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Abstract

We prove some geometric inequalities for pth-order chord power integrals $\mathcal{I}_p(P_d)$, $1 \le p \le d$, of d-parallelotopes P_d with positive volume $V_d(P_d)$. First, we derive upper and lower bounds of the ratio $\mathcal{I}_p(P_d)/V_d^2(P_d)$ which are attained by a d-cuboid C_d with the same volume resp. the same mean breadth as P_d. Second, we apply the device of Schur-convexity to obtain bounds of $\mathcal{I}_p(C_d)/V_d^2(C_d)$ which are attained by a d-cube with the same volume resp. the same mean breadth as C_d. Most of these inequalities are shown for a more general class of ovoid functionals containing, as by-product, a Pfiefer-type inequality for d-parallelotopes.

Keywords : Poisson hyperplane processes, mean breadth, Schur-convexity, Schur-criterion, Laplace transform, Carleman's inequality, Pfiefer-type inequality

MSC 2010: Primary 52A40 60D05 Secondary 52A20 52A22

1 Chord Power Integrals - General Facts and Motivation

Let K be a convex body in $\mathbb{R}^d$ with interior points and $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$ the boundary of the Euclidean unit ball $\mathbb{B}^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \le 1\}$. Further, let $\mathcal{H}^k$ denote the $k-$dimensional Hausdorff measure on $\mathbb{R}^d$ for $k = 1, \dots, d$ and, thus, $V_d(K) = \mathcal{H}^d(K)$ and $\mathcal{H}^{d-1}(\partial K)$ denote the volume and surface content of K, respectively. We recall that $\kappa_d := V_d(\mathbb{B}^d) = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ and $\mathcal{H}^{d-1}(\mathbb{S}^{d-1}) = d\,\kappa_d$ with $\Gamma(s) := \int_0^\infty e^{-x}\,x^{s-1}\mathrm{d}x$ for $s > 0$.

For any $p \ge 0$ we define the *pth-order chord power integral* (CPI) of K by

$$\mathcal{I}_p(K) = \frac{1}{2} \int\limits_{\mathbb{S}^{d-1}} \int\limits_{K|\mathbf{u}^\perp} \big(\mathcal{H}^1(K \cap \ell(\mathbf{x},\mathbf{u}))\big)^p \,\mathrm{d}\mathbf{x}\,\mathcal{H}^{d-1}(\mathrm{d}\mathbf{u}) \tag{1.1}$$

(with $0^0 := 0$), where $\ell(\mathbf{x},\mathbf{u}) := \{\mathbf{x} + \alpha\,\mathbf{u} : \alpha \in \mathbb{R}\}$ stands for the line in direction $\mathbf{u} \in \mathbb{S}^{d-1}$ through $\mathbf{x} \in \mathbb{R}^d$ and $K|\mathbf{u}^\perp$ is the orthogonal projection of K on $\mathbf{u}^\perp$ (= $(d-1)$-dimensional subspace orthogonal to $\mathbf{u}$). CPI's are of considerable interest in integral and stochastic geometry for a long time, see [9], [12], [13], [15], and have many applications in material sciences, physics and image analysis, see e.g. [1], [11], [3] and references therein. In textbooks of integral and convex geometry, see e.g. [9], [12], [13] the r.h.s. of (1.1) is mostly written as integral w.r.t. the *line measure* $\mu_1^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d,1)$ of one-dimensional affine subspaces of $\mathbb{R}^d$):

$$\mathcal{I}_p(K) = \frac{d\,\kappa_d}{2} \int\limits_{\mathbb{A}(d,1)} \big(\,\mathcal{H}^1(K\cap L)\,\big)^p\,\mu_1^{(d)}(\mathrm{d}L)\,, \tag{1.2}$$

where, for integers $p = 2,\ldots,d$, the Blaschke-Petkantschin formula, see [13] (p. 363), provides the representations

$$\mathcal{I}_{k+1}(K) = \frac{(k+1)\,d\,\kappa_d}{2\,\kappa_k} \int\limits_{\mathbb{A}(d,k)} \big(\mathcal{H}^k(K\cap L)\big)^2\,\mu_k^{(d)}(\mathrm{d}L) \tag{1.3}$$

for $k = 1,\ldots,d-1$ with the motion-invariant *k-flat measure* $\mu_k^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d,k)$ of k-dimensional affine subspaces of $\mathbb{R}^d$) satisfying the normalization $\mu_k^{(d)}(\{E\in\mathbb{A}(d,k) : E\cap\mathbb{B}^d\neq\emptyset\}) = \kappa_{d-k}$. From (1.1) for $p = 0,1$ and (1.3) for $k = d-1$ we get the following relations, see e.g. [11],

$$\mathcal{I}_0(K) = \frac{\kappa_{d-1}}{2}\,\mathcal{H}^{d-1}(\partial K)\ ,\ \mathcal{I}_1(K) = \frac{d\,\kappa_d}{2}\,V_d(K)\ ,\ \mathcal{I}_{d+1}(K) = \frac{d\,(d+1)}{2}\,V_d(K)^2\,.$$

Due to F. Piefke, see [11], the r.h.s. of (1.1) can be expressed for any $p > 1$ by the distribution of the interpoint distance of two randomly chosen points in K leading to

$$\mathcal{I}_p(K) = \frac{p\,(p-1)}{2} \int\limits_K\int\limits_K \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^{d+1-p}} \quad \text{for any} \quad p > 1. \tag{1.4}$$

Note that in the special $d = 3$ the third-order CPI $\mathcal{I}_3(K)$ coincides with Newton's self-potential of the body $K\subset\mathbb{R}^d$, see e.g. [9]. In stochastic geometry there are quite a few random functionals defined on an expanding domain $\varrho K\uparrow\infty$ (as $\varrho\to\infty$) whose asymptotic variances depend on the shape of K (which is assumed to be convex containing the origin $\mathbf{o}$ as inner point) expressed by $\mathcal{I}_p(K)$ for some $p = 1,\ldots,d-1$. Let us sketch a typical example - another one is discussed in [6]. To be precise we need some further notation, for details the reader is referred to [4].

Let $\Pi_\lambda = \{P_i : i\geq 1\}$ be a stationary Poisson process on the real line with $\mathbb{R}^1$ intensity $\lambda := \mathsf{E}\#\{i\geq 1 : P_i\in[0,1]\}$, and let Π_λ be independently marked with a sequence $\{U_i, i\geq 1\}$ of independent, uniformly on $\mathbb{S}^{d-1}$ distributed random vectors and $H(P_i,U_i) := U_i^\perp + P_iU_i$ defines a random (unoriented) hyperplane in $\mathbb{R}^d$ with orientation vector $U_i\in\mathbb{S}^{d-1}$ and signed perpendicular distance P_i from $\mathbf{o}$. The family $\{H(P_i,U_i) : i\geq 1\}$ represents a (motion-invariant) Poisson-hyperplane process in $\mathbb{R}^d$ with intensity λ. Further, we consider the associated (motion-invariant) *k-flat intersection processes* $\{\bigcap_{1\leq j\leq d-k} H(P_{i_j},U_{i_j}) : 1\leq i_1 < \cdots < i_{d-k}\}$ for $k = 0,1,\ldots,d-1$ and introduce the mean value functionals

$$\widehat{\zeta}_{k,d}(\lambda,\varrho K) := \frac{1}{V_d(\varrho K)} \sum_{1\leq i_1<\cdots<i_{d-k}} \mathcal{H}^k\big(\bigcap_{1\leq j\leq d-k} H(P_{i_j},U_{i_j})\cap\varrho K\big) \tag{1.5}$$

