrix} c_0 d_{\sigma(2)} - s(m_{\sigma(2)}-1)d_{\sigma(2)} & \dots & c_0 d_{\sigma(2)} - s(m_{\sigma(2)}-1)\sum_{i=2}^{\ell} d_{\sigma(i)} \\ -s(m_{\sigma(3)}-1)d_{\sigma(2)} & \dots & c_0 d_{\sigma(3)} - s(m_{\sigma(3)}-1)\sum_{i=2}^{\ell} d_{\sigma(i)} \\ \vdots & & \vdots \\ -s(m_{\sigma(\ell)}-1)d_{\sigma(2)} & \dots & c_0 d_{\sigma(\ell)} - s(m_{\sigma(\ell)}-1)\sum_{i=2}^{\ell} d_{\sigma(i)} \end{pmatrix}.$$

By factoring out $d_{\sigma(2)}$ and adding $-\sum_{i=2}^{k+1} d_{\sigma(i)}$ times column 1 to column $k \geq 2$, we get

$$D(\sigma) = d_{\sigma(2)} \det \begin{pmatrix} c_0 - s(m_{\sigma(2)}-1) & -c_0 d_{\sigma(3)} & \dots & -c_0 \sum_{i=3}^{\ell-1} d_{\sigma(i)} & -c_0 \sum_{i=3}^{\ell} d_{\sigma(i)} \\ -s(m_{\sigma(3)}-1) & c_0 d_{\sigma(3)} & \dots & c_0 d_{\sigma(3)} & c_0 d_{\sigma(3)} \\ \vdots & 0 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & c_0 d_{\sigma(\ell-1)} & c_0 d_{\sigma(\ell-1)} \\ -s(m_{\sigma(\ell)}-1) & 0 & \dots & 0 & c_0 d_{\sigma(\ell)} \end{pmatrix}.$$

Adding each row $k \geq 2$ to row 1 leads to

$$\begin{aligned} D(\sigma) &= d_{\sigma(2)} \det \begin{pmatrix} s(m_1-1) & 0 & \dots & 0 & 0 \\ -s(m_{\sigma(3)}-1) & c_0 d_{\sigma(3)} & \dots & c_0 d_{\sigma(3)} & c_0 d_{\sigma(3)} \\ \vdots & 0 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & c_0 d_{\sigma(\ell-1)} & c_0 d_{\sigma(\ell-1)} \\ -s(m_{\sigma(\ell)}-1) & 0 & \dots & 0 & c_0 d_{\sigma(\ell)} \end{pmatrix} \\ &= s(m_1-1) c_0^{\ell-2} \prod_{j=2}^{\ell} d_{\sigma(j)}. \end{aligned}$$

It follows that the modulus of the determinant is equal to

$$\frac{s(m_1-1)\prod_{j=2}^{\ell} d_{\sigma(j)}}{c_0 \prod_{j=2}^{\ell} c_1^{\sigma}(j)} = \frac{s(m_1-1)\prod_{j=2}^{\ell} d_j}{c_0 \prod_{j=2}^{\ell} c_1^{\sigma}(j)}.$$

When calculating $\mathrm{Vol}(Q_i)$ instead of $\mathrm{Vol}(Q_1)$, the result becomes

$$\frac{s(m_i-1)\prod_{j\neq i} d_j}{c_0 \prod_{j\neq i} c_i^{\sigma}(j)},$$

and summing the pieces as in the proof of Theorem 2.3 yields formula (2.10). **qed**

Consider a q-stationary divisor method, thus $s(k) = k + q$. Then the differences d_j in Theorem 2.4 are 1 for $m_k \geq 1$ and q for $m_k = 0$. This permits simplification of formula (2.10) to the result in Theorem 2.5, which shows that the volume of $P(\mathbf{m})$ depends only on $r(\mathbf{m}) = |\{i : m_i \neq 0\}|$.

Theorem 2.5 (Volumes of rounding polytopes for q-stationary rounding)
The volume of $P(\mathbf{m})$ depends only on $r := r(\mathbf{m})$ and is given by

$$\mathrm{Vol}(P(\mathbf{m})) = \frac{q^{\ell-r}\sqrt{\ell}}{\binom{\ell-1}{\ell-r}} \sum_{\mathbf{t}\in\left\{\ell-1 \atop \ell-r\right\}} \prod_{j=1}^{\ell-1} \frac{1}{M+(r+t_{(j)})q-(r-j+t_{(j)})}, \tag{2.11}$$

where we set $0^0 := 1$.

Remark 2.3 (Rounding up/down)
If $q = 0$ and $r(\mathbf{m}) < \ell$, then $q^{\ell-r(\mathbf{m})} = 0$ and therefore $\mathrm{Vol}(P(\mathbf{m})) = 0$.
If $q = 1$, then all rounding polytopes have the same volume.

Corollary 2.2 (Volumes of special rounding polytopes)
If $r(\mathbf{m}) \in \{1, \ell\}$, then the volume of $P(\mathbf{m})$ is given by, for $q \in [0,1]$,

$$\mathrm{Vol}(P(\mathbf{m})) = \sqrt{\ell} \times \begin{cases} q^{\ell-1}\prod_{j=2}^{\ell}\frac{1}{M+jq-1} & \textit{for } r(\mathbf{m}) = \ell, \\ \prod_{j=1}^{\ell-1}\frac{1}{M+\ell q-j} & \textit{for } r(\mathbf{m}) = 1. \end{cases} \tag{2.12}$$

The case $r(\mathbf{m}) = \ell$ in Corollary 2.2 is treated in [22, p. 204, Theorem 6.2.10].

Proof of Theorem 2.5. The case $q = 0$, $r(\mathbf{m}) < \ell$ is an immediate consequence of Remark 2.2. For $q > 0$ and $q = 0$, $r(\mathbf{m}) = \ell$ we obtain $s(m_i) > 0$ for all i. We assume that $\mathbf{m}$ is ordered as $m_1 \geq m_2 \geq \ldots \geq m_\ell$, which implies $s(m_1 - 1) > 0$. Hence, we can apply formula (2.10) from Theorem 2.4.

Then, $s(m_i - 1) = m_i + q - 1$ for all $i \leq r(\mathbf{m})$; $d_j = 1$ for $j \leq r(\mathbf{m})$ and $d_j = q$ for $j \geq r(\mathbf{m}) + 1$. Therefore, $c_0 = M + r(\mathbf{m})(q-1)$ and

$$\begin{aligned} c_i^\sigma(j) &= \sum_{k=1,k\neq i}^{j} s(m_{\sigma(k)}) + \sum_{k=j+1,k\neq i}^{\ell} s(m_{\sigma(k)} - 1) + s(m_i - 1) \\ &= \sum_{k=1,k\neq i}^{j} (m_{\sigma(k)} + q) + \sum_{k\in P_i^\sigma(j)} (m_k + q - 1) + (m_i + q - 1) \\ &= \begin{cases} M + jq + (1 + p_i^\sigma(j))(q-1) & \text{for } j < i, \\ M + (j-1)q + (1 + p_i^\sigma(j))(q-1) & \text{for } j > i, \end{cases} \end{aligned}$$

with $P_i^\sigma(j) := \sigma(\{j+1,\ldots,\ell\} \setminus \{i\}) \cap \{1,\ldots,r(\mathbf{m})\}$ and $p_i^\sigma(j) := |P_i^\sigma(j)|$. It follows from (2.10) that

$$\mathrm{Vol}(P(\mathbf{m})) = \frac{q^{\ell-r(\mathbf{m})}\sqrt{\ell}}{(\ell-1)!} \times \frac{\sum_{i=1}^{r(\mathbf{m})}(m_i+q-1)}{(M + r(\mathbf{m})(q-1))} \sum_{\sigma\in\Sigma_i}\prod_{j\neq i}\frac{1}{c_i^\sigma(j)} = \frac{q^{\ell-r(\mathbf{m})}\sqrt{\ell}}{(\ell-1)!} \sum_{\sigma\in\Sigma_i}\prod_{j\neq i}\frac{1}{c_i^\sigma(j)}.$$

We next prove that

$$Z_i := \sum_{\sigma\in\Sigma_i}\prod_{j\neq i}\frac{1}{c_i^\sigma(j)}$$

is independent of the index i by showing $Z_i = Z_1$ for all $1 \le i \le r(\mathbf{m})$, via the bijection $f : \Sigma_i \to \Sigma_1$. Since $f(\sigma) \in \Sigma_1$, it must hold that $f(\sigma)(1) = 1$. The remaining components of $f(\sigma)$ are defined as follows. If $\sigma^{-1}(1) \le i$ then

$$f(\sigma)(j) = \begin{cases} \sigma(j-1) & \text{for } 2 \le j \le i, j \neq \sigma^{-1}+1, \\ i & \text{for } j = \sigma^{-1}+1, \\ \sigma(j) & \text{for } i < j \le \ell, \end{cases}$$

otherwise if $\sigma^{-1}(1) > i$ then

$$f(\sigma)(j) = \begin{cases} \sigma(j-1) & \text{for } 2 \le j \le i, \\ \sigma(j) & \text{for } i < j \le \ell, j \neq \sigma^{-1}, \\ i & \text{for } j = \sigma^{-1}. \end{cases}$$

Consequently, $p_1^{f(\sigma)}(j) = p_i^\sigma(j-1)$ for $2 \le j \le i$, and $p_1^{f(\sigma)}(j) = p_i^\sigma(j)$ for $j > i$. It follows that $c_i^\sigma(j) = c_1^{f(\sigma)}(j+1)$ for $1 \le j < i$, and $c_i^\sigma(j) = c_1^{f(\sigma)}(j)$ for $i < j \le \ell$, which yields

$$\prod_{j\neq i}\frac{1}{c_i^\sigma(j)} = \prod_{j=2}^{\ell}\frac{1}{c_1^{f(\sigma)}(j)}$$

and $Z_i = Z_1$. Therefore the volume formula simplifies to

$$\operatorname{Vol}(P(\mathbf{m})) = \frac{q^{\ell-r(\mathbf{m})}\sqrt{\ell}}{(\ell-1)!}\sum_{\sigma\in\Sigma_1}\prod_{j=2}^{\ell}\frac{1}{c_1^\sigma(j)}.$$

We have $c_1^\sigma(j) = c_1^\tau(j)$ when $\sigma \sim \tau$ in the sense of the proof of Lemma 2.4. Thus, we index the equivalence classes by vectors $\mathbf{t} \in \{0,1\}^{\ell-1}$ with $\sum_{i=1}^{\ell-1} t_i = \ell - r(\mathbf{m})$, such that a permutation $\sigma \in \Sigma_1$ in an equivalence class associated with $\mathbf{t}$ satisfies $\sigma(i+1) \le r(\mathbf{m})$ if $t_i = 0$, and $\sigma(i+1) > r(\mathbf{m})$ if $t_i = 1$. We find

$$p_1^\sigma(j) = \sum_{i=j}^{\ell-1}(1-t_i) = (\ell-j) - \sum_{i=j}^{\ell-1} t_i.$$

This result leads to an expression for $c_1^\sigma(j)$, and we obtain

$$\operatorname{Vol}(P(\mathbf{m})) = \frac{q^{\ell-r}\sqrt{\ell}}{\binom{\ell-1}{\ell-r}}\sum_{\mathbf{t}\in\left\{\ell-1 \atop \ell-r\right\}}\prod_{j=2}^{\ell}\frac{1}{M+(j-1)q+(1+\ell-j-\sum_{i=j}^{\ell-1}t_i)(q-1)},$$

which implies the claimed formula (2.11). **qed**

To illustrate the results we again study the rounding polytope for

$$\mathbf{m} = (2,2,1)^t,$$

method	λ	$c(\lambda)$	$\mathbf{v}(\lambda)$
Adams ($q = 0$)	$(0,0,1)^t$	3	$\frac{1}{3}(1,1,1)^t$
	$(0,1,1)^t$	4	$\frac{1}{4}(1,2,1)^t$
	$(0,1,0)^t$	3	$\frac{1}{3}(1,2,0)^t$
	$(1,0,0)^t$	3	$\frac{1}{3}(2,1,0)^t$
	$(1,0,1)^t$	4	$\frac{1}{4}(2,1,1)^t$
Webster ($q = 0.5$)	$(0,0,1)^t$	4.5	$\frac{1}{9}(3,3,3)^t$
	$(0,1,1)^t$	5.5	$\frac{1}{11}(3,5,3)^t$
	$(0,1,0)^t$	4.5	$\frac{1}{9}(3,5,1)^t$
	$(1,1,0)^t$	5.5	$\frac{1}{11}(5,5,1)^t$
	$(1,0,0)^t$	4.5	$\frac{1}{9}(5,3,1)^t$
	$(1,0,1)^t$	5.5	$\frac{1}{11}(5,3,3)^t$
Jefferson ($q = 1$)	$(0,0,1)^t$	6	$\frac{1}{6}(2,2,2)^t$
	$(0,1,1)^t$	7	$\frac{1}{7}(2,3,2)^t$
	$(0,1,0)^t$	6	$\frac{1}{6}(2,3,1)^t$
	$(1,1,0)^t$	7	$\frac{1}{7}(3,3,1)^t$
	$(1,0,0)^t$	6	$\frac{1}{6}(3,2,1)^t$
	$(1,0,1)^t$	7	$\frac{1}{7}(3,2,2)^t$

Table 2.3: Vertices $\mathbf{v}(\lambda)$ of the rounding polytope $P(\mathbf{m} = (2,2,1)^t)$ for the q-stationary divisor methods with $q = 0$, $q = 0.5$, and $q = 1$, as determined by Theorem 2.3 ($M = 5$, $\ell = 3$).

which is highlighted for $q = 0$, $q = 0.5$, and $q = 1$ in Figure 2.1(b-d). By Theorem 2.3, we know for $q \neq 0$ that $\bar{r}(\mathbf{m}) = 3$ and $P(\mathbf{m})$ has $2^3 - 2^0 - 1 = 6$ vertices. For $q = 0$ we find only $2^3 - 2^1 - 1 = 5$ vertices, because $\bar{r}(\mathbf{m}) = 2$. To be more specific, $\lambda = (1,1,0)^t$ does not induce a vertex as $\lambda_j \neq 0$ for all $j \in \bar{R}(\mathbf{m}) = \{1,2\}$. Table 2.3 gives the coordinates of the vertices as they appear on a clockwise tour on the edges of the rounding polytope according to the sequence

$$(0,0,1)^t \to (0,1,1)^t \to (0,1,0)^t \to (1,1,0)^t \to (1,0,0)^t \to (1,0,1)^t$$

of vertex-inducing vectors. In the case $q = 0$, the vector $\lambda = (1,1,0)^t$ is skipped. Finally, Corollary 2.2 results in the volumes

$$\mathrm{Vol}(P(\mathbf{m})) = \begin{cases} \frac{\sqrt{3}}{12} & \approx 0.144 \qquad \text{for } q = 0, \\ \frac{\sqrt{3}}{24.75} & \approx 0.070 \qquad \text{for } q = 0.5, \\ \frac{\sqrt{3}}{42} & \approx 0.041 \qquad \text{for } q = 1. \end{cases}$$

Chapter 3

Combinatorial Approach to Seat Biases

In a proportional representation system, apportionment methods are needed to translate vote proportions into integer numbers of seats in parliament, see Balinski and Young [5], Kopfermann [22]. This rounding process leaves an inevitable gap between the ideal seat allocation based on essentially continuous fractions and the actual seat allocation based on the accuracy given by the size of the parliament. Of course, a suitable apportionment method should – on average – treat the smaller and larger parties equally, and not allow a systematic advantage in either direction.

Taking up original work by Pólya [27–29], Schuster *et al.* [32] introduce the notion of seat biases to quantify how much a given apportionment method favors smaller or larger parties. The shares of votes of a party are assumed to follow a uniform distribution, and seat biases are the expected differences between actual seat allocations and ideal shares of seats, under the condition that the parties are ordered by their vote proportions. For 3-party systems, Schuster *et al.* [32] derive seat bias formulas for popular apportionment methods. In addition, they provide numerical evidence about asymptotic seat biases for an arbitrary number of parties, as the number of seats in parliament grows.

Following [36], the present chapter first addresses seat allocation distributions, when the size of the parliament is given. Based on these distributions, a systematic method for calculating seat biases is developed subsequently.

3.1 Seat Allocation Distributions

We interpret the component w_i of the weight vector $\mathbf{w} \in S^\ell$ as the share of votes for the i-th largest party. For a given *district magnitude* or *house size* M, which is the number of seats to be allocated among the parties, the possible *seat allocations* $\mathbf{m}$ form the grid set $G^\ell(M)$. An *apportionment method* A maps a weight vector $\mathbf{w}$ into a seat allocation $\mathbf{m}$,

$$A : S^\ell \to G^\ell(M).$$

Rounding the weights w_i individually in general is not a feasible apportionment method, because the side condition

$$\sum_{i=1}^{\ell} m_i = M$$

is not enforced automatically, see Happacher [16, Section 1]. As apportionment methods, we address stationary divisor methods and the quota method of greatest remainders.

Let the q-stationary divisor method map a weight vector $\mathbf{w}$ into the integer vector

$$A^q_{\ell,M}(\mathbf{w}) \in G^\ell(M),$$

and assume that this weight vector follows the uniform distribution on S^ℓ. We are interested in the distribution of the random variable $A^q_{\ell,M}$ which we consider in Theorems 3.1 and 3.2.

Theorem 3.1 (Seat allocation distributions for divisor methods)
Suppose the weight vectors $\mathbf{w}$ are uniformly distributed on the probability simplex S^ℓ. Use the q-stationary divisor method with $q \in [0,1]$ to apportion the house size M.
Then the seat allocation $A^q_{\ell,M}$ is a discrete random variable, with values in the finite grid set $G^\ell(M)$, attaining a grid point $\mathbf{m} \in G^\ell(M)$ with probability

$$\begin{aligned} P(A^q_{\ell,M}(\mathbf{w}) = \mathbf{m}) &= \frac{q^{\ell-r}(\ell-1)!}{\binom{\ell-1}{\ell-r}} \sum_{\mathbf{t} \in \left\{ {\ell-1 \atop \ell-r} \right\}} \prod_{j=1}^{\ell-1} \frac{1}{M + (r + t_{(j)})q - (r - j + t_{(j)})} \\ &=: p_{q,\ell,M}(r), \end{aligned} \tag{3.1}$$

where r denotes the number of positive components of $\mathbf{m}$, and $0^0 := 1$ for $q = 0$, $r = \ell$.

Proof. Due to the uniform distribution assumption the probabilities are proportional to surface volumes. Since we know the constant of proportionality,

$$\mathrm{Vol}(S^\ell) = \frac{\sqrt{\ell}}{(\ell-1)!},$$

the result follows from Theorem 2.5. **qed**

For the specific parameter values $q = 0$ and $q = 1$ we can simplify formula (3.1) from Theorem 3.1 to obtain a more convenient representation.

