Theory of Relativistic Brownian Motion: The (1+3)-Dimensional Case

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A theory for (1+3)-dimensional relativistic Brownian motion under the influence of external force fields is put forward. Starting out from a set of relativistically covariant, but multiplicative Langevin equations we describe the relativistic stochastic dynamics of a forced Brownian particle. The corresponding Fokker-Planck equations are studied in the laboratory frame coordinates. In particular, the stochastic integration prescription, i.e. the discretization rule dilemma, is elucidated (pre-point discretization rule *vs.* mid-point discretization rule *vs.* post-point discretization rule). Remarkably, within our relativistic scheme we find that the post-point rule (or the transport form) yields the only Fokker-Planck dynamics from which the relativistic Maxwell-Boltzmann statistics is recovered as the stationary solution. The relativistic velocity effects become distinctly more pronounced by going from one to three spatial dimensions. Moreover, we present numerical results for the asymptotic mean square displacement of a free relativistic Brownian particle moving in (1+3) dimensions.

I. INTRODUCTION

The problem of formulating a consistent theory of Brownian motions [1, 2, 3, 4, 5, 6] in the framework of special relativity [7, 8] represents a longstanding issue in mathematical and statistical physics (classical references are [9, 10, 11]; more recent contributions include [12, 13, 14, 15, 16, 17, 18, 19, 20]; for a kinetic theory approach, see [21, 22]). In a preceding paper [23] – referred to as paper I hereafter – we have discussed in detail how one can construct Langevin equations for (1+1)-dimensional relativistic Brownian motions. In particular, it was demonstrated that for the relativistic Langevin equation *per se* cannot uniquely determine the corresponding Fokker-Planck equation (FPE). This ambiguity arises due to the fact that the relativistic Langevin equations, when e.g. written in laboratory coordinates, exhibit a multiplicative coupling between a function of the momentum coordinate and a Gaussian white noise process (laboratory frame ≡ rest frame of the heat bath). Thus, depending on the choice of the discretization rule, different forms of relativistic FPE are obtained [24, 25].

In paper I, we have analyzed the three most popular discretization rules for Langevin equations with multiplicative noise, which can be traced back to the proposals pioneered by Ito (pre-point discretization rule [26, 27]), by Fisk and Stratonovich (mid-point rule [25, 28, 29, 30, 31]), and by Hänggi and Klimontovich (post-point rule [32, 33, 34, 35]). In this context it could be shown that only the Hänggi-Klimontovich (HK) interpretation of the Langevin equation yields the transport form of the Fokker-Planck equation with state-dependent diffusion, whose stationary solution coincides with the one-dimensional relativistic Maxwell distribution. The latter is known from Jüttner's early work on the relativistic gas [36, 37] and also from the relativistic kinetic theory [22].

In paper I, we have focussed exclusively on the simplest situation, corresponding to *free* (1+1)-dimensional relativistic Brownian motions. Therefore, the present paper aims to extend the analysis to the physically relevant (1+3)-dimensional case. In particular, we wish to include as well the effects of additional, *external* force fields. To this end the paper is structured as follows: In Sec. II the relativistic Langevin equations are given in covariant 4-vector notation and also in laboratory frame coordinates. The corresponding Fokker-Planck equations and their stationary solutions are considered in Sec. III. Section IV contains a discussion of numerical results for the mean square displacement of free Brownian particles. The paper concludes with a resume of the main results in Sec. V.

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II. RELATIVISTIC LANGEVIN DYNAMICS

Let us first discuss the manifestly Lorentz-covariant 4-vector form of the relativistic Langevin equations (Sec. II A). For that purpose, we shall use of the results derived in Sec. II of paper I, which can readily be generalized to (1+3)-dimensions. Subsequently, the relativistic Langevin equations will be written in laboratory coordinates (Sec. II B). The latter form provides the basis for the numerical results of Sec. IV.

With regard to notation, the following conventions will be used throughout the paper: Upper and lower Greek indices $\alpha, \beta, \ldots$ can take values $0, 1, 2, 3$, where '0' refers to the time component. Upper and lower Latin indices $i, j, \ldots$ take values $1, 2, 3$ and are used to label the components of spatial 3-vectors, denoted by bold symbols. For example, we write $(x^\alpha) = (x^0, \boldsymbol{x}) = (ct, x^i)$ and $(p^\alpha) = (p^0, \boldsymbol{p}) = (E/c, p^i)$ with t denoting the coordinate time, E the energy, c the vacuum speed of light, x^i and p^i the spatial coordinates and relativistic momenta, respectively. Moreover, Einstein's summation convention is applied throughout. The (1+3)-dimensional Minkowski metric tensor with respect to Cartesian coordinates is taken as

$$(\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \text{diag}(-1, 1, 1, 1), \qquad (\eta^\alpha{}_\beta) = (\eta_\alpha{}^\beta) = \text{diag}(1, 1, 1, 1).$$

As commonly known, covariant vector components x_α can be calculated from the contravariant components x^α by virtue of $x_\alpha = \eta_{\alpha\beta} x^\beta$, which, in particular, means that for Cartesian coordinates $x_0 = -x^0$ and $x_i = x^i$ holds. Further, if in a certain inertial coordinate system Σ a Brownian particle has the 3-velocity $\boldsymbol{v}(t) \equiv \mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t$, then the differential $\mathrm{d}\tau$ of its proper-time is defined by

$$\mathrm{d}\tau \equiv \mathrm{d}t\sqrt{1 - \frac{v_i v^i}{c^2}}. \tag{1}$$

A. Langevin equations in 4-vector notation

Consider a Brownian particle with rest mass m, proper time τ and 4-velocity $u^\beta(\tau)$, i.e., the 4-momentum of the particle is given by $p^\alpha = mu^\alpha$, where $u_\alpha u^\alpha \equiv -c^2$. Assume that the particle is surrounded by an isotropic, homogeneous heat bath with constant 4-velocity U^β and, additionally, subject to an external 4-force $K^\alpha(x^\nu, p^\mu)$ such as, e.g., the Lorentz force. Then, according to the results in Sec. II of paper I, the relativistic Langevin equations of motion read

$$\mathrm{d}x^\alpha(\tau) = \frac{p^\alpha}{m}\mathrm{d}\tau \tag{2a}$$

$$\mathrm{d}p^\alpha(\tau) = \left\{K^\alpha - \nu^\alpha{}_\beta\left[p^\beta - mU^\beta\right]\right\}\mathrm{d}\tau + w^\alpha(\tau). \tag{2b}$$

For an isotropic homogeneous heat bath, the friction tensor $\nu^\alpha{}_\beta$ in (2b) is given by

$$\nu^\alpha{}_\beta = \nu\left(\eta^\alpha{}_\beta + \frac{u^\alpha u_\beta}{c^2}\right), \tag{2c}$$

with ν denoting the scalar viscous friction coefficient measured in the rest frame of the particle. Furthermore, the relativistic Wiener increments $w^\alpha(\tau) \equiv \mathrm{d}W^\alpha(\tau)$ are distributed according to the probability density

$$\mathcal{P}^{1+3}[w^\alpha(\tau)] = \frac{c}{(4\pi D\,\mathrm{d}\tau)^{3/2}}\,\exp\left[-\frac{w_\alpha(\tau)\,w^\alpha(\tau)}{4D\,\mathrm{d}\tau}\right] \times \delta\left[u_\alpha w^\alpha(\tau)\right], \tag{2d}$$

where D is the scalar noise amplitude parameter measured in the rest frame of the particle. Some useful comments concerning Eq. (2) are appropriate:

(i) The covariant friction tensor in Eq. (2c) carries the same structure as the covariant pressure tensor for an ideal fluid. In particular, this means that in each instantaneous rest frame Σ_* of the particle, where temporarily $(u_*^\alpha) = (c, \boldsymbol{0})$ holds, the tensor form Eq. (2c) reduces to the diagonal form

$$(\nu_*{}^\alpha{}_\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}. \tag{3}$$

This form of the friction tensor reflects the simplifying assumptions that the heat bath can, in good approximation, be considered as isotropic and homogeneous.