having the expectations $\mathsf{E}\widehat{\zeta}_{k,d}(\lambda,\varrho K) = \frac{\kappa_d}{\kappa_k}\binom{d}{k}\left(\frac{\lambda\,\kappa_{d-1}}{d\,\kappa_d}\right)^{d-k}$ and the asymptotic variances $\sigma^2_{k,d}(\lambda,K)$ are given by the limit

$$\lim_{\varrho\to\infty}\varrho\,\mathsf{Var}\big(\widehat{\zeta}_{k,d}(\lambda,\varrho K)\big) = \frac{2\,\kappa^2_{d-1}}{d\,\kappa^2_k}\binom{d-1}{k}^2\left(\frac{\lambda\,\kappa_{d-1}}{d\,\kappa_d}\right)^{2(d-k)-1}\frac{\mathcal{I}_d(K)}{V_d(K)^2} \tag{1.6}$$

for $k=0,1,\ldots,d-1$. Note that $\sqrt{\varrho}\,(\widehat{\zeta}_{k,d}(\lambda,\varrho K)-\mathsf{E}\widehat{\zeta}_{k,d}(\lambda,\varrho K))$ is asymptotically normally distributed with variance $\sigma^2_{k,d}(\lambda,K)$, see [4]. The dependence of $\sigma^2_{k,d}(\lambda,K)$ on the shape of K (not only on $V_d(K)$) is caused by the *long-range correlations* within the random union set $\bigcup_{i\geq 1}H(P_i,U_i)$, see similar results for Poisson cylinder processes in [6].

Statisticians aim at creating experimental designs such that estimators of model parameters have minimal variances. In our model this means to minimize the ratio $\mathcal{I}_d(K)/V_d(K)^2$ in (1.6) if another ovoid functional of K, e.g. the *mean breadth* $b_d(K)$ of K, is fixed. Here the mean breadth of K is defined by

$$b_d(K) := \frac{1}{d\,\kappa_d}\int\limits_{\mathbb{S}^{d-1}}\big(h(K,u)+h(K,-u)\big)\,\mathcal{H}^{d-1}(\mathrm{d}\mathbf{u}) = \frac{2\,\kappa_{d-1}}{d\,\kappa_d}\,V_1(K) \quad \text{, see [13] (p. 601)},$$

where $h(K,u)$ denotes the *support function* of K in direction $\mathbf{u}\in\mathbb{S}^{d-1}$ and $V_1(K)$ is the first intrinsic volume of K, see [13] (p. 600).

In the planar case best lower bounds of $\mathcal{I}_2(K)/V_2(K)^2$ have been proved for particular classes of convex discs in [5] when the perimeter $\mathcal{H}^1(\partial K)=\pi\,b_2(K)$ is given. In convex geometry, see [2], [12] or [13], one is mostly interested to maximize $\mathcal{I}_k(K)$ for $k=1,\ldots,d$ when $V_d(K)$ is fixed. Among all convex bodies the ball with radius $(V_d(K)/\kappa_d)^{1/d}$ is the unique maximizer due to *Carleman's inequality*, see [13] (p. 364),

$$\mathcal{I}_k(K) \leq (\geq) \frac{2^{k-1}\,d\,\kappa_d\,\kappa_{d-1+k}}{\kappa_k}\left(\frac{V_d(K)}{\kappa_d}\right)^{(d-1+k)/d} \quad \text{for } 1\leq k\leq d+1 \ \ (k\geq d+1)\,. \tag{1.7}$$

Upper and apparently best possible lower bounds of $\mathcal{I}_p(\mathbb{E}(\mathbf{a}))$, $1\leq p\leq d$, for d-dimensional ellipsoids $\mathbb{E}(\mathbf{a})$ with positive semi-axes $\mathbf{a}=(a_1,\ldots,a_d)$ have been obtained in [6].

2 Preliminaries and a Basic Lemma

In order to generalize the class of (motion-invariant) ovoid functionals (1.4) we consider integrals of the form

$$\int\limits_K\int\limits_K f(\|\mathbf{x}-\mathbf{y}\|^2)\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = \int\limits_{K\oplus(-K)} V_d(K\cap(K+\mathbf{z}))\,f(\|\mathbf{z}\|^2)\,\mathrm{d}\mathbf{z} \tag{2.1}$$

for any convex body $K \subset \mathbb{R}^d$ and any Borel-measurable function $f|(0,\infty) \to \mathbb{R}^1$ satisfying

$$\int_0^\tau x^{d-1} \, |f(x^2)| \, \mathrm{d}x < \infty \quad \text{for all} \quad \tau > 0 \, . \tag{2.2}$$

Since the difference body $K \oplus (-K) := \{x - y : x, y \in K\}$ is contained in a ball centred at $\mathbf{o}$ with radius $\mathrm{diam}(K) := \sup_{x,y \in K} \|x - y\|$ the condition (2.2) guarantees the existence of the above integrals over all convex bodies K. Hence, if additionally $V_d(K) > 0$, the functional

$$Q_d(f, K) := \frac{1}{V_d(K)^2} \int_K \int_K f(\|\mathbf{x} - \mathbf{y}\|^2) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} = \mathsf{E} f(\|X_K - Y_K\|^2) \tag{2.3}$$

is well-defined, where X_K, Y_K are independent random vectors uniformly distributed on K.

The rest of this paper is organized as follows: In the below Sections 3 and 4 we derive a lower (upper) bound of $Q_d(f, K)$ when K belongs to the class of d-parallelotopes with fixed mean breadth (volume) and f is convex (concave) or continuous and non-decreasing (non-increasing). For this, we need the below Lemma 1 which seems to be of interest for its own rights. In the final Section 5 we prove sharp bounds of $Q_d(f, K)$ for d-cuboids $K = \times_{i=1}^d [0, a_i]$ by applying the concept of Schur-convexity.

Lemma 1 *If the function $f|(0,\infty) \to \mathbb{R}^1$ is convex (concave), then*

$$f(x + c) + f(x - c) \geq \ (\leq) \ 2 \, f(x) \quad \textit{for all} \quad c \in (-x, x) \, , \ x > 0 \, . \tag{2.4}$$

If $f|(0,\infty) \to \mathbb{R}^1$ is continuous and non-increasing (non-decreasing) and $g|[0,1] \to [0,\infty)$ is non-increasing then the parameter integrals

$$J(f, g; a, b, c) := \int_0^1 g(x) \, f(a^2 + (b\,x + c)^2) \, \mathrm{d}x$$

satisfy the inequality

$$J(f, g; a, b, c) + J(f, g; a, b, -c) \leq \ (\geq) \ 2 \, J(f, g; a, b, 0) \quad \textit{for all} \quad a, b, c \in \mathbb{R}^1 \, . \tag{2.5}$$

For $a = 0$ we suppose in addition that $\int_0^\tau | \, f(x^2) \, | \, \mathrm{d}x < \infty$ for all $\tau > 0$.

Proof of Lemma 1. It suffices to prove (2.4) for $c \geq 0$. Due to the assumed convexity (concavity) of f on $(0,\infty)$ we have $f(x + c) - f(x) \ \geq \ (\leq) \ f(x) - f(x - c)$ for $0 \leq c < x$ which immediately yields the asserted inequality.

