56

Dynamical Systems and Linear Algebra

Fritz Colonius
Universität Augsburg

Wolfgang Kliemann
Iowa State University

Linear algebra plays a key role in the theory of dynamical systems, and concepts from dynamical systems allow the study, characterization, and generalization of many objects in linear algebra, such as similarity of matrices, eigenvalues, and (generalized) eigenspaces. The most basic form of this interplay can be seen as a matrix A gives rise to a continuous time dynamical system via the linear ordinary differential equation $\dot{\mathbf{x}} = A\mathbf{x}$, or a discrete time dynamical system via iteration $\mathbf{x}_{n+1} = A\mathbf{x}_n$. The properties of the solutions are intimately related to the properties of the matrix A. Matrices also define nonlinear systems on smooth manifolds, such as the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the Grassmann manifolds, or on classical (matrix) Lie groups. Again, the behavior of such systems is closely related to matrices and their properties. And the behavior of nonlinear systems, e.g., of differential equations $\dot{\mathbf{y}} = f(\mathbf{y})$ in $\mathbb{R}^d$ with a fixed point $\mathbf{y}_0 \in \mathbb{R}^d$, can be described locally around $\mathbf{y}_0$ via the linear differential equation $\dot{\mathbf{x}} = D_{\mathbf{y}} f(\mathbf{y}_0)\mathbf{x}$.

Since A. M. Lyapunov's thesis in 1892, it has been an intriguing problem how to construct an appropriate linear algebra for time varying systems. Note that, e.g., for stability of the solutions of $\dot{\mathbf{x}} = A(t)\mathbf{x}$, it is not sufficient that for all $t \in \mathbb{R}$ the matrices $A(t)$ have only eigenvalues with negative real part (see [Hah67], Chapter 62). Of course, Floquet theory (see [Flo83]) gives an elegant solution for the periodic case, but it is not immediately clear how to build a linear algebra around Lyapunov's "order numbers" (now called Lyapunov exponents). The multiplicative ergodic theorem of Oseledets [Ose68] resolves the issue for measurable linear systems with stationary time dependencies, and the Morse spectrum together with Selgrade's theorem [Sel75] clarifies the situation for continuous linear systems with chain transitive time dependencies.

This chapter provides a first introduction to the interplay between linear algebra and analysis/topology in continuous time. Section 56.1 recalls facts about d-dimensional linear differential equations $\dot{\mathbf{x}} = A\mathbf{x}$, emphasizing eigenvalues and (generalized) eigenspaces. Section 56.2 studies solutions in Euclidian space $\mathbb{R}^d$ from the point of view of topological equivalence and conjugacy with related characterizations of the matrix A. Section 56.3 presents, in a fairly general set-up, the concepts of chain recurrence and Morse decompositions for dynamical systems. These ideas are then applied in section 56.4 to nonlinear systems on Grassmannian and flag manifolds induced by a single matrix A, with emphasis on characterizations of the matrix A from this point of view. Section 56.5 introduces linear skew product flows as a way to model time varying linear systems $\dot{\mathbf{x}} = A(t)\mathbf{x}$ with, e.g., periodic, measurable ergodic, and continuous chain transitive time dependencies. The following sections 56.6, 56.7, and 56.8 develop generalizations of (real parts of) eigenvalues and eigenspaces as a starting point for a linear algebra for classes of time varying linear systems, namely periodic, random, and robust systems. (For the corresponding generalization of the imaginary parts of eigenvalues see, e.g., [Arn98] for the measurable ergodic case and [CFJ06] for the continuous, chain transitive case.) Section 56.9 introduces some basic ideas to study genuinely nonlinear systems via linearization, emphasizing invariant manifolds and Grobman–Hartman-type results that compare nonlinear behavior locally to the behavior of associated linear systems.

Notation:

In this chapter, the set of $d \times d$ real matrices is denoted by $gl(d, \mathbb{R})$ rather than $\mathbb{R}^{d\times d}$.

56.1 Linear Differential Equations

Linear differential equations can be solved explicitly if one knows the eigenvalues and a basis of eigenvectors (and generalized eigenvectors, if necessary). The key idea is that of the Jordan form of a matrix. The real parts of the eigenvectors determine the exponential behavior of the solutions, described by the Lyapunov exponents and the corresponding Lyapunov subspaces.

For information on matrix functions, including the matrix exponential, see Chapter 11. For information on the Jordan canonical form see Chapter 6. Systems of first order linear differential equations are also discussed in Chapter 55.

Definitions:

For a matrix $A \in gl(d, \mathbb{R})$, the **exponential** $e^A \in GL(d, \mathbb{R})$ is defined by $e^A = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n \in GL(d, \mathbb{R})$, where $I \in gl(d, \mathbb{R})$ is the identity matrix.

