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Abstract. – We study a one-dimensional Bose system with repulsive δ-function interaction in the presence of an $SU(2)$ intrinsic degree of freedom on the basis of the coordinate Bethe ansatz. The ground state and the low-lying excitations are determined by both numerical and analytical methods. It is shown that the ground state is an isospin-ferromagnetic state, and the excitations are composed of three elementary particles: holons, antiholons, and isospinons. The isospinon is a triplet coupled to the "ferromagnetic" background anti-parallelly.

Exactly solvable models [1–16] play an important role in physics, specifically in the investigation of one-dimensional (1D) interacting many-particle systems. They have served as a source of inspiration for the understanding of non-perturbative phenomena in correlated electronic systems; *e.g.*, the spinon was explicitly characterised on the basis of the exact solution of the Hubbard model [6]. Among these models, an earlier prototypical one is the model of 1D bosons with repulsive δ-function interaction, which was solved [7] by means of the Bethe ansatz. This method was also applied to solve the problem of spin-(1/2) fermions [8,9] with δ-function interaction; in fact, Yang [9] already suggested a general strategy for multi-component systems. Various extensions include the study of electrons on a crystalline lattice [6], the generalization to higher symmetries [10–12], and applications to different boundary conditions [13–16]. Nevertheless, $SU(2)$ bosons with δ-function interaction, as far as we are aware, have not been studied until now —in contrast, obviously, to the "spin 1/2" fermionic case, *i.e.* a two-component model with anti-symmetric permutation. Recently, however, a two-component Bose gas was created in magnetically trapped [87]Rb by rotating the two hyperfine states into each other with the help of a slightly detuned Rabi oscillation field [17,18]; and it was noticed [19] that the ground state of a Bose system can be surprisingly different from the scalar Bose system, once the particles acquire an intrinsic degree of freedom.

Therefore, in order to obtain non-perturbative insight into the features of one-dimensional $SU(2)$ bosons with repulsive δ-function interaction, we study a model which is integrable. Pointing out the connection with the coupled Gross-Pitaevski equation [20, 21], we solve the Bethe ansatz equations for bosons with an $SU(2)$ intrinsic degree of freedom. The ground-state properties and the low-lying excitations are studied both by a numerical calculation and

in the thermodynamic limit. Unlike a spin-(1/2) Fermi system, the ground state of the present model is a "ferromagnetic" state, consistent with the result of a variational approach [19]. The charge-isospin phase separation is confirmed, and the isospinon is a triplet instead of a doublet (the latter being well known for the spinon in the spin-(1/2) Fermi system).

The two-component Bose gas is known to satisfy the coupled Gross-Pitaevski equations:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix} = \begin{pmatrix}\hat{G}_1 & \hat{P}^*\\\hat{P} & \hat{G}_2\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix},$$

where $\hat{G}_b = -\frac{\hbar^2}{2m}\nabla_b^2 + V_b(\boldsymbol{r}) + \sum_a u_{ab}|\psi_a|^2$, $\hat{P} = \hbar\Omega/2$, and $a, b = 1, 2$; Ω is the Rabi oscillation field frequency, and $V(\boldsymbol{r})$ the trapping potential.

We consider the isotropic limit, in which the strengths of interaction between inter-species and intra-species are the same ($u_{ab} = c$); then the model is integrable. Considering further the system trapped in a 1D ring of length L, and introducing $\phi_1 = (\psi_1 + \psi_2)/\sqrt{2}$, $\phi_2 = (\psi_1 - \psi_2)/\sqrt{2}$, we obtain from the above equation (for real Ω) the following equivalent Hamiltonian:

$$H = \int \mathrm{d}x\left[\partial_x\phi_a^* \cdot \partial_x\phi_a + c\phi_a^*\phi_b^*\phi_b\phi_a - (-1)^a\Omega\phi_a^*\phi_a\right]$$

with summation over repeated indices implied; natural units are adopted for simplicity. The fields obey bosonic commutation relations, $[\phi_a^*(x), \phi_b(y)] = \sum_{n\in\mathbb{Z}}\delta_{ab}\delta(x-y-nL)$. The Rabi oscillation field contributes a Zeemann-type term. To avoid confusion with conventional spins, we denote the generators of the isospin $SU(2)$ by $\mathcal{I}$; correspondingly, $[\mathcal{I}^+, \mathcal{I}^-] = 2\mathcal{I}^z$. The intrinsic degree of freedom can be specified by the eigenvalues of $\mathcal{I}^z$, or by isospin up and down.