For proving the second inequality (2.5), let $b > 0$ and $c > 0$ without loss of generality. At first, let additionally $a^2 > 0$. By obvious rearrangements and the partial integration formula for Riemann - Stieltjes integrals we rewrite $J(f, g; a, b, c)$ as follows:

$$
\begin{aligned}
J(f,g;a,b,\pm c) &= \int_0^1 g(x)\,\mathrm{d}_x\Big(\int_0^x f(a^2+(b\,y\pm c)^2)\,\mathrm{d}y\Big)\\
&= g(1)\int_0^1 f(a^2+(b\,y\pm c)^2)\,\mathrm{d}y + \int_0^1\int_0^x f(a^2+(b\,y\pm c)^2)\,\mathrm{d}y\,\mathrm{d}(-g(x))\\
&= \frac{g(1)}{b}\int_{\pm c}^{b\pm c} f(a^2+z^2)\,\mathrm{d}z + \frac{1}{b}\int_0^1\int_{\pm c}^{b\,x\pm c} f(a^2+z^2)\,\mathrm{d}z\,\mathrm{d}(-g(x))\,.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\frac{\partial J(f,g;a,b,\pm c)}{\partial c} &= \frac{g(1)}{b}\big(\pm f(a^2+(b\pm c)^2)\mp f(a^2+c^2)\big)\\
&+ \frac{1}{b}\int_0^1\big(\pm f(a^2+(b\,x\pm c)^2)\mp f(a^2+c^2)\big)\,\mathrm{d}(-g(x))\,,
\end{aligned}
$$

whence we obtain that

$$
\frac{\partial J(f,g;a,b,c)}{\partial c}+\frac{\partial J(f,g;a,b,-c)}{\partial c} = \frac{g(1)}{b}\big(\,f(a^2+(b+c)^2)-f(a^2+(b-c)^2)\,\big)
$$

$$
+\frac{1}{b}\int_0^1\Big[\,f(a^2+(b\,x+c)^2) \;-\; f(a^2+(b\,x-c)^2)\Big]\,\mathrm{d}\big(-g(x)\big)\;\le\;(\ge)\;0\,.
$$

The latter is justified by $(b\,x+c)^2\ge(b\,x-c)^2$ and the monotonicity of f on $(0,\infty)$ and of g on $[0,1]$. Hence, the even function $c\mapsto J(f,g;a,b,c)+J(f,g;a,b,-c)$ is non-increasing (non-decreasing) for $c\ge 0$ attaining its maximum (minimum) at $c=0$. The inequalities (2.5) remain valid for $a=0$ by passing to the limit $a\to 0$ provided that $\int_0^\tau |\,f(x^2)\,|\,\mathrm{d}x<\infty$. □

To avoid ambiguity let us recall that a *d-parallelotope* is a convex body spanned by linearly independent vectors $\mathbf{a}_i=(a_i^{(1)},...,a_i^{(d)})$, $i=1,...,d$, in $\mathbb{R}^d$, i.e. $P_d(\mathbf{a}_1,...,\mathbf{a}_d):=\{\sum_{i=1}^d\lambda_i\,\mathbf{a}_i: 0\le\lambda_1,...,\lambda_d\le 1\}$. For brevity, we write P_d instead of $P_d(\mathbf{a}_1,...,\mathbf{a}_d)$ (if no confusion is possible). In what follows we often compare functionals of d-parallelotopes with corresponding functionals *d-cuboids* $C_d(a_1,\dots,a_d):=\times_{i=1}^d[0,a_i]$ having edge lengths $a_1,\dots,a_d>0$.

From analytic geometry it is well-known that the $d-$volume $V_d(P_d(\mathbf{a}_1,...,\mathbf{a}_d))$ coincides with the absolute value of the determinant

$$
\det\big(\,(a_j^{(i)})_{i,j=1}^d\,\big)=\det\big[\mathbf{a}_1^\mathsf{T},...,\mathbf{a}_d^\mathsf{T}\big]\,.
$$

Since any two distinct points $\mathbf{x}=(x_1,...,x_d)$, $\mathbf{y}=(y_1,...,y_d)\in P_d$ can be expressed as linear combination $\mathbf{x}=\lambda_1\,\mathbf{a}_1+\cdots+\lambda_d\,\mathbf{a}_d$ resp. $\mathbf{y}=\mu_1\,\mathbf{a}_1+\cdots+\mu_d\,\mathbf{a}_d$ with unique $\lambda_1,\mu_1,...,\lambda_d,\mu_d\in[0,1]$, we may apply the integral transformation formula with the Jacobian determinants

$$
\Big|\det\Big(\Big(\frac{\partial x_i}{\partial\lambda_j}\Big)_{i,j=1}^d\Big)\Big|=\Big|\det\Big(\Big(\frac{\partial y_i}{\partial\mu_j}\Big)_{i,j=1}^d\Big)\Big|=\Big|\det\big(\,(a_j^{(i)})_{i,j=1}^d\,\big)\Big|=V_d(P_d)\,,
$$

leading to the following representation of (2.3) for $K = P_d(\mathbf{a}_1, ..., \mathbf{a}_d)$:

$$Q_d(f, P_d) = \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^1 f\big(\| \sum_{i=1}^{d} (\lambda_i - \mu_i)\, \mathbf{a}_i \|^2\big)\, \mathrm{d}\lambda_d\, \mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\, \mathrm{d}\mu_1 \,. \tag{2.6}$$

Notice the remarkable fact that the mean breadth $b_d(P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d))$ only depends on the sum of the edge lengths $\|\mathbf{a}_1\|, \ldots, \|\mathbf{a}_d\|$, but not on the angles between the edges, see e.g. [9] (p. 227) for $d = 3$. More precisely, it holds $V_1(P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d)) = \|\mathbf{a}_1\| + \cdots + \|\mathbf{a}_d\|$ as can be seen from Steiner's formula, see [13] (p. 600), so that

$$b_d\big(P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d)\big) = \frac{2\,\kappa_{d-1}}{d\,\kappa_d} \Big(\sum_{i=1}^{d} \|\mathbf{a}_i\| \Big) = b_d\big(C_d(\|\mathbf{a}_1\|, \ldots, \|\mathbf{a}_d\|)\big) \,. \tag{2.7}$$

3 Lower bounds of $Q_d(f, P_d)$ for convex f

First we rewrite the $2d-$fold integral (2.6) as a sum of 2^d $d-$fold integrals which allow to estimate $Q_d(f, P_d)$ from below. By the following straightforward rearrangements

$$\begin{aligned} Q_d(f, P_d) &= \int_0^1 \int_{-\mu_1}^{1-\mu_1} \cdots \int_0^1 \int_{-\mu_d}^{1-\mu_d} f(\| \lambda_1\, \mathbf{a}_1 + \cdots + \lambda_d\, \mathbf{a}_d \|^2)\, \mathrm{d}\lambda_d\, \mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\, \mathrm{d}\mu_1 \\ &= \int_0^1 \int_0^{\mu_1} \cdots \int_0^1 \int_0^{\mu_d} \sum_{\nu_1, \ldots, \nu_d \in \{0,1\}} f\big(\| \sum_{i=1}^{d} (-1)^{\nu_i}\, \lambda_i\, \mathbf{a}_i \|^2\big)\, \mathrm{d}\lambda_d\, \mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\, \mathrm{d}\mu_1 \\ &= \int_0^1 \int_{\lambda_1}^1 \cdots \int_0^1 \int_{\lambda_d}^1 \sum_{\nu_1, \ldots, \nu_d \in \{0,1\}} f\big(\| \sum_{i=1}^{d} (-1)^{\nu_i}\, \lambda_i\, \mathbf{a}_i \|^2\big)\, \mathrm{d}\mu_d\, \mathrm{d}\lambda_d \cdots \mathrm{d}\mu_1\, \mathrm{d}\lambda_1 \end{aligned}$$

we arrive at

$$Q_d(f, P_d) = \int_0^1 \cdots \int_0^1 \sum_{\nu_1, \ldots, \nu_d \in \{0,1\}} f\big(\| \sum_{i=1}^{d} (-1)^{\nu_i}\, \lambda_i\, \mathbf{a}_i \|^2\big) \prod_{i=1}^{d} (1 - \lambda_i)\, \mathrm{d}\lambda_d \cdots \mathrm{d}\lambda_1 \,. \tag{3.1}$$

