Towards a Topological Classification of Bilinear Control Systems*

Victor Ayala
Departamento de Matematicas, Universidad Catolica del Norte,
Casilla 1280, Antofagasta, Chile

Fritz Colonius
Institut für Mathematik, Universität Augsburg,
86135 Augsburg, Germany

Wolfgang Kliemann
Department of Mathematics, Iowa State University,
Ames Iowa 50011, U.S.A.

May 15, 2006

Abstract

In the last twenty years, the problem of classifying control systems allowing state and feedback transformations has been extensively studied. This paper approaches the classification of bilinear control systems from a topological point of view, as is common in the theory of dynamical systems. As a first step towards a topological classification, we study conjugacy and equivalence for the flows associated with bilinear control systems. This allows us to characterize the controllability and (exponential) stability behavior of bilinear systems, using Morse decompositions on the projective bundle and the associated Morse spectrum.

1 Introduction

In the last twenty years, the problem to classify control systems allowing state and feedback transformations has been extensively studied. In particular, we mention the approach due to Kang and Krener [6] based on Taylor expansions and more geometric approaches to equivalence for (nonlinear) control systems that are based on equivalent distributions defined by a system on the tangent bundle. This point of view allows for the redefinition of controls (via feedback) and requires that the control range is a linear, unbounded space (see e.g. the recent survey by Respondek and Tall [7]). This paper approaches the classification of bilinear control systems from a topological point of view, as is common in the theory of dynamical systems, see, e.g., [5] and [8]; see also Baratchart, Chyba and Pomet [3] for a Grobman-Hartman result in the context of control systems. But while one is interested in trajectory-wise (equivalence and conjugacy) results in dynamical systems, for control systems such a concept has to be complemented by an analysis of the key concepts of controllability and stability/stabilization.

*Partially supported by Proyecto FONDECYT n^o 1020439 and Proyecto FONDECYT de Incentivo a la Cooperación Internacional n^o 7020439.

In this paper we take a few steps towards a topological classification of bilinear control systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = A(u(t))x(t) = [A_0 + \sum_{i=1}^{m} u_i(t)A_i]x(t) \quad (1)$$

with control range $U \subset \mathbb{R}^m$, which we assume to be a compact and convex set with $0 \in \mathrm{int}U$. The (open loop) control functions u are in $\mathcal{U} := \{u : \mathbb{R} \to U \text{ for all } t \in \mathbb{R}, \text{locally integrable}\}$. We aim at the following properties of a bilinear control system:

- controllability in $\mathbb{R}^d$ (and on the projective space $\mathbb{P}^{d-1}$),
- stability (and feedback stabilizability) at the origin,
- robust stability for all $u \in \mathcal{U}$, if the space $\mathcal{U}$ is interpreted as a space of time varying perturbations,
- spaces of equal exponential behavior, since these form the basis of results on invariant manifolds and Grobman-Hartman type theorems, if a bilinear system is obtained via linearization of a nonlinear control system at a fixed point.

As it turns out, see e.g. [4], the key concepts for these four issues are controllability on $\mathbb{P}^{d-1}$, the spectrum of (1), and the dimension of the spectral subbundles. This paper presents some results on the spectrum and the spectral subbundles.

2 Conjugacy and equivalence for bilinear control systems in $\mathbb{R}^d$

We denote by $\mathbf{B}(d, m, U)$ the set of bilinear control systems $\Sigma = (A_0, ..., A_m, U)$ in $\mathbb{R}^d$ with m controls and control range U. Associated with a control system is a dynamical system Φ (the control flow) in the following way, compare [4]:

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d, \ \Phi(t, u, x) = (\theta(t, u), \varphi(t, x, u)), \quad (2)$$

where $\varphi(t, x, u)$ is the trajectory corresponding to the control function u and the initial value x and we denote the shift on the base $\mathcal{U}$ by $\theta(t, u(\cdot)) = u(t + \cdot)$.