The Bethe ansatz equations for two-component bosons are given as follows:

$$\begin{aligned}e^{ik_jL} &= -\prod_{l=1}^{N}\Xi_1(k_j - k_l)\prod_{\nu=1}^{M}\Xi_{-1/2}(k_j - \lambda_\nu),\\1 &= -\prod_{l=1}^{N}\Xi_{-1/2}(\lambda_\gamma - k_l)\prod_{\nu=1}^{M}\Xi_1(\lambda_\gamma - \lambda_\nu),\end{aligned}\tag{1}$$

where $\Xi_\beta(x) = (x + i\beta c)/(x - i\beta c)$. Equation (1) determines the value of the quasi-momenta $\{k_j\}$ and the isospin rapidities $\{\lambda_\nu\}$ for a $N - 2M + 1$ fold multiplet characterised by the total isospin $\mathcal{I}_{\text{tot}} = (N - 2M)/2$. These equations are similar to the Bethe ansatz equation of [22], except for a variation in the exponential.

Equation (1) is obtained as follows. Applying the Hamiltonian to the Hilbert space of N particles, and considering its first quantized version on the domain $\mathbf{R} \setminus \{\mathbf{P}_{ij}\}$, where $\mathbf{P}_{ij} := \{x \mid x_i - x_j = 0\}$ is the hyperplane defined by the δ-function singularity, we see that only the N-dimensional Laplacian remains in the Schrödinger operator. Thus N-dimensional plane waves are solutions. We sum up all the plane waves with wave vectors which are just permutations of a definite $\boldsymbol{k} = (k_1, k_2, \ldots, k_N)$ according to the Bethe ansatz strategy. Integrating the Schrödinger equation across the hyperplanes, we obtain $S(k_i - k_j) = [k_i - k_j - ic\mathcal{P}]/[k_i - k_j + ic]$, which connects the wave functions defined on the regions separated by the hyperplanes, and $\check{S} := \mathcal{P}S$ ($\mathcal{P}$ stands for the spinor representation of the permutation group S_N) relates the coefficients of different plane waves in the same region. The bosonic permutation symmetry (instead of the antisymmetry) was imposed when solving for the S-matrix. Analogous to the case of spin-(1/2) fermions [9], the periodic boundary condition leads to an eigen-equation for the product of the S-matrices. As the S-matrix satisfies the Yang-Baxter equation, the quantum inverse scattering method [23] is applicable. After writing out the fundamental

commutation relations, and evaluating the eigenvalues of the reference state $|\omega\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$, one immediately recognises the differences to the case of spin-$(1/2)$ fermions. For example, $A(\xi)|\omega\rangle = \prod_l (\xi - \xi_l - ic)/(\xi - \xi_l + ic)|\omega\rangle$, and $D(\xi)|\omega\rangle = \prod_l (\xi - \xi_l)/(\xi - \xi_l + ic)|\omega\rangle$, in the notion of [23, 24]. Consequently, we obtain eq. (1). We note that the Bethe ansatz strategy implies the existence of infinitely many constants of motion, $\sum_j k_j^n = \text{const}$, in addition to the usual energy $E = \sum_{l=1}^{N} k_l^2 + \Omega(N - 2M)$, and momentum $P = \sum_l^N k_l$.

Taking the logarithm of eq. (1) leads to

$$
\begin{aligned}
k_j &= \frac{2\pi}{L} I_j + \frac{1}{L}\sum_{l=1}^{N} \Theta_1(k_j - k_l) + \frac{1}{L}\sum_{\nu=1}^{M} \Theta_{-1/2}(k_j - \lambda_\nu), \\
2\pi J_\gamma &= \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\nu=1}^{M} \Theta_1(\lambda_\gamma - \lambda_\nu),
\end{aligned}
\tag{2}
$$

where $\Theta_\beta(x) := -2\tan^{-1}(x/\beta c)$; both the quantum numbers I_j and J_γ take *integer* or *half-integer* values, depending on whether $N - M$ is *odd* or *even*. For comparison, the Bethe ansatz equation for spin-$(1/2)$ fermions does not only lack the first summation, but also has an opposite sign in the second summation in the first line of eq. (2). The momentum is easily obtained from eq. (2), $P = \sum_l k_l = (\sum_l I_l - \sum_\nu J_\nu)2\pi/L$.