By means of the identity

$$\|z_1\, \mathbf{a}_1 + \cdots + z_d\, \mathbf{a}_d\|^2 = \sum_{i=1}^{d} z_i^2\, \|\mathbf{a}_i\|^2 + 2 \sum_{1 \le i < j \le d} z_i\, z_j\, \langle\, \mathbf{a}_i, \mathbf{a}_j \,\rangle$$

for $z_1, \ldots, z_d \in [-1, 1]$, where $\langle\, \mathbf{a}_i, \mathbf{a}_j \,\rangle$ denotes the scalar product of $\mathbf{a}_i$ and $\mathbf{a}_j$, we deduce from (3.1) for pairwise orthogonal vectors $\mathbf{a}_i$ that

$$Q_d(f, C_d(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|)) = 2^d \int_0^1 \cdots \int_0^1 f\big(\sum_{i=1}^d \lambda_i^2 \, \|\mathbf{a}_i\|^2\big) \prod_{i=1}^d (1-\lambda_i)\, \mathrm{d}\lambda_d \cdots \mathrm{d}\lambda_1 \,. \tag{3.2}$$

Next, under the assumption that $x \mapsto f(x)$ is convex for $x > 0$, we get a lower bound of the $d-$fold integral on the r.h.s of (3.1). (Note that, if $x \mapsto f(x)$ is concave for $x > 0$, then $-f(x)$ is convex for $x > 0$ leading to an upper bound.) For this purpose we apply the elementary inequality (2.4) for $x = \| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \, \mathbf{a}_i \|^2 + \lambda_d^2 \, \|\mathbf{a}_d\|^2$ and $c = 2 \, \langle \, \lambda_d \, \mathbf{a}_d \, , \, \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \, \mathbf{a}_i \, \rangle$ (satisfying $-x \le c \le x$) implying that

$$\begin{aligned} & \sum_{\nu_d \in \{0,1\}} f(\| \, (-1)^{\nu_1} \lambda_1 \, \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \, \mathbf{a}_d \, \|^2) \\ = \; & \sum_{\nu_d \in \{0,1\}} f(\| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \, \mathbf{a}_i \|^2 + \lambda_d^2 \, \|\mathbf{a}_d\|^2 + 2 \, (-1)^{\nu_d} \lambda_d \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \, \langle \, \mathbf{a}_i, \mathbf{a}_d \, \rangle) \\ \ge \; & 2 \, f(\| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \, \mathbf{a}_i \|^2 + \lambda_d^2 \, \|\mathbf{a}_d\|^2) \,. \end{aligned}$$

Proceeding in this way leads to

$$\begin{aligned} & \sum_{\nu_k \in \{0,1\}} f(\| \, (-1)^{\nu_1} \lambda_1 \, \mathbf{a}_1 + \cdots + (-1)^{\nu_k} \lambda_k \, \mathbf{a}_k \, \|^2 + \lambda_{k+1}^2 \, \|\mathbf{a}_{k+1}\|^2 + \cdots + \lambda_d^2 \, \|\mathbf{a}_d\|^2) \\ \ge \; & 2 \, f(\| \sum_{i=1}^{k-1} (-1)^{\nu_i} \lambda_i \, \mathbf{a}_i \|^2 + \lambda_k^2 \, \|\mathbf{a}_k\|^2 + \cdots + \lambda_d^2 \, \|\mathbf{a}_d\|^2) \end{aligned}$$

for $k = d-1, ..., 2$. Summarizing all these inequalities yields

$$\sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f(\| \, (-1)^{\nu_1} \lambda_1 \, \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \, \mathbf{a}_d \, \|) \ge 2^d \, f(\lambda_1^2 \, \|\mathbf{a}_1\|^2 + \cdots + \lambda_d^2 \, \|\mathbf{a}_d\|^2)$$

whence it follows together with (3.2) the assertion of

Theorem 1 *If the function $f | (0, \infty) \to \mathbb{R}^1$ is convex (concave) satisfying (2.2) then*

$$Q_d(f, P_d(\mathbf{a}_1, ..., \mathbf{a}_d)) \; \ge \; (\le) \; Q_d(f, C_d(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|)) \,. \tag{3.3}$$

4 Lower Bounds of $Q_d(f, P_d)$ for non-decreasing f

The volume $V_d(P_d)$ as well as the integral defined in (2.1) are invariant under rigid motions of P_d. In particular, we have $Q_d(f, P_d) = Q_d(f, P_d \, \mathbf{O})$ for any orthogonal $d \times d$-matrix $\mathbf{O}$. We define such an orthogonal matrix by the equations $\mathbf{a}_j \, \mathbf{O} = \mathbf{b}_j = (b_j^{(1)}, ..., b_j^{(j)}, 0, ..., 0)$ and put

$a_j := |b_j^{(j)}| > 0$ for $j = 1, ..., d$ with $a_1 = \|\mathbf{a}_1\|$, where the components $b_j^{(i)}, 1 \le i \le j \le d$ can be calculated step by step from the equations

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \mathbf{b}_i, \mathbf{b}_j \rangle = \sum_{k=1}^{i} b_i^{(k)} \, b_j^{(k)} \quad \text{for} \quad 1 \le i \le j \le d$$

which are equivalent to the recursive relations

$$b_j^{(i)} = \frac{1}{b_i^{(i)}} \Big(\langle \mathbf{a}_i, \mathbf{a}_j \rangle - \sum_{k=1}^{i-1} b_i^{(k)} \, b_j^{(k)} \Big) \ \text{ for } \ i = 1, \dots, j \ \text{ and } \ j = 1, ..., d\,. \tag{4.1}$$

It is immediately clear that $V_d(P_d) = \big|\det\big[\mathbf{b}_1^\intercal, ..., \mathbf{b}_d^\intercal\big]\big| = \prod_{j=1}^d a_j = V_d(C_d(a_1, \dots, a_d))\,.$

$$\|(-1)^{\nu_1}\lambda_1\,\mathbf{a}_1 + \cdots + (-1)^{\nu_d}\lambda_d\,\mathbf{a}_d\|^2 = \sum_{j,k=1}^{d} (-1)^{\nu_j+\nu_k}\lambda_j\,\lambda_k\langle\mathbf{a}_j, \mathbf{a}_k\rangle$$

$$= \sum_{j,k=1}^{d} (-1)^{\nu_j+\nu_k}\lambda_j\,\lambda_k\langle\mathbf{b}_j, \mathbf{b}_k\rangle = \sum_{j=1}^{d}\lambda_j^2\sum_{i=1}^{j}(b_j^{(i)})^2 + 2\sum_{1\le j<k\le d}(-1)^{\nu_j+\nu_k}\lambda_j\,\lambda_k\sum_{i=1}^{j} b_j^{(i)} b_k^{(i)}$$

$$= \sum_{i=1}^{d}\sum_{j=i}^{d}(b_j^{(i)}\,\lambda_j^2)^2 + 2\sum_{i=1}^{d}\sum_{i\le j<k\le d}(-1)^{\nu_j+\nu_k}\,b_j^{(i)} b_k^{(i)}\,\lambda_j\,\lambda_k = \sum_{i=1}^{d}\Big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\Big)^2$$