A **linear differential equation** (with constant coefficients) is given by a matrix $A \in gl(d, \mathbb{R})$ via $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where $\dot{\mathbf{x}}$ denotes differentiation with respect to t. Any function $\mathbf{x} : \mathbb{R} \longrightarrow \mathbb{R}^d$ such that $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ for all $t \in \mathbb{R}$ is called a solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

The **initial value problem** for a linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ consists in finding, for a given **initial value** $\mathbf{x}_0 \in \mathbb{R}^d$, a solution $\mathbf{x}(\cdot, \mathbf{x}_0)$ that satisfies $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$.

The distinct (complex) eigenvalues of $A \in gl(d, \mathbb{R})$ will be denoted $\mu_1, \dots, \mu_r$. (For definitions and more information about eigenvalues, eigenvectors, and eigenspaces, see Section 4.3. For information about generalized eigenspaces, see Chapter 6.) The real version of the generalized eigenspace is denoted by $E(A, \mu_k) \subset \mathbb{R}^d$ or simply E_k for $k = 1, \dots, r \leq d$.

The **real Jordan form** of a matrix $A \in gl(d, \mathbb{R})$ is denoted by $J_A^{\mathbb{R}}$. Note that for any matrix A there is a matrix $T \in GL(d, \mathbb{R})$ such that $A = T^{-1} J_A^{\mathbb{R}} T$.

Let $\mathbf{x}(\cdot, \mathbf{x}_0)$ be a solution of the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Its **Lyapunov exponent** for $\mathbf{x}_0 \neq 0$ is defined as $\lambda(\mathbf{x}_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|\mathbf{x}(t, \mathbf{x}_0)\|$, where log denotes the natural logarithm and $\|\cdot\|$ is any norm in $\mathbb{R}^d$.

Let $\mu_k = \lambda_k + i\nu_k$, $k = 1, \dots, r$, be the distinct eigenvalues of $A \in gl(d, \mathbb{R})$. We order the distinct real parts of the eigenvalues as $\lambda_1 < \dots < \lambda_l$, $1 \leq l \leq r \leq d$, and define the **Lyapunov space** of λ_j as $L(\lambda_j) = \bigoplus E_k$, where the direct sum is taken over all generalized real eigenspaces associated to eigenvalues with real part equal to λ_j. Note that $\bigoplus_{j=1}^{l} L(\lambda_j) = \mathbb{R}^d$.

The **stable, center, and unstable subspaces** associated with the matrix $A \in gl(d,\mathbb{R})$ are defined as $L^- = \bigoplus\{L(\lambda_j), \lambda_j < 0\}$, $L^0 = \bigoplus\{L(\lambda_j), \lambda_j = 0\}$, and $L^+ = \bigoplus\{L(\lambda_j), \lambda_j > 0\}$, respectively.

The zero solution $\mathbf{x}(t,0) \equiv 0$ is called **exponentially stable** if there exists a neighborhood $U(0)$ and positive constants $a, b > 0$ such that $\mathbf{x}(t,\mathbf{x}_0) \leq a\|\mathbf{x}_0\|e^{-bt}$ for all $t \in \mathbb{R}$ and $\mathbf{x}_0 \in U(0)$.

Facts:

Literature: [Ama90], [HSD04].

1. For each $A \in gl(d,\mathbb{R})$ the solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ form a d-dimensional vector space $sol(A) \subset C^\infty(\mathbb{R},\mathbb{R}^d)$ over $\mathbb{R}$, where $C^\infty(\mathbb{R},\mathbb{R}^d) = \{f : \mathbb{R} \longrightarrow \mathbb{R}^d, f \text{ is infinitely often differentiable}\}$. Note that the solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ are even real analytic.
2. For each initial value problem given by $A \in gl(d,\mathbb{R})$ and $\mathbf{x}_0 \in \mathbb{R}^d$, the solution $\mathbf{x}(\cdot,\mathbf{x}_0)$ is unique and given by $\mathbf{x}(t,\mathbf{x}_0) = e^{At}\mathbf{x}_0$.
3. Let $\mathbf{v}_1,\ldots,\mathbf{v}_d \in \mathbb{R}^d$ be a basis of $\mathbb{R}^d$. Then the functions $\mathbf{x}(\cdot,\mathbf{v}_1),\ldots,\mathbf{x}(\cdot,\mathbf{v}_d)$ form a basis of the solution space $sol(A)$. The matrix function $X(\cdot) := [\mathbf{x}(\cdot,\mathbf{v}_1),\ldots,\mathbf{x}(\cdot,\mathbf{v}_d)]$ is called a fundamental matrix of $\dot{\mathbf{x}} = A\mathbf{x}$, and it satisfies $\dot{X}(t) = AX(t)$.
4. Let $A \in gl(d,\mathbb{R})$ with distinct eigenvalues $\mu_1,\ldots,\mu_r \in \mathbb{C}$ and corresponding multiplicities $n_k = \alpha(\mu_k)$, $k = 1,\ldots,r$. If E_k are the corresponding generalized real eigenspaces, then $\dim E_k = n_k$ and $\bigoplus_{k=1}^r E_k = \mathbb{R}^d$, i.e., every matrix has a set of generalized real eigenvectors that form a basis of $\mathbb{R}^d$.
5. If $A = T^{-1}J_A^{\mathbb{R}}T$, then $e^{At} = T^{-1}e^{J_A^{\mathbb{R}}t}T$, i.e., for the computation of exponentials of matrices it is sufficient to know the exponentials of Jordan form matrices.
6. Let $\mathbf{v}_1,\ldots,\mathbf{v}_d$ be a basis of generalized real eigenvectors of A. If $\mathbf{x}_0 = \sum_{i=1}^d \alpha_i\,\mathbf{v}_i$, then $\mathbf{x}(t,\mathbf{x}_0) = \sum_{i=1}^d \alpha_i\mathbf{x}(t,\mathbf{v}_i)$ for all $t \in \mathbb{R}$. This reduces the computation of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ to the computation of solutions for Jordan blocks; see the examples below or [HSD04, Chap. 5] for a discussion of this topic.
7. Each generalized real eigenspace E_k is invariant for the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$, i.e., for $\mathbf{x}_0 \in E_k$ it holds that $\mathbf{x}(t,\mathbf{x}_0) \in E_k$ for all $t \in \mathbb{R}$.
8. The Lyapunov exponent $\lambda(\mathbf{x}_0)$ of a solution $\mathbf{x}(\cdot,\mathbf{x}_0)$ (with $\mathbf{x}_0 \neq 0$) satisfies $\lambda(\mathbf{x}_0) = \lim_{t\to\pm\infty} \frac{1}{t}\log\|\mathbf{x}(t,\mathbf{x}_0)\| = \lambda_j$ if and only if $\mathbf{x}_0 \in L(\lambda_j)$. Hence, associated to a matrix $A \in gl(d,\mathbb{R})$ are exactly l Lyapunov exponents, the distinct real parts of the eigenvalues of A.
9. The following are equivalent:

   (a) The zero solution $\mathbf{x}(t,0) \equiv 0$ of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable.

   (b) The zero solution is exponentially stable

   (c) All Lyapunov exponents are negative.

   (d) $L^- = \mathbb{R}^d$.

Examples:

1. Let $A = \mathrm{diag}(a_1,\ldots,a_d)$ be a diagonal matrix. Then the solution of the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ with initial value $\mathbf{x}_0 \in \mathbb{R}^d$ is given by $\mathbf{x}(t,\mathbf{x}_0) = e^{At}\mathbf{x}_0 = \begin{bmatrix} e^{a_1t} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & e^{a_dt} \end{bmatrix}\mathbf{x}_0.$
2. Let $\mathbf{e}_1 = (1,0,\ldots,0)^T,\ldots,\mathbf{e}_d = (0,0,\ldots,1)^T$ be the standard basis of $\mathbb{R}^d$. Then $\{\mathbf{x}(\cdot,\mathbf{e}_1),\ldots,\mathbf{x}(\cdot,\mathbf{e}_d)\}$ is a basis of the solution space $sol(A)$.
3. Let $A = \mathrm{diag}(a_1,\ldots,a_d)$ be a diagonal matrix. Then the standard basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ of $\mathbb{R}^d$ consists of eigenvectors of A.

4. Let $A \in gl(d, \mathbb{R})$ be diagonalizable, i.e., there exists a transformation matrix $T \in GL(d, \mathbb{R})$ and a diagonal matrix $D \in gl(d, \mathbb{R})$ with $A = T^{-1}DT$. Then the solution of the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ with initial value $x_0 \in \mathbb{R}^d$ is given by $\mathbf{x}(t, \mathbf{x}_0) = T^{-1}e^{Dt}T\mathbf{x}_0$, where e^{Dt} is given in Example 1.
5. Let $B = \begin{bmatrix} \lambda & -\nu \\ \nu & \lambda \end{bmatrix}$ be the real Jordan block associated with a complex eigenvalue $\mu = \lambda + i\nu$ of the matrix $A \in gl(d, \mathbb{R})$. Let $\mathbf{y}_0 \in E(A, \mu)$, the real eigenspace of μ. Then the solution $\mathbf{y}(t, \mathbf{y}_0)$ of $\dot{\mathbf{y}} = B\mathbf{y}$ is given by $\mathbf{y}(t, \mathbf{y}_0) = e^{\lambda t}\begin{bmatrix} \cos \nu t & -\sin \nu t \\ \sin \nu t & \cos \nu t \end{bmatrix}\mathbf{y}_0$. According to Fact 6 this is also the $E(A, \mu)$-component of the solutions of $\dot{\mathbf{x}} = J_A^{\mathbb{R}}\mathbf{x}$.
6. Let B be a Jordan block of dimension n associated with the real eigenvalue μ of a matrix $A \in gl(d, \mathbb{R})$. Then for

$$B = \begin{bmatrix} \mu & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & \cdot & 1 \\ & & & & \mu \end{bmatrix} \text{ one has } e^{Bt} = e^{\mu t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdot & \cdot & \frac{t^{n-1}}{(n-1)!} \\ & \cdot & \cdot & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \frac{t^2}{2!} \\ & & & & \cdot & t \\ & & & & & 1 \end{bmatrix}.$$