Definition 1 *For $i = 1, 2$ let $\Sigma_i \in \mathbf{B}(d, m, U_i)$, be bilinear control systems and denote by $\Phi_i = (\theta_i, \varphi_i) : \mathbb{R} \times \mathcal{U}_i \times \mathbb{R}^d \to \mathcal{U}_i \times \mathbb{R}^d$ the associated control flows. We say that Φ_1 and Φ_2 are*
(i) skew conjugate if there exists a skew homeomorphism $h = (f, g) : \mathcal{U}_1 \times \mathbb{R}^d \to \mathcal{U}_2 \times \mathbb{R}^d$ such that $h(\Phi_1(t, u, x)) = \Phi_2(t, h(u, x))$, i.e., $f : \mathcal{U}_1 \to \mathcal{U}_2$ and $g : \mathcal{U}_1 \times \mathbb{R}^d \to \mathbb{R}^d$ with

$$f(\theta_1(t, u)) = \theta_2(t, f(u)) \text{ for all } (t, u) \in \mathbb{R} \times \mathcal{U}_1, \text{ and}$$
$$g(\theta_1(t, u), \varphi_1(t, x, u)) = \varphi_2(t, g(u, x), f(u)) \text{ for all } (t, u, x) \in \mathbb{R} \times \mathcal{U}_1 \times \mathbb{R}^d;$$

(ii) skew equivalent if there exists a skew homeomorphism $h = (f, g) : \mathcal{U}_1 \times \mathbb{R}^d \to \mathcal{U}_2 \times \mathbb{R}^d$ as above that maps trajectories of Φ_1 onto trajectories of Φ_2, preserving the orientation, but possibly with a time shift, i.e., for each $(u, x) \in \mathcal{U}_1 \times \mathbb{R}^d$ there exists a continuous, strictly increasing time parametrization $\tau_{x,u} : \mathbb{R} \to \mathbb{R}$ such that $h(\Phi_1(t, u, x)) = \Phi_2(\tau_{x,u}(t), h(u, x))$;
(iii) shift conjugate if the shift flows in the bases are conjugate, i.e., there exists a homeomorphism $f : \mathcal{U}_1 \to \mathcal{U}_2$ such that $f(\theta_1(t, u)) = \theta_2(t, f(u))$ for all $(t, u) \in \mathbb{R} \times \mathcal{U}_1$.

For control systems skew conjugation of the flows requires, in particular, conjugation of the shift flows in the bases. We obtain the following result.

Theorem 2 *Let $\Sigma_1 \in \mathbf{B}(d, m, U_1)$ and $\Sigma_2 \in \mathbf{B}(d, m, U_2)$ be bilinear control systems.*
(i) If U_1 and U_2 are affi nely isomorphic, i.e.$U_2 = H[U_1]$ with $H(v) = Mv + b$ and M invertible, then Σ_1 and Σ_2 are shift conjugate.
(ii) If U_1 and U_2 are the convex hull of $2m$ points in $\mathbb{R}^m$ of the form $U_i = co\{v_i^1, ...v_i^m, -v_i^1, ... - v_i^m\}$, then Σ_1 and Σ_2 are shift conjugate.

For bilinear control systems that are shift conjugate, the following theorem shows that skew conjugacy is a very weak condition which-in the hyperbolic situation-only depends on the dimension of the stable subbundle.

Theorem 3 *Consider two bilinear control systems of the form (1) which are shift conjugate.*
(i) If both fbws are exponentially (un)stable, then they are skew conjugate.
(ii) Let both fbws be hyperbolic, i.e. the vector bundles $\mathcal{U}_i \times \mathbb{R}^d$ can be written as the Whitney sums of exponentially stable and unstable subbundles. Then they are skew conjugate iff the dimensions of their stable (and unstable) subbundles coincide.

Theorem 3 generalizes the well-known result for hyperbolic matrices to bilinear control systems. The proof follows from the fact that the control fbw associated with a bilinear control system is a linear fbw on a vector bundle. Then a general theorem for such fbws [2] can be applied. The next section is devoted to the study of the number of spectral subbundles and their dimensions under skew conjugacy.