(ii) Note that the probability density of the increments in Eq. (2d) can equivalently be written as

$$\mathcal{P}^{1+3}[w^\alpha] = \frac{c}{(4\pi D \, d\tau)^{3/2}} \, \exp\left[-\frac{1}{2d\tau}\hat{D}_{\alpha\beta}w^\alpha \, w^\beta\right] \times \delta\left[u_\alpha \, w^\alpha\right], \tag{4a}$$

where the tensor

$$\hat{D}^\alpha{}_\beta = \frac{1}{2D}\left(\eta^\alpha{}_\beta + \frac{u^\alpha u_\beta}{c^2}\right) \tag{4b}$$

carries the same isotropic structure as the friction tensor from Eq. (2c). The δ-function in Eqs. (4a) accounts for the fact that the Minkowski scalar-product of the 4-force and the 4-velocity must identically vanish.

(iii) The increment density in Eq. (2d) is normalized so that

$$1 = \left\{\prod_{\alpha=0}^{3}\int_{-\infty}^{\infty}d[w^\alpha(\tau)]\right\}\mathcal{P}^{1+3}[w^\alpha(\tau)] \tag{5a}$$

holds. For the first two moments one finds

$$\langle w^\alpha(\tau)\rangle = 0, \tag{5b}$$

$$\langle w^\alpha(\tau)\, w^\beta(\tau')\rangle = \begin{cases} 0, & \tau \neq \tau'; \\ D^{\alpha\beta}d\tau, & \tau = \tau', \end{cases} \tag{5c}$$

where

$$D^\alpha{}_\beta = 2D\left(\eta^\alpha{}_\beta + \frac{u^\alpha u_\beta}{c^2}\right). \tag{5d}$$

The easiest way to validate this is to perform the calculations leading to Eqs. (5) in a co-moving Lorentz frame Σ_*, where, at a given instant of time $t_*(\tau)$, the particle is at rest. In such a co-moving frame Σ_* the marginal distribution of the spatial momentum increments, defined by

$$\mathcal{P}^3[\boldsymbol{w}_*(t_*)] = \int_{-\infty}^{\infty}dw_*^0 \, \mathcal{P}^{1+3}[w_*^\alpha], \tag{6}$$

reduces to a Gaussian. One thus recovers, as it should hold true, from Eqs. (2) the nonrelativistic Brownian motion in the Newtonian limit case $\boldsymbol{v}^2 \ll c^2$.

B. Langevin dynamics in the laboratory frame

In this section the covariant Langevin equations (2) will be rewritten in laboratory coordinates. A laboratory frame Σ_0 is, by definition, an inertial system, in which the heat bath is at rest. That is, in Σ_0 the 4-velocity of the heat bath is given by $(U^\beta) = (c, \boldsymbol{0})$ for all times t, where t is the coordinate time of Σ_0.

From Eq. (2a), we obtain three differential equations for the position coordinates

$$dx^i(t) = v^i \, dt, \tag{7a}$$

where

$$v^i = \frac{cp^i}{\sqrt{m^2c^2 + p_ip^i}}. \tag{7b}$$

Furthermore, since we have $(U^\beta) = (c, \boldsymbol{0})$ in Σ_0, the four stochastic differential Eqs. (2b) can be rewritten as [23, 38]

$$dp^i = (\gamma^{-1}K^i - \nu\,p^i)\,dt + w^i, \tag{7c}$$

$$dE = (\gamma^{-1}K^i - \nu\,p^i)v_i\,dt + cw^0, \tag{7d}$$

where the relativistic (kinetic) energy is here defined by $E \equiv cp^0$, and

$$\gamma \equiv \left(1 - \frac{v_iv^i}{c^2}\right)^{-1/2} = \left(1 + \frac{p_ip^i}{m^2c^2}\right)^{1/2}. \tag{7e}$$

Before discussing the stochastic increments w^α, let us briefly consider the deterministic force components K^i. If $\boldsymbol{F} = (F^i)$ is the nonrelativistic (Newtonian) force, then the corresponding relativistic force 3-vector $\boldsymbol{K} = (K^i)$ is given by (see Chap. 2 of [38])

$$\boldsymbol{K} = \boldsymbol{F} + (\gamma - 1)\frac{(\boldsymbol{v} \cdot \boldsymbol{F})}{v^2}\, \boldsymbol{v}. \tag{8}$$

If the dynamics is confined to one spatial dimension only, then Eq. (8) simplifies to $\boldsymbol{K} = \gamma \boldsymbol{F}$.

Next, let us turn to the stochastic force components, appearing on the rhs. of Eqs. (7c) and (7d). According to Eq. (2d), the distribution of the stochastic momentum increments w^α, also depends on the particle's velocity $\boldsymbol{v}$. By virtue of the relation

$$(u_\alpha) = (-\gamma c, \gamma v_i), \qquad (w^\alpha) = (w^0, w^i), \tag{9}$$

we can rewrite the increment density (2d) in laboratory coordinates as follows

$$\mathcal{P}^{1+3}[w^\alpha] = c\left(\frac{\gamma}{4\pi D\, \mathrm{d}t}\right)^{3/2} \exp\left[-\frac{w_i w^i - (w^0)^2}{4D\, \mathrm{d}t/\gamma}\right] \times \delta\left(c\gamma w^0 - \gamma v_i w^i\right). \tag{10}$$

As we already pointed out above, the δ-function in Eq. (10) reflects the fact that the energy increment w^0 is coupled to the spatial momentum increments w^i via

$$0 = u_\alpha w^\alpha = -c\gamma w^0 + \gamma v_i w^i \qquad \Rightarrow \qquad w^0 = \frac{v_i w^i}{c}. \tag{11}$$

This is just the stochastic analogue of the well-known deterministic identity

$$u_\alpha K^\alpha \equiv 0.$$

Hence, similar to K^0, also w^0 can be eliminated from the Langevin equation (7d), yielding

$$\mathrm{d}E = (\gamma^{-1}K^i - \nu\, p^i)v_i\, \mathrm{d}t + v_i w^i(t) = v_i\, \mathrm{d}p^i. \tag{12}$$

Using the identity (7b), we can further rewrite Eq. (12) as

$$\mathrm{d}E = \frac{cp_i}{\sqrt{m^2 c^2 + \boldsymbol{p}^2 c^2}}\, \mathrm{d}p^i, \tag{13}$$

where $\boldsymbol{p}^2 \equiv p_i p^i$. From the preceding equation we regain the well-known energy momentum law

$$E(t) = \sqrt{m^2 c^4 + \boldsymbol{p}(t)^2 c^2}. \tag{14}$$

It might be worthwhile to remark that in the presence of force fields the relativistic energy $E = cp^0$ is generally different from the relativistic Hamiltonian [39]. Note that Eq. (14) remains valid also for our stochastic model.