In other words, for $\mathbf{y}_0 = [y_1, \ldots, y_n]^T \in E(A, \mu)$, the jth component of the solution of $\dot{\mathbf{y}} = B\mathbf{y}$ reads $y_j(t, \mathbf{y}_0) = e^{\mu t}\sum_{k=j}^{n} \frac{t^{k-j}}{(k-j)!}y_k$. According to Fact 6 this is also the $E(A, \mu)$-component of $e^{J_A^{\mathbb{R}}t}$.
7. Let B be a real Jordan block of dimension $n = 2m$ associated with the complex eigenvalue $\mu = \lambda + i\nu$ of a matrix $A \in gl(d, \mathbb{R})$. Then with $D = \begin{bmatrix} \lambda & -\nu \\ \nu & \lambda \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for

$$B = \begin{bmatrix} D & I & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & \cdot & I \\ & & & & D \end{bmatrix} \text{ one has } e^{Bt} = e^{\lambda t}\begin{bmatrix} \widehat{D} & t\widehat{D} & \frac{t^2}{2!}\widehat{D} & \cdot & \cdot & \frac{t^{n-1}}{(n-1)!}\widehat{D} \\ & \cdot & \cdot & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \frac{t^2}{2!}\widehat{D} \\ & & & & \cdot & t\widehat{D} \\ & & & & & \widehat{D} \end{bmatrix},$$

where $\widehat{D} = \begin{bmatrix} \cos \nu t & -\sin \nu t \\ \sin \nu t & \cos \nu t \end{bmatrix}$. In other words, for $\mathbf{y}_0 = [y_1, z_1, \ldots, y_m, z_m]^T \in E(A, \mu)$, the jth components, $j = 1, \ldots, m$, of the solution of $\dot{\mathbf{y}} = B\mathbf{y}$ read

$$y_j(t, \mathbf{y}_0) = e^{\mu t}\sum_{k=j}^{m} \frac{t^{k-j}}{(k-j)!}(y_k \cos \nu t - z_k \sin \nu t),$$

$$z_j(t, \mathbf{y}_0) = e^{\mu t}\sum_{k=j}^{m} \frac{t^{k-j}}{(k-j)!}(z_k \cos \nu t + y_k \sin \nu t).$$

According to Fact 6, this is also the $E(A, \mu)$-component of $e^{J_A^{\mathbb{R}}t}$.
8. Using these examples and Facts 5 and 6, it is possible to compute explicitly the solutions to any linear differential equation in $\mathbb{R}^d$.
9. Recall that for any matrix A there is a matrix $T \in GL(d, \mathbb{R})$ such that $A = T^{-1}J_A^{\mathbb{R}}T$, where $J_A^{\mathbb{R}}$ is the real Jordan canonical form of A. The exponential behavior of the solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be read off from the diagonal elements of $J_A^{\mathbb{R}}$.

56.2 Linear Dynamical Systems in $\mathbb{R}^d$

The solutions of a linear differential equation $\dot{x} = Ax$, where $A \in gl(d, \mathbb{R})$, define a (continuous time) dynamical system, or linear flow in $\mathbb{R}^d$. The standard concepts for comparison of dynamical systems are equivalences and conjugacies that map trajectories into trajectories. For linear flows in $\mathbb{R}^d$ these concepts lead to two different classifications of matrices, depending on the smoothness of the conjugacy or equivalence.

Definitions:

The real square matrix A is **hyperbolic** if it has no eigenvalues on the imaginary axis.

A **continuous dynamical system** over the "time set" $\mathbb{R}$ with state space M, a complete metric space, is defined as a map $\Phi : \mathbb{R} \times M \longrightarrow M$ with the properties

(i) $\Phi(0, x) = x$ for all $x \in M$,
(ii) $\Phi(s + t, x) = \Phi(s, \Phi(t, x))$ for all $s, t \in \mathbb{R}$ and all $x \in M$,
(iii) Φ is continuous (in both variables).

The map Φ is also called a (continuous) flow.

For each $x \in M$ the set $\{\Phi(t, x), t \in \mathbb{R}\}$ is called the **orbit (or trajectory)** of the system through x.

For each $t \in \mathbb{R}$ the **time-t map** is defined as $\varphi_t = \Phi(t, \cdot) : M \longrightarrow M$. Using time-$t$ maps, the properties (i) and (ii) above can be restated as (i)$'$ $\varphi_0 = id$, the identity map on M, (ii)$'$ $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$.

A **fixed point (or equilibrium)** of a dynamical system Φ is a point $x \in M$ with the property $\Phi(t, x) = x$ for all $t \in \mathbb{R}$.