Corollary 3.1 (Rounding up/down)
For the divisor method with rounding up we have

$$P(A^0_{\ell,M} = \mathbf{m}) = \begin{cases} \binom{M-1}{\ell-1}^{-1} & \text{for } \mathbf{m} \in G^\ell(M) \text{ with } r = \ell, \\ 0 & \text{for } \mathbf{m} \in G^\ell(M) \text{ with } r < \ell. \end{cases} \tag{3.2}$$

For the divisor method with rounding down we have

$$P(A^1_{\ell,M} = \mathbf{m}) = \binom{M+\ell-1}{\ell-1}^{-1} \quad \text{for } \mathbf{m} \in G^\ell(M). \tag{3.3}$$

Proof. For $q = 1$, Theorem 3.1 implies that $p_{q,\ell,M}(r)$ is constant in r and hence equals the inverse of the cardinality $|G^\ell(M)|$. For $q = 0$, $p_{q,\ell,M}(r) = 0$ when $r < \ell$, and it is the inverse of the cardinality $|\{\mathbf{m} \in G^\ell(M) : m_i \geq 1 \text{ for } i = 1, \ldots, \ell\}|$ when $r = \ell$. **qed**

As already indicated by Pólya [29, p. 367], all seat allocations $\mathbf{m}$ occur under the divisor method with rounding down with the same probability, see Remark 2.3.

Let the quota method of greatest remainders map a weight vector $\mathbf{w}$ into the integer vector

$$A_{\ell,M}(\mathbf{w}) \in G^\ell(M).$$

The following Theorem deals with the distribution of this random variable.

Theorem 3.2 (Seat allocation distributions for the Hamilton method)
Suppose the weight vectors $\mathbf{w}$ *are uniformly distributed on the probability simplex* S^ℓ. *Use the quota method of greatest remainders to apportion the house size* M.
Then the seat allocation $A_{\ell,M}$ *is a discrete random variable, with values in the finite grid set* $G^\ell(M)$, *attaining a grid point* $\mathbf{m} \in G^\ell(M)$ *with probability*

$$\begin{aligned} P(A_{\ell,M}(\mathbf{w}) = \mathbf{m}) \quad &= \quad \frac{(\ell-1)!}{\binom{\ell}{r} M^{\ell-1}} \sum_{\mathbf{t} \in \left\{ {\ell-1 \atop \ell-r} \right\}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{r + t_{(j)}}\right)^{t_{j+1}} \\ &=: \quad p_{\ell,M}(r), \end{aligned} \tag{3.4}$$

where r *denotes the number of positive components of* $\mathbf{m}$.

Proof. By means of Theorem 2.2, the result can be obtained analogously to the proof of Theorem 3.1. **qed**

3.2 Combinatorial Method

Let the ℓ parties be decreasingly ordered by votes, and give the largest party index 1 and the smallest party index ℓ. We now consider the weight vectors in the ordered subset of the probability simplex

$$S^\ell_\geq := \left\{\mathbf{w} \in S^\ell : w_1 \geq \cdots \geq w_\ell\right\},$$

see Figure 3.1, and define the grid set of ordered seat allocation vectors in $G^\ell(M)$

$$G^\ell_\geq(M) := \left\{\mathbf{m} \in G^\ell(M) : m_1 \geq \cdots \geq m_\ell\right\}.$$

Note that the definition of stationary divisor methods as well as of the quota method of greatest remainders entails

$$\mathbf{w} \in S^\ell_\geq \implies A(\mathbf{w}) \in G^\ell_\geq(M).$$

Conditional on $w_1 \geq \cdots \geq w_\ell$, i.e. $\mathbf{w} \in S^\ell_\geq$, the weight vectors are uniformly distributed in $S^\ell_\geq$. Hence, probabilities are still proportional to volumes, where the constant of proportionality becomes, by symmetry,

$$\mathrm{Vol}(S^\ell_\geq) = \frac{\mathrm{Vol}(S^\ell)}{\ell!} = \frac{\sqrt{\ell}}{\ell!(\ell-1)!}.$$

Define the expected ideal share of seats to be

$$\mathbf{I}^\ell(M) := \mathrm{E}[\mathbf{w}M | w_1 \geq \cdots \geq w_\ell],$$

and the expected number of seats to be

$$\mathbf{E}^\ell(M) := \mathrm{E}[A(\mathbf{w}) | w_1 \geq \cdots \geq w_\ell].$$

The *vector of seat biases*, sorted from the largest to the smallest party, then becomes

$$\mathbf{B}^\ell(M) := \mathrm{E}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell] = \mathbf{E}^\ell(M) - \mathbf{I}^\ell(M).$$

Of course, the components $B^\ell_1(M), \ldots, B^\ell_\ell(M)$ of $\mathbf{B}^\ell(M)$ must sum to zero

$$\sum_{i=1}^{\ell} B^\ell_i(M) = E\left[\sum_{i=1}^{\ell} m_i - M \sum_{i=1}^{\ell} w_i \middle| w_1 \geq \cdots \geq w_\ell\right] = M - M = 0.$$

The expected ideal seat allocation $\mathbf{I}^\ell(M) = (I^\ell_1(M), \ldots, I^\ell_\ell(M))^t$ results from geometrical considerations.

Lemma 3.1 (Expected ideal share of seats)

The expected ideal share of seats of the i-th largest party equals

$$I_i(M) = \frac{M}{\ell} \sum_{j=0}^{\ell-i} \frac{1}{\ell - j} = \frac{M}{\ell} \sum_{j=i}^{\ell} \frac{1}{j}. \tag{3.5}$$

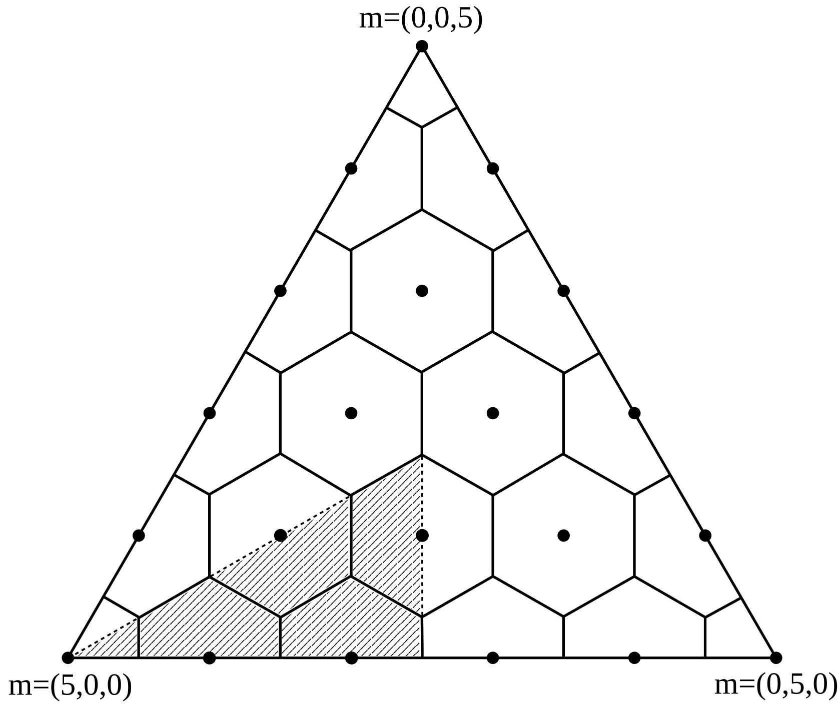


Figure 3.1: Ordered subset of the probability simplex.

Proof. The vector $\mathbf{I}(M)$ equals M times

$$\mathrm{E}[\mathbf{w}|w_1 \geq \ldots \geq w_\ell],$$

which is the center of mass of the ordered probability simplex $S^\ell_\geq$. The center of mass of a simplex is the (arithmetic) mean of its vertices, and the vertices of $S^\ell_\geq$ are given by the vectors $\mathbf{w}^j$, $j = 0, \ldots, \ell - 1$, with i-th component

$$w_i^j = \begin{cases} \frac{1}{\ell - j} & \text{if } i \leq \ell - j, \\ 0 & \text{if } i > \ell - j, \end{cases}$$

see [8]. Multiplying the mean of these vertices by M yields (3.5). **qed**

The expected ideal seat proportions $\mathbf{I}^\ell(M)/M$ do not depend on the house size, but only on the number of parties in the system. In particular, one obtains

$$\frac{\mathbf{I}^2(M)}{M} = \left(\frac{3}{4}, \frac{1}{4}\right)^t, \quad \frac{\mathbf{I}^3(M)}{M} = \left(\frac{11}{18}, \frac{5}{18}, \frac{2}{18}\right)^t, \text{ and } \frac{\mathbf{I}^4(M)}{M} = \left(\frac{25}{48}, \frac{13}{48}, \frac{7}{48}, \frac{3}{48}\right)^t.$$

Let $\mathbf{e}_i \in \mathbb{R}^\ell$ be the Euclidean unit vector with i-th component one and all other components zero. Defining

$$\mathbf{v}_r := \frac{1}{r} \sum_{j=1}^{r} \mathbf{e}_j$$

for $r = 1, \ldots, \ell$, we have

$$\mathbf{I}^\ell(M) = \frac{M}{\ell} \sum_{r=1}^{\ell} \mathbf{v}_r.$$

For obtaining seat biases, it remains the task of determining the expected number of seats $\mathbf{E}^\ell(M) = (E_1^\ell(M), \ldots, E_\ell^\ell(M))^t$, for the various apportionment methods. We state

$$\mathbf{E}^\ell(M) = \sum_{\mathbf{m} \in G_\geq^\ell(M)} \mathbf{m}\, P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \cdots \geq w_\ell).$$

Let S be the permutation group on $\{1, \ldots, \ell\}$, and define

$$\sigma(\mathbf{x}) := \left(x_{\sigma(1)}, \ldots, x_{\sigma(\ell)}\right)^t, \quad \sigma \in S,\ \mathbf{x} \in \mathbb{R}^\ell.$$

Then

$$b(\mathbf{m}) := |\{\sigma \in S : \sigma(\mathbf{m}) = \mathbf{m}\}|$$

counts the number of permutations leaving the seat allocation $\mathbf{m}$ invariant. We will call $b(\mathbf{m})$ the *boundary factor* for $\mathbf{m}$, since we have $b(\mathbf{m}) \neq 1$ only if the weight vector $\mathbf{m}/M$ is located on the boundary of $S_\geq^\ell$.

Stationary divisor methods as well as the quota method of greatest remainders map weight vectors with permuted entries to permuted seat allocations

$$A(\mathbf{w}) = \mathbf{m} \implies A(\sigma(\mathbf{w})) = \sigma(\mathbf{m}) \quad \text{for all } \sigma \in S,$$

a property called "anonymity" by Balinski and Young [5]. For $\mathbf{m} \in G_\geq^\ell(M)$, we have by symmetry

$$P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \cdots \geq w_\ell) = \frac{\ell!}{b(\mathbf{m})}\, P(A(\mathbf{w}) = \mathbf{m}),$$

where the unconditional probability $P(A(\mathbf{w}) = \mathbf{m})$ is given in Theorem 3.1 or 3.2. Note that $P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \cdots \geq w_\ell) = 0$ for $\mathbf{m} \notin G_\geq^\ell(M)$.

We decompose the grid set $G_\geq^\ell(M)$ into disjoint subsets, for $r = 1, \ldots, \ell$,

$$K_r(M) := \left\{\mathbf{m} \in G_\geq^\ell(M) : m_r > 0 = m_{r+1}\right\}.$$

Therefore, a seat allocation $\mathbf{m} \in K_r(M)$ has the first r components positive and the last $\ell - r$ components zero, and the subset $K_r(M)$ comprises the grid points in the polytope generated by the vertices $\mathbf{v}_1, \ldots, \mathbf{v}_r$. The probability

$$p(r) := P(A(\mathbf{w}) = \mathbf{m})$$

is constant for $\mathbf{m} \in K_r(M)$, see Theorems 3.1 and 3.2. Furthermore, the boundary factor $b(\mathbf{m})$ decomposes according to

$$b(\mathbf{m}) = (\ell - r)!\, b_r(\mathbf{m}),$$

where $b_r(\mathbf{m}) := b((m_1,\ldots,m_r)^t)$. As a consequence, we obtain

$$\mathbf{E}^\ell(M) = \sum_{r=1}^{\ell} \frac{\ell!}{(\ell-r)!}\, p(r) \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})}\, \mathbf{m}.$$

This leaves us with the task of determining

$$\mathbf{S}^r(M) = (S_1^r(M),\ldots,S_\ell^r(M))^t := \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})}\, \mathbf{m}.$$

Because the components of $\mathbf{S}^r(M)$ are polynomials in M, they are called *apportionment polynomials* in the following.

The definition of $K_r(M)$ implies that the last $\ell - r$ components are zero

$$\mathbf{S}^r(M) = (S_1^r(M),\ldots,S_r^r(M),0,\ldots,0)^t\,.$$

The non-vanishing polynomials $S_i^r(M)$, $i \le r$, reflect the combinatorial-geometric structure of the boundary classes $K_r(M)$ of the ordered grid set $G_{\geq}^{\ell}(M)$; they do not depend on the size ℓ of the system, nor on the apportionment method. Seat biases therefore can be represented as follows.

Theorem 3.3 (Calculation of seat biases)

Under the assumptions of Theorems 3.1 and 3.2 the seat biases satisfy

$$\mathbf{B}^\ell(M) = \left(\sum_{r=1}^{\ell} \frac{\ell!}{(\ell-r)!}\, p(r)\mathbf{S}^r(M)\right) - \frac{M}{\ell}\sum_{r=1}^{\ell} \mathbf{v}_r.$$

Proof. The assertion is a consequence of the decomposition of $G_{\geq}^{\ell}(M)$ into boundary classes, the definition of the apportionment polynomials, and Lemma 3.1. **qed**

3.3 Apportionment Polynomials

In view of Theorem 3.3 there still remains the task to efficiently handle the polynomials $S_i^r(M)$ for $i \le r$, $r = 1, 2, \ldots$. Since we frequently will have to distinguish different cases according to the divisibility of M, it is convenient to introduce the notation

$$[y_1, y_2, \ldots, y_k]_k^M := \begin{cases} y_1 & \text{for } \frac{M}{k} \in \mathbb{N}, \\ y_2 & \text{for } \frac{M-1}{k} \in \mathbb{N}, \\ \vdots & \vdots \\ y_k & \text{for } \frac{M-(k-1)}{k} \in \mathbb{N}. \end{cases}$$

For $r = 1$, the class $K_1(M)$ contains only $\mathbf{m} = (M, 0, \ldots, 0)^t$, and we immediately find

$$S_1^1(M) = M.$$

For $r = 2$, we obtain

$$\begin{aligned} S_1^2(M) &= \left(\sum_{m_2=1}^{\lfloor \frac{M-1}{2} \rfloor} (M - m_2) \right) + \left[\frac{M}{4}, 0 \right]_2^M, \\ S_2^2(M) &= \left(\sum_{m_2=1}^{\lfloor \frac{M-1}{2} \rfloor} m_2 \right) + \left[\frac{M}{4}, 0 \right]_2^M. \end{aligned}$$

While the polynomial $S_1^2(M)$ sums over the seats $m_1 = M - m_2$ of the largest party, the polynomial $S_2^2(M)$ sums over the seats m_2 of the second-largest party. When M is even the tied seat allocation $\mathbf{m} = (M/2, M/2, 0, \ldots, 0)^t$ generates the additional term $M/4$.

While it is difficult to calculate the polynomials $S_1^r(M), \ldots, S_r^r(M)$ individually, their sum has a simple form not depending on the divisibility of the house size M.

Lemma 3.2 (Sum of apportionment polynomials)

The apportionment polynomials $S_1^r(M), \ldots, S_r^r(M)$ fulfill, for all $r = 1, 2, \ldots$,

$$\sum_{i=1}^r S_i^r(M) = \frac{M}{r!} \binom{M-1}{r-1} = \frac{1}{(r-1)!} \binom{M}{r}.$$

Proof. Choose $\ell = r$ and recall $E_i^\ell(M) = \mathrm{E}[m_i | w_1 \ge \cdots \ge w_\ell]$. Generally, we have for the stationary divisor method with parameter $q \in [0, 1]$

$$M = \sum_{i=1}^\ell E_i^\ell(M) = \sum_{i=1}^\ell \sum_{j=1}^\ell \frac{\ell!}{(\ell - j)!} \, p_{q,\ell,M}(j) \sum_{\mathbf{m} \in K_j(M)} \frac{m_i}{b_j(\mathbf{m})}.$$

Specifically, selecting $q = 0$, Corollary 3.1 leads to $p_{0,\ell,M}(j) = 0$ for $j < \ell = r$, and

$$M = \sum_{i=1}^r r! p_{0,r,M}(r) \sum_{\mathbf{m} \in K_r(M)} \frac{m_i}{b_r(\mathbf{m})} = \sum_{i=1}^r r! p_{0,r,M}(r) S_i^r(M) = r! \binom{M-1}{r-1}^{-1} \sum_{i=1}^r S_i^r(M).$$

From this relation the claimed formula for $\sum_{i=1}^{r} S_i^r(M)$ follows. **qed**

In general, we obtain $\mathbf{S}^r(M)$ from recursions on M and r. According to our previous definition, the first r components of the vectors $r\mathbf{v}_r \in \left\{ {\ell \atop r} \right\}$ are equal to one, and the last $\ell - r$ components are equal to zero.