The relativistic Brownian motion is therefore completely described by the three Langevin equations (7c). The statistics of the increments w^i in (7c) is determined by the marginal distribution $\mathcal{P}^3[\boldsymbol{w}]$, defined in Eq. (6). Performing the integration over the δ-function in Eq. (10), we find

$$\begin{aligned}
\mathcal{P}^3[\boldsymbol{w}] &= \int_{-\infty}^{\infty} \mathrm{d}w^0\, \mathcal{P}^{1+3}[w^\alpha] \\
&= c\int_{-\infty}^{\infty} \mathrm{d}w^0\, \left(\frac{\gamma}{4\pi D\, \mathrm{d}t}\right)^{3/2} \exp\left[-\frac{w_i w^i - (w^0)^2}{4D\, \mathrm{d}t/\gamma}\right] \times \delta\left(c\gamma w^0 - \gamma v_i w^i\right) \\
&= \frac{1}{\gamma}\left(\frac{\gamma}{4\pi D\mathrm{d}t}\right)^{3/2} \exp\left[-\frac{\gamma}{4D\mathrm{d}t}\left(\delta_{ij} - \frac{v_i v_j}{c^2}\right)w^i w^j\right],
\end{aligned} \tag{15}$$

where δ_{ij} denotes the Kronecker-symbol (defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise).

In principle, it is straightforward to perform computer simulations on the basis of Eqs. (7c) and (15). In Sec. IV we will discuss several numerical findings. Before doing so, however, it is worthwhile to consider in greater detail the Fokker-Planck equations of the relativistic Brownian motion in the laboratory frame Σ_0. By doing so it will become clear that the choice $v^i = v^i(t)$ in Eq. (15) is consistent with an Ito-interpretation [24, 26, 27] of the stochastic differential equations (7c). However, we shall also see that alternative interpretations – such as e.g. the post-point discretization rule, $v^i = v^i(t + dt)$ – lead to physically reliable results as well.

III. RELATIVISTIC FOKKER-PLANCK EQUATIONS

The objective here is to discuss relativistic Fokker-Planck equations (FPE) for the one-particle momentum density $f(t, \boldsymbol{p})$ and, as well, for the phase space density $f(t, \boldsymbol{p}, \boldsymbol{x})$. In the remainder, we will exclusively refer to the coordinates of the laboratory frame Σ_0. Before turning to the relativistic FPE in Sects. III B and III C, it is useful to briefly recall the nonrelativistic case.

A. Nonrelativistic case

Consider the nonrelativistic Langevin equations

$$\mathrm{d}x^i = v^i \,\mathrm{d}t \tag{16a}$$

$$\mathrm{d}p^i = (K^i - \nu \, p^i)\mathrm{d}t + w^i, \tag{16b}$$

where $p^i(t) = m v^i(t)$ denotes the nonrelativistic momentum components, $K^i = -\partial^i U$ represents the vector components of a conservative force with time-independent potential $U(\boldsymbol{x})$, and where the increments $w^i \equiv \mathrm{d}W^i$ are distributed according to

$$\mathcal{P}[\boldsymbol{w}] = \left(\frac{1}{4\pi D \,\mathrm{d}t}\right)^{3/2} \exp\left[-\frac{w_i \, w^i}{4D \,\mathrm{d}t}\right]. \tag{16c}$$

Equations (16) govern the nonrelativistic motions of a Brownian particle in the rest frame of the heat bath. As one readily observes, in the case of conservative force fields, Eqs. (16) can be obtained from the relativistic equations (7a), (7c) and (15) by formally taking the limit $c \to \infty$ (Newtonian limit case). It is well known that the phase space density $f(t, \boldsymbol{p}, \boldsymbol{x})$, associated with the stochastic process (16), is governed by the FPE [24, 25, 40]

$$\frac{\partial}{\partial t}f + \frac{p^i}{m}\frac{\partial}{\partial x^i}f + \frac{\partial}{\partial p^i}(K^i f) = \frac{\partial}{\partial p^i}\left(\nu p^i f + D\delta^{ij}\frac{\partial}{\partial p^j}f\right). \tag{17}$$

The stationary solution of Eq. (17) is the (nonrelativistic) Maxwell-Boltzmann distribution, i.e.,

$$f(\boldsymbol{p}, \boldsymbol{x}) = C \exp\left[-\frac{\boldsymbol{p}^2 + 2m \, U(\boldsymbol{x})}{2m \, k_{\mathrm{B}} T}\right], \tag{18a}$$

where C is a normalization constant. The temperature T of the bath is defined by the Einstein relation (k_{B} denotes the Boltzmann constant)

$$k_{\mathrm{B}} T \equiv \frac{D}{m\nu}. \tag{18b}$$

The related marginal momentum distribution is the usual Maxwellian probability density

$$f(\boldsymbol{p}) = \left(\frac{\nu}{2\pi D}\right)^{3/2} \exp\left(-\frac{\nu \boldsymbol{p}^2}{2D}\right). \tag{19}$$

B. Relativistic FPE for free Brownian particles

We next reason three different types of relativistic Fokker-Planck equations for the momentum density $f(t, \boldsymbol{p})$ of a free Brownian particle, whose dynamics is governed by the stochastic process given by Eqs. (7c) and (15). The corresponding equations for the phase-space density $f(t, \boldsymbol{p}, \boldsymbol{x})$ will be considered separately in Sec. III C, where we will also include the influence of an external force field.

As our starting point serves the relativistic Langevin equations (7c), which hold in the laboratory frame Σ_0 (i.e., in the rest frame of the heat bath). In absence of external force fields the stochastic differential equations (7c) reduce to

$$\mathrm{d}p^i = -\nu \, p^i \,\mathrm{d}t + w^i, \tag{20a}$$

where $p^i = \gamma m v^i$ is now the relativistic momentum and γ has been defined in Eq. (7e). According to Eq. (15) – and in distinct contrast to Eq. (16c) – the distribution of the relativistic increments reads

$$\mathcal{P}^3[\boldsymbol{w}] = \frac{1}{\gamma} \left(\frac{\gamma}{4\pi D \mathrm{d}t}\right)^{3/2} \exp\left[-\frac{w_i A^i{}_j w^j}{4D\mathrm{d}t}\right] \tag{20b}$$

with matrix elements given by

$$A^i{}_j \equiv \left(\delta^i{}_j - \frac{v^i v_j}{c^2}\right)\gamma = \left(\delta^i{}_j - \frac{p^i p_j}{\gamma^2 m^2 c^2}\right)\gamma. \tag{20c}$$

Following the reasoning of paper I, the next aim is to rewrite the Langevin equations (20) in such a form, that the resulting equations exhibit multiplicative Gaussian white noise, governed by a velocity-independent normal distribution of the form (16c). In order to achieve this objective we first note that the matrix $\boldsymbol{A}(\boldsymbol{p}) = (A^i{}_j)$ is symmetric. Its eigenvalues and determinant are given by

$$\mathrm{spec}(\boldsymbol{A}) = \{\gamma, \gamma, \gamma^{-1}\}, \qquad \det(\boldsymbol{A}) = \gamma. \tag{21}$$

Thus, the matrix $\boldsymbol{A}$ is positive definite for velocities $\boldsymbol{v}^2 < c^2$, and the elements of the inverse matrix $\boldsymbol{A}^{-1}$ read

$$(A^{-1})^j{}_k = \left(\frac{\delta^j{}_k}{\gamma^2} + \frac{v^j v_k}{c^2}\right)\gamma = \left(\delta^j{}_k + \frac{p^j p_k}{m^2 c^2}\right)\frac{1}{\gamma}. \tag{22}$$

Furthermore, there exists a unique Cholesky-decomposition [41]

$$\boldsymbol{A} = \boldsymbol{L}^\top \boldsymbol{L} = \begin{pmatrix} L^1{}_1 & 0 & 0 \\ L^2{}_1 & L^2{}_2 & 0 \\ L^3{}_1 & L^3{}_2 & L^3{}_3 \end{pmatrix} \begin{pmatrix} L^1{}_1 & L^2{}_1 & L^3{}_1 \\ 0 & L^2{}_2 & L^3{}_2 \\ 0 & 0 & L^3{}_3 \end{pmatrix}, \tag{23}$$

where the matrix $\boldsymbol{L}(\boldsymbol{p})$ is *nonsingular* with elements given by

$$\begin{aligned} L^1{}_1 &= \sqrt{A^1{}_1}, \\ L^2{}_1 &= A^2{}_1/L^1{}_1, \\ L^2{}_2 &= \sqrt{A^2{}_2 - (L^2{}_1)^2}, \\ L^3{}_1 &= A^3{}_1/L^1{}_1, \\ L^3{}_2 &= (A^3{}_2 - L^3{}_1 L^2{}_1)/L^2{}_2, \\ L^3{}_3 &= \sqrt{A^3{}_3 - (L^3{}_1)^2 - (L^3{}_2)^2}. \end{aligned} \tag{24}$$