An orbit $\{\Phi(t, x), t \in \mathbb{R}\}$ of a dynamical system Φ is called **periodic** if there exists $\widehat{t} \in \mathbb{R}, \widehat{t} > 0$ such that $\Phi(\widehat{t} + s, x) = \Phi(s, x)$ for all $s \in \mathbb{R}$. The infimum of the positive $\widehat{t} \in \mathbb{R}$ with this property is called the **period** of the orbit. Note that an orbit of period 0 is a fixed point.

Denote by $C^k(X, Y)$ $(k \geq 0)$ the set of k-times **differentiable functions** between C^k-manifolds X and Y, with C^0 denoting continuous.

Let $\Phi, \Psi : \mathbb{R} \times M \longrightarrow M$ be two continuous dynamical systems of class C^k $(k \geq 0)$, i.e., for $k \geq 1$ the state space M is at least a C^k-manifold and Φ, Ψ are C^k-maps. The flows Φ and Ψ are:

(i) C^k**-equivalent** $(k \geq 1)$ if there exists a (local) C^k-diffeomorphism $h : M \to M$ such that h takes orbits of Φ onto orbits of Ψ, **preserving the orientation** (but not necessarily parametrization by time), i.e.,

  (a) For each $x \in M$ there is a strictly increasing and continuous parametrization map $\tau_x : \mathbb{R} \to \mathbb{R}$ such that $h(\Phi(t, x)) = \Psi(\tau_x(t), h(x))$ or, equivalently,

  (b) For all $x \in M$ and $\delta > 0$ there exists $\varepsilon > 0$ such that for all $t \in (0, \delta)$, $h(\Phi(t, x)) = \Psi(t', h(x))$ for some $t' \in (0, \varepsilon)$.

(ii) C^k**-conjugate** $(k \geq 1)$ if there exists a (local) C^k-diffeomorphism $h : M \to M$ such that $h(\Phi(t, x)) = \Psi(t, h(x))$ for all $x \in M$ and $t \in \mathbb{R}$.

Similarly, the flows Φ and Ψ are C^0**-equivalent** if there exists a (local) homeomorphism $h : M \to M$ satisfying the properties of (i) above, and they are C^0**-conjugate** if there exist a (local) homeomorphism $h : M \to M$ satisfying the properties of (ii) above. Often, C^0-equivalence is called **topological equivalence,** and C^0-conjugacy is called **topological conjugacy** or simply **conjugacy**.

Warning: While this terminology is standard in dynamical systems, the terms *conjugate* and *equivalent* are used differently in linear algebra. *Conjugacy* as used here is related to matrix similarity (cf. Fact 6), *not* to matrix conjugacy, and *equivalence* as used here is *not* related to matrix equivalence.

Facts:

Literature: [HSD04], [Rob98].

1. If the flows Φ and Ψ are C^k-conjugate, then they are C^k-equivalent.
2. Each time-t map φ_t has an inverse $(\varphi_t)^{-1} = \varphi_{-t}$, and $\varphi_t : M \longrightarrow M$ is a homeomorphism, i.e., a continuous bijective map with continuous inverse.
3. Denote the set of time-t maps again by $\Phi = \{\varphi_t, t \in \mathbb{R}\}$. A dynamical system is a group in the sense that $(\Phi, \circ)$, with $\circ$ denoting composition of maps, satisfies the group axioms, and $\varphi : (\mathbb{R}, +) \longrightarrow (\Phi, \circ)$, defined by $\varphi(t) = \varphi_t$, is a group homomorphism.
4. Let M be a C^∞-differentiable manifold and X a C^∞-vector field on M such that the differential equation $\dot{x} = X(x)$ has unique solutions $x(t, x_0)$ for all $x_0 \in M$ and all $t \in \mathbb{R}$, with $x(0, x_0) = x_0$. Then $\Phi(t, x_0) = x(t, x_0)$ defines a dynamical system $\Phi : \mathbb{R} \times M \longrightarrow M$.
5. A point $x_0 \in M$ is a fixed point of the dynamical system Φ associated with a differential equation $\dot{x} = X(x)$ as above if and only if $X(x_0) = 0$.
6. For two linear flows Φ (associated with $\dot{\mathbf{x}} = A\mathbf{x}$) and Ψ (associated with $\dot{\mathbf{x}} = B\mathbf{x}$) in $\mathbb{R}^d$, the following are equivalent:
   - Φ and Ψ are C^k-conjugate for $k \geq 1$.
   - Φ and Ψ are linearly conjugate, i.e., the conjugacy map h is a linear operator in $\mathrm{GL}(\mathbb{R}^d)$.
   - A and B are similar, i.e., $A = TBT^{-1}$ for some $T \in GL(d, \mathbb{R})$.
7. Each of the statements in Fact 6 implies that A and B have the same eigenvalue structure and (up to a linear transformation) the same generalized real eigenspace structure. In particular, the C^k-conjugacy classes are exactly the real Jordan canonical form equivalence classes in $gl(d, \mathbb{R})$.
8. For two linear flows Φ (associated with $\dot{\mathbf{x}} = A\mathbf{x}$) and Ψ (associated with $\dot{\mathbf{x}} = B\mathbf{x}$) in $\mathbb{R}^d$, the following are equivalent:
   - Φ and Ψ are C^k-equivalent for $k \geq 1$.
   - Φ and Ψ are linearly equivalent, i.e., the equivalence map h is a linear map in $\mathrm{GL}(\mathbb{R}^d)$.
   - $A = \alpha TBT^{-1}$ for some positive real number α and $T \in GL(d, \mathbb{R})$.
9. Each of the statements in Fact 8 implies that A and B have the same real Jordan structure and their eigenvalues differ by a positive constant. Hence, the C^k-equivalence classes are real Jordan canonical form equivalence classes modulo a positive constant.
10. The set of hyperbolic matrices is open and dense in $gl(d, \mathbb{R})$. A matrix A is hyperbolic if and only if it is structurally stable in $gl(d, \mathbb{R})$, i.e., there exists a neighborhood $U \subset gl(d, \mathbb{R})$ of A such that all $B \in U$ are topologically equivalent to A.
11. If A and B are hyperbolic, then the associated linear flows Φ and Ψ in $\mathbb{R}^d$ are C^0-equivalent (and C^0-conjugate) if and only if the dimensions of the stable subspaces (and, hence, the dimensions of the unstable subspaces) of A and B agree.