Let $x \mapsto f(x)$ be non-decreasing (non-increasing) and continuous for $x > 0$ and $\int_0^\tau x^{d-1}\,|f(x)|\,\mathrm{d}x < \infty$ for all $\tau > 0$. Under this assumption we derive a lower (upper) bound of the $d-$fold integral

$$Q_d\big(f, P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)\big) = \int_0^1 \cdots \int_0^1 \sum_{\nu_1,\dots,\nu_d\in\{0,1\}} f\big(\|\sum_{i=1}^{d}(-1)^{\nu_i}\,\lambda_i\,\mathbf{b}_i\|^2\big)\prod_{i=1}^{d}(1-\lambda_i)\,\mathrm{d}\lambda_d\cdots\mathrm{d}\lambda_1\,.$$

Using the inequality (2.5) of Lemma 1 with $g(x) = 1 - x$ for $a^2 = \sum_{i=2}^{d}\big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\big)^2$, $b = b_1^{(1)}$ (and $a_1 = |b_1^{(1)}|$) and $c = (-1)^{\nu_1}\sum_{j=2}^d(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j$ we find that

$$\sum_{\nu_1\in\{0,1\}}\int_0^1 f\Big(\sum_{i=1}^{d}\big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\big)^2\Big)\,(1-\lambda_1)\,\mathrm{d}\lambda_1$$

$$\ge \quad (\le)\ 2\int_0^1 f\Big(a_1^2\lambda_1^2 + \sum_{i=2}^{d}\big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\big)^2\Big)\,(1-\lambda_1)\,\mathrm{d}\lambda_1$$

Analogously, we get successively for $k = 2, \dots, d$ that

$$\sum_{\nu_k\in\{0,1\}}\int_0^1 f\Big(a_1^2\lambda_1^2 + \cdots + a_{k-1}^2\lambda_{k-1}^2 + \sum_{i=k}^{d}\big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\big)^2\Big)\,(1-\lambda_k)\,\mathrm{d}\lambda_k$$

$$\ge \quad (\le)\ 2\int_0^1 f\Big(a_1^2\lambda_1^2 + \cdots + a_k^2\lambda_k^2 + \sum_{i=k+1}^{d}\big(\sum_{j=i}^{d}(-1)^{\nu_j}\,b_j^{(i)}\,\lambda_j\big)^2\Big)\,(1-\lambda_k)\,\mathrm{d}\lambda_k\,.$$

In this way we obtain

Theorem 2 *If the function* $f|(0,\infty)\to\mathbb{R}^1$ *is continuous and non-decreasing (non-increasing) satisfying (2.2) then*

$$Q_d(f,P_d(\mathbf{a}_1,...,\mathbf{a}_d)) \;\ge\; (\le)\; Q_d(f,C_d(a_1,\dots,a_d))\,, \tag{4.2}$$

where the edge lengths $a_j=|b_j^{(j)}|$, $j=1,\dots,d$, *are defined by (4.1).*

In Section 5 we establish lower resp. upper bounds of the pth-order CPI of d-cuboids C_d in terms of $b_d(C_d)$ resp. $V_d(C_d)$ and the pth-order CPI of the unit cube $[0,1]^d$.

5 Lower and upper bounds of $\mathcal{I}_p(C_d)$ for $1<p\le d$

From (1.4), (2.3) and (3.2) it is easily seen that in case of the d-cuboid $C_d=\times_{i=1}^d[0,a_i]$ the ratio $\mathcal{I}_p(C_d)/V^2(C_d)$ is equal to $(d-q)(d-q+1)\,2^{d-1}\,\mathcal{J}_q(a_1,\dots,a_d)$ with the d-fold parameter integral

$$\mathcal{J}_q(a_1,\dots,a_d):=\int_0^1\cdots\int_0^1\frac{(1-x_1)\,\cdots\,(1-x_d)}{(a_1^2\,x_1^2+\cdots+a_d^2\,x_d^2)^{q/2}}\,\mathrm{d}x_d\cdots\mathrm{d}x_1 \tag{5.1}$$

for $q=d+1-p\in[0,d)$, i.e., $1<p\le d+1$. We mostly write shorthand $\mathcal{J}_q(\mathbf{a})$ with $\mathbf{a}=(a_1,...,a_d)$ instead of $\mathcal{J}_q(a_1,\dots,a_d)$.

Definition *(see [16]) For* $-\infty\le a<b\le\infty$ *and* $d\ge 2$, *a symmetric function* $F|(a,b)^d\to\mathbb{R}^1$ *is said to be Schur-convex (Schur-concave) if for every doubly stochastic matrix* $\mathbf{S}=(s_{ij})_{i,j=1}^d$ *(i.e.,* $s_{ij}\ge 0$ *such that* $s_{i1}+\cdots+s_{id}=s_{1j}+\cdots+s_{dj}=1$ *for* $1\le i,j\le d$,

$$F(\mathbf{x}\,\mathbf{S})\;\le\;(\ge)\;F(\mathbf{x})\;\; \textit{for all}\;\; \mathbf{x}=(x_1,\dots,x_d)\in(a,b)^d\,. \tag{5.2}$$

Obviously, F *is Schur-concave if and only if* $-F$ *is Schur-convex.*

The following condition which goes back to I. Schur provides a useful criterion to prove Schur-convexity.

Lemma 2 *(see [16]) A symmetric function* $F(\mathbf{x})=F(x_1,\dots,x_d)$ *with continuous partial derivatives on* $(a,b)^d$ *is Schur-convex (Schur-concave) if and only if*

$$(x_1-x_2)\Big(\frac{\partial F(\mathbf{x})}{\partial x_1}-\frac{\partial F(\mathbf{x})}{\partial x_2}\Big)\;\ge\;(\le)\;0\;\; \textit{for all}\;\; \mathbf{x}=(x_1,\dots,x_d)\in(a,b)^d\,. \tag{5.3}$$

For alternative definitions, historical background and further details related with Schur-convexity the reader is referred to the monographs [7], [8], and [14].

Theorem 3 *For $1 \le q < d$ the mapping $(a_1,\dots,a_d) \mapsto \mathcal{J}_q(a_1,\dots,a_d)$ is Schur-convex on $(0,\infty)^d$. Furthermore, the mapping*

$$\mathbf{b} = (b_1,\dots,b_d) \mapsto \mathcal{J}(f;\mathbf{b}) := \int_0^1 \cdots \int_0^1 f\big(\sum_{i=1}^d x_i^2\, e^{2\,b_i}\big) \prod_{i=1}^d (1-x_i)\,\mathrm{d}x_d \cdots \mathrm{d}x_1 \qquad (5.4)$$

is Schur-convex (Schur-concave) on $\mathbb{R}^d$ if the function $f|(0,\infty) \to \mathbb{R}^1$ is continuous and non-decreasing (non-increasing) satisfying (2.2).