The inverse matrix $\boldsymbol{L}(\boldsymbol{p})^{-1}$ reads

$$\boldsymbol{L}^{-1} = \frac{1}{\det(\boldsymbol{L})} \begin{pmatrix} L^2{}_2 L^3{}_3 & -L^2{}_1 L^3{}_3 & L^2{}_1 L^3{}_2 - L^3{}_1 L^2{}_2 \\ 0 & L^3{}_3 L^1{}_1 & -L^3{}_2 L^1{}_1 \\ 0 & 0 & L^1{}_1 L^2{}_2 \end{pmatrix}, \tag{25a}$$

where

$$\det(\boldsymbol{L}) = L^1{}_1 L^2{}_2 L^3{}_3. \tag{25b}$$

Let us next introduce a stochastic vector variable $\boldsymbol{y}(t) = (y^i(t))$ by

$$y^i \equiv L^i{}_j w^j. \tag{26}$$

Then, by taking into account that for $\boldsymbol{w}^\top \equiv (w_i)$ and $\boldsymbol{y}^\top \equiv (y_i)$ the relation

$$\boldsymbol{w}^\top \boldsymbol{A} \boldsymbol{w} = \boldsymbol{w}^\top \boldsymbol{L}^\top \boldsymbol{L} \boldsymbol{w} = (\boldsymbol{L}\boldsymbol{w})^\top \boldsymbol{L}\boldsymbol{w} = \boldsymbol{y}^\top \boldsymbol{y} \tag{27}$$

holds, we can rewrite the Langevin equation (20) in the form

$$\mathrm{d}\boldsymbol{p} = -\nu\,\boldsymbol{p}\,\mathrm{d}t + \boldsymbol{L}(\boldsymbol{p})^{-1}\boldsymbol{y}, \tag{28a}$$

where $\boldsymbol{y}(t)$ is distributed according to the momentum-independent Gaussian probability density

$$\mathcal{P}_y^3[\boldsymbol{y}] \;=\; \left(\frac{1}{4\pi D\,\mathrm{d}t}\right)^{3/2} \exp\left[-\frac{y_i\,y^i}{4D\,\mathrm{d}t}\right].$$ (28b)

Put differently, because the inverse matrix $\boldsymbol{L}(\boldsymbol{p})^{-1}$ depends on the momentum coordinate $\boldsymbol{p}$, the random vector $\boldsymbol{y}(t)$ enters in relativistic Langevin equation (28a) as an ordinary 'multiplicative' Gaussian white noise process with noise strength D.

As is well known, for multiplicative stochastic processes of the type (28) the Langevin equation *per se* does *not* uniquely determine a corresponding Fokker-Planck equation [24, 25]. In the following subsections, we shall discuss the three most popular choices of resulting Fokker-Planck equations for a Langevin equation of the form (28). These choices are rooted in the different proposals put forward by Ito [24, 25, 26, 27], by Stratonovich and Fisk [25, 28, 29, 30, 31], and by Hänggi [32, 33, 34] and Klimontovich [35], respectively. All three approaches have in common that the related Fokker-Planck equation can be written as a continuity equation (conservation of probability) of the form [34]

$$\frac{\partial}{\partial t}f(t,\boldsymbol{p}) + \frac{\partial}{\partial p^i}j^i(t,\boldsymbol{p}) = 0,$$ (29)

but with distinct different expressions for the probability current $\boldsymbol{j}(t,\boldsymbol{p})$. It is worthwhile to anticipate here that only the Hänggi-Klimontovich approach (see Sec. III B 3) yields a stationary distribution, which can be identified with Jüttner's relativistic Maxwell distribution [36].

1. Ito approach

According to Ito's interpretation of the Langevin equation (28a), the coefficient matrix before $\boldsymbol{y}(t)$ is to be evaluated at the lower boundary of the interval $[t, t+\mathrm{d}t]$, i.e.,

$$\boldsymbol{L}(\boldsymbol{p})^{-1} = \boldsymbol{L}\big(\boldsymbol{p}(t)\big)^{-1}.$$ (30)

Ito's choice is also known as the pre-point discretization rule [24, 25, 26, 27] and leads to the following explicit expression for the current

$$j_{\mathrm{I}}^i(t,\boldsymbol{p}) = -\left\{\nu p^i f + D\frac{\partial}{\partial p_j}\big[\boldsymbol{L}^{-1}(\boldsymbol{L}^{-1})^\top\big]{}^i{}_j\, f\right\}.$$ (31)

In view of the identity

$$\boldsymbol{L}^{-1}(\boldsymbol{L}^{-1})^\top = \boldsymbol{L}^{-1}(\boldsymbol{L}^\top)^{-1} = (\boldsymbol{L}^\top\boldsymbol{L})^{-1} = \boldsymbol{A}^{-1},$$ (32)

Eq. (31) can be rewritten more conveniently as

$$j_{\mathrm{I}}^i(t,\boldsymbol{p}) = -\left[\nu p^i f + D\frac{\partial}{\partial p_j}(A^{-1})^i{}_j f\right],$$ (33)

where the matrix $\boldsymbol{A}(\boldsymbol{p})^{-1}$ is given in Eq. (22). The related relativistic Fokker-Planck equation is obtained by inserting this current into the conservation law (29). As elucidated in the Appendix A, the stationary solution of the resulting Fokker-Planck equation reads

$$f_{\mathrm{I}}(\boldsymbol{p}) = C_{\mathrm{I}}\left(1 + \frac{\boldsymbol{p}^2}{m^2c^2}\right)^{-3/2} \exp\left(-\chi\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2c^2}}\right),$$ (34)

which contains in the prefactor an explicit (non-Maxwell like) dependence on velocity and where C_{I} is a normalization constant. In the solution (34), the dimensionless parameter

$$\chi \equiv \frac{\nu m^2 c^2}{D}$$ (35)

is related to the scalar temperature T of the heat bath via the Einstein relation

$$k_{\mathrm{B}}T \equiv \frac{mc^2}{\chi} = \frac{D}{m\nu}.$$ (36)

Thus, the quantity $\chi = mc^2/(k_{\mathrm{B}}T)$ gives the ratio between rest energy and thermal energy of the Brownian particle. It should be mentioned that we used in paper I the notation β instead of χ. However, in order to avoid a possible confusion with the commonly used abbreviation $\beta = (k_{\mathrm{B}}T)^{-1}$ we opted here for this slight change of notation.

2. Stratonovich-Fisk approach

According to the Stratonovich-Fisk prescription, the coefficient matrix before $\boldsymbol{y}(t)$ in (28a) is to be evaluated with the mid-point discretization rule, i.e.,

$$\boldsymbol{L}(\boldsymbol{p})^{-1} = \boldsymbol{L}\left(\frac{\boldsymbol{p}(t) + \boldsymbol{p}(t + \mathrm{d}t)}{2}\right)^{-1}. \tag{37}$$

This choice [25, 28, 29, 30] leads to a different expression for the current [25, 28, 29, 30]:

$$j_{\mathrm{SF}}^i(t, \boldsymbol{p}) = -\left\{\nu p^i f + D\,(L^{-1})^i{}_k \frac{\partial}{\partial p_j}\left[(L^{-1})^\top\right]^k{}_j\,f\right\}. \tag{38}$$

The explicit stationary solution of Stratonovich's Fokker-Planck equation reads (see Appendix A 2)

$$f_{\mathrm{SF}}(\boldsymbol{p}) = C_{\mathrm{SF}}\left(1 + \frac{\boldsymbol{p}^2}{m^2 c^2}\right)^{-3/4}\exp\left(-\chi\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2 c^2}}\right), \tag{39}$$

and thus differs from Eq. (34) in the power of the velocity-dependent prefactor.