It is instructive to analyse eq. (2) in the strong- and weak-coupling regimes. For a strong interaction, $c \to \infty$, the wave function vanishes for any $x_i = x_j$, and hence the bosons avoid each other like fermions, which is in agreement with the discussion of quantum degeneracy in trapped 1D gases [25]. On the other hand, in the weak-coupling limit, $c \to 0$, using $\Theta_1(x) \to -\pi\,\text{sgn}(x)$, and $\Theta_{-1/2}(x) \to \pi\,\text{sgn}(x)$ for $x \gg 1$, eq. (2) becomes

$$
\begin{aligned}
k_j + \frac{\pi}{L}\sum_{l=1}^{N}\text{sgn}(k_j - k_l) - \frac{\pi}{L}\sum_{\nu=1}^{M}\text{sgn}(k_j - \lambda_\nu) &= \frac{2\pi}{L} I_j\,, \\
\sum_{l=1}^{N}\text{sgn}(\lambda_\gamma - k_l) - \sum_{\nu=1}^{M}\text{sgn}(\lambda_\gamma - \lambda_\nu) &= 2J_\gamma\,.
\end{aligned}
\tag{3}
$$

The subscript of the isospin rapidity λ_γ can be chosen in such a way that J_γ is arranged in increasing order; then the second equation of (3) turns into

$$
\sum_{l=1}^{N}\text{sgn}(\lambda_\gamma - k_l) = 2J_\gamma + 2\gamma - M - 1.
\tag{4}
$$

Because $|J_\gamma| < (N - M + 1)/2$ for a given M and $M \leq N/2$ due to the restriction given by the Young tableau, the minimum value of the right-hand side of eq. (4) is $-N + 2$. This requires that the smallest k_l must be smaller than the smallest λ_ν, otherwise the left-hand side would be $-N$. Equation (4) also implies

$$
\sum_{l=1}^{N}\left[\text{sgn}(\lambda_{\gamma+1} - k_l) - \text{sgn}(\lambda_\gamma - k_l)\right] = 2(J_{\gamma+1} - J_\gamma + 1).
\tag{5}
$$

Thus, for $J_{\gamma+1} - J_\gamma = m$, there must exist exactly $m + 1$ solutions of k_l satisfying $\lambda_\gamma < k_l < \lambda_{\gamma+1}$. Furthermore, from the first equation of (3), we obtain

$$
k_{j+1} - k_j - \frac{\pi}{L}\sum_{\nu=1}^{M}\left[\text{sgn}(k_{j+1} - \lambda_\nu) - \text{sgn}(k_j - \lambda_\nu)\right] = \frac{2\pi}{L}(I_{j+1} - I_j - 1).
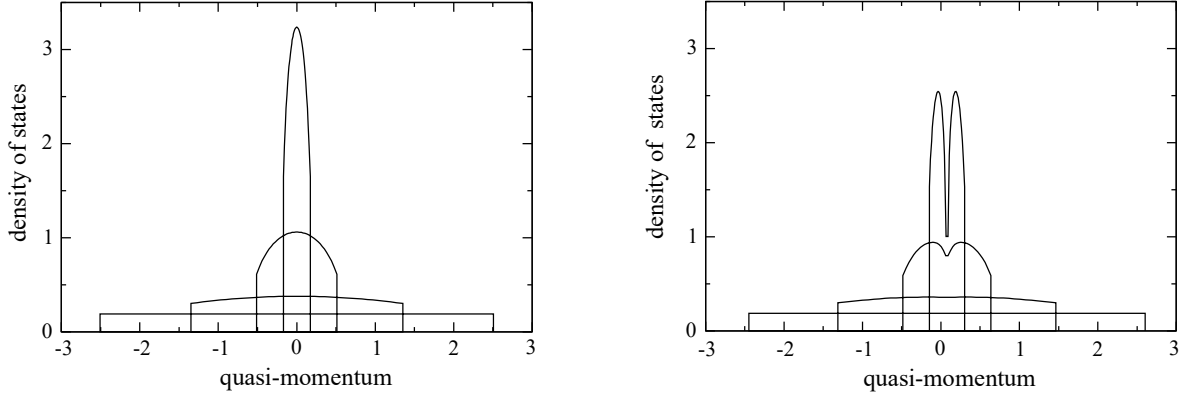\tag{6}
$$