Examples:

1. Linear differential equations: For $A \in gl(d, \mathbb{R})$ the solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ form a continuous dynamical system with time set $\mathbb{R}$ and state space $M = \mathbb{R}^d$: Here $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined by $\Phi(t, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}_0) = e^{At}\mathbf{x}_0$.
2. Fixed points of linear differential equations: A point $\mathbf{x}_0 \in \mathbb{R}^d$ is a fixed point of the dynamical system Φ associated with the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$ if and only if $\mathbf{x}_0 \in \ker A$, the kernel of A.
3. Periodic orbits of linear differential equations: The orbit $\Phi(t, \mathbf{x}_0) := \mathbf{x}(t, \mathbf{x}_0), t \in \mathbb{R}$ is periodic with period $\widehat{t} > 0$ if and only if $\mathbf{x}_0$ is in the eigenspace of a nonzero complex eigenvalue with zero real part.

4. For each matrix $A \in gl(d, \mathbb{R})$ its associated linear flow in $\mathbb{R}^d$ is C^k-conjugate (and, hence, C^k-equivalent) for all $k \geq 0$ to the dynamical system associated with the Jordan form $J_A^{\mathbb{R}}$.

56.3 Chain Recurrence and Morse Decompositions of Dynamical Systems

A matrix $A \in gl(d, \mathbb{R})$ and, hence, a linear differential equation $\dot{x} = Ax$ maps subspaces of $\mathbb{R}^d$ into subspaces of $\mathbb{R}^d$. Therefore, the matrix A also defines dynamical systems on spaces of subspaces, such as the Grassmann and the flag manifolds. These are nonlinear systems, but they can be studied via linear algebra, and vice versa; the behavior of these systems allows for the investigation of certain properties of the matrix A. The key topological concepts for the analysis of systems on compact spaces like the Grassmann and flag manifolds are chain recurrence, Morse decompositions, and attractor–repeller decompositions. This section concentrates on the first two approaches, the connection to attractor–repeller decompositions can be found, e.g., in [CK00, App. B2].

Definitions:

Given a dynamical system $\Phi : \mathbb{R} \times M \longrightarrow M$, for a subset $N \subset M$ the **α-limit set** is defined as $\alpha(N) = \{y \in M$, there exist sequences x_n in N and $t_n \to -\infty$ in $\mathbb{R}$ with $\lim_{n\to\infty} \Phi(t_n, x_n) = y\}$, and similarly the **$\omega$-limit set** of N is defined as $\omega(N) = \{y \in M$, there exist sequences x_n in N and $t_n \to \infty$ in $\mathbb{R}$ with $\lim_{n\to\infty} \Phi(t_n, x_n) = y\}$.

For a flow Φ on a complete metric space M and $\varepsilon, T > 0$, an **(ε, T)-chain** from $x \in M$ to $y \in M$ is given by

$$n \in \mathbb{N},\ x_0 = x, \ldots, x_n = y,\ T_0, \ldots, T_{n-1} > T$$

with

$$\mathrm{d}(\Phi(T_i, x_i), x_{i+1}) < \varepsilon \text{ for all } i,$$

where d is the metric on M.

A set $K \subset M$ is **chain transitive** if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y.

The **chain recurrent set** $\mathcal{CR}$ is the set of all points that are chain reachable from themselves, i.e., $CR = \{x \in M$, for all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to $x\}$.

A set $\mathcal{M} \subset M$ is a **chain recurrent component**, if it is a maximal (with respect to set inclusion) chain transitive set. In this case $\mathcal{M}$ is a connected component of the chain recurrent set $\mathcal{CR}$.

For a flow Φ on a complete metric space M, a compact subset $K \subset M$ is called **isolated invariant**, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int} N$, such that $\Phi(t, x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$.