3 Conjugacy and equivalence for bilinear control systems in projective space

A system $\Sigma \in \mathbf{B}(d, m, U)$ induces a (nonlinear) control system $\mathbb{P}\Sigma$ on the projective space $\mathbb{P}^{d-1}$ in the following way:

$$\dot{s}(t) = \mathbb{P}A(u(t), s(t)) = \mathbb{P}A_0(s) + \sum_{i=1}^{m} u_i \mathbb{P}A_i(s), \tag{3}$$
$$\mathbb{P}A_i(u, s) = (A_i - s^T A_i s \cdot I)s \text{ for all } i = 0, ..., m.$$

Here ^{T} denotes transposition and I is the $d \times d$ identity matrix. For all $(u, s) \in \mathcal{U} \times \mathbb{P}^{d-1}$ the system has a unique solution, denoted by $\mathbb{P}\varphi(t, s, u)$ for all $t \in \mathbb{R}$ with $\mathbb{P}\varphi(0, s, u) = s$. The associated dynamical system reads

$$\mathbb{P}\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U} \times \mathbb{P}^{d-1},\ \mathbb{P}\Phi(t, u, x) = (\theta(t, u), \mathbb{P}\varphi(t, x, u)). \tag{4}$$

The *Morse spectrum* of the system Σ is $\Sigma_{Mo} = \bigcup_{j=1}^{l} \Sigma_{Mo}(E_j)$, where the E_j are the chain control sets of $\mathbb{P}\Sigma$. The Morse spectrum contains all Lyapunov exponents and leads to a corresponding subbundle decomposition, see [4].

Theorem 4 *For $i = 1, 2$, let $\Sigma_i \in \mathbf{B}(d, m, U_i)$ be two bilinear control systems with associated fbws Φ_i in $\mathcal{U}_i \times \mathbb{R}^d$ and projected fbws $\mathbb{P}\Phi_i$ in $\mathcal{U}_i \times \mathbb{P}^{d-1}$. Denote the associated spectral bundle decompositions by $\bigoplus_{j=1}^{l_i} \mathcal{V}_i^j = \mathcal{U}_i \times \mathbb{R}^d$. Suppose that there exists a skew equivalence $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ between $\mathbb{P}\Phi_1$ and $\mathbb{P}\Phi_2$. Then*
(i) h maps chain recurrent components of $\mathbb{P}\Phi_1$ onto chain recurrent components of $\mathbb{P}\Phi_2$ and $l_1 = l_2$, and hence the chain control sets in $\mathbb{P}^{d-1}$ are mapped onto chain control sets,
(ii) h respects the order of the chain recurrent components, and hence of the chain control

sets,
(iii) Σ_1 *and* Σ_2 *have the same number of spectral intervals and* h *respects the order between these intervals,*
(iv) h *maps the associated bundle decompositions into each other, and the dimensions of corresponding fibers agree.*

The next section provides a converse of this theorem.

4 The Lyapunov index of bilinear control systems

In order to characterize skew conjugate bilinear control systems we introduce the following notion of Lyapunov indices.

Definition 5 *The Lyapunov index* $L(\Sigma)$ *of* $\Sigma \in \mathbf{B}(d, m, U)$ *is the diagonal matrix*

$$\begin{bmatrix} \Lambda_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \Lambda_l \end{bmatrix} \quad \text{with } \Lambda_i = \begin{bmatrix} \kappa^*(E_i),\ \kappa(E_i) & & 0 \\ & \cdot & \\ 0 & & \kappa^*(E_i),\ \kappa(E_i) \end{bmatrix},$$

where $\kappa^*(E_i) = \inf \Sigma_{Mo}(E_i)$, $\kappa(E_i) = \sup \Sigma_{Mo}(E_i)$, *and the block size of* Λ_i *is the multiplicity* $m(E_i)$ *of the corresponding chain control set. The blocks are arranged according to the order* $E_1 \prec ... \prec E_l$. *Two bilinear control systems* $\Sigma_i \in \mathbf{B}(d, m, U_i)$, $i = 1, 2$, *are called Lyapunov equivalent if* $S(\Sigma_1) = S(\Sigma_2)$.