Fig. 1 – The density of state per length in k-space for the ground state (left) and for the state in the presence of one isospin rapidity λ by choosing $J_1 = 0$ (right). The distribution gradually changes from a "histogram" to a narrow peak for strong to weak coupling, $c = 10, 1, 0.1, 0.01$. The calculation is done for $L = 40$, $N = 40$. The left panel is similar to fig. 2 of [7].

Obviously, for $I_{j+1} - I_j = n$, there will be $k_{j+1} - k_j = 2n\pi/L$ if there is a λ_γ such that $k_j < \lambda_\gamma < k_{j+1}$, otherwise $k_{j+1} - k_j = (n-1)2\pi/L$. Thus an isospin rapidity of value λ_μ always repels the quasi-momenta away from that value. As a result, an existing λ_μ will suppress the density of states in k-space at the point $k = \lambda_\mu$. The more isospin rapidities there are, the higher the energy will be. Thus, the ground state of $SU(2)$ interacting bosons is no longer a $SU(2)$ singlet, but an isospin "ferromagnetic" state which differs from the Fermi case considerably.

For N particles, the ground state is characterised by a one-row N-column Young tableau $[N]$, of which the quantum numbers are $\{I_j\} := \{-(N-1)/2, \ldots, (N-1)/2\}$ and $\{J_\gamma\} =$ empty. For this state, eq. (2) reduces to the case studied in [7], but the ground state of the present model is an $(N+1)$-fold multiplet with $\mathcal{I}^2 = N(N+2)/4$. The density of states per length for the ground state is plotted in fig. 1 (left) for various coupling constants. The "particle"-hole (or maybe to be called holon-antiholon) excitation is defined by the quantum numbers $I_1 = -(N-1)/2 + \delta_{1,j_1}$ (for $1 \leq j_1 \leq N$), $I_j = I_{j-1} + 1 + \delta_{j,j_1}$ (for $j = 2, \ldots, N-1$), and $|I_N| \geq (N+1)/2$. Figure 2 (left) shows the corresponding excitation spectrum. The isospinon-holon excitation is characterised by the Young tableau $[N-1, 1]$, i.e., $M = 1$. In comparison to those of the ground state, the quantum numbers $\{I_j\}$ change from half-integer to integer or
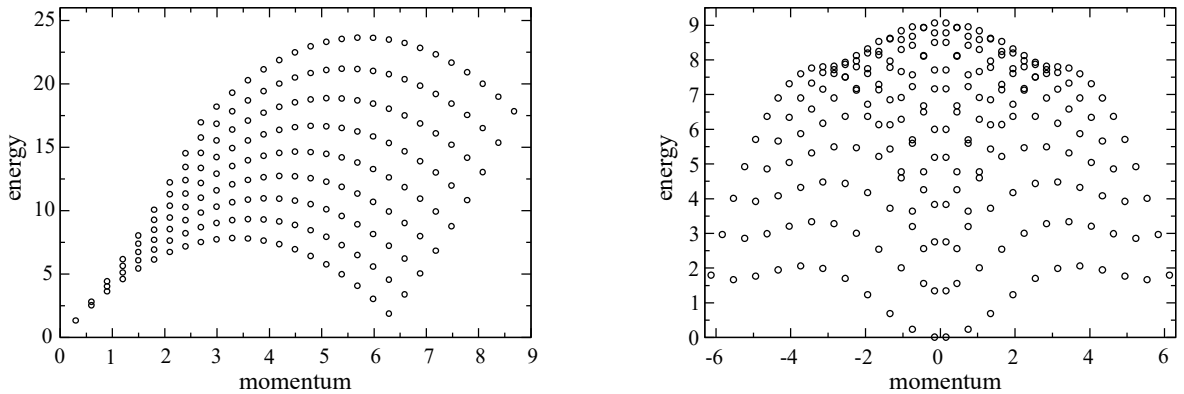