Proof of Theorem 3. First we apply Schur's criterion (5.3) to show that the symmetric function $\mathcal{J}_q(\mathbf{a})$ is Schur-convex. This means that, for $a_1 \ge a_2 > 0$ and any fixed $a_3,\dots,a_d > 0$, we have to verify the inequality

$$\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} \ge \frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2}\,. \qquad (5.5)$$

After differentiation and partial integration w.r.t. x_1 we arrive at

$$\begin{aligned}
\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} &= -q \int_0^1\int_0^1\cdots\int_0^1 \frac{a_1\,x_1^2\,(1-x_1)\,(1-x_2)\cdots(1-x_d)}{(\,a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2\,)^{q/2+1}}\,\mathrm{d}x_d \cdots \mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \frac{1}{a_1}\int_0^1\int_0^1\cdots\int_0^1 x_1 \prod_{i=1}^d (1-x_i)\,\mathrm{d}_{x_1}\big(\big(\sum_{i=1}^d a_i^2\,x_i^2\,\big)^{-q/2}\big)\,\mathrm{d}x_2 \cdots \mathrm{d}x_d \\
&= -\frac{1}{a_1}\int_0^1\int_0^1\cdots\int_0^1 \frac{(1-2\,x_1)\,(1-x_2)\,(1-x_3)\cdots(1-x_d)}{(a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2)^{q/2}}\,\mathrm{d}x_d \cdots \mathrm{d}x_2\,\mathrm{d}x_1
\end{aligned}$$

and, likewise, we get that

$$\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2} = -\frac{1}{a_2}\int_0^1\int_0^1\cdots\int_0^1 \frac{(1-x_1)\,(1-2\,x_2)\,(1-x_3)\cdots(1-x_d)}{(a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2)^{q/2}}\,\mathrm{d}x_d \cdots \mathrm{d}x_2\,\mathrm{d}x_1\,.$$

Unfortunately, to the best of the authors knowledge, it seems that there is no direct way to prove the relation (5.5). For this reason we rewrite the derivatives $\frac{\partial \mathcal{J}_q}{\partial a_1}$ and $\frac{\partial \mathcal{J}_q}{\partial a_2}$ by means of Laplace transforms. Setting $r := a_1/a_2 \ge 1$ and $r_i := a_i/a_2$ for $i = 2,\dots,d$) and using the identity

$$\frac{\Gamma(q/2)}{s^{q/2}} = \int_0^\infty e^{-st}t^{q/2-1}\mathrm{d}t = 2\int_0^\infty e^{-s\,t^2}t^{q-1}\mathrm{d}t$$

for $s = a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2$ with the Laplace transforms

$$u(t) = \int_0^1 e^{-t^2\,x^2}(1-x)\mathrm{d}x \quad , \quad v(t) = \int_0^1 e^{-t^2\,x^2}(1-2\,x)\mathrm{d}x = u(t) - \frac{1-e^{-t^2}}{2\,t^2} \quad , \quad t \ge 0$$

we obtain that

$$\Gamma\big(\frac{q}{2}\big)\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} = -\frac{2}{a_1}\int_0^\infty v(a_1\,t)\prod_{i=2}^d u(a_i\,t)\,t^{q-1}\mathrm{d}t = -\frac{2}{a_1\,a_2^q}\int_0^\infty v(r\,t)\,u(t)\prod_{i=3}^d u(r_i\,t)\,t^{q-1}\mathrm{d}t$$

and

$$\Gamma\big(\tfrac{q}{2}\big)\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2} = -\frac{2}{a_2}\int_0^\infty v(a_2\,t)\prod_{\substack{i=1\\ i\neq 2}}^{d} u(a_i\,t)\,t^{q-1}\mathrm{d}t = -\frac{2}{a_2\,a_1^q}\int_0^\infty v\big(\frac{t}{r}\big)\,u(t)\prod_{i=3}^{d} u\big(\frac{r_i}{r}t\big)\,t^{q-1}\mathrm{d}t\,.$$

Hence, (5.5) can be equivalently expressed by

$$\int_0^\infty v(r\,t)\,u(t)\prod_{i=3}^{d} u(r_i\,t)\,t^{q-1}\,\mathrm{d}t \le r^{1-q}\int_0^\infty v\big(\frac{t}{r}\big)\,u(t)\prod_{i=3}^{d} u\big(\frac{r_i}{r}t\big)\,t^{q-1}\,\mathrm{d}t\,. \tag{5.6}$$

The function $u(t)$ can be calculated by partial integration as follows

$$t\,u(t) = \int_0^1 \frac{1-x}{2\,t\,x}\,\mathrm{d}_x(1-e^{-t^2x^2}) = -\int_0^1 \frac{1-e^{-t^2x^2}}{2\,t}\,\mathrm{d}_x\Big(\frac{1-x}{x}\Big) = \int_0^1 \frac{1-e^{-t^2x^2}}{2\,t\,x^2}\,\mathrm{d}x$$

so that

$$u(t) = \frac{1}{2\,t}\int_0^t \frac{1-e^{-x^2}}{x^2}\,\mathrm{d}x \quad\text{and}\quad v(t) = \frac{1}{2\,t}\int_0^t \Big(\frac{1-e^{-x^2}}{x^2} - \frac{1-e^{-t^2}}{t^2}\Big)\,\mathrm{d}x \ge 0\,.$$

The latter holds since the mapping $t\mapsto(1-e^{-t^2})/t^2$ is strictly decreasing for $t>0$. Obviously, the Laplace transform $u(t)$ is strictly decreasing whereas the function $\hat{u}(t) := t\,u(t)$ is strictly increasing for $t>0$. Since the derivative $(t\,v(t))' = [1-(1+t^2)\,e^{-t^2})]\,t^{-2}$ is strict positive for $t>0$ the function $\hat{v}(t) := t\,v(t)$ turns out strictly increasing.

In view of $r\ge 1$ and the monotonicity of $u(t)$ we have $u(r_i\,t)\le u\big(\frac{r_i}{r}t\big)$ for all $t>0$, $r_i>0$ and $i=3,\dots,d$. Thus, for proving (5.6) it suffices to show that

$$\int_0^\infty v(r\,t)\,u(t)\,t^{q-1}\,\mathrm{d}t \le r^{1-q}\int_0^\infty v\big(\frac{t}{r}\big)\,u(t)\,t^{q-1}\,\mathrm{d}t$$

which is just the desired inequality for $d=2$. By substituting $t=s/r$ on the l.h.s. and $t=s\,r$ on the r.h.s. of the latter inequality we get that

$$\int_0^\infty v(s)\,u\big(\frac{s}{r}\big)\,s^{q-1}\,\mathrm{d}s \le r^{q+1}\int_0^\infty v(s)\,u(s\,r)\,s^{q-1}\,\mathrm{d}s$$

which in turn is equivalent to

$$\int_0^\infty v(s)\,\hat{u}\big(\frac{s}{r}\big)\,s^{q-2}\,\mathrm{d}s \le r^{q-1}\int_0^\infty v(s)\,\hat{u}(s\,r)\,s^{q-2}\,\mathrm{d}s\,. \tag{5.7}$$

Since $\hat{u}(s/r)\le\hat{u}(s\,r)$, the monotonicity of $\hat{u}(t)$ reveals that (5.7) and therefore (5.6) hold at least for $q\ge 1$. In other words, Schur's criterion (5.5) is satisfied for $q\ge 1$.