3. Hänggi-Klimontovich approach

Ultimately, let us next consider the Hänggi-Klimontovich (HK) stochastic integral interpretation, sometimes referred to as the transport form [32, 33, 34] or also as the kinetic form [35]. According to this interpretation, the coefficient matrix in front of $\boldsymbol{y}(t)$ in (28a) is to be evaluated at the upper boundary of the interval $[t, t + \mathrm{d}t]$; i.e., within the post-point discretization we set

$$\boldsymbol{L}(\boldsymbol{p})^{-1} = \boldsymbol{L}\big(\boldsymbol{p}(t + \mathrm{d}t)\big)^{-1}. \tag{40}$$

This choice leads to the following expression for the current [33, 34, 35]

$$j_{\mathrm{HK}}^i(t, \boldsymbol{p}) = -\left[\nu p^i f + D\,(A^{-1})^i{}_j \frac{\partial}{\partial p_j} f\right], \tag{41}$$

and the stationary solution of the related FPE reads (see Appendix Eq. (A 3))

$$f_{\mathrm{HK}}(\boldsymbol{p}) = C_{\mathrm{HK}}\exp\left(-\chi\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2 c^2}}\right). \tag{42a}$$

Note that this solution contains no velocity dependence in the prefactor. Using the temperature definition in (36) and the relativistic kinetic energy formula $E = \sqrt{m^2 c^4 + \boldsymbol{p}^2 c^2}$, we can recast (42a) as

$$f_{\mathrm{HK}}(\boldsymbol{p}) = C_{\mathrm{HK}}\exp\left(-\frac{E}{k_{\mathrm{B}}T}\right). \tag{42b}$$

The normalization constant is given by [11]

$$C_{\mathrm{HK}}^{-1} = \int \mathrm{d}^3 \boldsymbol{p}\,\exp\left(-\chi\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2 c^2}}\right) = 4\pi \int_0^\infty \mathrm{d}p\,p^2\,\exp\left(-\chi\sqrt{1 + \frac{p^2}{m^2 c^2}}\right) = 4\pi(mc)^3 \frac{K_2(\chi)}{\chi}, \tag{42c}$$

where $K_2(\chi)$ denotes the modified Bessel function of the second kind. The distribution function (42) is known as the *relativistic Maxwell distribution*. It was first derived by F. Jüttner [36] in 1911, when he investigated the velocity distribution of non-interacting relativistic gas particles (see also [37]). By comparing (34), (39) and (42a) one readily observes that the stationary solutions $f_{\mathrm{I/SF}}$ differ from the Jüttner function f_{HK} through additional $\boldsymbol{p}$-dependent prefactors. The quantitative difference between these three stationary solutions becomes significant in the relativistic limit, corresponding to a low rest energy-to-temperature ratio χ.

As already mentioned in paper I, for the one-dimensional case the relativistic Maxwell distribution has also been obtained by Schay, see Eq. (3.63) and (3.64) in Ref. [9], who studied relativistic diffusions employing a transfer probability method. Moreover, the distribution (42) can also be derived in the framework of the relativistic kinetic theory [22]. This suggests that the HK-discretization rule is physically preferable, if one wishes to use the above Langevin equations for numerical simulations of relativistic kinetic processes. In general, however, additional information about the microscopic structure of the heat bath is required, in order to decide which discretization rule is physically reasonable (see, e.g., the discussion of Ito-Stratonovich dilemma in the context of 'internal/external' noise as given in Chap. IX.5 of van Kampen's textbook [24]).

Finally, we still note that the related velocity probability density functions $\phi_{\mathrm{I/SF/HK}}(\boldsymbol{v})$ are obtained by applying the general transformation law

$$\phi(\boldsymbol{v}) \equiv f(\boldsymbol{p}(\boldsymbol{v})) \left| \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} \right|, \tag{43}$$

where, as usual,

$$\boldsymbol{p}(\boldsymbol{v}) = \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2/c^2}}.$$

The determinant factor

$$\left| \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{v}} \right| = m^3 \left(1 - \frac{\boldsymbol{v}^2}{c^2} \right)^{-5/2}, \tag{44}$$

appearing on the rhs. of Eq. (43), ensures that the velocity density functions $\phi_{\mathrm{I/SF/HK}}(\boldsymbol{v})$ drop to zero for $\boldsymbol{v}^2 \to c^2$. For example, in the case of the Jüttner function (42) one explicitly obtains

$$\phi_{\mathrm{HK}}(\boldsymbol{v}) = \frac{\chi}{4\pi c^3 K_2(\chi)} \exp\left(-\frac{\chi}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \right) \left(1 - \frac{\boldsymbol{v}^2}{c^2} \right)^{-5/2}. \tag{45}$$

C.   The inclusion of external force fields

The preceding section concentrated on Fokker-Planck equations for the momentum density $f(t, \boldsymbol{p})$. In this part we shall discuss the corresponding resulting equations for the one-particle phase space density $f(t, \boldsymbol{p}, \boldsymbol{x})$. As before, we refer to the coordinates of the laboratory frame Σ_0, in which the heat bath is at rest.

If an external force field is present, then Eq. (28a) generalizes to

$$\mathrm{d}\boldsymbol{p} = (\gamma^{-1}\boldsymbol{K} - \nu\,\boldsymbol{p})\,\mathrm{d}t + \boldsymbol{L}(\boldsymbol{p})^{-1}\boldsymbol{y}, \tag{46}$$

where $\boldsymbol{y}$ is distributed according to the momentum-independent Gaussian density from Eq. (28b). The related relativistic Fokker-Planck equation for the full phase space density $f(t, \boldsymbol{p}, \boldsymbol{x})$ thus reads

$$\frac{\partial}{\partial t}f + \frac{p^i}{m\gamma}\frac{\partial}{\partial x^i}f + \frac{\partial}{\partial p^i}\left(\frac{K^i}{\gamma}f \right) = -\frac{\partial}{\partial p^i}j^i_{\mathrm{I/SF/HK}} \tag{47}$$

where the current densities $\boldsymbol{j}_{\mathrm{I/SF/HK}}(t, \boldsymbol{p}, \boldsymbol{x})$ are obtained by replacing $f(t, \boldsymbol{p})$ with $f(t, \boldsymbol{p}, \boldsymbol{x})$ in the above expressions for $\boldsymbol{j}_{\mathrm{I/SF/HK}}(t, \boldsymbol{p})$, respectively. In the limiting case that $\nu \to 0$, $D \to 0$, the rhs. of Eq. (47) becomes equal to zero and one regains the relativistic Liousville equation or, equivalently, the collision-less Boltzmann-Vlasov equation [22, 42].