Theorem 3.4 (Recursive scheme for apportionment polynomials)
Starting with $\mathbf{S}^1(M) = (M, 0, \ldots, 0)^t$ *the vector* $\mathbf{S}^r(M)$ *of the apportionment polynomials for the class* $r = 2, 3, \ldots$ *obeys the recursive scheme*

$$\begin{aligned} \mathbf{S}^r(M) &= \left(\sum_{h=1}^{r-1} \frac{1}{(r-h)!} \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \left(\mathbf{S}^h(M - m_r r) + \frac{\binom{M-m_r r-1}{h-1} m_r r}{h!} \mathbf{v}_r \right) \right) \\ &\quad + \left[\frac{M}{r!}, 0, \ldots, 0 \right]_r^M \mathbf{v}_r. \end{aligned} \tag{3.6}$$

Proof. The case $r = 1$ has been considered previously. In the case $r \geq 2$, we split the sum over $\mathbf{m} \in K_r(M)$ into an iterated sum; first over the last non-vanishing component m_r, and then over $m_1, \ldots, m_{r-1}$. We decompose the ordered seat allocation $\mathbf{m} \in G_{\geq}^{\ell}(M)$ as

$$\mathbf{m} = (\mathbf{m} - m_r r \mathbf{v}_r) + m_r r \mathbf{v}_r.$$

From $\mathbf{m} \in K_r(M)$ we conclude that $(\mathbf{m} - m_r r \mathbf{v}_r) \in K_h(M - m_r r)$, for some $h \leq r - 1$, and obtain

$$\mathbf{S}^r(M) = \left(\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \sum_{h=1}^{r-1} \sum_{\substack{\mathbf{m} \in \\ K_h(M-m_r r)}} \frac{m_r r \mathbf{v}_r + \mathbf{m}}{(r-h)! b_h(\mathbf{m})} \right) + \left[\frac{M}{r!}, 0, \ldots, 0 \right]_r^M \mathbf{v}_r.$$

Applying the definition of the apportionment polynomials and

$$\sum_{\substack{\mathbf{m} \in \\ K_h(M-m_r r)}} \frac{1}{b_h(\mathbf{m})} = \frac{1}{M - m_r r} \sum_{\substack{\mathbf{m} \in \\ K_h(M-m_r r)}} \sum_{i=1}^{h} \frac{m_i}{b_h(\mathbf{m})} = \sum_{i=1}^{h} S_i^h(M - m_r r),$$

Lemma 3.2 results in the expression claimed in the assertion. **qed**

The recursion of Theorem 3.4 permits us to write the apportionment polynomials as iterated sums. These sums may be simplified via the well known formulas for

$$\sum_{k=1}^{s} k^i, \quad i, s \in \mathbb{N},$$

see Burrows and Talbot [9] or Edwards [14], which yields polynomial expressions for the components of the vectors $\mathbf{S}^r(M)$. For $r = 2$ we have

$$\begin{aligned} S_1^2(M) &= \tfrac{3}{8} M^2 - \tfrac{1}{2} M + \left[0, \tfrac{+1}{8}\right]_2^M, \\ S_2^2(M) &= \tfrac{1}{8} M^2 \qquad\qquad + \left[0, \tfrac{-1}{8}\right]_2^M. \end{aligned}$$

For $r = 3$ we obtain

$$\begin{aligned}
S_1^3(M) &= \tfrac{11}{216}M^3 - \tfrac{3}{16}M^2 + \tfrac{13}{72}M + \big[0, \tfrac{+1}{54}, \tfrac{-1}{54}\big]_3^M + \big[0, \tfrac{-1}{16}\big]_2^M\,, \\
S_2^3(M) &= \tfrac{5}{216}M^3 - \tfrac{1}{16}M^2 + \tfrac{1}{72}M + \big[0, \tfrac{-2}{54}, \tfrac{+2}{54}\big]_3^M + \big[0, \tfrac{+1}{16}\big]_2^M\,, \\
S_3^3(M) &= \tfrac{2}{216}M^3 \qquad\qquad - \tfrac{2}{72}M + \big[0, \tfrac{+1}{54}, \tfrac{-1}{54}\big]_3^M\,,
\end{aligned}$$

and for $r = 4$ the result is

$$\begin{aligned}
S_1^4(M) &= \tfrac{25}{6912}M^4 - \tfrac{11}{432}M^3 + \tfrac{203}{3456}M^2 - \tfrac{7}{144}M + \big[0, \tfrac{+3}{256}, \tfrac{-1}{96}, \tfrac{+3}{256}\big]_4^M + \big[0, 0, \tfrac{+1}{54}\big]_3^M\,, \\
S_2^4(M) &= \tfrac{13}{6912}M^4 - \tfrac{5}{432}M^3 + \tfrac{71}{3456}M^2 - \tfrac{1}{144}M + \big[0, \tfrac{-1}{256}, \tfrac{+3}{96}, \tfrac{-1}{256}\big]_4^M + \big[0, 0, \tfrac{-2}{54}\big]_3^M\,, \\
S_3^4(M) &= \tfrac{7}{6912}M^4 - \tfrac{2}{432}M^3 + \tfrac{5}{3456}M^2 + \tfrac{2}{144}M + \big[0, \tfrac{-3}{256}, \tfrac{-3}{96}, \tfrac{-3}{256}\big]_4^M + \big[0, 0, \tfrac{+1}{54}\big]_3^M\,, \\
S_4^4(M) &= \tfrac{3}{6912}M^4 \qquad\qquad - \tfrac{15}{3456}M^2 \qquad\qquad + \big[0, \tfrac{+1}{256}, \tfrac{+1}{96}, \tfrac{+1}{256}\big]_4^M\,.
\end{aligned}$$

The divisibility of M affects only the constant terms of the apportionment polynomials. By Lemma 3.2, the sum $\sum_{i=1}^r S_i^r(M)$ has no constant term, as readily verified for $r \leq 4$.

Chapter 4

Seat Bias Results

We now are able to turn to the calculation of seat biases by means of the combinatorial approach. Focussing on the divisor method with standard rounding and rounding down, the conjecture of Schuster *et al.* [32, Section A.3] about seat biases in systems with four parties can be confirmed. Moreover, analogous results for the quota method of greatest remainders are given. Following [37], the combinatorial approach allows us to study the seat biases in proportional representation systems with thresholds, see also [18]. Finally, probabilities for the violation of majority and minority criteria are calculated.

4.1 Seat Bias Formulas

With the apportionment polynomials $\mathbf{S}^r(M)$, $r \leq 4$, we are able to determine the vector of seat biases $\mathbf{B}^\ell(M)$ for $\ell \leq 4$ parties by Theorem 3.3. Schuster *et al.* [32] give seat bias formulas for $\ell \leq 3$ parties, as confirmed in the present approach. They study the divisor methods with standard rounding as well as rounding down, for which we now derive seat bias formulas in the case of $\ell = 4$ parties. The results are given up to the highest order in $1/M$ not depending explicitly on the divisibility of M. Theorems 3.1 and 3.3 yield for general q-stationary divisor methods

$$\begin{aligned}
\mathbf{B}^4(M) &= \frac{24q^3}{(M+4q-1)(M+3q-1)(M+2q-1)}\,\mathbf{S}^1(M) \\
&+ \frac{72(M^2+M+2)q^2}{(M+4q-1)(M+4q-2)(M+3q-1)(M+3q-2)(M+2q-1)}\,\mathbf{S}^2(M) \\
&+ \frac{48(3M^2+21qM-12M+37q^2-42q+11)q}{(M+4q-1)(M+4q-2)(M+4q-3)(M+3q-1)(M+3q-2)}\,\mathbf{S}^3(M) \\
&+ \frac{144}{(M+4q-1)(M+4q-2)(M+4q-3)}\,\mathbf{S}^4(M) \\
&- \frac{M}{\ell}\sum_{r=1}^{\ell}\mathbf{v}_r
\end{aligned}$$

$$= \begin{pmatrix} \frac{13}{12}(q-\frac{1}{2}) + \left(\frac{11}{3}(q-q^2) - \frac{25}{48}\right)\frac{1}{M} \\ \frac{1}{12}(q-\frac{1}{2}) + \left(\frac{5}{3}(q-q^2) - \frac{13}{48}\right)\frac{1}{M} \\ -\frac{5}{12}(q-\frac{1}{2}) + \left(\frac{2}{3}(q-q^2) - \frac{7}{48}\right)\frac{1}{M} \\ -\frac{9}{12}(q-\frac{1}{2}) - \left(\frac{18}{3}(q-q^2) - \frac{45}{48}\right)\frac{1}{M} \end{pmatrix} + \mathcal{O}\left(\frac{1}{M^2}\right).$$

For the divisor method with standard rounding ($q = 1/2$), the seat biases in the case of $\ell = 4$ parties are given by

$$\mathbf{B}^4(M) = \begin{pmatrix} \frac{19}{48}\frac{1}{M} \\ \frac{7}{48}\frac{1}{M} \\ \frac{1}{48}\frac{1}{M} \\ -\frac{27}{48}\frac{1}{M} \end{pmatrix} + \mathcal{O}\left(\frac{1}{M^3}\right).$$

The leading terms are of the order of magnitude $1/M$, and there is no term proportional to $1/M^2$. As is visible from Figure 2.1(a), the divisor method with standard rounding is asymptotically unbiased when the number of available seats M increases.

For the divisor method with rounding down ($q = 1$), it follows that the seat biases in the case of $\ell = 4$ parties are given by

$$\mathbf{B}^4(M) = \begin{pmatrix} \frac{13}{24} - \frac{25}{48}\frac{1}{M} + \frac{25}{24}\frac{1}{M^2} \\ \frac{1}{24} - \frac{13}{48}\frac{1}{M} + \frac{13}{24}\frac{1}{M^2} \\ -\frac{5}{24} - \frac{7}{48}\frac{1}{M} + \frac{7}{24}\frac{1}{M^2} \\ -\frac{9}{24} + \frac{45}{48}\frac{1}{M} - \frac{45}{24}\frac{1}{M^2} \end{pmatrix} + \mathcal{O}\left(\frac{1}{M^3}\right).$$

Here we obtain noticeable seat biases in favor of the larger parties. When the number of available seats M tends to infinity the asymptotic seat biases are

$$\lim_{M\to\infty} \mathbf{B}^4(M) = \left(\frac{13}{24}, \frac{1}{24}, -\frac{5}{24}, -\frac{9}{24}\right)^t.$$

The largest party thus can expect an extra seat – beyond its ideal share of seats – every other election. Non-zero asymptotes stand out in Figure 2.1(b).

The previous results confirm the conjecture of Schuster *et al.* [32] on asymptotic seat biases for $\ell \to \infty$ parties, which will be proved in the next chapter. Seat bias formulas in the case of $\ell = 5$ parties are addressed in [35].

For the quota method of greatest remainders we compute the seat biases analogously with the q-stationary divisor methods. For $\ell = 4$ parties, Theorems 3.2 and 3.3 yield

$$\begin{aligned} \mathbf{B}^4(M) &= \frac{1}{M^3}\left(\mathbf{S}^1(M) + 14\,\mathbf{S}^2(M) + 72\,\mathbf{S}^3(M) + 144\,\mathbf{S}^4(M)\right) - \frac{M}{\ell}\sum_{r=1}^{\ell}\mathbf{v}_r \\ &= \begin{pmatrix} \frac{5}{24}\frac{1}{M} + \frac{1}{M^3}[0, \frac{-17}{16}, \frac{-24}{16}, \frac{-17}{16}] + \frac{1}{M^3}[0, \frac{+4}{3}, \frac{+4}{3}] \\ \frac{5}{24}\frac{1}{M} + \frac{1}{M^3}[0, \frac{+35}{16}, \frac{+72}{16}, \frac{+35}{16}] + \frac{1}{M^3}[0, \frac{-8}{3}, \frac{-8}{3}] \\ \frac{5}{24}\frac{1}{M} + \frac{1}{M^3}[0, \frac{-27}{16}, \frac{-72}{16}, \frac{-27}{16}] + \frac{1}{M^3}[0, \frac{+4}{3}, \frac{+4}{3}] \\ -\frac{15}{24}\frac{1}{M} + \frac{1}{M^3}[0, \frac{+9}{16}, \frac{+24}{16}, \frac{+9}{16}] \end{pmatrix}, \end{aligned}$$

as depicted in Figure 2.1(c).

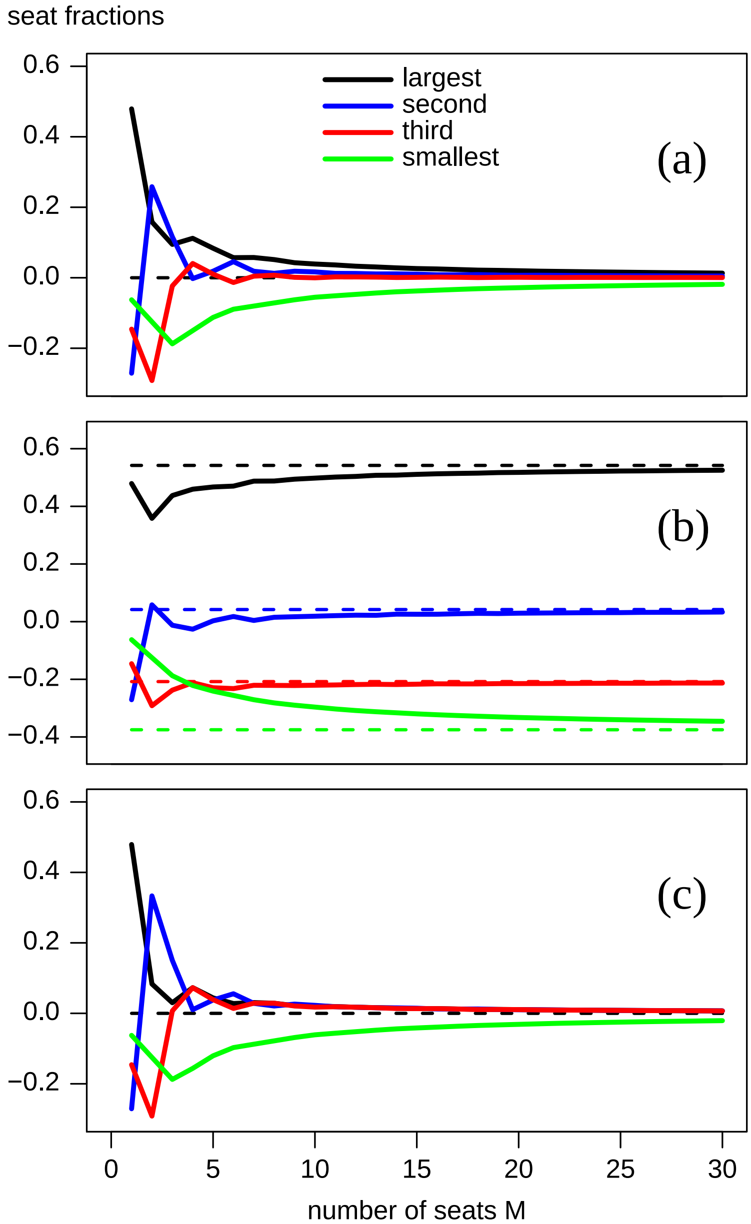


Figure 4.1: Seat biases B_1^4, B_2^4, B_3^4, and B_4^4 for $\ell = 4$ parties, as functions of the house size $M \leq 30$. (a) Divisor method with standard rounding (Webster, Sainte-Laguë). (b) Divisor method with rounding down (Jefferson, D'Hondt). (c) Quota method of greatest remainders (Hamilton, Hare). The seat biases of the Webster and Hamilton method are tiny and quickly converge to zero; thus the methods legitimately are termed "practically unbiased". In contrast, the Jefferson method comes with strong seat biases, favoring the two larger at the expense of the two smaller parties. Moreover, dashed lines indicate the asymptotic behaviour of the various seat bias curves.

4.2 Systems with Thresholds

There are many proportional representation systems and plenty of methods are available to determine the number of representatives proportionally to some electoral input data, see Taagepera and Shugart [33] for a review from the point of view of political science. In the following we extend our previous results on seat biases to electoral systems where, in order to be eligible to participate in the apportionment process, the vote proportion of a party has to exceed a certain *threshold* t. Many systems impose a five percent threshold.

There exists an extensive literature on thresholds in electoral systems, see Taagepera [34], Palomares and Ramirez [26], and the references given there. However, those authors do not study the impact of thresholds on seat biases, but calculate minimum thresholds a party must pass in order to possibly be allocated some number of seats, and maximum thresholds beyond which a party is certain to be allocated that many seats. In contrast, we address thresholds that are fixed by the applicable electoral law, so that the smallest party needs a vote proportion larger than t. In such systems, the vector of seat biases

$$\mathbf{B}^\ell(M,t) = \mathrm{E}[\mathbf{m} - M\mathbf{w} | w_1 \geq \cdots \geq w_\ell \geq t]$$

depends on the threshold. In other words, while the parties are still ordered from largest to smallest, we condition on the event that the last party has a weight which exceeds the threshold, $w_\ell \geq t$, and cannot be arbitrarily close to zero.

The threshold t can range from 0 to $1/\ell$, where ℓ denotes the number of parties that are eligible to participate in the apportionment process. When the threshold is equal or close to 0, the disparity between the largest and smallest party is most pronounced. On the other hand, when the threshold is close to $1/\ell$, all parties have their vote proportion close to $1/\ell$, and thus are more or less equal. It is therefore to be expected that, if at all an apportionment method suffers from non-zero seat biases, they will be largest for small thresholds and wear away as the threshold t grows close to $1/\ell$. Indeed, the dependence on t turns out to be practically linear.

Theorem 4.1 (Seat biases in systems with thresholds)
Under the condition that the vote proportions exceed a threshold t, $w_1 \geq \ldots \geq w_\ell \geq t$, the seat biases satisfy

$$\mathbf{B}^\ell(M,t) = (1 - \ell t) \cdot \mathbf{B}(M,0), \tag{4.1}$$

where $\mathbf{B}^\ell(M,0)$ is the vector of seat biases for zero threshold.

In deriving formula (4.1) some mild approximations are used. However, it transpires that these approximations are practically negligible. Figure 4.3 exhibits the straight-line decrease of the seat biases in systems of 2, 3, and 4 parties. Dots for thresholds of 5, 10, and 15 percent represent results from computer simulations (100 000 realizations). If the approximations in deriving formula (4.1) were not negligible, the dots would show some deviation from the straight lines, which is not the case.

Figure 4.3 was generated using the divisor method with rounding down, which yields the most prominent seat biases among the traditional apportionment methods. Schuster

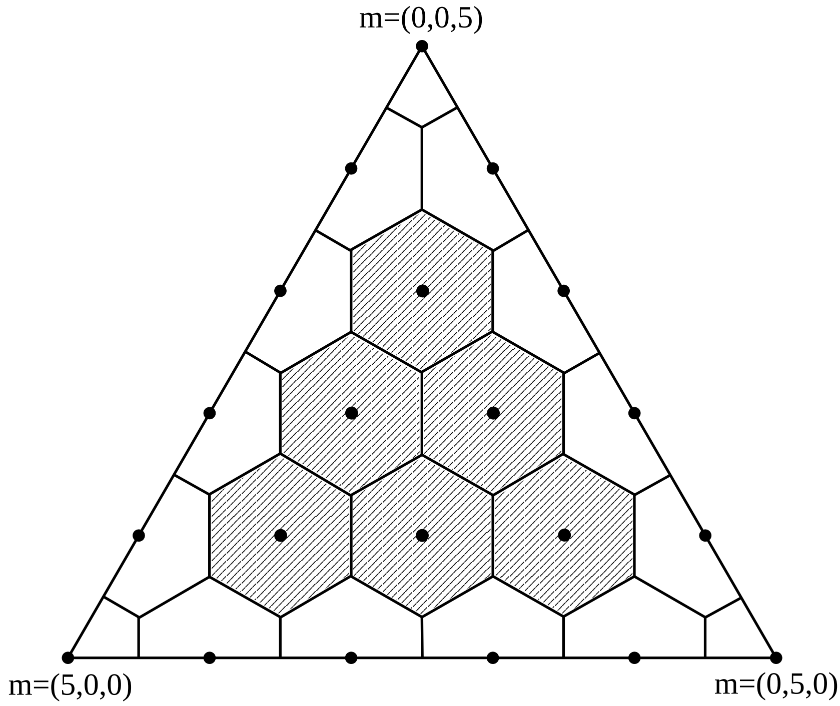


Figure 4.2: Restriction of the probability simplex due to the condition that every party wins at least $s = 1$ seat under the quota method of greatest remainders ($M = 5$, $\ell = 3$).

et al. [32] investigate two empirical data sets: one refers to the Swiss Kanton Solothurn, where thresholds were never implemented, one comes from Bavaria, where the threshold was at 5 percent throughout. Theoretical seat biases in a three-party system are a gain of $5/12 = 0.42$ seats for the largest, a loss of $-1/12 = -0.08$ seats for the middle, and a loss of $-4/12 = -0.33$ seats for the smallest party. Applying a 5 percent threshold, the seat biases need to be multiplied by a factor of $1 - 3/20 = 0.85$. This change is so small that the concordance with the empirical data set from Bavaria, which after all embraces just 49 apportionments, persists.