Fig. 2 – The holon-antiholon excitation spectrum (left) and holon-isospinon excitation spectrum (right), calculated for $L = 20$, $N = 40$, and $c = 10$.

vice versa; accordingly, $I_1 = -N/2 + \delta_{1,j_1}$ (for $1 \leq j_1 \leq N+1$), $I_j = I_{j-1} + 1 + \delta_{j,j_1}$ (for $j = 2, \ldots, N$), while $J_1 = I_1 + n$ so that $I_1 < J_1 < I_N$. This is an $(N-1)$-fold multiplet with $\mathcal{I}^2 = N(N-2)/4$. The excitation spectrum is plotted in fig. 2. In fact two branches of the quasi-particle excitations were recently observed in a two-component condensate [26] by means of techniques from nonlinear optics. The density of states for $J_1 = 0$, $j_1 = 1$ is plotted in fig. 1. In comparison to the ground state where no isospin rapidity exists, a rift emerges at the position of the isospin rapidity for small c that is consistent with our previous analysis for weak coupling.

In the thermodynamic limit, the Bethe ansatz equations lead to the following integral equations for the density of roots and the density of holes, respectively, in quasi-momentum and isospin rapidity spaces:

$$\rho(k) + \rho_h(k) = \frac{1}{2\pi} + \int_Q^Q dk' \rho(k') K_1(k - k') - \int_{-B}^B d\lambda' \sigma(\lambda') K_{1/2}(k - \lambda'),$$

$$\sigma(\lambda) + \sigma_h(\lambda) = \int_{-Q}^Q dk' \rho(k') K_{1/2}(\lambda - k') - \int_{-B}^B d\lambda' \sigma(\lambda') K_1(\lambda - \lambda'), \tag{7}$$

where $K_\mu(x) = \pi^{-1} \mu c/(\mu^2 c^2 + x^2)$. The limits of integration, Q and B, are determined to be consistent with $\int_{-Q}^Q \rho(k)\, dk = N/L$, and $\int_{-B}^B \sigma(\lambda)\, d\lambda = M/L$. It is easy to check by Fourier transformation that the state with $B = \infty$ and $\sigma_h = 0$ is an isospin singlet, but it is not the ground state. The ground state corresponds to $\sigma = \rho_h = 0$ in eq. (7), *i.e.* it is an isospin "ferromagnetic" state, in agreement with the result of mean-field theory [19]. The two-particle case is a pedagogical example: For the two-body Schrödinger equation in the center-of-mass frame, the permutation of particle coordinates equals the parity reflection of their relative coordinate. The oscillation theorem in quantum mechanics tells that the spatial wave function without nodes, an even parity solution, yields the lowest energy. If it possesses an $SU(2)$ intrinsic degree of freedom, the intrinsic wave function must be symmetric (anti-symmetric) to keep the total wave function with the lowest energy being symmetric (anti-symmetric). Then the ground state of the Bose system (Fermi system) is of "ferromagnetic" ("anti-ferromagnetic") character.

The highly degenerate ferromagnetic ground state, obtained in the case of a vanishing Zeemann term, will split up into Zeemann sublevels once the external field is applied. The ground state hence becomes a polarized state once the Rabi field, which breaks the $SU(2)$ symmetry, is turned on.

In order to evaluate the excitation energies we put $\rho(k) = \rho_0(k) + \rho_1(k)/L$ (ρ_0 refers to the ground state). In the presence of the isospin degree of freedom, there will be a holon-isospinon excitation, in addition to the holon-antiholon excitation. The latter is created by a hole inside the quasi Fermi sea $\bar{k} \in [-k_F, k_F]$, and an additional k_p outside it, *i.e.*

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} dk' \rho_1 K_1(k - k') + K_1(k - k_p).$$