As a particular example, let us consider a relativistic Brownian particle with rest charge q, being subject to a static electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$, measured with respect to Σ_0. Then, in addition to the stochastic interaction with the heat bath, the deterministic Lorentz-force [43]

$$\frac{\boldsymbol{K}(\boldsymbol{p}, \boldsymbol{x})}{\gamma} = q\left[\boldsymbol{E}(\boldsymbol{x}) + \frac{\boldsymbol{p}}{m\gamma c} \times \boldsymbol{B}(\boldsymbol{x}) \right] \tag{48}$$

is acting on the particle, where '×' denotes the exterior vector product. For simplicity, let us confine ourselves to the HK-form of the FPE and let $\boldsymbol{E}(\boldsymbol{x}) = \nabla\Phi(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x}) \equiv \boldsymbol{0}$ in the laboratory system. In this case, the stationary solution of Eq. (47) emerges as

$$f_{\mathrm{HK}}(\boldsymbol{p}, \boldsymbol{x}) = C_{\mathrm{HK}} \exp\left\{-\chi\left[\gamma(\boldsymbol{p}) - \frac{q\Phi(\boldsymbol{x})}{mc^2}\right]\right\} \tag{49a}$$

$$= C_{\mathrm{HK}} \exp\left[-\frac{E(\boldsymbol{p}) - q\Phi(\boldsymbol{x})}{k_{\mathrm{B}}T}\right], \tag{49b}$$

where C_{HK} is a normalization constant, and $E(\boldsymbol{p}) = (m^2c^4 + \boldsymbol{p}^2c^2)^{1/2}$ denotes the relativistic (kinetic) energy. It is reassuring to see that the solution (49) just represents the relativistic generalization of the nonrelativistic Maxwell-Boltzmann distribution from Eq. (18a).

IV. NUMERICAL INVESTIGATIONS

The numerical results presented in this section were obtained on the basis of the relativistic Langevin equations (28), which hold in the laboratory frame Σ_0. For simplicity, we confined ourselves to considering free Brownian particles (i.e. $\boldsymbol{K} = \boldsymbol{0}$) and employed the Ito-discretization scheme with fixed time step dt, see Sec. III B 1. In all simulation we have used an ensemble size of $N = 1000$ particles. A characteristic unit system was fixed by setting $m = c = \nu = 1$. Formally, this corresponds to considering re-scaled dimensionless quantities, such as $\tilde{p}^i = p^i/(mc)$, $\tilde{x}^i = x^i\nu/c$, $\tilde{t} = t\nu$, $\tilde{v}^i = v^i/c$ etc.. The simulation time-step was always chosen as $dt = 0.001\nu^{-1}$, and the Gaussian random variables $y^i(t)$ were generated with the pseudo-random number generator of Mathematica [44].

A. Distribution functions

In the simulations we have numerically measured the stationary cumulative distribution function F of the absolute velocity values $v \equiv \sqrt{v_i v^i}$ in the laboratory frame Σ_0. Given, e.g., a spherically symmetric probability density $\phi(\boldsymbol{v}) \equiv \tilde{\phi}(v)$ with normalization

$$1 = \int d^3\boldsymbol{v}\ \phi(\boldsymbol{v}) = 4\pi \int_0^1 dv\ v^2\tilde{\phi}(v), \tag{50}$$

the respective cumulative distribution function is defined by

$$F(v) = 4\pi \int_0^v du\ u^2\ \tilde{\phi}(u). \tag{51}$$

In order to obtain $F(v)$ from numerical simulations, one simply measures the relative fraction of particles with absolute velocities in the interval $[0, v)$. Figure 1 depicts the numerically determined stationary distribution functions, taken at time $t = 100\nu^{-1}$ and also the corresponding analytical curves $F_{\mathrm{I/SF/HK}}(v)$. The latter were obtained by numerically integrating the formula (51), using the three different stationary density functions $\phi_{\mathrm{I/SF/HK}}(\boldsymbol{v}) = \tilde{\phi}_{\mathrm{I/SF/HK}}(v)$ found in Sec. III.

As one can see in diagram 1 (a), for low temperature values, corresponding to $\chi \gg 1$, the three stationary distribution functions approach each other, since they all converge to the nonrelativistic Maxwell distribution in the limit $\chi \to \infty$. For high temperatures, corresponding to $\chi \leq 1$, the stationary solutions exhibit significant quantitative differences, cf. Fig. 1 (b). Since our simulations are based on an Ito-discretization scheme, the numerical data points agree best with the Ito solution (solid line). Similar to the (1+1)-dimensional case [23], the quality of the fit remains satisfactory over several orders of magnitudes of the parameter χ. One may therefore conclude that the numerical simulations of Langevin equations provide a useful tool for studying relativistic Brownian motions in (1+3) dimensions. It should, however, be stressed again, that the appropriate choice of the discretization rule is particularly important with regard to potential applications to physical situations.

Furthermore, Fig. 2 illustrates the influence of the number of spatial dimensions on the occurrence of relativistic effects. In the two diagrams we depicted the ratio $F_{\mathrm{HK}}(v)/F_{\mathrm{M}}(v)$ for three different values of the characteristic parameter χ, with $F_{\mathrm{HK/M}}(v)$ denoting the cumulative velocity distribution function of the relativistic and nonrelativistic Maxwell distribution, respectively. Figure 2 (a) corresponds to the case of (1+3)-dimensional free Brownian motions, while Fig. 2 (b) refers to the (1+1)-dimensional case, discussed in paper I. The comparison of the two diagrams reveals that relativistic effects become significantly enhanced at lower temperatures (or larger values of χ, respectively), if the Brownian particle moves in (1+3)-dimensions.
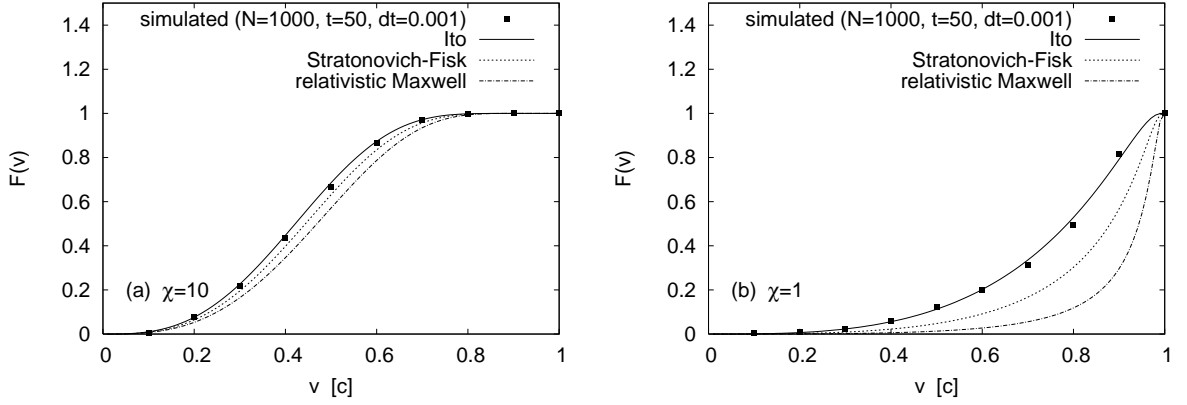
FIG. 1: These diagrams depict a comparison among the numerical Ito-prrescription and the three analytical results for the corresponding stationary cumulative distribution function $F(v)$ in the laboratory frame Σ_0. (a) In the nonrelativistic limit $\chi \gg 1$ the stationary solutions of the three different FPE are nearly indistinguishable. (b) In the strong relativistic limit case $\chi \leq 1$, however, the stationary solutions exhibit deviations from each other. Because our simulations are based on an Ito-discretization scheme, the numerical data points do agree with the Ito solution (solid line).
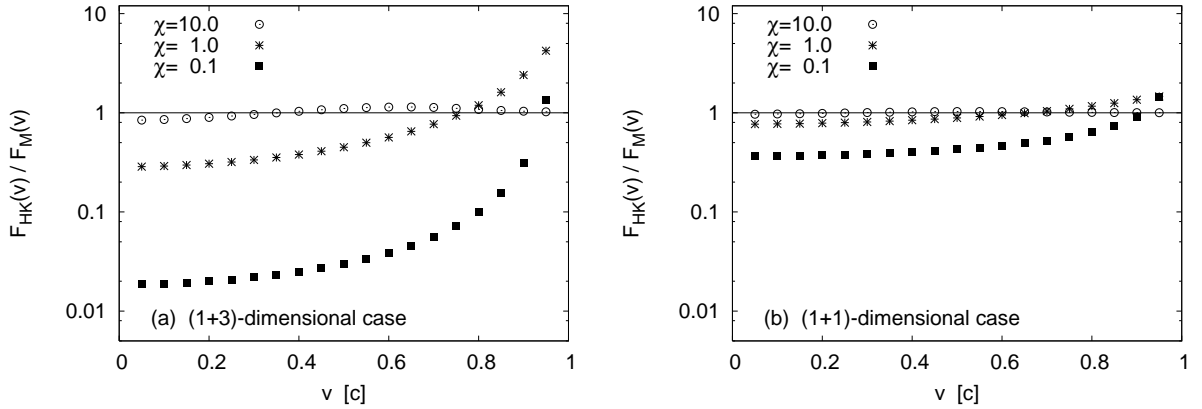