Proof of Theorem 4.1. The mentioned approximation is a transition from the vote region to the seat region. Theoretically, we restrict our attention to situations where the smallest party weight exceeds the threshold. Practically, we substitute this condition by demanding that the smallest party wins at least s seats, where s fulfills $s/M \approx t$. While the threshold t is a continuous variable, the proportion of seats s/M is discrete. For all district magnitudes M which are practically relevant this approximation works perfectly well, and it appears to be a substantial simplification to condition on the event that the smallest party wins at least s seats. Figure 4.2 gives an example.

Therefore we suppose $m_\ell \geq s$ and replace the threshold seat bias by

$$\mathbf{B}(M, s) = \mathrm{E}[\mathbf{m} - M\mathbf{w} | w_1 \geq \ldots \geq w_\ell, m_\ell \geq s].$$

Except for constants not depending on s, Theorem 3.3 yields that the i-th component of

$\mathbf{B}(M, s)$ is the quotient of two sums

$$B_i(M, s) = \sum_{\mathbf{m}} \frac{m_i}{b(\mathbf{m})} \Big/ \sum_{\mathbf{m}} \frac{1}{b(\mathbf{m})},$$

where the sums extend over all ordered seat allocations $\mathbf{m} \in G^{\ell}_{\geq}(M)$ satisfying $m_\ell \geq s$.

Let n denote the integer part of $(M-1)/\ell$. Except for constants not depending on s, Theorem 3.4 then implies for the numerator of the above relation

$$\sum_{\mathbf{m}} \frac{m_i}{b(\mathbf{m})} = \sum_{j=s}^{n} \left(j^{\ell-1} + \mathcal{O}(j^{\ell-2}) \right) = n^{\ell} - (s-1)^{\ell} + \mathcal{O}(n^{\ell-1}).$$

Similarly, the sum in the denominator is seen to equal

$$\sum_{\mathbf{m}} \frac{1}{b(\mathbf{m})} = \sum_{j=s}^{n} \left(j^{\ell-2} + \mathcal{O}(j^{\ell-3}) \right) = n^{\ell-1} - (s-1)^{\ell-1} + \mathcal{O}(n^{\ell-2}).$$

Being the quotient of polynomials in s, of degree ℓ in the numerator and of degree $\ell - 1$ in the denominator, $B_i(M, s)$ thus is linear in s, except for lower order remainder terms. Neglecting these terms leads to $B_i(M, s) = as + b$, where we have $b = B_i(M, 0)$. Finally, $B_i(M, M/\ell) = 0$ results in $a = -(\ell/M)B_i(M, 0)$ and it follows

$$B_i(M, s) = \left(1 - \ell \frac{s}{M}\right) \cdot B_i(M, 0).$$

Replacing the absolute seat threshold s by the corresponding proportional threshold $t = s/M$, we obtain equation (4.1). **qed**

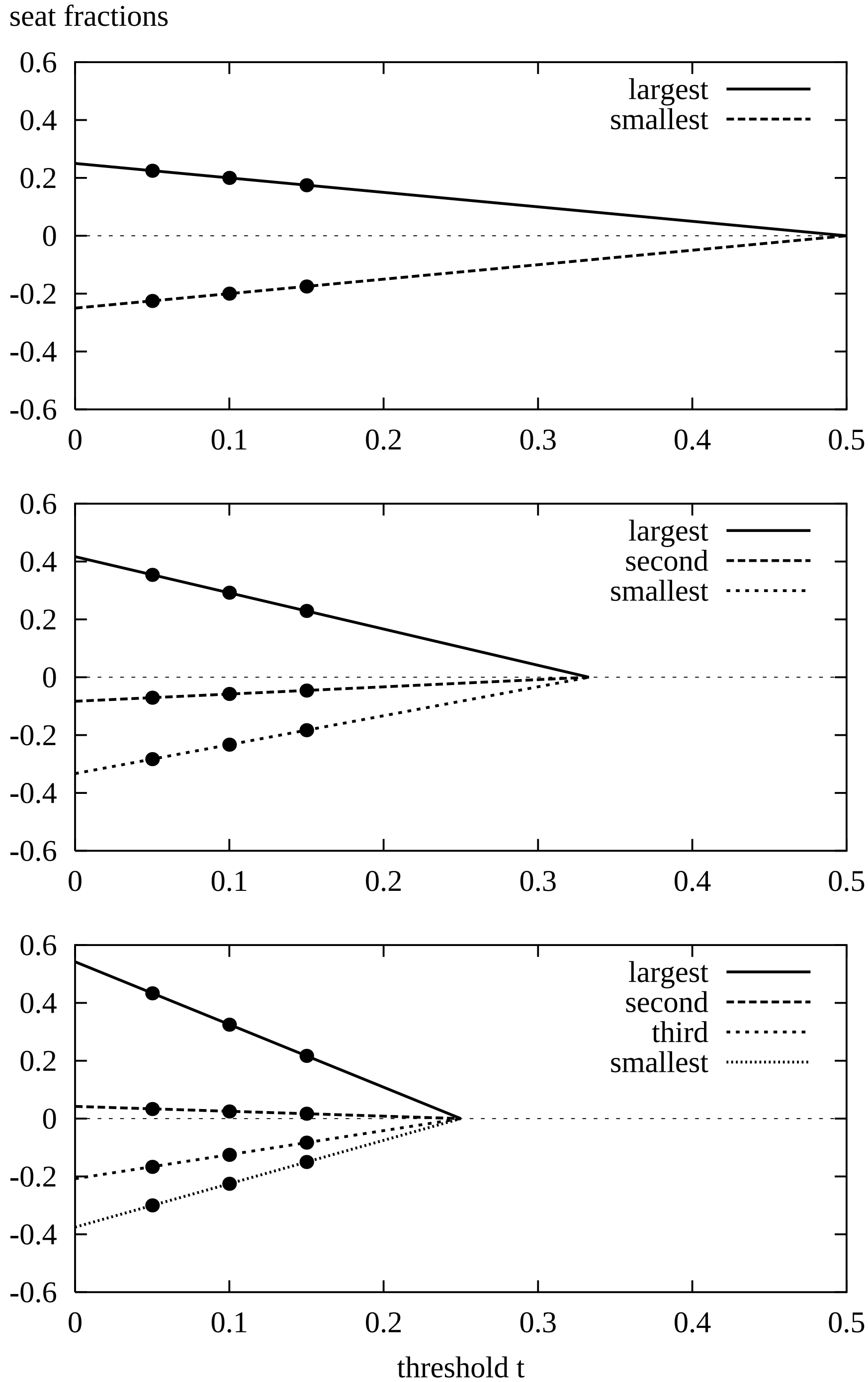


Figure 4.3: Linear decrease of seat biases in a system of $\ell = 2, 3, 4$ parties and $M = 598$ seats for the divisor method with rounding down. With threshold t growing from zero to $1/\ell$, the linear decrease is seen to be in perfect agreement with the simulated seat biases, indicated by bold dots, for thresholds of 5, 10, and 15 percent.

4.3 Majority and Minority Criteria

There are several criteria to decide whether an apportionment method is suitable for use in proportional representation [22, 24, 25]. We may expect that the following two criteria should be fulfilled:

1.) A majority of votes implies a majority of seats in parliament.

2.) A minority of votes implies a minority of seats in parliament.

However, in general these criteria are violated. An example for such a violation is shown in Figure 4.4, where the marked set of weight vectors represents a majority of votes – of course for the largest party. Still, because the divisor method with rounding up leads to the seat allocation $\mathbf{m} = (2, 2, 1)^t$, there is no majority of seats. In the present section we aim at calculating the probability that the majority or minority criterion is not fulfilled.

Therefore, we introduce the set of weight vectors violating the majority criterion

$$V_+(A, \ell) := \left\{ \mathbf{w} \in S^\ell_\geq : w_1 > \frac{1}{2} \text{ and } A(w_1) \leq \frac{M}{2} \right\},$$

and the set of weight vectors violating the minority criterion

$$V_-(A, \ell) := \left\{ \mathbf{w} \in S^\ell_\geq : w_1 < \frac{1}{2} \text{ and } A(w_1) \geq \frac{M}{2} \right\}.$$

Assuming a uniform distribution of the weight vectors $\mathbf{w}$ on $S^\ell_\geq$, we are interested in the probabilities

$$P_+(A, \ell) := P(\mathbf{w} \in V_+(A, \ell) | w_1 \geq \ldots \geq w_\ell)$$

and

$$P_-(A, \ell) := P(\mathbf{w} \in V_-(A, \ell) | w_1 \geq \ldots \geq w_\ell).$$

To calculate these probabilities, the case that the house size M is even has to be distinguished from the case that M is odd.

For M even, under the condition that $w_1 = 1/2$ implies $A(w_1) \leq M/2$ we have

$$P_+(A, \ell) = P\left(w_1 > \frac{1}{2} \middle| w_1 \geq \ldots \geq w_\ell \right) - \sum_{\substack{\mathbf{m} \in G^\ell_\geq(M), \\ m_1 > \frac{M}{2}}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell),$$

and under the condition that $w_1 = 1/2$ implies $A(w_1) \geq M/2$ we have

$$P_-(A, \ell) = \sum_{\substack{\mathbf{m} \in G^\ell_\geq(M), \\ m_1 \geq \frac{M}{2}}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell) - P\left(w_1 > \frac{1}{2} \middle| w_1 \geq \ldots \geq w_\ell \right).$$

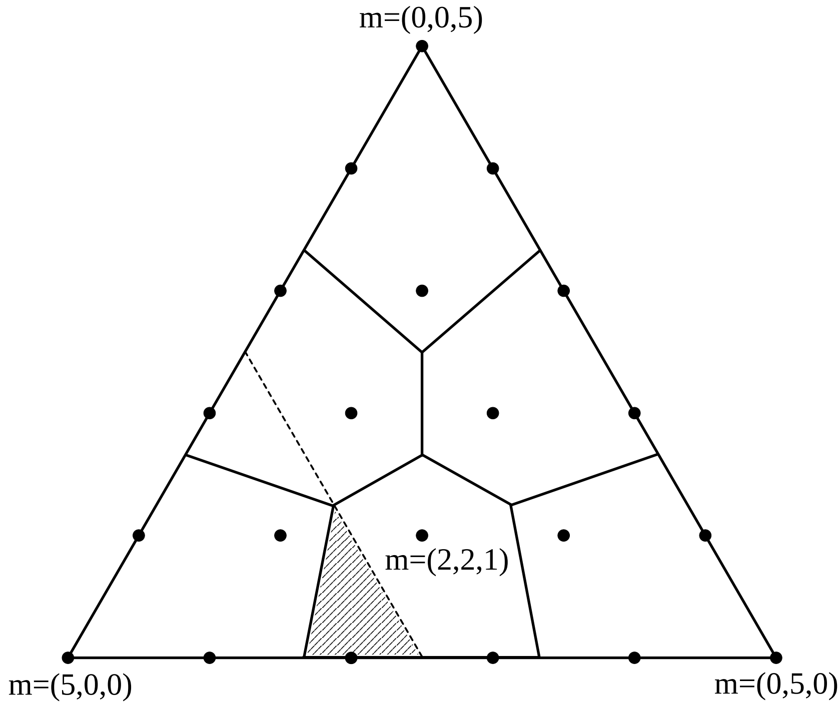


Figure 4.4: Set of weight vectors $V_+(A, \ell)$ violating the majority criterion for the divisor method with rounding up ($M = 5$, $\ell = 3$).

The quota method of greatest remainders fulfills both conditions, since the definition of $z(\lambda)$ and $e(\lambda)$ implies $z(\lambda) + e(\lambda) \leq \ell$, and therefore

$$\frac{e(\lambda)}{\ell - z(\lambda)} \leq 1.$$

By (2.3), this guarantees that a weight vector $\mathbf{w}$ with $w_1 = 1/2$ leads to a seat allocation $\mathbf{m}$ with $m_1 = M/2$.

For the q-stationary divisor methods we examine whether the first component of the vertex $\mathbf{v}(\lambda = (0, 1, \ldots, 1)^t)$ of the rounding polytope $P(\mathbf{m} = (M/2 + 1, m_2, \ldots, m_\ell)^t)$ is at least 1/2, see (2.9),

$$\frac{\left(\frac{M}{2} + 1\right) + q - 1}{M + \ell q - 1} \geq \frac{1}{2} \iff q \leq \frac{1}{\ell - 2},$$

as well as whether the first component of the vertex $\mathbf{v}(\lambda = (1, 0, \ldots, 0)^t)$ of the rounding polytope $P(\mathbf{m} = (M/2 - 1, m_2, \ldots, m_\ell)^t)$ is at most 1/2,

$$\frac{\left(\frac{M}{2} - 1\right) + q}{M + \ell q - (\ell - 1)} \leq \frac{1}{2} \iff q \geq \frac{\ell - 3}{\ell - 2}.$$

For system size $\ell = 3$ we therefore can compute both $P_+(A, \ell)$ and $P_-(A, \ell)$, particularly for the divisor methods with rounding up ($q = 0$) and down ($q = 1$). For $\ell = 4$, one can calculate $P_+(A,\ell)$ for the divisor method with rounding up, and $P_-(A,\ell)$ for the divisor method with rounding down.

Turning to the case M odd, we have the relation

$$\begin{aligned} P_+(A,\ell) - P_-(A,\ell) &= P\left(w_1 > \frac{1}{2}\middle| w_1 \geq \ldots \geq w_\ell\right) \\ &\quad - \sum_{\substack{\mathbf{m}\in G^\ell_\geq(M),\\ m_1 > \frac{M}{2}}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell), \end{aligned}$$

which we can evaluate only if $P_+(A,\ell) = 0$ or $P_-(A,\ell) = 0$; except for the quota method of greatest remainders where we have additional geometrical insight.

For the q-stationary divisor methods we examine whether the first component of the vertex $\mathbf{v}(\lambda = (0,1,\ldots,1)^t)$ of the rounding polytope $P(\mathbf{m} = ((M+1)/2, m_2, \ldots, m_\ell)^t)$ is at least 1/2, see (2.9),

$$\frac{\frac{M+1}{2} + q - 1}{M + \ell q - 1} \geq \frac{1}{2} \iff q = 0,$$

as well as whether the first component of the vertex $\mathbf{v}(\lambda = (1,0,\ldots,0)^t)$ of the rounding polytope $P(\mathbf{m} = ((M-1)/2, m_2, \ldots, m_\ell)^t)$ is at most 1/2,

$$\frac{\frac{M-1}{2} + q}{M + \ell q - (\ell - 1)} \leq \frac{1}{2} \iff q = 1.$$

For any system size, it follows that $P_-(A,\ell) = 0$ for the divisor method with rounding up ($q = 0$) and that $P_+(A,\ell) = 0$ for the divisor method with rounding down ($q = 1$).

By integration we obtain

$$P\left(w_1 > \frac{1}{2}\middle| w_1 \geq \ldots \geq w_\ell\right) = \frac{\int_0^{\frac{1/2}{\ell-1}} dw_\ell \int_{w_\ell}^{\frac{1/2-w_\ell}{\ell-2}} dw_{\ell-1} \ldots \int_{w_3}^{\frac{1/2-w_\ell-\ldots-w_3}{1}} dw_2\, 1}{\int_0^{\frac{1}{\ell}} dw_\ell \int_{w_\ell}^{\frac{1-w_\ell}{\ell-1}} dw_{\ell-1} \ldots \int_{w_3}^{\frac{1-w_\ell-\ldots-w_3}{2}} dw_2\, 1},$$

which yields for $\ell = 3$ the value 3/4 and for $\ell = 4$ the value 1/2. The summations

$$\sum_{\substack{\mathbf{m}\in G^\ell_\geq(M),\\ m_1 > \frac{M}{2}}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell) \quad \text{and} \quad \sum_{\substack{\mathbf{m}\in G^\ell_\geq(M),\\ m_1 \geq \frac{M}{2}}} P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell)$$

can be evaluated analogous with the calculation of seat biases by apportionment polynomials. In particular, we apply the decomposition of $G^\ell_\geq(M)$ into the disjoint subsets, for $r = 1,\ldots,\ell$,

$$K_r(M) = \left\{\mathbf{m} \in G^\ell_\geq(M) : m_r > 0 = m_{r+1}\right\},$$

and account for the additional condition $m_1 > \frac{M}{2}$ or $m_1 \geq \frac{M}{2}$. Resulting probabilities of violated majority and minority criteria are presented in Table 4.5, for the quota method of greatest remainders and the divisor methods with rounding up and down.

method	ℓ	M	$P_+(A,\ell)$	$P_-(A,\ell)$
Hamilton	3	even	$\frac{12M-8}{8M^2}$	$\frac{12M-16}{8M^2}$
	3	odd	$\frac{M-1}{8M^2}$	$\frac{M+1}{8M^2}$
	4	even	$\frac{12M^2-16M+8}{8M^3}$	$\frac{12M^2+16M-48}{8M^3}$
	4	odd	$\frac{M^2-2M+1}{8M^3}$	$\frac{M^2+2M-1}{8M^3}$
Adams ($q=0$)	3	even	$\frac{3M-6}{\frac{4}{3}(M-1)(M-2)}$	$\frac{M-2}{\frac{4}{3}(M-1)(M-2)}$
	3	odd	$\frac{M-1}{\frac{4}{3}(M-1)(M-2)}$	0
	4	even	$\frac{4M^2-22M+28}{\frac{4}{3}(M-1)(M-2)(M-3)}$	—
	4	odd	$\frac{2M^2-8M+6}{\frac{4}{3}(M-1)(M-2)(M-3)}$	0
Jefferson ($q=1$)	3	even	$\frac{M+2}{\frac{4}{3}(M+1)(M+2)}$	$\frac{3M-2}{\frac{4}{3}(M+1)(M+2)}$
	3	odd	0	$\frac{M+1}{\frac{4}{3}(M+1)(M+2)}$
	4	even	—	$\frac{4M^2+22M-20}{\frac{4}{3}(M+1)(M+2)(M+3)}$
	4	odd	0	$\frac{2M^2+8M+6}{\frac{4}{3}(M+1)(M+2)(M+3)}$

Table 4.5: Probabilities of violated majority and minority criteria for several apportionment methods in systems of $\ell = 3$ and $\ell = 4$ parties.

Chapter 5

Asymptotic Seat Biases

The combinatorial approach to seat biases enables us to prove the mentioned conjecture by Schuster *et al.* [32, Section A.3] on asymptotic seat biases for an arbitrary number of parties, as the house size tends to infinity. Considered apportionment methods are both the stationary divisor methods and the quota method of greatest remainders. Following [13], the asymptotic seat bias formulas are given in the first section and proved applying results from the second section of this chapter.

5.1 Asymptotic Seat Bias Formulas

General seat bias formulas holding for arbitrary system size ℓ are stated in the following two Theorems.

Theorem 5.1 (Seat biases of divisor methods)
Under the q-stationary divisor method, the seat bias $B_i^\ell(M)$, $i = 1, \ldots, \ell$, fulfills

$$B_i^\ell(M) = \left(\frac{1}{2} - q\right)\left(1 - \sum_{j=i}^{\ell} \frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right).$$

Theorem 5.2 (Seat biases of the Hamilton method)
Under the quota method of greatest remainders, the seat bias $B_i^\ell(M)$, $i = 1, \ldots, \ell$, fulfills

$$B_i^\ell(M) = \mathcal{O}\left(\frac{1}{M}\right).$$

Due to Theorem 5.1 we find that the divisor method with standard rounding ($q = 0.5$) is the only q-stationary divisor method which is asymptotically unbiased as the house size M tends to infinity. In addition, Theorem 5.2 shows that the quota method of greatest remainders is asymptotically unbiased. Schuster *et al.* [32] give a detailed interpretation of these results.