Note that the Lyapunov index of a bilinear system characterizes the stability behavior. We remark that it can also be used to characterize (exponential) feedback stabilizability and controllability properties of the system in $\mathbb{R}^d$ and in $\mathbb{P}^{d-1}$ as well as the robust stability of linear differential equations under time varying perturbations.

Definition 6 *The short Lyapunov index* $SL(\Sigma)$ *of* $\Sigma \in \mathbf{B}(d, m, U)$ *is given by the vector of the multiplicities* d_i *of the subbundles (in the natural order of their chain control sets):* $SL(\Sigma) = (l, d_1, ..., d_l)$ *where* $l \leq d$ *is the number of distinct chain control sets. The short zero-Lyapunov index additionally includes the dimension of the stable subbundle.*

Note that two bilinear control systems Σ_1 and Σ_2 on $\mathbb{R}^d$ have the same short Lyapunov index if and only if the (ordered) blocks of the Lyapunov indices $L(\Sigma_1)$ and $L(\Sigma_2)$ have the same dimensions.

Theorem 7 *Consider two bilinear control systems* $\Sigma_i \in \mathbf{B}(d, m, U_i)$ *which are shift conjugate via* $f : \mathcal{U}_1 \to \mathcal{U}_2$. *Then* Σ_1 *and* Σ_2 *have the same short Lyapunov index iff there is a skew homeomorphism* $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ *with* $g : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ *that maps the finest Morse decomposition of* $\mathbb{P}\Phi_1$ *into the finest Morse decomposition of* $\mathbb{P}\Phi_2$, *i.e.* h *maps Morse sets into Morse sets and preserves their order.*

Corollary 8 *Consider two bilinear control systems* $\Sigma_i \in \mathbf{B}(d, m, U_i)$ *which are shift conjugate via* $f : \mathcal{U}_1 \to \mathcal{U}_2$ *and have hyperbolic linear flows* Φ_i. *Then* Σ_1 *and* Σ_2 *have the same short zero-Lyapunov index iff their linear flows* Φ_i *in* $\mathcal{U} \times \mathbb{R}^d$ *are skew conjugate and there is a skew homeomorphism* $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ *respecting the finest Morse decompositions of the projected flows.*

This follows by combining Theorem 7 with Theorem 3(ii). In [1] we introduced the Grassmann graphs of flows associated with linear differential equations. These graphs can be generalized to bilinear control systems. It turns out that for two shift equivalent systems the short Lyapunov indices coincide iff the Grassmann graphs of the two systems are isomorphic.

References

[1] V. Ayala, F. Colonius, and W. Kliemann, *Dynamical characterization of the Lyapunov form of matrices,* Linear Algebra and Its Applications 420(2005), 272 - 290.

[2] V. Ayala, F. Colonius, and W. Kliemann, *On Topological Equivalence of Linear Flows with Applications to Bilinear Control Systems,* submitted.

[3] B. Baratchart, M. Chyba and J.P. Pomet, *On the Grobman-Hartman theorem for control systems.* Report INRIA 5040 (http://www-sop.inria.fr/rapports/sophia/RR-5040.html).

[4] F. Colonius and W. Kliemann, *The Dynamics of Control,* Birkhäuser 2000.

[5] M. Hirsch, S. Smale and R.L. Devaney, *Differential Equations, Dynamical Systems and an Introduction to Chaos*, Second Edition, Academic Press, 2004.

[6] W. Kang and A.J. Krener, *Extended quadratic controller normal form and dynamic feedback linearization of nonlinear systems*, SIAM J. Control Optim. 30(1992), 1319-1337.

[7] W. Respondek and I. Tall, *Feedback equivalence of nonlinear control systems: a survey on formal approach*, to appear in: *Normal Forms, Bifurcations and Chaos in Automatic: From Theory Towards Applications,* W. Perruquetti and J.-P. Barbot, eds., Marcel Dekker, 2005.

[8] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Applications,* Springer-Verlag 1996.