A **Morse decomposition** of a flow Φ on a complete metric space M is a finite collection $\{\mathcal{M}_i,\ i = 1, \ldots, l\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that

(i) For all $x \in M$, $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{l} \mathcal{M}_i$; and

(ii) Suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_n}$ and $x_1, \ldots, x_n \in M \setminus \bigcup_{i=1}^{l} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \ldots, n$; then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_n}$.

The elements of a Morse decomposition are called **Morse sets.**

A Morse decomposition $\{\mathcal{M}_i,\ i = 1, \ldots, l\}$ is **finer** than another decomposition $\{\mathcal{N}_j,\ j = 1, \ldots, n\}$, if for all $\mathcal{M}_i$ there exists an index $j \in \{1, \ldots, n\}$ such that $\mathcal{M}_i \subset \mathcal{N}_j$.

Facts:

Literature: [Rob98], [CK00], [ACK05].

1. For a Morse decomposition $\{\mathcal{M}_i,\ i = 1, \ldots, l\}$ the relation $\mathcal{M}_i \prec \mathcal{M}_j$, given by $\alpha(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_j$ for some $x \in M \backslash \cup_{i=1}^{l} \mathcal{M}_i$, induces an order.
2. Let $\Phi, \Psi : \mathbb{R} \times M \longrightarrow M$ be two dynamical systems on a state space M and let $h : M \to M$ be a topological equivalence for Φ and Ψ. Then
   (i) The point $p \in M$ is a fixed point of Φ if and only if $h(p)$ is a fixed point of Ψ;
   (ii) The orbit $\Phi(\cdot, p)$ is closed if and only if $\Psi(\cdot, h(p))$ is closed;
   (iii) If $K \subset M$ is an α-(or ω-) limit set of Φ from $p \in M$, then $h\,[K]$ is an α-(or ω-) limit set of Ψ from $h(p) \in M$.
   (iv) Given, in addition, two dynamical systems $\Theta_{1,2} : \mathbb{R} \times N \longrightarrow N$, if $h : M \to M$ is a topological conjugacy for the flows Φ and Ψ on M, and $g : N \to N$ is a topological conjugacy for Θ_1 and Θ_2 on N, then the product flows $\Phi \times \Theta_1$ and $\Psi \times \Theta_2$ on $M \times N$ are topologically conjugate via $h \times g : M \times N \longrightarrow M \times N$. This result is, in general, not true for topological equivalence.
3. Topological equivalences (and conjugacies) on a compact metric space M map chain transitive sets onto chain transitive sets.
4. Topological equivalences map invariant sets onto invariant sets, and minimal closed invariant sets onto minimal closed invariant sets.
5. Topological equivalences map Morse decompositions onto Morse decompositions.

Examples:

1. Dynamical systems in $\mathbb{R}^1$: Any limit set $\alpha(x)$ and $\omega(x)$ from a single point x of a dynamical system in $\mathbb{R}^1$ consists of a single fixed point. The chain recurrent components (and the finest Morse decomposition) consist of single fixed points or intervals of fixed points. Any Morse set consists of fixed points and intervals between them.
2. Dynamical systems in $\mathbb{R}^2$: A nonempty, compact limit set of a dynamical system in $\mathbb{R}^2$, which contains no fixed points, is a closed, i.e., a periodic orbit (Poincaré–Bendixson). Any nonempty, compact limit set of a dynamical system in $\mathbb{R}^2$ consists of fixed points, connecting orbits (such as homoclinic or heteroclinic orbits), and periodic orbits.
3. Consider the following dynamical system Φ in $\mathbb{R}^2 \backslash \{0\}$, given by a differential equation in polar form for $r > 0, \theta \in [0, 2\pi)$, and $a \neq 0$:
$$\dot{r} = 1 - r,\ \dot{\theta} = a.$$
   For each $\mathbf{x} \in \mathbb{R}^2 \backslash \{0\}$ the ω-limit set is the circle $\omega(\mathbf{x}) = \mathbb{S}^1 = \{(r, \theta), r = 1, \theta \in [0, 2\pi)\}$. The state space $\mathbb{R}^2 \backslash \{0\}$ is not compact, and α-limit sets exist only for $\mathbf{y} \in \mathbb{S}^1$, for which $\alpha(\mathbf{y}) = \mathbb{S}^1$.
4. Consider the flow Φ from the previous example and a second system Ψ, given by
$$\dot{r} = 1 - r,\ \ \dot{\theta} = b$$
   with $b \neq 0$. Then the flows Φ and Ψ are topologically equivalent, but not conjugate if $b \neq a$.
5. An example of a flow for which the limit sets from points are strictly contained in the chain recurrent components can be obtained as follows: Let $M = [0, 1] \times [0, 1]$. Let the flow Φ on M be defined such that all points on the boundary are fixed points, and the orbits for points $(x, y) \in (0, 1) \times (0, 1)$ are straight lines $\Phi(\cdot, (x, y)) = \{(z_1, z_2), z_1 = x, z_2 \in (0, 1)\}$ with $\lim_{t \to \pm\infty} \Phi(t, (x, y)) = (x, \pm 1)$. For this system, each point on the boundary is its own α- and ω-limit set. The α-limit sets for points in the interior $(x, y) \in (0, 1) \times (0, 1)$ are of the form $\{(x, -1)\}$, and the ω-limit sets are $\{(x, +1)\}$.