The proofs of Theorems 5.1 and 5.2 rely on the knowledge that the i-th component of $\mathbf{B}^\ell(M)$ is equal to (Theorem 3.3)

$$B_i^\ell(M) = \left(\sum_{r=1}^{\ell} \binom{\ell}{r} p(r)\, \bar{S}_i^r(M)\right) - \frac{M}{\ell} \sum_{j=i}^{\ell} \frac{1}{j},$$

where we introduce the notation, for $i = 1, \ldots, r$,

$$\bar{S}_i^r(M) := r! S_i^r(M) = \sum_{\mathbf{m} \in K_r(M)} \frac{r! m_i}{b_r(\mathbf{m})}.$$

Till now apportionment polynomials were defined without the factor $r!$ in the numerator. However, the factor is advantageous to cast the subsequent findings into a more pleasing format. Recall that $b_r(\mathbf{m})$ counts the permutations leaving $\mathbf{m}$ invariant, and

$$K_r(M) = \left\{\mathbf{m} \in G_{\geq}^\ell(M) : m_r > 0 = m_{r+1}\right\}.$$

For $i \geq r+1$, we have $\bar{S}_i^r(M) = 0$. In addition, Theorem 3.4 yields $\bar{S}_1^1(M) = M$, and for $r \geq 2$ and $i \leq r$

$$\bar{S}_i^r(M) = \left(\sum_{h=1}^{r-1} \binom{r}{h} \sum_{m_r=1}^{\left\lfloor \frac{M-1}{r} \right\rfloor} \left(\bar{S}_i^h(M - m_r r) + \binom{M - m_r r - 1}{h-1} m_r\right)\right) + \mathcal{O}(M).$$

Proving Theorems 5.1 and 5.2 requires knowledge about the two leading terms of the apportionment polynomials. Thus, we need to calculate the coefficients $s_i(r)$ and $t_i(r)$ in

$$\bar{S}_i^r(M) = s_i(r) M^r + t_i(r) M^{r-1} + \mathcal{O}\left(M^{r-2}\right),$$

which is done in Lemma 5.1.

Lemma 5.1 (Leading terms of apportionment polynomials)
The coefficients of M^r and M^{r-1} of the apportionment polynomial $\bar{S}_i^r(M)$ are

$$s_i(r) = \begin{cases} \frac{1}{r!} \sum_{j=i}^{r} \frac{1}{j} & \text{for } i \leq r, \\ 0 & \text{for } i > r, \end{cases} \tag{5.1}$$

and

$$t_i(r) = -\frac{r}{2}\, s_i(r-1) = \begin{cases} -\frac{r}{2(r-1)!} \sum_{j=i}^{r-1} \frac{1}{j} & \text{for } i < r, \\ 0 & \text{for } i \geq r. \end{cases} \tag{5.2}$$

Proof. The assertion results from Lemmata 5.2, 5.3, and 5.4. **qed**

Lemma 5.1 and the seat allocation distributions given in Theorems 3.1 and 3.2 enable us to prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Fix $q \in [0,1]$. For obtaining the leading term of $B_i^\ell(M)$ in an expansion in powers of M^{-1}, we have to calculate $p(\ell)$ and $p(\ell-1)$ by (3.1). We find

$$\begin{aligned} p(\ell) &= (\ell-1)! \sum_{\mathbf{t} \in \left\{ {\ell-1 \atop 0} \right\}} \prod_{j=1}^{\ell-1} \frac{1}{M + (\ell + t_{(j)})q - (\ell - j + t_{(j)})} \\ &= \frac{(\ell-1)!}{M^{\ell-1} + M^{\ell-2} \sum_{j=1}^{\ell-1} (\ell q - \ell + j) + \mathcal{O}\left(M^{\ell-3}\right)} \end{aligned}$$

and

$$\begin{aligned} p(\ell-1) &= q(\ell-2)! \sum_{\mathbf{t} \in \left\{ {\ell-1 \atop 1} \right\}} \prod_{j=1}^{\ell-1} \frac{1}{M + (\ell - 1 + t_{(j)})q - (\ell - 1 - j + t_{(j)})} \\ &= \frac{q(\ell-1)!}{M^{\ell-1} + \mathcal{O}\left(M^{\ell-2}\right)}. \end{aligned}$$

On the other hand, by Lemma 5.1, we have

$$\bar{S}_i^\ell(M) = \frac{M^\ell}{\ell!} \left(\sum_{j=i}^{\ell} \frac{1}{j} \right) - \frac{\ell M^{\ell-1}}{2(\ell-1)!} \left(\sum_{j=i}^{\ell-1} \frac{1}{j} \right) + \mathcal{O}\left(M^{\ell-2}\right)$$

and

$$\bar{S}_i^{\ell-1}(M) = \frac{M^{\ell-1}}{(\ell-1)!} \left(\sum_{j=i}^{\ell-1} \frac{1}{j} \right) + \mathcal{O}\left(M^{\ell-2}\right).$$

By polynomial division we obtain

$$p(\ell)\bar{S}_i^\ell(M) = \frac{1}{\ell} \left(\sum_{j=i}^{\ell} \frac{1}{j} \right) \left(M - \sum_{j=1}^{\ell-1} (\ell q - \ell + j) \right) - \frac{\ell}{2} \left(\sum_{j=i}^{\ell-1} \frac{1}{j} \right) + \mathcal{O}\left(\frac{1}{M} \right)$$

and

$$p(\ell-1)\bar{S}_i^{\ell-1}(M) = q \left(\sum_{j=i}^{\ell-1} \frac{1}{j} \right) + \mathcal{O}\left(\frac{1}{M} \right).$$

This finally leads to

$$\begin{aligned} B_i^\ell(M) &= \left(\sum_{j=i}^{\ell} \frac{1}{j} \right) \left(\sum_{j=1}^{\ell-1} \left(1 - q - \frac{j}{\ell} \right) \right) + \left(\ell q - \frac{\ell}{2} \right) \left(\sum_{j=i}^{\ell-1} \frac{1}{j} \right) + \mathcal{O}\left(\frac{1}{M} \right) \\ &= \left(\sum_{j=i}^{\ell} \frac{1}{j} \right) \left(\frac{1}{2} - q \right) (\ell - 1) + \left(\frac{1}{2} - q \right) \left(1 - \ell \sum_{j=i}^{\ell} \frac{1}{j} \right) + \mathcal{O}\left(\frac{1}{M} \right), \end{aligned}$$

which yields the expression claimed in the assertion. **qed**

Proof of Theorem 5.2. Proceeding analogously to the proof of Theorem 5.1 we get from (3.4)

$$p(\ell) = \frac{(\ell-1)!}{M^{\ell-1}} \sum_{\mathbf{t}\in\left\{ {\ell-1 \atop 0} \right\}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{\ell + t_{(j)}}\right)^{t_{j+1}} = \frac{(\ell-1)!}{M^{\ell-1}}$$

and

$$\begin{aligned} p(\ell-1) &= \frac{(\ell-1)!}{\ell M^{\ell-1}} \sum_{\mathbf{t}\in\left\{ {\ell-1 \atop 1} \right\}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{\ell - 1 + t_{(j)}}\right)^{t_{j+1}} = \frac{(\ell-1)!}{\ell M^{\ell-1}} \sum_{j=0}^{\ell-2} \left(1 - \frac{j}{\ell-1}\right) \\ &= \frac{(\ell-1)!}{2M^{\ell-1}}. \end{aligned}$$

This results in

$$p(\ell)\bar{S}_i^\ell(M) = \frac{M}{\ell}\left(\sum_{j=i}^{\ell} \frac{1}{j}\right) - \frac{\ell}{2}\left(\sum_{j=i}^{\ell-1} \frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right)$$

and

$$p(\ell-1)\bar{S}_i^{\ell-1}(M) = \frac{1}{2}\left(\sum_{j=i}^{\ell-1} \frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right),$$

leading to the expression claimed in the assertion. **qed**

An alternative proof of Theorem 5.1 for the stationary divisor methods results from the asymptotic treatment of the *seat excess*

$$A(\mathbf{w}) - \mathbf{w}M$$

in Heinrich *et al.* [18, Corollary 3.2], see also [17, 19]. Assuming that the truncated weight vector $(w_1, \ldots, w_{\ell-1})^t$ has a Riemann integrable Lebesgue density on its domain

$$\left\{ (w_1, \ldots, w_{\ell-1})^t \in [0,1]^{\ell-1} : \sum_{i=1}^{\ell-1} w_i < 1 \right\},$$

one can show that the (conditional) expectation of the seat excess satisfies

$$\lim_{M\to\infty} \mathrm{E}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell \geq t] = \left(\frac{1}{2} - q\right)(1 - \ell \cdot \mathrm{E}[\mathbf{w}|w_1 \geq \cdots \geq w_\ell \geq t]), \tag{5.3}$$

where t is the threshold that parties have to pass in order to be eligible to participate in the apportionment process. Under the assumption that the weight vector $\mathbf{w}$ is uniformly distributed on the probability simplex S^ℓ, this leads to the asymptotic seat bias formula of Theorem 5.3.

Theorem 5.3 (Seat biases of divisor methods in systems with thresholds) *In systems with threshold t, under the q-stationary divisor method, the seat bias $B_i^\ell(M,t)$, $i = 1, \ldots, \ell$, fulfills*

$$\lim_{M\to\infty} B_i^\ell(M,t) = (1-\ell t)\left(\frac{1}{2} - q\right)\left(1 - \sum_{j=i}^{\ell} \frac{1}{j}\right).$$

Proof of Theorem 5.3. On the conditioning event $\{w_1 \geq \cdots \geq w_\ell \geq t\}$, with some threshold $t \leq 1/\ell$, the transformed variables, for $i = 1, \ldots, \ell$,

$$w_i^* := \frac{w_i - t}{1 - \ell t}$$

are non-negative and sum to unity. Therefore, they inherit the uniform distribution of $\mathbf{w}$ and we get

$$\begin{aligned} \mathrm{E}[w_i | w_1 \geq \cdots \geq w_\ell \geq t] &= (1-\ell t) \cdot \mathrm{E}[w_i^* | w_1^* \geq \cdots \geq w_\ell^* \geq 0] + t \\ &= (1-\ell t) \cdot \mathrm{E}[w_i | w_1 \geq \cdots \geq w_\ell] + t. \end{aligned}$$

From Lemma 3.1, see also Johnson *et al.* [20, p. 500], we know that

$$\mathrm{E}[w_i | w_1 \geq \cdots \geq w_\ell] = \frac{1}{\ell} \sum_{j=i}^{\ell} \frac{1}{j}.$$

This leads to

$$1 - \ell \cdot \mathrm{E}[w_i | w_1 \geq \cdots \geq w_\ell \geq t] = (1-\ell t)\left(1 - \sum_{j=i}^{\ell} \frac{1}{j}\right),$$

which completes the proof due to (5.3). **qed**

5.2 Leading Terms of Apportionment Polynomials

We prove Lemma 5.1 via the following Lemmata about the coefficients $s_i(r)$ and $t_i(r)$ of the apportionment polynomials $\bar{S}_i^r(M)$.

Lemma 5.2 (Coefficients of highest order)
For $i = 1, \ldots, r$, the coefficient of order M^r of the apportionment polynomial $\bar{S}_i^r(M)$ is

$$s_i(r) = \frac{1}{r!} \sum_{j=i}^{r} \frac{1}{j}. \tag{5.4}$$

Proof. For the stationary divisor method with parameter $q \in [0,1]$ and for $\mathbf{w} \in S^\ell$ we have

$$\lim_{M\to\infty} \frac{m_i(\mathbf{w})}{M} = w_i.$$

Selecting $q = 0$, it follows from (3.2), for all $i = 1, \ldots, \ell$,

$$0 = \lim_{M\to\infty} \frac{1}{M}\, B_i^\ell(M) = \lim_{M\to\infty} \frac{1}{M\binom{M-1}{\ell-1}}\, \bar{S}_i^\ell(M) - \frac{1}{\ell} \sum_{j=i}^{\ell} \frac{1}{j}.$$

Now

$$\binom{M-1}{\ell-1} = \frac{M^{\ell-1}}{(\ell-1)!} + \mathcal{O}\left(M^{\ell-2}\right)$$

implies

$$(\ell-1)!\, s_i(\ell) - \frac{1}{\ell} \sum_{j=i}^{\ell} \frac{1}{j} = 0,$$

from which the claim is derived. **qed**

Lemma 5.3 (Vanishing coefficient of second highest order)
The apportionment polynomial $\bar{S}_r^r(M)$ has no term of order M^{r-1},

$$t_r(r) = 0. \tag{5.5}$$

Proof. For $r \leq 4$ the result has been established previously. For $r \geq 3$ we have

$$\bar{S}_r^r(M) = \left(\sum_{h=1}^{r-1} \binom{r}{h} \sum_{m_r=1}^{\left\lfloor \frac{M-1}{r} \right\rfloor} \binom{M - m_r r - 1}{h-1} m_r \right) + \mathcal{O}(M).$$

Using

$$\binom{M - m_r r - 1}{h-1} = \frac{1}{(h-1)!}\, (M - m_r r)^{h-1} - \frac{h}{2(h-2)!}\, (M - m_r r)^{h-2} + \mathcal{O}\left((M - m_r r)^{h-3}\right)$$

for $h = r-1$ and $h = r-2$ leads to cancellation of terms involving $(M - m_r r)^{r-3}$, and yields

$$\bar{S}_r^r(M) = \left(\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \frac{(M - m_r r)^{r-2} m_r r}{(r-2)!} \right) + \mathcal{O}\left(M^{r-2}\right).$$

Applying the Binomial Theorem, we write

$$\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} (M - m_r r)^{r-2} m_r r = \sum_{i=0}^{r-2} \binom{r-2}{i} (-1)^i r^{i+1} M^{r-2-i} \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} m_r^{i+1}.$$

By Burrows and Talbot [9] or Edwards [14], the sum of powers of integers amounts to

$$\sum_{m_r=1}^{s} m_r^{i+1} = \frac{s^{i+2}}{i+2} + \frac{s^{i+1}}{2} + \mathcal{O}\left(s^i\right),$$

for all $i \geq 0$, where

$$s := \frac{M}{r} + c_M := \left\lfloor \frac{M-1}{r} \right\rfloor.$$

Here, $c_M = \mathcal{O}(1)$ depends on M but is bounded. Applying the Binomial Theorem to the term s^{i+2} yields

$$\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} m_r^{i+1} = \frac{M^{i+2}}{(i+2) r^{i+2}} + \frac{(i+2) M^{i+1} c_M}{r^{i+1}(i+2)} + \frac{M^{i+1}}{2 r^{i+1}} + \mathcal{O}\left(M^i\right).$$

By this result we obtain

$$\begin{aligned} \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} (M - m_r r)^{r-2} m_r r \quad &= \quad \frac{M^r}{r} \sum_{i=0}^{r-2} \binom{r-2}{i} \frac{(-1)^i}{i+2} \\ &\quad + M^{r-1} \left(c_M + \frac{1}{2} \right) \sum_{i=0}^{r-2} \binom{r-2}{i} (-1)^i + \mathcal{O}\left(M^{r-2}\right). \end{aligned}$$

The fact that

$$\sum_{i=0}^{r-2} \binom{r-2}{i} (-1)^i = 0,$$

for $r \geq 3$, then implies the claim. **qed**

One can show that

$$\sum_{i=0}^{r-2} \binom{r-2}{i} \frac{(-1)^i}{i+2} = \frac{1}{r(r-1)}.$$

Thus, the proof of Lemma 5.3 gives rise to an alternative proof of Lemma 5.2 for $i = r$ because it follows

$$S_r^r(M) = \frac{M^r}{r \cdot r!} + \mathcal{O}\left(M^{r-1}\right).$$

Lemma 5.4 (Coefficients of second highest order)
For $i = 1, \ldots, r$, the coefficient of order M^{r-1} of the apportionment polynomial $\bar{S}_i^r(M)$ is

$$t_i(r) = -\frac{r}{2}\, s_i(r-1) = -\frac{r}{2(r-1)!} \sum_{j=i}^{r-1} \frac{1}{j}. \tag{5.6}$$

Proof. Lemma 5.3 establishes the claim for $i = r$ as then the sum in (5.6) is empty. Moreover, for $r \leq 4$ the claim has been established previously. By virtue of these results we can proceed by induction and assume

$$t_i(r-1) = -\frac{r-1}{2}\, s_i(r-2).$$

The recursion of the apportionment polynomials is rewritten as

$$\bar{S}_i^r(M) = \bar{S}_r^r(M) + \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \sum_{h=1}^{r-1} \binom{r}{h} S_i^h(M - m_r r).$$

By Lemma 5.3, the polynomial $\bar{S}_r^r(M)$ does not have a term of order M^{r-1}. Thus, $t_i(r)$ can be determined from the second term in the above relation, which we rewrite as

$$\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \left(r\bar{S}_i^{r-1}(M - m_r r) + \frac{r(r-1)}{2} \bar{S}_i^{r-2}(M - m_r r) + \mathcal{O}\left((M - m_r r)^{r-3}\right)\right).$$

Due to our assumption we know

$$r\bar{S}_i^{r-1}(M) + \frac{r(r-1)}{2} \bar{S}_i^{r-2}(M) = r s_i(r-1) M^{r-1} + \mathcal{O}\left(M^{r-3}\right),$$

and thus obtain

$$\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} \sum_{h=1}^{r-1} \binom{r}{h} \bar{S}_i^h(M - m_r r) = r s_i(r-1) \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} (M - m_r r)^{r-1} + \mathcal{O}\left(M^{r-2}\right).$$

The Binomial Theorem yields

$$\sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} (M - m_r r)^{r-1} = \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i r^i M^{r-1-i} \sum_{m_r=1}^{\lfloor \frac{M-1}{r} \rfloor} m_r^i.$$

Defining

$$s := \frac{M}{r} + c_M := \left\lfloor \frac{M-1}{r} \right\rfloor,$$

we can evaluate $\sum_{m_r=1}^{s} m_r^i$ for $i \geq 1$ as in the proof of Lemma 5.3, but for $i = 0$ we have $\sum_{m_r=1}^{s} 1 = s$. Hence, we find

$$\begin{aligned}
\sum_{m_r=1}^{\left\lfloor \frac{M-1}{r} \right\rfloor} (M - m_r r)^{r-1} &= \frac{M^r}{r} \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{(-1)^i}{i+1} + \frac{M^{r-1}}{2} \sum_{i=1}^{r-1} \binom{r-1}{i} (-1)^i \\
&\quad + M^{r-1} c_M \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i + \mathcal{O}\left(M^{r-2}\right) \\
&= \frac{M^r}{r^2} - \frac{M^{r-1}}{2} + \mathcal{O}\left(M^{r-2}\right).
\end{aligned}$$

After all, the coefficient of order M^{r-1} in $\bar{S}_i^r(M)$ equals $t_i(r) = -\frac{r}{2}\, s_i(r-1)$. **qed**

Chapter 6

Seat Excess Variances

The present chapter complements the work on seat biases, and addresses the conditional variance of the seat excess. The first section gives analytical results, that are proved via facts from the second section. Moreover, numerical simulations and a study of Bavarian electoral data are discussed. These findings have been published in a recent paper [38].