The excitation energy consists of two terms: $\Delta E = \int k^2 \rho\, dk + k_p^2 = \varepsilon_h(\bar{k}) + \varepsilon_a(k_p)$, where the holon energy ε_h, and the antiholon energy $\varepsilon_a(k_p) = -\varepsilon_h(k_p)$, are given by

$$\varepsilon_h(y) = -y^2 + \int_{-k_F}^{k_F} k^2 \rho_1^h(k, y)\, dk,$$

$$\rho_1^h(k, y) + K_1(k - y) = \int_{-k_F}^{k_F} dk' K_1(k - k') \rho_1^h(k', y). \tag{8}$$
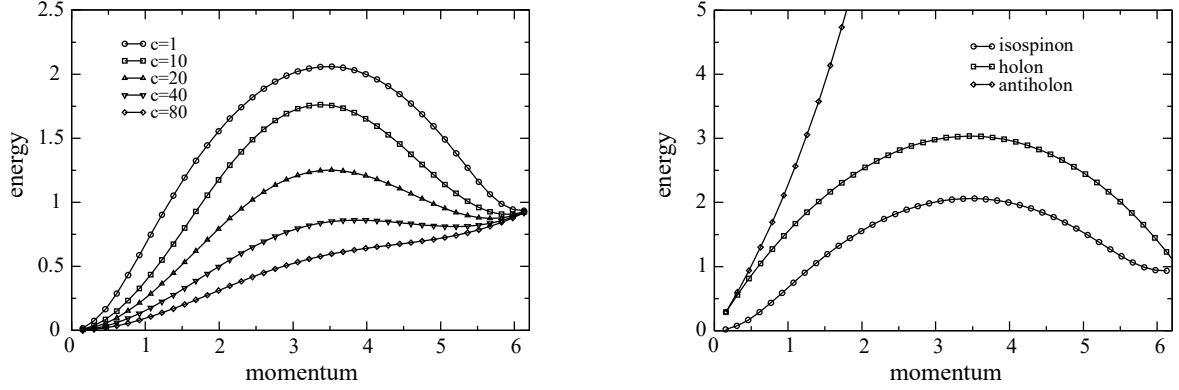
Fig. 3 – Dispersion relations of the isospinon for different coupling constants (left); the curves from top to bottom correspond to $c = 1, 10, 20, 40$, and 80, respectively. Right: dispersion of antiholon, holon, and isospinon, for $c = 1$ ($L = 20$, $N = 40$).

Flipping one isospin corresponds to adding one isospin rapidity to the background of the ferromagnetic ground state, which inevitably brings about one hole in the k-sector. The excitation energy $\Delta E = \int k^2 \rho_1 \mathrm{d}k$ is obtained from

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_\mathrm{F}}^{k_\mathrm{F}} \mathrm{d}k' K_1(k - k')\rho_1(k') - K_{1/2}(k - \lambda);$$

consequently, $\Delta E = \varepsilon_h(\bar{k}) + \varepsilon_i(\lambda)$. Here ε_h is given by eq. (8), and ε_i by $\varepsilon_i(\lambda) = \int k^2 \rho_1^i(k, \lambda)\, \mathrm{d}k$, with

$$\rho_1^i(k, \lambda) + K_{1/2}(k - \lambda) = \int_{-k_\mathrm{F}}^{k_\mathrm{F}} \mathrm{d}k' K_1(k - k')\rho_1^i(k', \lambda).$$

In conclusion we found three elementary quasi-particles: holon, antiholon and isospinon. From the asymptotic behaviour of those basic modes, for $\bar{k}$, k_p, and λ tending to k_F, we find that both the holon-antiholon and holon-isospinon excitations are gapless. The related dispersions for finite N are plotted in fig. 3. Different from the spinons in a Fermi system, the isospinon here is a triplet which always couples to the "ferromagnetic" background anti-parallelly. Although it is always accompanied by a charge excitation (holon) for single or odd number of isospinons, the isospinons can be excited in pairs without exciting the $U(1)$ charge mode. Because of the coupling between the charge sector and the isospin sector in eq. (7), both cases bring about changes in the quasi-momentum distribution and hence lead to an excitation energy. The charge-isospin phase separation predicted by mean-field theory [27] is clearly confirmed in the present case due to the structure of eq. (7). The holon and antiholon are quasi-particles created in momentum space, while the isospinon behaves like a dark soliton [28] in the isospin sector that tends to decrease the total isospin $\mathcal{I}$ eigenvalue by one.