In the second part we prove that the function $\mathbf{b}\mapsto\mathcal{J}(f;\mathbf{b})$ is Schur-convex on $\mathbb{R}^d$ if $f|(0,\infty)\to\mathbb{R}^1$ is continuous and non-decreasing. Since $\mathcal{J}(f;b_1,\dots,b_d)$ is symmetric and has continuous partial derivatives (as seen from the below formula (5.9)) we may apply Lemma 2 in the case of Schur-convexity which means to verify that

$$\frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_1} \geq \frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_2}\,. \tag{5.8}$$

for $-\infty < b_2 \leq b_1 < \infty$ and any fixed $b_3,\ldots,b_d \in \mathbb{R}^1$, For brevity put $A_1 = e^{2\,b_1}$, $A_2 = e^{2\,b_2}$ with $A_1 \geq A_2 > 0$ and $B = e^{2\,b_3}\,x_3^2 + \cdots + e^{2\,b_d}\,x_d^2 \geq 0$. To avoid the differentiation of the function f we apply the partial integration formula for Riemann-Stieltjes integrals yielding

$$\begin{aligned}
\int_0^1 (1-x_1)\, f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1 &= \int_0^1 (1-x_1)\,\mathrm{d}_{x_1}\Big(\int_0^{x_1} f(A_1\,y^2 + A_2\,x_2^2 + B)\,\mathrm{d}y\Big)\\
= \int_0^1\int_0^{x_1} f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}y\,\mathrm{d}x_1 &= e^{-b_1}\int_0^1\int_0^{e^{b_1}x_1} f(y^2 + A_2\,x_2^2 + B)\,\mathrm{d}y\,\mathrm{d}x_1\,.
\end{aligned}$$

After differentiating w.r.t. b_1 and partial integration w.r.t. x_1 we get the relations

$$\begin{aligned}
&\frac{\partial}{\partial b_1}\Big(e^{-b_1}\int_0^1\int_0^{e^{b_1}x_1} f(y^2 + A_2\,x_2^2 + B)\,\mathrm{d}y\,\mathrm{d}x_1\Big)\\
= \;& -e^{-b_1}\int_0^1\int_0^{e^{b_1}x_1} f(y^2 + A_2\,x_2^2 + B)\,\mathrm{d}y\,\mathrm{d}x_1 + e^{-b_1}\int_0^1 e^{b_1}\,x_1\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\\
= \;& -e^{-b_1}\int_0^{e^{b_1}} f(y^2 + A_2\,x_2^2 + B)\,\mathrm{d}y + 2\int_0^1 x_1\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\\
= \;& \int_0^1 (2\,x_1 - 1)\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\,.
\end{aligned}$$

This leads to the partial derivatives

$$\frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_1} = \int_0^1\cdots\int_0^1\int_0^1 (2\,x_1 - 1)\prod_{i=2}^d (1-x_i)\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_d\,. \tag{5.9}$$

and likewise

$$\frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_2} = \int_0^1\cdots\int_0^1\int_0^1 (2\,x_2 - 1)\prod_{\substack{i=1\\ i\neq 2}}^d (1-x_i)\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_d\,.$$

Hence,

$$\frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_1} - \frac{\partial \mathcal{J}(f;\mathbf{b})}{\partial b_2} = \int_0^1\cdots\int_0^1\int_0^1 (x_1 - x_2)\prod_{i=3}^d (1-x_i)\,f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_d\,.$$

In order to prove that the $d-$fold integral on the r.h.s. takes non-negative values it suffices to show that

$$h(f;A_1,A_2) := \int_0^1\int_0^1 (x_1-x_2)\, f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\,\mathrm{d}x_2 \geq 0 \quad \text{iff} \quad A_1 \geq A_2\,.$$

For this we rewrite $h(f;A_1,A_2)$ as follows:

$$\begin{aligned}
h(f;A_1,A_2) &= \int_0^1\int_0^{x_1} (x_1-x_2)\, f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&+ \int_0^1\int_0^{x_2} (x_1-x_2)\, f(A_1\,x_1^2 + A_2\,x_2^2 + B)\,\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_0^1\int_0^1 (x_1-x_1\,y)\,x_1\, f(A_1\,x_1^2 + A_2\,x_1^2\,y^2 + B)\,\mathrm{d}y\,\mathrm{d}x_1 \\
&+ \int_0^1\int_0^1 (x_2\,x-x_2)\,x_2\, f(A_1\,x_2^2\,x^2 + A_2\,x_2^2 + B)\,\mathrm{d}x\,\mathrm{d}x_2
\end{aligned}$$

$$= \int_0^1\int_0^1 x^2(1-y)\,\big(f(A_1\,x^2 + A_2\,x^2\,y^2 + B) - f(A_1\,x^2\,y^2 + A_2\,x^2 + B)\big)\,\mathrm{d}y\,\mathrm{d}x\,.$$

Obviously, $f(A_1\,x^2 + A_2\,x^2\,y^2 + B) \geq f(A_1\,x^2\,y^2 + A_2\,x^2 + B)$ for all $x, y \in [0,1]$ iff $A_1 \geq A_2$ which confirms (5.10) and hence (5.8) for a non-decreasing function f. The reverse inequality (5.8) for a non-increasing function f follows by applying the above arguments to $-f$. Thus, Theorem 3 is completely proved. □

Corollary 1 *For $1 \leq q < d$ the parameter integral (5.1) allows the inclusion*

$$\frac{\mathcal{J}_q(1,\ldots,1)}{(\mathrm{AM}(\mathbf{a}))^q} \leq \mathcal{J}_q(\mathbf{a}) \leq \frac{\mathcal{J}_q(1,\ldots,1)}{(\mathrm{GM}(\mathbf{a}))^q} \quad \textit{for all } \mathbf{a} = (a_1,\ldots,a_d) \in (0,\infty)^d, \tag{5.10}$$

where $\mathrm{AM}(\mathbf{a}) := (a_1 + \cdots + a_d)/d$ *and* $\mathrm{GM}(\mathbf{a}) := (a_1\cdot\ldots\cdot a_d)^{1/d}$.
In particular, $\inf\{\mathcal{J}_q(r_1,\ldots,r_d) : r_1,\ldots,r_d \geq 0, r_1+\cdots+r_d = 1\} = \mathcal{J}_q(1/d,\ldots,1/d)$.

Proof of Corollary 1. Since $\mathcal{J}_q(t\mathbf{a}) = t^{-q}\,\mathcal{J}_q(\mathbf{a})$ for $t > 0$ we have

$$\mathcal{J}_q(a_1,\ldots,a_d) = \frac{\mathcal{J}_q(r_1,\ldots,r_d)}{(a_1+\cdots+a_d)^q} \quad \text{with} \quad r_i = a_i/(a_1+\cdots+a_d)\ ,\ \ i = 1,\ldots,d\,.$$

Choosing a doubly stochastic matrix $\mathbf{S}^*$ with identical entries equal to $s^*_{ij} = 1/d$ the Schur-convexity of $a \mapsto \mathcal{J}_q(\mathbf{a})$ implies that $\mathcal{J}_q(\mathbf{r}) \geq \mathcal{J}_q(\mathbf{r}\,\mathbf{S}^*) = \mathcal{J}_q(1/d,\ldots,1/d) = d^q\,\mathcal{J}_q(1,\ldots,1)$ for all $\mathbf{r} = (r_1,\ldots,r_d)$ satisfying $r_1,\ldots,r_d \geq 0$ and $r_1+\cdots+r_d = 1$. Combining this with the foregoing equality yields the lower bound of (5.10). The upper bound of (5.10) follows from the second assertion of Theorem 3 for the strictly decreasing function $f(x) = x^{-q/2}$, and $b_i = \log a_i$ for $i = 1,\ldots,d$ and $\bar{b} := \mathrm{AM}(\mathbf{b}) = (b_1+\cdots+b_d)/d = \log(\mathrm{GM}(\mathbf{a}))$. $\mathcal{J}(f;\mathbf{b}) \leq \mathcal{J}(f;\mathbf{b}\,\mathbf{S}^*) = \mathcal{J}_q(\exp\{\bar{b}\},\ldots,\exp\{\bar{b}\}) = \exp\{-q\,\bar{b}\}\,\mathcal{J}_q(1,\ldots,1) = (\mathrm{GM}(\mathbf{a}))^{-q}\,\mathcal{J}_q(1,\ldots,1)$. □

Next, we formulate a Pfiefer-type inequality for d- parallelotopes. Pfiefer's original result says that, for given $V_d(K) > 0$ and strictly decreasing f on $(0, \infty)$ satisfying (2.2), the functional (2.3) yields the maximum for balls with radius $V_d(K)^{1/d}$, see [10] or [13] (p. 363).