6.1 Analytical and Numerical Results

Let the *seat excess* be the difference between the number of allocated seats and the ideal share of seats,

$$A(\mathbf{w}) - \mathbf{w}M.$$

The previously studied *seat bias* then is the (conditional) expectation of the seat excess,

$$\mathbf{B}^\ell(M) := \mathrm{E}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell],$$

where $\mathbf{w}$ is assumed to be uniformly distributed on S^ℓ. Now we turn to the (conditional) *seat excess variance* $\mathbf{V}^\ell(M) = (V_1^\ell, \ldots, V_\ell^\ell)^t$, which is defined as

$$\mathbf{V}^\ell(M) := \mathrm{Var}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell].$$

The following two Theorems give the asymptotic value of the seat excess variance as the house size M tends to infinity in systems with two or three parties.

Theorem 6.1 (Seat excess variances of divisor methods)

For system size $\ell = 2$ and $\ell = 3$, the asymptotic seat excess variances for the q-stationary divisor methods are given by

$$\begin{aligned}
\mathbf{V}^2(M) &= \left(\tfrac{1}{12} + \tfrac{1}{12}\left(q - \tfrac{1}{2}\right)^2, \tfrac{1}{12} + \tfrac{1}{12}\left(q - \tfrac{1}{2}\right)^2\right)^t + \mathcal{O}\left(\tfrac{1}{M}\right), \qquad (6.1)\\
\mathbf{V}^3(M) &= \left(\tfrac{301}{2592} + \tfrac{13}{72}\left(q - \tfrac{1}{2}\right)^2, \tfrac{235}{2592} + \tfrac{7}{72}\left(q - \tfrac{1}{2}\right)^2, \tfrac{220}{2592} + \tfrac{4}{72}\left(q - \tfrac{1}{2}\right)^2\right)^t + \mathcal{O}\left(\tfrac{1}{M}\right).
\end{aligned}$$

method	ℓ	V_1	V_2	V_3	$\ldots$	V_ℓ	average
Hamilton	2	0.083	0.083				0.083
	3	0.093	0.093	0.093			0.093
	4	0.094	0.094	0.094	$\ldots$		0.094
	5	0.093	0.093	0.093	$\ldots$	0.093	0.093
	10	0.090	0.090	0.090	$\ldots$	0.090	0.090
	50	0.085	0.085	0.085	$\ldots$	0.085	0.085
Webster ($q = 0.5$)	2	0.083	0.083				0.083
	3	0.117	0.091	0.085			0.097
	4	0.131	0.096	0.088	$\ldots$		0.100
	5	0.138	0.101	0.090	$\ldots$	0.084	0.100
	10	0.145	0.110	0.097	$\ldots$	0.083	0.096
	50	0.118	0.104	0.099	$\ldots$	0.083	0.086
Jefferson ($q = 1$)	2	0.104	0.104				0.104
	3	0.161	0.116	0.099			0.126
	4	0.200	0.122	0.106	$\ldots$		0.130
	5	0.227	0.127	0.109	$\ldots$	0.090	0.130
	10	0.301	0.149	0.120	$\ldots$	0.083	0.123
	50	0.413	0.198	0.147	$\ldots$	0.083	0.099

Table 6.1: Seat excess variances $V_1, \ldots, V_\ell$ for traditional apportionment methods, as the size of parliament M tends to infinity, which is simulated using $M = 50,000$. Systems of $\ell = 2, 3, 4, 5, 10$ and 50 parties are considered.

Theorem 6.2 (Seat excess variances of the Hamilton method)
For system size $\ell = 2$ and $\ell = 3$, the asymptotic seat excess variance for the quota method of greatest remainders has the same value for all parties,

$$\begin{aligned} \mathbf{V}^2(M) &= \left(\tfrac{1}{12}, \tfrac{1}{12}\right)^t + \mathcal{O}\left(\tfrac{1}{M}\right), \\ \mathbf{V}^3(M) &= \left(\tfrac{5}{54}, \tfrac{5}{54}, \tfrac{5}{54}\right)^t + \mathcal{O}\left(\tfrac{1}{M}\right). \end{aligned} \tag{6.2}$$

Proof of Theorems 6.1 and 6.2. The seat excess variance can be written as

$$\mathbf{V}^\ell(M) = \bar{\mathbf{E}}^\ell(M) + \bar{\mathbf{I}}^\ell(M) - \left(\mathbf{B}^\ell(M)\right)^2.$$

Here,

$$\bar{\mathbf{I}}^\ell(M) := \mathrm{E}\left[\mathbf{w}^2 M^2 | w_1 \geq \ldots \geq w_\ell\right]$$

is independent of the apportionment method,

$$\bar{\mathbf{E}}^\ell(M) := \mathrm{E}\left[A(\mathbf{w})^2 - 2A(\mathbf{w})\mathbf{w}M | w_1 \geq \ldots \geq w_\ell\right],$$

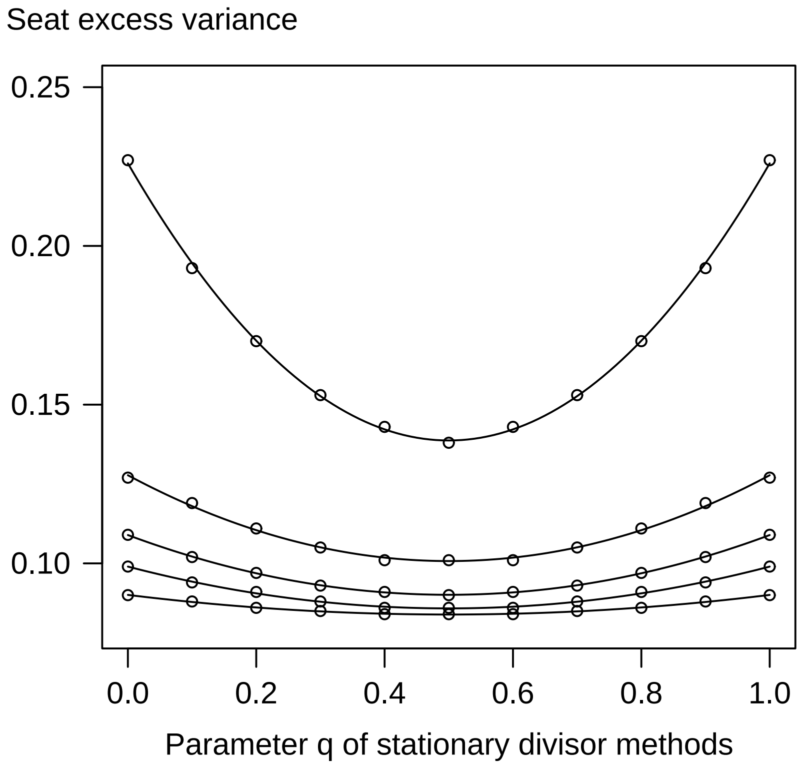


Figure 6.2: Simulated seat excess variances for q-stationary divisor methods, where $\ell = 5$ and $M = 50,000$. Smaller variances correspond to smaller parties.

and the square of a vector is to be read as the vector of squared components. The values for the seat bias for $\ell = 2$ and $\ell = 3$ can be found in Schuster *et al.* [32], see also [35].

Integration over $S^\ell_\geq$ yields

$$\begin{aligned}\bar{\mathbf{I}}^2(M) &= M^2 \left(\tfrac{7}{12}, \tfrac{1}{12}\right)^t, \\ \bar{\mathbf{I}}^3(M) &= M^2 \left(\tfrac{85}{216}, \tfrac{19}{216}, \tfrac{4}{216}\right)^t.\end{aligned}$$

Furthermore, we obtain

$$\bar{\mathbf{E}}^\ell(M) = \sum_{\mathbf{m} \in G^\ell_\geq(M)} \left(\mathbf{m}^2 - 2\mathbf{m}\mathrm{C}^\ell(\mathbf{m})\right) P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell),$$

where

$$\mathrm{C}^\ell(\mathbf{m}) := \mathrm{E}\left[\mathbf{w}M | A(\mathbf{w}) = \mathbf{m}, w_1 \geq \ldots \geq w_\ell\right].$$

The conditional probabilities $P(A(\mathbf{w}) = \mathbf{m} | w_1 \geq \ldots \geq w_\ell)$ have been analyzed in section 3.2, and the expectation $\mathrm{C}^\ell(\mathbf{m})$ equals M times the barycenter of the polytope

$$\bar{P}(\mathbf{m}) := \mathrm{cl}\left\{\mathbf{w} \in S^\ell_\geq : A(\mathbf{w}) = \mathbf{m}\right\}.$$

It is computed using formulas for the barycenters from the subsequent section. Thus the sum in the above relation for $\bar{\mathbf{E}}^\ell(M)$ can be evaluated, leading to $\mathbf{V}^\ell(M)$. In this lengthy

method	$\hat{V}_1^3$	$\hat{V}_2^3$	$\hat{V}_3^3$	average	SE($\hat{V}_1^3$)	SE($\hat{V}_2^3$)	SE($\hat{V}_3^3$)
Hamilton	0.078	0.092	0.077	0.082	0.013	0.013	0.013
Webster ($q = 0.5$)	0.095	0.082	0.077	0.085	0.019	0.013	0.011
Jefferson ($q = 1$)	0.115	0.073	0.129	0.106	0.030	0.019	0.015

Table 6.3: Empirical seat excess variances in three-party systems, based on 49 Bavarian elections (1966-1998), their average, and their simulated standard errors.

evaluation, a decomposition of $G^\ell_\geq(M)$ into the disjoint subsets, for $r = 1, \ldots, \ell$,

$$K_r(M) = \left\{\mathbf{m} \in G^\ell_\geq(M) : m_r > 0 = m_{r+1}\right\}$$

is helpful; compare the calculation of seat biases and the following sections. **qed**

Systems of more than three parties can be investigated by the numerical simulations summarized in Table 6.1. Asymptotic seat excess variances for $M \to \infty$ are simulated by the house size $M = 50,000$. Each value in Table 6.1 is computed from $50,000$ simulated seat excesses. For the quota method of greatest remainders the results confirm Theorem 6.2. The fact that the seat excess variance is asymptotically equal for all parties appears to be a general feature of the method, observed in all the simulations. Interestingly, the variance becomes maximal for $\ell = 4$ and decreases monotonously for larger ℓ. One may conjecture that the asymptotic seat excess variances converge to 1/12 as $\ell \to \infty$.

The outcome of the simulations for q-stationary divisor methods confirms Theorem 6.1. Generally, the seat excess variance is found to be maximal in the case of the largest party and to decrease monotonously for smaller parties, where its average value again is maximal for $\ell = 4$ and descreases thereafter. The simulations show that the asymptotic variance, as a function of the parameter q, is a parabola with the minimum at $q = 1/2$. Figure 6.2 illustrates this conjecture for $\ell = 5$ parties, presenting results of simulations for $q = 0.0, 0.1, \ldots, 1.0$ together with a least squares parabola.

Finally, using data from Schuster *et al.* [32] on 49 elections with three parties in the German State of Bavaria (1966-1998), empirical seat excess variances can be computed, see Table 2.3. Ignoring that the house size M of the elections was moderate and varied from 19 to 65, these empirical variances can be compared to the theoretical asymptotic values from Theorems 6.1 and 6.2; they are of the same order of magnitude. To judge if the deviations between empirical and theoretical results are within chance variability, 49 weight vectors from a uniform distribution on $S^3_\geq$ were repeatedly simulated, and the seat excess variances for the large house size $M = 50,000$ were computed. Using 10,000 repetitions, the standard errors SE($\hat{V}_i^3$), $i = 1, 2, 3$, of the empirical seat excess variance take the values stated in Table 2.3. They confirm nice agreement between empirical and theoretical variances which deviate by two standard errors or less, except for the case of $\hat{V}_2^3$ for the Jefferson method ($q = 1$) in which the deviation is slightly larger. Comparing empirical seat excess variances and simulated variances for $M \to \infty$ is justified since the dependence of $\mathbf{V}^\ell(M)$ on the house size turns out to be very small. As a function of M, the seat excess variances rapidly approach their asymptotic values.

6.2 Barycenters of Rounding Polytopes

The proof of Theorems 6.1 and 6.2 in the former section was based on the barycenters of the polytopes, for $\mathbf{m} \in G^\ell_\geq(M)$,

$$\bar{P}(\mathbf{m}) = \mathrm{cl}\left\{\mathbf{w} \in S^\ell_\geq : A(\mathbf{w}) = \mathbf{m}\right\}.$$

Set closure circumvents ambiguities in the definition of seat allocations when the weight vector $\mathbf{w}$ contains ties. The polytope $\bar{P}(\mathbf{m})$ is the intersection of the rounding polytope

$$P(\mathbf{m}) = \mathrm{cl}\left\{\mathbf{w} \in S^\ell : A(\mathbf{w}) = \mathbf{m}\right\}$$

with the ordered probability simplex $S^\ell_\geq$. Note that these notations suppress the dependence on the chosen apportionment method. Rounding polytopes for the quota method of greatest remainders and for divisor methods have been addressed previously, resulting in their vertices and surface volumes.

Recall that $\mathbf{0}_\ell$ and $\mathbf{1}_\ell$ are the row vectors in $\mathbb{R}^\ell$ with all components equal to zero or one, respectively. For $\mathbf{m} \in G^\ell(M)$,

$$R(\mathbf{m}) = \{i : m_i \neq 0\} \quad \text{and} \quad r(\mathbf{m}) = |R(\mathbf{m})|.$$

The quota method of greatest remainders and the divisor methods give rise to rounding polytopes with $2^\ell - 2^{\ell - r(\mathbf{m})} - 1$ vertices $\mathbf{v}(\lambda)$, induced by the vectors $\lambda \in \{0,1\}^\ell \setminus \{\mathbf{0}_\ell, \mathbf{1}_\ell\}$ with $\lambda_j = 0$ for some index $j \in R(\mathbf{m})$. If $r(\mathbf{m}) = 1$, then $\mathbf{v}(\mathbf{0}_\ell)$ is an additional vertex.

By (2.3), for the quota method of greatest remainders the components of the vertices are, for $i = 1, \ldots, \ell$,

$$v_i(\lambda) = \begin{cases} \frac{1}{M}\left(m_i + 1 - \frac{e(\lambda)}{\ell - z(\lambda)}\right) & \text{if } \lambda_i = 1, \\ \frac{1}{M}\left(m_i - \frac{e(\lambda)}{\ell - z(\lambda)}\right) & \text{if } \lambda_i = 0 \text{ and } i \in R(\mathbf{m}), \\ 0 & \text{if } \lambda_i = 0 \text{ and } i \notin R(\mathbf{m}), \end{cases}$$

where $z(\lambda) = |\{i \notin R(\mathbf{m}) : \lambda_i = 0\}|$ and $e(\lambda) := |\{1 \leq i \leq \ell : \lambda_i = 1\}|$.

By (2.9), for the q-stationary divisor methods (with $q \in [0,1]$) the components of the vertices are, for $i = 1, \ldots, \ell$,

$$v_i(\lambda) = \begin{cases} \frac{m_i + q}{c(\lambda)} & \text{if } \lambda_i = 1, \\ \frac{m_i + q - 1}{c(\lambda)} & \text{if } \lambda_i = 0 \text{ and } i \in R(\mathbf{m}), \\ 0 & \text{if } \lambda_i = 0 \text{ and } i \notin R(\mathbf{m}), \end{cases}$$

with the normalization $c(\lambda) = M + \ell q - |\{i : \lambda_i = 0\}|$.

From the proofs of Lemma 2.4 and Theorem 2.3 we know

$$\bar{P}(\mathbf{m}) = S^\ell_\geq \cap \bigcup_{k=1}^{r(\mathbf{m})} U_k,$$

where the convex hulls

$$U_k := \mathrm{ch}\left\{\mathbf{v}(\lambda) : \lambda \in \{0,1\}^\ell, \lambda_k = 0\right\}$$

are pairwise disjoint. In the following we decompose the sets U_k into simplices, compare the proof of Lemma 2.4, and compute the barycenters of the simplices, which we can do because for a simplex the barycenter is the arithmetic mean of its vertices. The simplex barycenters then yield the barycenter of $\bar{P}(\mathbf{m})$.

Let S_k be the group of permutations on $\{1,\ldots,\ell\}$ leaving k fix, and define

$$\sigma(\mathbf{x}) := \left(x_{\sigma(1)},\ldots,x_{\sigma(\ell)}\right)^t, \quad \sigma \in S_k,\ \mathbf{x} \in \mathbb{R}^\ell.$$

Let

$$\bar{S}_k := \{\sigma \in S_k : \sigma(\lambda_{k,i})_j \geq \sigma(\lambda_{k,i})_{j+1} \text{ for all } i \text{ and all } j \in B(\mathbf{m})\},$$

where $B(\mathbf{m}) := \{j : 1 \leq j \leq \ell-1, m_j = m_{j+1}\}$. Defining

$$\lambda_{k,i} := (\mathbf{1}_{k-1}, 0, \mathbf{1}_{i-k+1}, \mathbf{0}_{\ell-i-1})^t \in \mathbb{R}^\ell,$$

the definition of U_k implies

$$U_k = \bigcup_{\sigma\in S_k} \Delta_{k,\sigma} \quad \text{and} \quad \bar{U}_k := U_k \cap S^\ell_{\geq} = \bigcup_{\sigma\in\bar{S}_k} \Delta_{k,\sigma},$$

where

$$\Delta_{k,\sigma} := \mathrm{ch}\{\mathbf{v}(\sigma(\lambda_{k,i})) : i = 0,\ldots,\ell-1\}.$$

By using only the permutations in $\bar{S}_k$ one takes into account exactly the ordered weight vectors because

$$\sigma(\lambda_{k,i})_j \geq \sigma(\lambda_{k,i})_{j+1} \iff v_j(\sigma(\lambda_{k,i})) \geq v_{j+1}(\sigma(\lambda_{k,i})),$$

compare the computed vertices for the quota method of greatest remainders and for the q-stationary divisor methods.