Considering the experiment on a two-component Bose gas whose transverse excitations are frozen out, such that the dynamics becomes essentially one-dimensional [29], it is expected that the above results can be confirmed through a careful measurement of the excitation spectra, which in particular should show the isospin excitations. However, the question has to remain open of how much can be learned from our model —which is an integrable one— about the phenomenon of Bose-Einstein condensation, as, *e.g.*, the inclusion of a realistic trapping potential into the model should be important.

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REFERENCES

[1] TSVELICK A. W. and WIEGMANN P. W., *Adv. Phys.*, **32** (1983) 453.

[2] ANDREI N., FURUYA K. and LOWENSTEIN L. H., *Rev. Mod. Phys.*, **55** (1983) 331.

[3] HALDANE F. D. M., *Phys. Lett. A*, **93** (1983) 464; *Phys. Rev. Lett.*, **50** (1983) 1153.

[4] DEVEGA H. J. and WOYNAROVICH F., *Nucl. Phys. B*, **251** (1985) 439.

[5] KOREPIN V. E., BOGOLIUBOV N. M. and IZERGIN A. G., *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge) 1993.

[6] LIEB E. H. and WU F., *Phys. Rev. Lett.*, **20** (1968) 1444.

[7] LIEB E. H. and LINIGER W., *Phys. Rev.*, **130** (1963) 1605.

[8] GAUDIN M., *Phys. Lett. A*, **24** (1967) 55.

[9] YANG C. N., *Phys. Rev. Lett.*, **19** (1967) 1312.

[10] SUTHERLAND B., *Phys. Rev. Lett.*, **20** (1968) 98.

[11] CHOY T. C., *Phys. Lett. A*, **80** (1980) 49; HALDANE F. D. M., *Phys. Lett. A*, **80** (1980) 281; **81** (1981) 545; CHOY T. C. and HALDANE F. D. M., *Phys. Lett. A*, **90** (1982) 83.

[12] SCHLOTTMANN P., *Phys. Rev. B*, **43** (1991) 3101.

[13] GAUDIN M., *Phys. Rev. A*, **4** (1971) 386.

[14] SCHULZ H., *J. Phys. C*, **18** (1985) 581.

[15] WOYNAROVICH F., *Phys. Lett. A*, **108** (1985) 401.

[16] LI Y. Q. and GRUBER C., *Phys. Rev. Lett.*, **80** (1998) 1034.

[17] WILLIAMS J. E. and HOLLAND M. J., *Nature*, **401** (1999) 568.

[18] WILLIAMS J. E., HOLLAND M. J., WIEMAN C. E. and CORNELL E. A., *Phys. Rev. Lett.*, **83** (1999) 3358.

[19] HO T. L., *Phys. Rev. Lett.*, **81** (1998) 742.

[20] GROSS E. P., *Nuovo Cimento*, **20** (1961) 454.

[21] PITAEVSKII L. P., *Zh. Eksp. Teor. Fiz.*, **40** (1961) 646 (*Sov. Phys. JETP*, **13** (1961) 451).

[22] SUTHERLAND B., *Phys. Rev. B*, **12** (1975) 3795.

[23] FADDEEV L. D., *Recent Advances in Field Theory and Statistical Mechanics*, edited by ZUBER J. and STORA R. (Elsevier, Amsterdam) 1984, p. 569.

[24] KULISH P. P., *Physica D*, **3** (1981) 246.

[25] PETROV D. S., SHLYAPNIKOV G. V. and WALRAVEN J. T. M., *Phys. Rev. Lett.*, **85** (2000) 3745.

[26] GOLDSTEIN E. V., MOORE M. G., PU H. and MEYSTRE P., *Phys. Rev. Lett.*, **85** (2000) 5030.

[27] HO T.-L. and SHENOY V. B., *Phys. Rev. Lett.*, **77** (1996) 3276; TIMMERMANS E., *Phys. Rev. Lett.*, **81** (1998) 5718.

[28] CARR L. D., BRAND J., BURGER S. and SANPERA A., *Phys. Rev. A*, **63** (2001) 051601(R).

[29] KETTERLE W. and VAN DRUTEN N. J., *Phys. Rev. A*, **54** (1996) 656.