Corollary 2 *If $f|(0,\infty) \to \mathbb{R}^1$ is continuous and non-increasing satisfying (2.2), then*

$$Q_d(f, P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d)) \;\leq\; Q_d(f, V_d(P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d))^{1/d}\,[0,1]^d)\,. \tag{5.11}$$

In other words, among all $d-$parallelotopes P_d with given volume $V_d(P_d) > 0$, precisely the cubes provide the maximum of the functional $Q_d(f, P_d)$.

Proof of Corollary 2. In view of (4.2) and $V_d(P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d)) = V_d(C_d(a_1, \ldots, a_d))$, where the edge lengths $a_j = |b_j^{(j)}|$, $j = 1, \ldots, d$, are defined by (4.1), it suffices to show that $Q_d(f, C_d(a_1, ..., a_d)) \;\leq\; Q_d(f, (a_1 \cdot \ldots \cdot a_d)^{1/d}\,[0,1]^d)$. Since $f|(0,\infty) \to \mathbb{R}^1$ is continuous and non-increasing we may apply Theorem 3 to the Schur-concave mapping $(\log a_1, \ldots, \log a_d) = \mathbf{b} \mapsto \mathcal{J}(f; \mathbf{b})$ and take $\mathbf{S}^*$ as in the proof of Corollary 1. Thus, we get the desired inequality

$$Q_d(f, C_d(a_1, ..., a_d)) = 2^d\,\mathcal{J}(f; \mathbf{b}) \leq 2^d\,\mathcal{J}(f; \mathbf{b}\,\mathbf{S}^*) = Q_d(f, (a_1 \cdot \ldots \cdot a_d)^{1/d}\,[0,1]^d)\,.$$

□

Corollary 3 *Let $P_d = P_d(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ be a d-parallelotope spanned by linearly independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{R}^d$. Then the inclusion*

$$\Big(\frac{2\kappa_{d-1}}{\kappa_d}\Big)^{d+1-p}\frac{\mathcal{I}_p([0,1]^d)}{b_d(P_d)^{d+1-p}} = \frac{d^{d+1-p}\,\mathcal{I}_p([0,1]^d)}{(\|\mathbf{a}_1\| + \ldots + \|\mathbf{a}_d\|)^{d+1-p}} \leq \frac{\mathcal{I}_p(P_d)}{V_d(P_d)^2} \leq \frac{\mathcal{I}_p([0,1]^d)}{V_d(P_d)^{(d+1-p)/d}}$$

holds for $1 \leq p \leq d$. This means that for given mean breadth $b_d(P_d)$ (resp. volume $V_d(P_d)$) the ratio $\mathcal{I}_p(P_d)/V_d(P_d)^2$ attains its minimum (resp. maximum) for cubes with edge length $(\|\mathbf{a}_1\| + \ldots + \|\mathbf{a}_d\|)/d$ (resp. $V_d(P_d)^{1/d}$). Moreover, P_d satisfies the inequalities

$$\mathcal{I}_p(P_d) \begin{cases} \leq V_d(P_d)^{(d+p-1)/d}\,\mathcal{I}_p([0,1]^d) & \text{for} \quad 1 \leq p \leq d+1\,, \\ \geq V_d(P_d)^{(d+p-1)/d}\,\mathcal{I}_p([0,1]^d) & \text{for} \quad p \geq d+1\,. \end{cases} \tag{5.12}$$

with equality for a cube with edge length $V_d(P_d)^{1/d}$.

Proof of Corollary 3. The equality of the lower bounds in the asserted inclusion follows from (2.7). For $p = 1$, the r.h.s. of the inclusion is trivial since $\mathcal{I}_1(K) = \frac{1}{2}\,d\,\kappa_d\,V_d(K)$ for any convex body K, whereas the l.h.s. is just the volume inequality $V_d(P_d) \le ((\|\mathbf{a}_1\| + \ldots + \|\mathbf{a}_d\|)/d)^d$ which follows directly by comparing the lower and upper bound for $p = d$. For $1 < p \le d$, the desired lower bound of $\mathcal{I}_p(P_d)/V_d(P_d)^2$ is obtained by combining the inequality (3.3) applied to the convex function $f(x) = x^{-(d+1-p)/2}$ (which satisfies (2.2)) with the lower bound of (5.10) and the fact that $\mathcal{I}_p(C_d) = V_d(C_d)^2\, p\,(p-1)\, 2^{d-1}\, \mathcal{J}_{d+1-p}(a_1, \ldots, a_d)$ for $C_d = \times_{i=1}^{d}[0, a_i]$. Similarly, the upper bound of $\mathcal{I}_p(P_d)/V_d(P_d)^2$ follows by combining (4.2) applied to the non-increasing function $f(x) = x^{-(d+1-p)/2}$ with the upper bound of (5.10). Finally, the bounds in (5.12) are obtained from (5.11) applied to the non-increasing functions $f(x) = x^{-(d+1-p)/2}$ for $1 \le p \le d+1$ and $f(x) = -x^{(p-d-1)/2}$ for $p \ge d+1$. □

Remark *Both inequalities of (5.12) are stronger than those of (1.7) for* $K = P_d$ *since* $\mathcal{I}_k([0,1]^d) \le (\ge)\ \mathcal{I}_k(\mathbb{B}^d)/\kappa_d^{(d+k-1)/d}$ *for* $1 \le k \le d+1\,(k \ge d+1)$ *which also follows from (1.7). Note that the lower bound of* $\mathcal{I}_p(P_d)/V_d(P_d)^2$ *in Corollary 3 is a still unproved for* $d < p < d+1$. *The crucial point is to show the first assertion of Theorem 3 for* $0 < q < 1$.

To conclude with we give the explicit values for the second-order CPI of squares $[0,a]^2$ and the third-order CPI of cubes $[0,a]^3$ with edge-length $a > 0$. Using (1.4) and (5.1) we obtain after rather lengthy calculations that

$$\mathcal{I}_2([0,a]^2) = \int_{[0,a]^2}\int_{[0,a]^2} \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} = 4\,a^3\,\mathcal{J}_1(1,1) \approx 0.97881799\,a^3\,,$$

$$\mathcal{I}_3([0,a]^3) = 3\int_{[0,a]^3}\int_{[0,a]^3} \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} = 24\,a^5\,\mathcal{J}_1(1,1,1) \approx 5.64693794\,a^5\,,$$

where

$$\mathcal{J}_1(1,1) = \int_0^1\int_0^1 \frac{(1-x_1)(1-x_2)\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2}} = \log(1+\sqrt{2}) - \frac{\sqrt{2}-1}{3} \approx 0.2447045\,,$$

$$\mathcal{J}_1(1,1,1) = \int_0^1\int_0^1\int_0^1 \frac{(1-x_1)(1-x_2)(1-x_3)\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} = \arcsin\big(\frac{1+\sqrt{3}}{2\sqrt{2}}\big) - \frac{\pi}{2}$$

$$+ \frac{\log\big((2+\sqrt{3})(1+\sqrt{2})\big)}{4} + \frac{1+\sqrt{2}-2\sqrt{3}}{20} \approx 0.2352891\,.$$

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