The barycenter of the simplex $\Delta_{k,\sigma}$ equals

$$\mathrm{bc}(\Delta_{k,\sigma}) = \frac{1}{\ell}\sum_{i=0}^{\ell-1} \mathbf{v}(\sigma(\lambda_{k,i})).$$

As the interior of $U_i \cap U_j$ is empty for $i \neq j$ and the interior of $\Delta_{k,\sigma} \cap \Delta_{k,\tau}$ is empty for $\sigma \neq \tau$, the barycenter of $\bar{P}(\mathbf{m})$ therefore equals

$$\mathrm{bc}(\bar{P}(\mathbf{m})) = \frac{1}{\mathrm{Vol}(\bar{P}(\mathbf{m}))}\sum_{k=1}^{r(\mathbf{m})}\sum_{\sigma\in\bar{S}_k}\frac{\mathrm{Vol}(\Delta_{k,\sigma})}{\ell}\sum_{i=0}^{\ell-1}\mathbf{v}(\sigma(\lambda_{k,i})).$$

Considering the effects of a permutation $\sigma \in \bar{S}_k$ on the vector $\lambda_{k,\ell-1}$ yields that the sum over $k = 1,\ldots,r(\mathbf{m})$ in the latter relation can be replaced by a sum over $k \notin B(\mathbf{m})$. The special form of $\lambda_{k,i}$ then enables us to combine the sums over k and σ, leading to

$$\mathrm{bc}(\bar{P}(\mathbf{m})) = \frac{1}{\mathrm{Vol}(\bar{P}(\mathbf{m}))}\sum_{\sigma\in\bar{S}^\ell(\mathbf{m})}\frac{\mathrm{Vol}(\Delta_\sigma)}{\ell}\sum_{i=0}^{\ell-1}\mathbf{v}(\sigma(\lambda_{1,i})),$$

where

$$\begin{aligned}\bar{S}^\ell(\mathbf{m}) \;&:=\; \big\{\sigma \in S^\ell : \sigma(1) \le r(\mathbf{m}), \sigma(1) \notin B(\mathbf{m}),\\ &\qquad \text{and } \sigma^{-1}(j) < \sigma^{-1}(j+1) \text{ for all } j \in B(\mathbf{m}), j \neq \sigma(1)-1\big\}.\end{aligned}$$

This equation allows us to compute $\mathrm{bc}(\bar{P}(\mathbf{m}))$ because we can extract the volumes of the simplices Δ_σ from the proofs of Theorems 2.2 and 2.5.

We obtain for the quota method of greatest remainders

$$\mathrm{Vol}(\Delta_\sigma) = \frac{\sqrt{\ell}}{\ell! M^{\ell-1}} \prod_{j=1}^{\ell-2} \left(1 - \frac{j}{r + t^\sigma_{(j)}}\right)^{t^\sigma_{j+1}},$$

and for the q-stationary divisor methods it follows

$$\mathrm{Vol}(\Delta_\sigma) = \frac{q^{\ell-r}\sqrt{\ell}(m_{\sigma(1)} + q - 1)}{(\ell-1)!(M + rq - r)} \prod_{j=1}^{\ell-1} \frac{1}{M + (r + t^\sigma_{(j)})q - (r - j + t^\sigma_{(j)})}.$$

Here the vector $\mathbf{t}^\sigma = (t^\sigma_1, \ldots, t^\sigma_{\ell-1})^t$ is defined by $t^\sigma_i = 1$ for $\sigma(i) \le r(\mathbf{m})$, $i < \sigma(1)$ and for $\sigma(i+1) \le r(\mathbf{m})$, $i > \sigma(1)$; $t^\sigma_i = 0$ otherwise. Finally, for $j \le \ell - 1$ we define

$$t^\sigma_{(j)} := \sum_{i=1}^{j} t^\sigma_i.$$

The volume of the polytope $\bar{P}(\mathbf{m})$ results from

$$\mathrm{Vol}(\bar{P}(\mathbf{m})) = \frac{\sqrt{\ell}}{(\ell-1)! b(\mathbf{m})}\, P(A(\mathbf{w}) = \mathbf{m})$$

and the seat allocation distributions stated in Theorems 3.1 and 3.2.

6.3 Generalized Apportionment Polynomials

The seat allocations in the subset $K_r(M)$ of $G^\ell_{\geq}(M)$ have the first r components positive and the last $\ell - r$ components zero. If the probability $P(A(\mathbf{w}) = \mathbf{m})$ is constant for these seat allocations we can write

$$\bar{\mathbf{E}}(M) = \sum_{r=1}^{\ell} \frac{\ell!}{(\ell-r)!}\, p(r) \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})} \left(\mathbf{m}^2 - 2\mathbf{m}C^\ell(\mathbf{m})\right).$$

To deal with the sum over $\mathbf{m} \in K_r(M)$ we introduce generalized apportionment polynomials, for $r = 1, \ldots, \ell$,

$$\bar{\mathbf{S}}^r(M) := \sum_{\mathbf{m}\in K_r(M)} \frac{1}{b_r(\mathbf{m})} \left(\mathbf{m}^2 - 2\mathbf{m}C^\ell(\mathbf{m})\right).$$

They are determined from the apportionment polynomials $\mathbf{S}^r(M)$ via the function

$$f(\mathbf{x}) := \mathbf{x}^2 - 2\mathbf{x}C^\ell(\mathbf{x})$$

as follows. Define

$$\begin{aligned}
\mathbf{x}_1 &:= (M, 0, \ldots, 0)^t,\\
\mathbf{x}_2 &:= (M - m_2, m_2, 0, \ldots, 0)^t,\\
\mathbf{x}_3 &:= \left(\tfrac{M}{2}, \tfrac{M}{2}, 0, \ldots, 0\right)^t,\\
\mathbf{x}_4 &:= (M - 2m_3, m_3, m_3, 0, \ldots, 0)^t,\\
\mathbf{x}_5 &:= (M - m_2 - m_3, m_2, m_3, 0, \ldots, 0)^t,\\
\mathbf{x}_6 &:= \left(\tfrac{M-m_3}{2}, \tfrac{M-m_3}{2}, m_3, 0, \ldots, 0\right)^t,\\
\mathbf{x}_7 &:= \left(\tfrac{M}{3}, \tfrac{M}{3}, \tfrac{M}{3}, 0, \ldots, 0\right)^t.
\end{aligned}$$

For $r = 1, 2, 3$, the generalized apportionment polynomials then are given by

$$\begin{aligned}
\bar{\mathbf{S}}^1(M) &= f(\mathbf{x}_1),\\
\bar{\mathbf{S}}^2(M) &= \sum_{m_2=1}^{\lfloor \frac{M-1}{2} \rfloor} f(\mathbf{x}_2) + \left[\frac{f(\mathbf{x}_3)}{2}, 0\right]_2^M,\\
\bar{\mathbf{S}}^3(M) &= \sum_{m_3=1}^{\lfloor \frac{M-1}{3} \rfloor} \left(\frac{f(\mathbf{x}_4)}{2} + \sum_{m_2=1+m_3}^{\lfloor \frac{M-m_3-1}{2} \rfloor} f(\mathbf{x}_5) + \left[\frac{f(\mathbf{x}_6)}{2}, 0\right]_2^M \right) + \left[\frac{f(\mathbf{x}_7)}{6}, 0, 0\right]_3^M.
\end{aligned}$$

Now we calculate the seat excess variance for systems of two and three parties via the relation

$$\mathbf{V}(M) = \bar{\mathbf{E}}(M) + \bar{\mathbf{I}}(M) - \mathbf{B}(M)^2.$$

By Schuster *et al.* [32], the seat biases of the quota method of greatest remainders are

$$\mathbf{B}^2(M) = \frac{1}{M}\left(\left[0, \frac{+1}{4}\right]_2^M, \left[0, \frac{-1}{4}\right]_2^M\right)^t$$

and

$$\mathbf{B}^3(M) = \frac{1}{M}\left(\frac{1}{6} + \left[0, \frac{+2}{9M}, \frac{-2}{9M}\right]_3^M, \frac{1}{6} + \left[0, \frac{-4}{9M}, \frac{+4}{9M}\right]_3^M, \frac{-2}{6} + \left[0, \frac{+2}{9M}, \frac{-2}{9M}\right]_3^M\right)^t.$$

Calculating the barycenters of the rounding polytopes $\bar{P}(\mathbf{m})$ for all classes of seat allocations yields for $\ell = 2$ parties

$$\begin{aligned}
\mathrm{bc}(\mathbf{x}_1) &= \tfrac{1}{M}\left(M - \tfrac{1}{4}, \tfrac{1}{4}\right)^t, \\
\mathrm{bc}(\mathbf{x}_2) &= \tfrac{1}{M}\left(M - m_2, m_2\right)^t, \\
\mathrm{bc}(\mathbf{x}_3) &= \tfrac{1}{M}\left(\tfrac{M}{2} + \tfrac{1}{4}, \tfrac{M}{2} - \tfrac{1}{4}\right)^t,
\end{aligned}$$

and for $\ell = 3$ parties

$$\begin{aligned}
\mathrm{bc}(\mathbf{x}_1) &= \tfrac{1}{M}\left(M - \tfrac{7}{18}, \tfrac{5}{18}, \tfrac{2}{18}\right)^t, \\
\mathrm{bc}(\mathbf{x}_2) &= \tfrac{1}{M}\left(M - m_2 - \tfrac{7}{54}, m_2 - \tfrac{7}{54}, \tfrac{14}{54}\right)^t, \\
\mathrm{bc}(\mathbf{x}_3) &= \tfrac{1}{M}\left(\tfrac{M}{2} + \tfrac{5}{54}, \tfrac{M}{2} - \tfrac{19}{54}, \tfrac{14}{54}\right)^t, \\
\mathrm{bc}(\mathbf{x}_4) &= \tfrac{1}{M}\left(M - 2m_3, m_3 + \tfrac{2}{9}, m_3 - \tfrac{2}{9}\right)^t, \\
\mathrm{bc}(\mathbf{x}_5) &= \tfrac{1}{M}\left(M - m_2 - m_3, m_2, m_3\right)^t, \\
\mathrm{bc}(\mathbf{x}_6) &= \tfrac{1}{M}\left(\tfrac{M-m_3}{2} + \tfrac{2}{9}, \tfrac{M-m_3}{2} - \tfrac{2}{9}, m_3\right)^t, \\
\mathrm{bc}(\mathbf{x}_7) &= \tfrac{1}{M}\left(\tfrac{M}{3} + \tfrac{1}{3}, \tfrac{M}{3}, \tfrac{M}{3} - \tfrac{1}{3}\right)^t.
\end{aligned}$$

Recall that $C(\mathbf{m})$ equals M times the barycenter of $\bar{P}(\mathbf{m})$.

Concerning the probabilities $p_{\ell,M}(r)$, we obtain for $\ell = 2$ parties

$$p_{2,M}(1) = \frac{1}{2M} \quad \text{and} \quad p_{2,M}(2) = \frac{1}{M},$$

whereas for $\ell = 3$ parties we find

$$p_{3,M}(1) = \frac{1}{3M^2}, \quad p_{3,M}(2) = \frac{1}{M^2}, \quad \text{and} \quad p_{3,M}(3) = \frac{2}{M^2}.$$

The seat excess variance for the q-stationary divisor methods is calculated analogous with the preceeding considerations. By Schuster *et al.* [32], we have the seat biases

$$\mathbf{B}^2(M, q) = \left(\frac{1}{2}\left[q - \frac{1}{2}\right] + \mathcal{O}\left(\frac{1}{M}\right), \frac{-1}{2}\left[q - \frac{1}{2}\right] + \mathcal{O}\left(\frac{1}{M}\right)\right)^t$$

and

$$\mathbf{B}^3(M,q) = \left(\frac{5}{6}\left[q-\frac{1}{2}\right]+\mathcal{O}\left(\frac{1}{M}\right), \frac{-1}{6}\left[q-\frac{1}{2}\right]+\mathcal{O}\left(\frac{1}{M}\right), \frac{-4}{6}\left[q-\frac{1}{2}\right]+\mathcal{O}\left(\frac{1}{M}\right)\right)^t.$$

Turning to the probabilities $p_{q,\ell,M}(r)$, we find for $\ell = 2$ parties

$$p_{q,2,M}(1) = \frac{q}{M+2q-1} \quad \text{and} \quad p_{q,2,M}(2) = \frac{1}{M+2q-1},$$

whereas the result for $\ell = 3$ parties is

$$\begin{aligned} p_{q,3,M}(1) &= \frac{2q^2}{(M+3q-1)(M+2q-1)}, \\ p_{q,3,M}(2) &= \frac{q(2M+5q-1)}{(M+3q-2)(M+3q-1)(M+2q-1)}, \\ p_{q,3,M}(3) &= \frac{2}{(M+3q-2)(M+3q-1)}. \end{aligned}$$

Chapter 7

Asymptotic Equivalence of Seat Bias Models

Seat biases are defined as averages of the difference between the seats apportioned to the parties and their ideal shares of seats. To evaluate this expectation, Schuster *et al.* [32] assumed a uniform distribution of the electoral vote proportions. Motivated by Balinski and Young [5], this chapter introduces an alternative probabilistic model for evaluating averages of the difference between actual and ideal seat allocations. The model stresses the importance of the rounding process underlying the allocation of seats, as it is based on an assumption other than that of uniformly distributed vote proportions. However, it turns out that the asymptotic seat biases of stationary divisor methods follow the same formula as found for the distributional assumption in Schuster *et al.* [32].

Following [39], the present chapter first gives a comparison of the seat bias models of interest. Then we will turn to the calculation of asymptotic seat biases, where the proof of the central result uses facts derived in the last section of this chapter.

7.1 Apportionment-oriented Seat Bias Model

Schuster *et al.* [32] introduced their concept of seat biases in order to investigate how an apportionment method treats smaller and larger parties, in the process of allocating the M seats available in a parliament to the ℓ competing parties. Assuming that the parties are ordered according to their vote counts, the vector of seat biases is given by

$$\mathbf{B}^\ell(M) := \mathrm{E}\left[A(\mathbf{w}) - \mathbf{w}M \mid w_1 \geq \cdots \geq w_\ell\right].$$

In this model the weight vector $\mathbf{w} = (w_1, \ldots, w_\ell)^t$ represents the vote proportions of the competing parties. It is assumed to be uniformly distributed on the probability simplex

$$S^\ell := \left\{ \mathbf{w} \in [0,1]^\ell : \sum_{i=1}^{\ell} w_i = 1 \right\}.$$

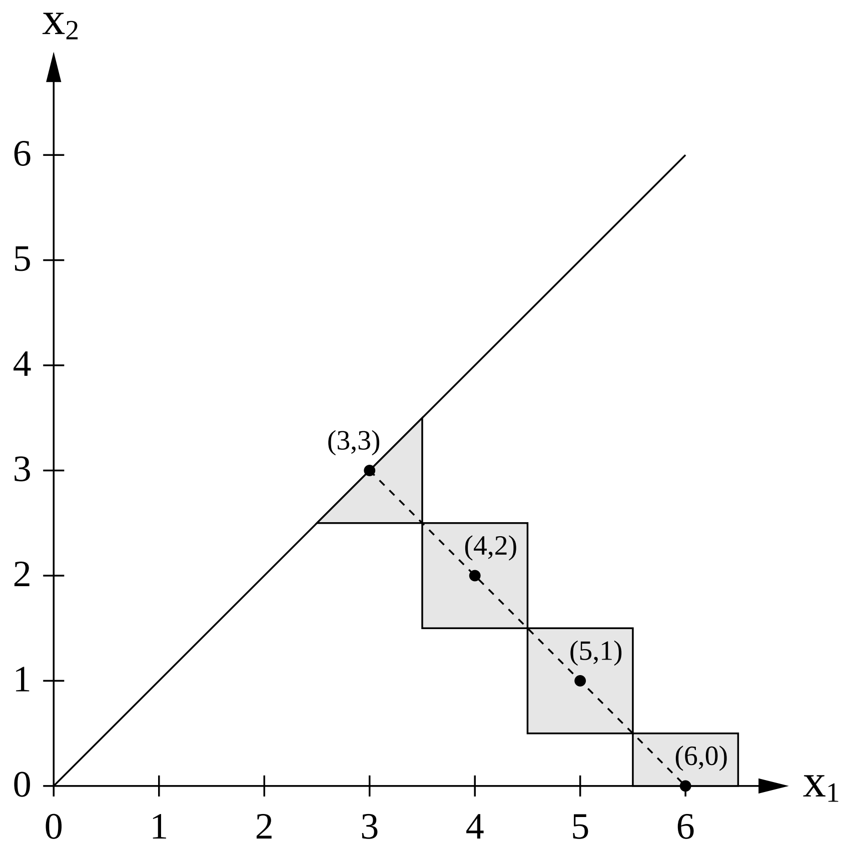


Figure 7.1: Rounding polytopes for the apportionment-oriented seat bias model, for the divisor method with standard rounding, $\ell = 2$ competing parties, and house size $M = 6$. In this model, it is assumed that the vote outcomes $\mathbf{x} = (x_1, x_2)^t$ are equally distributed over the shaded area, indicating the regions of scaled weight vectors $\mathbf{x} \in [0, \infty)^2$ that are rounded to $M = 6$ seats.

The seat bias model of Schuster *et al.* [32] stresses the voter's point of view, because it is based on vote proportions resulting from the electoral process. The most important assumption entering their model is the uniform distribution of these vote proportions on the probability simplex. Alternatively, it is reasonable to approach the problem from the point of view of the apportionment process itself, very much in the spirit of Balinski and Young [5, p. 120f]. For this reason, we restrict our considerations to q-stationary divisor methods, which allocate seats by means of the rounding function $r_q(x)$, with parameter $q \in [0, 1]$. The quantity entering this function is a weight vector scaled by some divisor D, that is $x_i = w_i/D$, for all $i \in \{1, \ldots, \ell\}$. For a given set of vote proportions, we have, in general, not only a single divisor leading to the correct total of M allocated seats but an interval of possible divisors. Therefore, we introduce the set $X^\ell(M)$ of scaled weight vectors which result in seat allocations with a total of M seats,

$$X^\ell(M) := \left\{ \mathbf{x} \in [0, \infty)^\ell : \sum_{i=1}^{\ell} r_q(x_i) = M \right\}.$$

Because any scaled weight vector $\mathbf{x} \in X^\ell(M)$ may enter the rounding process, we assume these weights to be uniformly distributed on the set $X^\ell(M)$, see Figure 7.1.

By means of this probabilistic assumption, it is possible to deal with seat biases in a similar way as discussed in the previous chapters for the model of Schuster *et al.* [32]. In particular, we assume the parties to be ordered according to their vote count. Again the vector of seat biases $\mathbf{B}^\ell(M)$ is the conditional expectation of the difference between the number of allocated seats $A(\mathbf{x})$ and the ideal share of seats. However, the latter now is given by a projection of the scaled weight vector according to

$$\mathbf{x} = (x_1, \ldots, x_\ell)^t \mapsto \frac{M}{\sum_{i=1}^\ell x_i}\, \mathbf{x},$$

for $\mathbf{x} \in X^\ell(M)$. Therefore, we have to deal with the expectation

$$\mathbf{B}^\ell(M) := \mathrm{E}\left[A(\mathbf{x}) - \frac{M}{\sum_{i=1}^\ell x_i}\, \mathbf{x} \,\middle|\, x_1 \geq \cdots \geq x_\ell \right], \tag{7.1}$$

where the weights $\mathbf{x} \in X^\ell(M)$ are assumed to be equally probable. We call this concept the *apportionment-oriented model* of seat biases.

7.2 Calculation of Asymptotic Seat Biases

In the following, we address the asymptotic behaviour of the seat biases when the house size increases, $M \to \infty$. This case is practically relevant for proportional representation systems because the number of seats in a parliament almost always exceeds the number of parties by far. The asymptotic seat biases in the apportionment-oriented model fulfill the formula of Theorem 7.1.

Theorem 7.1 (Asymptotic seat biases in the apportionment-oriented model) *In the apportionment-oriented seat bias model, under the q-stationary divisor method with parameter $q \in [0,1]$, the seat bias of the i-th largest party is*

$$B_i^\ell(M) = \left(\frac{1}{2} - q\right)\left(1 - \sum_{j=i}^{\ell} \frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right), \tag{7.2}$$

for $i = 1, \ldots, \ell$.