FIG. 2: The diagrams show the ratio $F_{\mathrm{HK}}(v)/F_{\mathrm{M}}(v)$ for three different values of the characteristic parameter $\chi = mc^2/(k_{\mathrm{B}}T)$, where F_{M} and F_{HK} denote the cumulative distribution functions of absolute velocity for the nonrelativistic and relativistic Maxwell distributions, respectively. Note that the deviation between the nonrelativistic and the corresponding relativistic distribution functions are more pronounced (at fixed value of χ) for the (1+3)-dimensional situation, i.e. in Fig. (a). Therefore, compared with the (1+1)-dimensional case in Fig. (b), relativistic effects become detectable best at finite, lower temperatures, if the Brownian dynamics proceeds in (1+3)-dimensions.

B. Mean square displacement

Next, the spatial mean square displacement of the free relativistic Brownian motion is investigated. Since this quantity is easily accessible in experiments, it has played an important role in the verification of the nonrelativistic theory.

As before, we consider an ensemble of N independent, free Brownian particles with coordinates $\boldsymbol{x}_{(i)}(t)$ in Σ_0 and initial conditions $\boldsymbol{x}_{(i)}(0) = \boldsymbol{0}, \boldsymbol{v}_{(i)}(0) = \boldsymbol{0}$ for $i = 1, 2, \ldots, N$. The position mean value is obtained as

$$\overline{\boldsymbol{x}}(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}_{(i)}(t), \tag{52}$$

and the related second moment is given by

$$\overline{\boldsymbol{x}^2}(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \left[\boldsymbol{x}_{(i)}(t) \right]^2. \tag{53}$$

The empirical mean square displacement can then be defined as follows

$$\sigma^2(t) \equiv \overline{\boldsymbol{x}^2}(t) - [\overline{\boldsymbol{x}}(t)]^2 \,. \tag{54}$$

Important results of the nonrelativistic theory of the three-dimensional Brownian motion read

$$\lim_{t \to +\infty} \overline{\boldsymbol{x}}(t) \;\to\; \boldsymbol{0}, \tag{55a}$$

$$\lim_{t \to +\infty} \frac{\sigma^2(t)}{t} \;\to\; 3 \cdot 2D^x, \tag{55b}$$

where the constant

$$D^x = \frac{k_{\mathrm{B}} T}{m\nu} = \frac{D}{m^2 \nu^2} \tag{56}$$

is the nonrelativistic coefficient of diffusion in coordinate space [not to be confused with noise strength $D = kT/(m\nu)$].

It is therefore interesting to consider the asymptotic behavior of the quantity $\sigma^2(t)/t$ for relativistic Brownian motions. In Fig. 3 (a) one can see the corresponding numerical results for different values of χ, evaluated on the basis of the Ito-scheme from Sec. III B 1. As evident from this diagram, for each value of χ, the quantity $\sigma^2(t)/t$ converges to a constant value. This means that (with respect to the laboratory frame Σ_0) the asymptotic mean square displacement of the free relativistic Brownian motions is again *normal*, i.e. it increases linearly with t. For completeness, we mention that according to our simulations the asymptotic relation (55a) holds in the relativistic case, too.
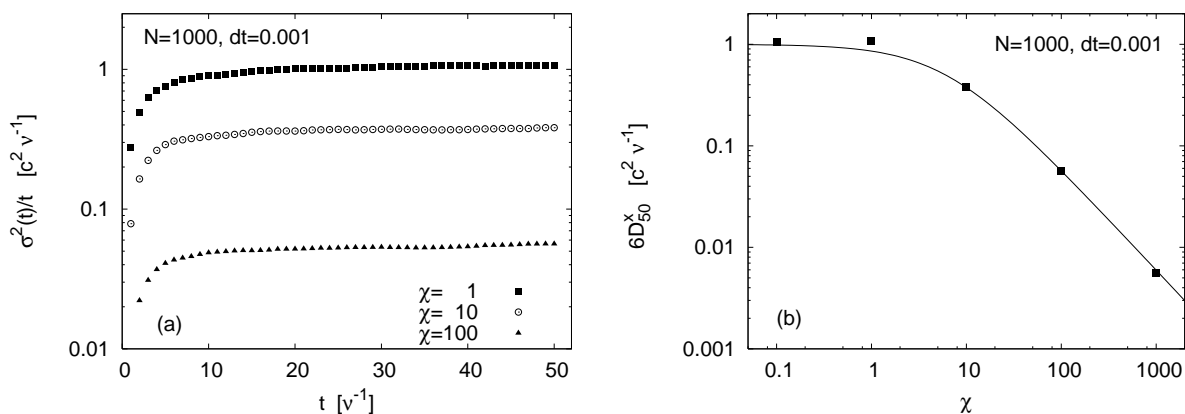

FIG. 3: (a) Mean square displacement, divided by elapsed time t, as numerically calculated for different χ-values in the laboratory frame Σ_0 (rest frame of the heat bath). As evident from this diagram, for the relativistic Brownian motion the related asymptotic mean square displacement grows linearly with t. (b) The coordinate space diffusion constant $D_{50}^x(\chi)$ was numerically determined at time $t = 50\nu^{-1}$. The solid line corresponds to the empirical fitting formula $D^x(\chi) = c^2\nu^{-1}(\chi+6)^{-1}$, which reduces to the classical nonrelativistic result $D^x \simeq c^2/(\nu\chi) = kT/(m\nu)$ for $\chi \gg 6$.

In spite of these similarities between nonrelativistic and relativistic theory, an essential difference consists in the explicit temperature dependence of the limit value $6D^x$. As illustrated in Fig. 3 (b), the numerical limit values $6D_{50}^x$, measured at time $t = 50\nu^{-1}$, are reasonably well fitted by the formula

$$D^x = \frac{c^2}{\nu(\chi+6)}, \tag{57}$$

which reduces to the nonrelativistic result (56) in the limit case $\chi \gg 6$ (low-temperature limit case). It remains as an open problem for the future to find an analytic expression for the relativistic diffusion constant D^x. Note also that, with the position being the integral over the velocity degree of freedom, the different discretization rules do not impact the asymptotic (long time) result for the position diffusion coefficient.

V. SUMMARY

The challenge of this work has been to extend our previous work [23] on (1+1)-dimensional relativistic Brownian motions to the (1+3)-dimensional case. To this end, we have introduced in Sec. II a (1+3)-dimensional relativistic

generalization of the nonrelativistic Langevin equations (LE). Analogous to the nonrelativistic Ornstein-Uhlenbeck theory of Brownian motion [3, 24, 25, 45], it is implicitly assumed that the heat bath, (which causes the stochastic motions of the particle) can be regarded as an isotropic, homogeneous fluid. Based on this assumption, the relativistic equations of motions are constructed such that they reduce to the well-known nonrelativistic LE in the limit case $c \to \infty$.

In our relativistic version of the LE, the viscous friction between Brownian particle and heat bath is modelled by a friction tensor $\nu_{\alpha\beta}$, exhibiting the same formal structure as the pressure tensor of a perfect fluid [38]. In particular, this means that the friction tensor is uniquely determined by the value of the (scalar) viscous friction coefficient ν, as measured in the instantaneous rest frame of the particle. Similarly, the amplitude of the stochastic force is also governed by a single parameter D, specifying the Gaussian fluctuations of the heat bath, as seen in the instantaneous rest frame of the particle.

In Sec. II B we have rewritten the relativistic LE in laboratory coordinates, corresponding to a specific class of Lorentz frames, in which the heat bath is assumed to be at rest at all times. Further, it was shown that the relativistic equations can be recast such that they contain ordinary 'multiplicative' Gaussian white noise. Analogous to nonrelativistic stochastic processes with 'multiplicative' noise, this leads to an ambiguity regarding the interpretation of the stochastic differential equation; i.e., different discretization rules yield different Fokker-Planck equations (Sec. III). Similar to the previous paper [23], we concentrated here on the three most popular discretization schemes, corresponding to the pre-point rule proposed by Ito [26, 27], the mid-point rule by Stratonovich and Fisk [28, 29, 30, 31], and the post-point rule of Klimontovich and Hänggi [32, 33, 34, 35]. It was then shown in Sec. III C that only the latter prescription, i.e. the post-point discretization scheme, yields a Fokker-Planck equation, whose stationary solution coincides with the relativistic Maxwell-Boltzmann distribution, as e.g. known from the work of Jüttner [36], Schay [9] and de Groot *et al.* [22].

In Sec. IV we presented several numerical results (based on the Ito scheme), including numerically obtained velocity distribution functions and furthermore, the mean square displacement of free Brownian particles. According to our findings, the relativistic mean square displacement grows linearly with the laboratory coordinate time; compared with nonrelativistic diffusions, however, the temperature dependence of the spatial diffusion constant becomes more intricate.

Finally, we would like to emphasize that, so far, our approach of constructing a relativistic Brownian motion dynamics is merely based on the condition that – given the (a priori prescribed) stochastic force of the heat bath – the relativistic LE has to converge to the well-known nonrelativistic dynamical equation, if the Brownian particle moves sufficiently slow. In particular, there presently remains the open (and seemingly very difficult) problem of how to tackle in a relativistically consistent manner the dynamics of all those particles that constitute the heat bath (including their coupling to the relativistic Brownian particle).

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APPENDIX A: STATIONARY SOLUTIONS OF RELATIVISTIC FOKKER-PLANCK EQUATIONS

We seek the stationary solutions $f(\boldsymbol{p})$ of the FPE (29), which lead to vanishing currents

$$j^i_{\mathrm{I/SF/HK}}(\boldsymbol{p}) \equiv 0. \tag{A1}$$

We use the following Ansatz

$$f(\boldsymbol{p}) = C\gamma^{-\alpha}\exp(-\chi\gamma), \tag{A2}$$

where $C > 0$ is a normalization constant and

$$\gamma = \left(1 + \frac{p_i p^i}{m^2 c^2}\right)^{1/2}. \tag{A3}$$

The parameters α and χ have to be determined from the condition (A1). Therefore we have the following partial derivatives

$$\frac{\partial\gamma}{\partial p^j} = \frac{p_j}{\gamma m^2 c^2}, \tag{A4a}$$

$$\frac{\partial f}{\partial p^j} = \frac{-p_j}{\gamma m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right)f, \tag{A4b}$$

and furthermore the divergence

$$\frac{\partial}{\partial p^j}(A^{-1})^{ij} = \frac{\partial}{\partial p^j}\left[\left(\delta^{ij} + \frac{p^i p^j}{m^2 c^2}\right)\frac{1}{\gamma}\right] = \frac{3p^i}{\gamma m^2 c^2}. \tag{A4c}$$

1. Ito current

For the Ito current $\boldsymbol{j}_{\mathrm{I}}$ from (33), the condition (A1) yields

$$
\begin{aligned}
0 \;\equiv\; & -j_{\mathrm{I}}^i(\boldsymbol{p}) \\
= \; & \nu p^i f + Df \frac{\partial}{\partial p^j}(A^{-1})^{ij} + D\,(A^{-1})^{ij}\frac{\partial f}{\partial p^j} \\
= \; & \nu p^i f + Df \frac{3p^i}{\gamma m^2 c^2} + D\left(\delta^{ij} + \frac{p^i p^j}{m^2 c^2}\right)\frac{-p_j}{\gamma^2 m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right) f \\
= \; & p^i f\left[\nu + \frac{3D}{\gamma m^2 c^2} - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right)\right],
\end{aligned}
\tag{A5}
$$

which is fulfilled for $\alpha = 3$ and $\chi = \nu m^2 c^2 / D$.

2. Stratonovich-Fisk current

For the Stratonovich-Fisk current $\boldsymbol{j}_{\mathrm{SF}}$ from (38), the condition (A1) becomes

$$
\begin{aligned}
0 \;\equiv\; & -j_{\mathrm{SF}}^i(\boldsymbol{p}) \\
= \; & \nu p^i f + D\,(L^{-1})^i{}_k \frac{\partial}{\partial p_j}\left[(L^{-1})^\top\right]^k{}_j\, f \\
= \; & \nu p^i f + D\,(L^{-1})^i{}_k\, f\frac{\partial}{\partial p_j}\left[(L^{-1})^\top\right]^k{}_j + D\,(A^{-1})^i{}_j\frac{\partial f}{\partial p_j},
\end{aligned}
\tag{A6}
$$

where $\boldsymbol{L}(\boldsymbol{p})^{-1}$ has been given in (25). A lengthy, though straightforward calculation shows that

$$
(L^{-1})^i{}_k \frac{\partial}{\partial p_j}\left[(L^{-1})^\top\right]^k{}_j = \frac{3p^i}{2\gamma m^2 c^2}.
\tag{A7}
$$

Inserting this into (A6), one finds

$$
\begin{aligned}
0 \;=\; & \nu p^i f + Df \frac{3p^i}{2\gamma m^2 c^2} + D\left(\delta^{ij} + \frac{p^i p^j}{m^2 c^2}\right)\frac{-p_j}{\gamma^2 m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right) f \\
= \; & p^i f\left[\nu + \frac{3D}{2\gamma m^2 c^2} - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right)\right],
\end{aligned}
\tag{A8}
$$

which is fulfilled for $\alpha = 3/2$ and $\chi = \nu m^2 c^2 / D$.

3. HK-current

For the HK-current $\boldsymbol{j}_{\mathrm{K}}$ from (41), the condition (A1) yields

$$
\begin{aligned}
0 \;\equiv\; & -j_{\mathrm{HK}}^i(\boldsymbol{p}) \\
= \; & \nu p^i f + D\,(A^{-1})^{ij}\frac{\partial f}{\partial p^j} \\
= \; & \nu p^i f + D\left(\delta^{ij} + \frac{p^i p^j}{m^2 c^2}\right)\frac{-p_j}{\gamma^2 m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right) f \\
= \; & p^i f\left[\nu - \frac{D}{m^2 c^2}\left(\frac{\alpha}{\gamma} + \chi\right)\right],
\end{aligned}
\tag{A9}
$$

which is fulfilled for $\alpha = 0$ and $\chi = \nu m^2 c^2 / D$.