This is just the same asymptotic formula as obtained under the assumption of uniformly distributed weights $\mathbf{w} \in S^\ell$, see Theorem 5.1, and therefore the apportionment-oriented model is asymptotically equivalent to the model of Schuster *et al.* [32] when the number of seats available for apportionment increases, $M \to \infty$. One would not expect this kind of equivalence as a uniform distribution on the set $X^\ell(M)$ corresponds to a non-uniform distribution on the probability simplex S^ℓ, which follows from studying the projection of $X^\ell(M)$ onto S^ℓ,

$$\mathbf{x} = (x_1, \ldots, x_\ell) \mapsto \frac{1}{\sum_{i=1}^{\ell} x_i}\,\mathbf{x}.$$

For the case of $\ell = 2$ competing parties the situation is illustrated in Figure 7.1.

Formula (7.2) in Theorem 7.1 states that standard rounding (Webster/Sainte-Laguë, $q = 0.5$) yields the unique q-stationary divisor method which is asymptotically unbiased. Balinski and Young [5, p. 120f] have a different concept of biasedness, in which a divisor method is (pairwise) unbiased when the probability that party 1 is favored over party 2 is equal to the probability that party 2 is favored over party 1. For this model, the method of Webster/Sainte-Laguë is found to be (pairwise) unbiased even for finite house size M.

The proof of Theorem 7.1 will be given after some preparations which we address in the following. The method of calculating the seat biases for the apportionment-oriented model is similar to the geometric-combinatorial approach which we used for the model of Schuster *et al.* [32], see the previous chapters. We again start from knowledge about the vertices of rounding polytopes,

$$P(\mathbf{m}) := \mathrm{cl}\left\{\mathbf{x} \in X^\ell(M) : A(\mathbf{x}) = \mathbf{m}\right\},$$

where cl denotes set closure. A rounding polytope now is the set of scaled weight vectors $\mathbf{x}$ resulting in the seat allocation $\mathbf{m}$ under the apportionment method A. Recall that the quantity $r(\mathbf{m}) := |\{i : m_i \neq 0\}|$ denotes the number of positive components of $\mathbf{m}$. The vertices of a rounding polytope $P(\mathbf{m})$ with $r(\mathbf{m}) = 0$ and $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$ are given by

$$P(\mathbf{m}) = \left\{ \mathbf{x} \in X^\ell(M) : m_i + q - 1 \leq x_i \leq m_i + q \text{ for all } i = 1, \ldots, \ell \right\}. \tag{7.3}$$

If $m_{i-1} = m_i$, for some $i = 2, \ldots, \ell$, the scaled weight vector $\mathbf{x}$ additionally has to fulfill the condition $x_{i-1} \leq x_i$, in order to lie in $P(\mathbf{m})$. If $r(\mathbf{m}) = k$, we have to account for the extra conditions $x_i \geq 0$, for $i > \ell - k$.

For determining asymptotic seat biases for $M \to \infty$ we subsequently have to account only for the highest and second-highest order term in M. Thus, the following three types of rounding polytopes will be of interest. As a consequence of (7.3), a rounding polytope P($\mathbf{m}$) with $m_{i-1} \neq m_i$ for all $i = 2, \ldots, \ell$ has the ℓ-dimensional volume $\text{Vol}(P(\mathbf{m})) = 1$ if $r(\mathbf{m}) = \ell$. It has the volume $\text{Vol}(P(\mathbf{m})) = q$ if $r(\mathbf{m}) = \ell - 1$. Moreover, if $r(\mathbf{m}) = \ell$ and $m_{i_0-1} = m_{i_0}$ for only one $i_0 = 2, \ldots, \ell$ the volume amounts to $\text{Vol}(P(\mathbf{m})) = 0.5$, which is proved by simple integration.

For a predetermined house size M, we define $V^\ell(M)$ to be the sum of the volumes of all rounding polytopes,

$$V^\ell(M) := \sum_{\mathbf{m} \in G^\ell_{\geq}(M)} \text{Vol}(P(\mathbf{m})),$$

where $G^\ell_{\geq}(M)$ is still the grid set of ordered seat allocations. The leading term of $V^\ell(M)$ in the house size M is established in the following.

Lemma 7.1 (Sum of volumes of rounding polytopes)

The sum of the volumes of all rounding polytopes fulfills

$$V^\ell(M) = \frac{M^{\ell-1}}{\ell!(\ell-1)!} + \mathcal{O}(M^{\ell-2}). \tag{7.4}$$

Proof. By Corollary 3.1, we know for the divisor method with rounding up, i.e. $q = 0$, that the probability to have a seat allocation $\mathbf{m} \in G^\ell_{\geq}(M)$ with $r(\mathbf{m}) = \ell$ and $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$ is

$$P(A = \mathbf{m}) = \ell! \binom{M-1}{\ell-1}^{-1}.$$

In leading order in the house size M, seat allocations with $r(\mathbf{m}) \neq \ell$ or $m_{i-1} = m_i$ for at least one $i = 2, \ldots, \ell$ are not of interest, and we obtain

$$V^\ell(M) = P(A = \mathbf{m})^{-1}$$

since $\text{Vol}(P(\mathbf{m})) = 1$ for all polytopes $P(\mathbf{m})$ contributing in leading order. Therefore, we can write

$$V^\ell(M) = \frac{1}{\ell!}\binom{M-1}{\ell-1} = \frac{(M-1)!}{\ell!(M-\ell)!(\ell-1)!} = \frac{M^{\ell-1}}{\ell!(\ell-1)!} + \mathcal{O}(M^{\ell-2}),$$

which completes the proof. **qed**

Remark 7.1 (Leading term of expansion of partition numbers)

By Lemma 7.1, we established the leading term of the expansion of the partition numbers $p(n,k)$, see Jacobs [21, p. 249],

$$p(n,k) = \frac{n^{k-1}}{k!(k-1)!} + \mathcal{O}(n^{k-2}),$$

where $n, k \in \mathbb{N}$.

Now we can turn to the proof of the asymptotic seat bias formula of Theorem 7.1 for the apportionment-oriented model.

Proof of Theorem 7.1. In order to evaluate (7.1), we write the conditional expectation as a sum over all seat allocations $\mathbf{m} \in G^\ell_{\geq}(M)$,

$$\mathbf{B}^\ell(M) = \frac{1}{V^\ell(M)} \sum_{\mathbf{m}\in G^\ell_{\geq}(M)} \mathbf{I}(\mathbf{m}), \tag{7.5}$$

where $\mathbf{I}(\mathbf{m})$ denotes the seat bias resulting from the rounding polytope $P(\mathbf{m})$, weighted by the volume of $P(\mathbf{m})$. In general, the class of rounding polytopes with $r(\mathbf{m}) = \ell$ and $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$ contributes terms of order $\mathcal{O}(M^\ell)$, after summation in (7.5). The class of polytopes with $r(\mathbf{m}) = \ell$ and $m_{i_0-1} = m_{i_0}$ for only one $i_0 = 2, \ldots, \ell$ yields terms of order $\mathcal{O}(M^{\ell-1})$, which is also true for the class of polytopes with $r(\mathbf{m}) = \ell - 1$ and $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$. Furthermore, contributions of other classes of rounding polytopes are at most of order $\mathcal{O}(M^{\ell-2})$ and thus have not to be taken into account.

For the class of seat allocations $\mathbf{m} \in G^\ell_{\geq}(M)$ with $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$ we get, for $r(\mathbf{m}) = \ell$,

$$\mathbf{I}(\mathbf{m}) = \mathbf{m} - \int\limits_{m_\ell+q-1}^{m_\ell+q} dx_\ell \cdots \int\limits_{m_1+q-1}^{m_1+q} dx_1 \, \frac{M\mathbf{x}}{\sum_{i=1}^\ell x_i} = \left(\frac{1}{2} - q\right)\left(1 - \frac{\ell}{M}\,\mathbf{m}\right) + \mathcal{O}(\mathbf{m}M^{-2}), \tag{7.6}$$

while for $r(\mathbf{m}) = \ell - 1$ we have

$$\mathbf{I}(\mathbf{m}) = q\mathbf{m} - \int\limits_0^q dx_\ell \int\limits_{m_{\ell-1}+q-1}^{m_{\ell-1}+q} dx_{\ell-1} \cdots \int\limits_{m_1+q-1}^{m_1+q} dx_1 \, \frac{M\mathbf{x}}{\sum_{i=1}^\ell x_i} = \mathcal{O}(\mathbf{m}M^{-1}). \tag{7.7}$$

For $r(\mathbf{m}) = \ell$ and $m_{i_0-1} = m_{i_0}$ for only one $i_0 = 2, \ldots, \ell$ we obtain

$$\mathbf{I}(\mathbf{m}) = \frac{\mathbf{m}}{2} - \int\limits_{m_\ell+q-1}^{m_\ell+q} dx_\ell \cdots \int\limits_{x_{i_0}}^{m_{i_0}+q} dx_{i_0-1} \cdots \int\limits_{m_1+q-1}^{m_1+q} dx_1 \, \frac{M\mathbf{x}}{\sum_{i=1}^\ell x_i} = \mathcal{O}(\mathbf{m}M^{-1}). \tag{7.8}$$

An evaluation of the above integrals is presented in the subsequent section. As (7.7) and (7.8) miss terms of higher order than $\mathcal{O}(\mathbf{m}M^{-1})$, the corresponding classes of rounding polytopes contribute only terms of order $\mathcal{O}(M^{\ell-2})$, after summation in (7.5). Therefore, only the rounding polytopes with $r(\mathbf{m}) = \ell$ and $m_{i-1} \neq m_i$ for $i = 2, \ldots, \ell$ contribute in leading order in the house size M, and we can rewrite (7.5) as

$$\mathbf{B}^\ell(M) = \frac{\frac{1}{2} - q}{V^\ell(M)} \sum_{m_\ell=1}^{\lfloor \frac{M-1}{\ell} \rfloor} \sum_{m_{\ell-1}=m_\ell+1}^{\lfloor \frac{M-m_\ell-1}{\ell-1} \rfloor} \cdots \sum_{m_2=m_3+1}^{\lfloor \frac{M-\sum_{i=3}^{\ell} m_i - 1}{2} \rfloor} \left(1 - \frac{\ell}{M}\,\mathbf{m}\right) + \mathcal{O}\left(\frac{1}{M}\right),$$

where $\lfloor \cdot \rfloor = r_1(\cdot)$ denotes rounding down and $m_1 = M - \sum_{i=2}^{\ell} m_i$. In leading order in M, the summation over 1 yields $V^\ell(M)$ and the result of the summation over $\mathbf{m}$ is given by the highest order term of the apportionment polynomial $\mathbf{S}^\ell(M)$ as determined in Lemma 5.1. The i-th component of this polynomial reads

$$S_i^\ell(M) = \frac{M^\ell}{(\ell!)^2} \sum_{j=i}^{\ell} \frac{1}{j} + \mathcal{O}(M^{\ell-1}).$$

Therefore, we obtain for the i-th component ($i = 1, \ldots, \ell$) of the seat bias vector $\mathbf{B}^\ell(M)$, using (7.4),

$$B_i^\ell(M) = \left(\frac{1}{2} - q\right) \left(1 - \frac{\ell}{M} \cdot \frac{\ell!(\ell-1)!}{M^{\ell-1}} \cdot \frac{M^\ell}{(\ell!)^2} \sum_{j=i}^{\ell} \frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right).$$

Simplifying this relation leads to the assertion of Theorem 7.1. **qed**

7.3 Integral Formulas

The proof of Theorem 7.1 is based on formulas (7.6), (7.7), and (7.8), which we derive in the following. Focussing on (7.6), the i-th component of the vector $\mathbf{I}(\mathbf{m})$ may be written as, for $i = 1, \ldots, \ell$,

$$I_i(\mathbf{m}) = m_i - \int_{q-1}^{q} dx_\ell \cdots \int_{q-1}^{q} dx_2 \int_{q-1+M+\sum_{i=2}^{\ell} x_i}^{q+M+\sum_{i=2}^{\ell} x_i} dx_1 \; M(x_i + m_i)\,\frac{1}{x_1}. \tag{7.9}$$

Because we need from the result of these integrations only the terms of order $\mathcal{O}(m_i)$ and $\mathcal{O}(m_i M^{-1})$, we use the approximation

$$\int_{a-1}^{a} dx \left(\frac{1}{x} + \frac{b}{x^2}\right) = \frac{1}{a} + \frac{b+\frac{1}{2}}{a^2} + \mathcal{O}(a^{-3})$$

in order to successively evaluate the integrals. In each step the structure of the integrals in (7.9) is reproduced. For example, for $i > 2$, we obtain

$$I_i(\mathbf{m}) = m_i - \int_{q-1}^{q} dx_\ell \cdots \int_{q-1}^{q} dx_3 \int_{2q-1+M+\sum_{i=3}^{\ell} x_i}^{2q+M+\sum_{i=3}^{\ell} x_i} dx_1 \; M(x_i + m_i)\left(\frac{1}{x_2} + \frac{\frac{1}{2}}{x_2^2} + \mathcal{O}(x_2^{-3})\right).$$

When integrating over x_i we are led to the additional factor $\left(q - \frac{1}{2} + m_i\right)$. For example, for $i \neq \ell$, we have in the last step

$$I_i(\mathbf{m}) = m_i - \int_{\ell q-1+M}^{\ell q+M} dx_\ell \; M\left(q - \frac{1}{2} + m_i\right)\left(\frac{1}{x_\ell} + \frac{\frac{\ell-1}{2}}{x_\ell^2} + \mathcal{O}(x_\ell^{-3})\right).$$

This finally leads to

$$\begin{aligned} I_i(\mathbf{m}) &= m_i - \frac{M}{\ell q + M}\left(q - \frac{1}{2} + m_i + \frac{\ell}{2}\frac{m_i}{M}\right) + \mathcal{O}(m_i M^{-2}) \\ &= \left(\frac{1}{2} - q\right)\left(1 - \frac{\ell}{M}\,m_i\right) + \mathcal{O}(m_i M^{-2}). \end{aligned}$$

Formulas (7.7) and (7.8) are derived similarly, except that only terms of order $\mathcal{O}(m_i)$ are needed. However, in the case of formula (7.8), the bookkeeping of the leading terms during the successive integration is a bit more cumbersome, because the integration over x_{i_0-1} introduces additional terms by virtue of the approximation

$$\int_{a-c}^{a} dx\,\frac{1}{x} = \frac{c}{a} + \mathcal{O}(a^{-2}).$$

Chapter 8

Summary and Outlook

In the previous chapters we have discussed a geometric-combinatorial approach in order to study biases resulting from the apportionment of seats in proportional representation systems. Special attention has been paid both to stationary divisor methods and to the quota method of greatest remainders, for which we have obtained quantitative seat bias results. We have started from a decomposition of the probability simplex into rounding polytopes, and have assumed that the vote proportions are uniformly distributed on the probability simplex. For the investigated apportionment methods, it has turned out that the volume of a rounding polytope depends only on the number of parties obtaining at least one seat, which has allowed us to calculate the volumes of all rounding polytopes, and has paved the way for determining the seat bias

$$\mathbf{B}^\ell(M) = \mathrm{E}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell], \tag{8.1}$$

by means of systematic summation over the possible seat allocations. The general seat bias formula

$$\mathbf{B}^\ell(M) = \left(\sum_{r=1}^{\ell} \frac{\ell!}{(\ell-r)!}\, p(r) \mathbf{S}^r(M) \right) - \frac{M}{\ell} \sum_{r=1}^{\ell} \mathbf{v}_r,$$

of Theorem 3.3 needs as geometrical input the correct seat allocation distribution $p(r)$, see the results of Theorems 3.1 and 3.2, and as combinatorial input the apportionment polynomial $\mathbf{S}^r(M)$ resulting from the recursive scheme of Theorem 3.4. The seat biases due to the divisor method with standard rounding (Webster, Sainte-Laguë) are tiny and quickly converge to zero, when the size of parliament grows large. The same is true for the quota method of greatest remainders (Hamilton). These two methods therefore are "practically unbiased", while the divisor methods with rounding up (Adams) and down (Jefferson, D'Hondt) lead to rather strong seat biases, in favor of the smaller and larger parties, respectively.

Apparently for the first time, it has been possible to study the seat biases in proportional representation systems imposing a threshold t. We have found that the disparity between the larger and smaller parties is most pronounced when the threshold t is equal or close to zero. In addition, all seat biases wear away when t grows close to $1/\ell$, where the dependence on t is linear. In perfect agreement with numerical simulations, we have obtained the relation, see Theorem 4.1,

$$\mathbf{B}^\ell(M,t) = (1-\ell t)\cdot \mathbf{B}(M,0).$$

Comparison with empirical data is difficult due to a lack of sufficiently large data sets.

We have addressed the probability for a violation of the majority/minority criterion, that is, the probability that a majority/minority of votes in the election does not imply a majority/minority of seats in parliament. For evaluating these probabilities, we have proceeded analogous with the calculation of seat biases via apportionment polynomials, accounting for additional restrictions of the probability simplex and the set of possible seat allocations. The findings given in Table 4.5 reveal that the probability for violating the majority or minority criterion vanishes when the size of parliament grows large.

As a central result, the geometric-combinatorial approach has enabled us to prove a previous conjecture on asymptotic seat biases for an arbitrary number of parties, as the size of parliament grows large. It has been necessary to establish the leading coefficients of an expansion of the apportionment polynomials in the house size, see Lemma 5.1. As stated in Theorems 5.1 and 5.2, under the q-stationary divisor methods the asymptotic seat bias of the i-th largest party fulfills

$$B_i^\ell(M) = \left(\frac{1}{2}-q\right)\left(1-\sum_{j=i}^{\ell}\frac{1}{j}\right) + \mathcal{O}\left(\frac{1}{M}\right), \tag{8.2}$$

whereas the quota method of greatest remainders is asymptotically unbiased.

We have complemented the investigation of seat biases by addressing the seat excess variance

$$\mathbf{V}^\ell(M) := \mathrm{Var}[A(\mathbf{w}) - \mathbf{w}M | w_1 \geq \cdots \geq w_\ell],$$

based on knowledge about the barycenters of rounding polytopes. For systems with two and three parties, the calculated variances are summarized in Theorems 6.1 and 6.2. In addition, asymptotic seat excess variances for large house size have been simulated, see Table 6.3. For the quota method of greatest remainders, this variance is asymptotically equal for every party. We have conjectured that the asymptotic value converges to 1/12 when the number of competing parties grows large. For the stationary divisor methods, the asymptotic variance, as function of the parameter q, is a parabola with minimum at $q = 1/2$. Moreover, a comparison with empirical data from Bavarian elections results in nice agreement with the theoretical variances.

Finally, we have discussed an alternative probabilistic model stressing the importance of the rounding process inherent in all apportionment methods. This model replaces the assumption of a uniform distribution of the vote proportions on the probability simplex by assuming a uniform distribution on the set of weight vectors rounded to a fixed house size. Despite such diverse assumptions, we have been able to show for stationary divisor methods that the apportionment-oriented model yields the same asymptotic seat biases as denoted in equation (8.2) for the model of Schuster *et al.* [32]. This may suggest that the distribution entering the conditional expectation of (8.1) plays a minor role for the asymptotic behaviour of the seat bias. A detailed study of this dependence would be of great interest, though at present it seems almost out of reach.

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Figures and Tables

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