The only chain recurrent component for this system is $M = [0,1] \times [0,1]$, which is also the only Morse set.

56.4 Linear Systems on Grassmannian and Flag Manifolds

Definitions:

The **kth Grassmannian** $\mathbb{G}_k$ of $\mathbb{R}^d$ can be defined via the following construction: Let $F(k,d)$ be the set of k-frames in $\mathbb{R}^d$, where a **k-frame** is an ordered set of k linearly independent vectors in $\mathbb{R}^d$. Two k-frames $X = [\mathbf{x}_1,\ldots,\mathbf{x}_k]$ and $Y = [\mathbf{y}_1,\ldots,\mathbf{y}_k]$ are said to be equivalent, $X \sim Y$, if there exists $T \in GL(k,\mathbb{R})$ with $X^T = TY^T$, where X and Y are interpreted as $d \times k$ matrices. The quotient space $\mathbb{G}_k = F(k,d)/\sim$ is a compact, $k(d-k)$-dimensional differentiable manifold. For $k = 1$, we obtain the projective space $\mathbb{P}^{d-1} = \mathbb{G}_1$ in $\mathbb{R}^d$.

The **kth flag** of $\mathbb{R}^d$ is given by the following k−sequences of subspace inclusions,

$$\mathbb{F}_k = \{F_k = (V_1,\ldots,V_k), V_i \subset V_{i+1} \text{ and } \dim V_i = i \text{ for all } i\}.$$

For $k = d$, this is the **complete flag** $\mathbb{F} = \mathbb{F}_d$.

Each matrix $A \in gl(d,\mathbb{R})$ defines a map on the subspaces of $\mathbb{R}^d$ as follows: Let $V = \text{Span}(\{\mathbf{x}_1,\ldots,\mathbf{x}_k\})$. Then $AV = \text{Span}(\{A\mathbf{x}_1,\ldots,A\mathbf{x}_k\})$.

Denote by $\mathbb{G}_k\Phi$ and $\mathbb{F}_k\Phi$ the induced flows on the Grassmannians and the flags, respectively.

Facts:

Literature: [Rob98], [CK00], [ACK05].

1. Let $\mathbb{P}\Phi$ be the projection onto $\mathbb{P}^{d-1}$ of a linear flow $\Phi(t,x) = e^{At}x$. Then $\mathbb{P}\Phi$ has l chain recurrent components $\{\mathcal{M}_1,\ldots,\mathcal{M}_l\}$, where l is the number of different Lyapunov exponents (i.e., of different real parts of eigenvalues) of A. For each Lyapunov exponent λ_i, $\mathcal{M}_i = \mathbb{P}L_i$, the projection of the ith Lyapunov space onto $\mathbb{P}^{d-1}$. Furthermore $\{\mathcal{M}_1,\ldots,\mathcal{M}_l\}$ defines the finest Morse decomposition of $\mathbb{P}\Phi$ and $\mathcal{M}_i \prec \mathcal{M}_j$ if and only if $\lambda_i < \lambda_j$.
2. For A, $B \in gl(d,\mathbb{R})$, let $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence h of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$. Then the chain recurrent components $\mathcal{N}_1,\ldots,\mathcal{N}_n$ of $\mathbb{P}\Psi$ are of the form $\mathcal{N}_i = h[\mathcal{M}_i]$, where $\mathcal{M}_i$ is a chain recurrent component of $\mathbb{P}\Phi$. In particular, the number of chain recurrent components of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ agree, and h maps the order on $\{\mathcal{M}_1,\ldots,\mathcal{M}_l\}$ onto the order on $\{\mathcal{N}_1,\ldots,\mathcal{N}_l\}$.
3. For A, $B \in gl(d,\mathbb{R})$ let $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological equivalence h of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$. Then the projected subspaces corresponding to real Jordan blocks of A are mapped onto projected subspaces corresponding to real Jordan blocks of B preserving the dimensions. Furthermore, h maps projected eigenspaces corresponding to real eigenvalues and to pairs of complex eigenvalues onto projected eigenspaces of the same type. This result shows that while C^0-equivalence of projected linear flows on $\mathbb{P}^{d-1}$ determines the number l of distinct Lyapunov exponents, it also characterizes the Jordan structure within each Lyapunov space (but, obviously, not the size of the Lyapunov exponents nor their sign). It imposes very restrictive conditions on the eigenvalues and the Jordan structure. Therefore, C^0-equivalences are not a useful tool to characterize l. The requirement of mapping orbits into orbits is too strong. A weakening leads to the following characterization.
4. Two matrices A and B in $gl(d,\mathbb{R})$ have the same vector of the dimensions d_i of the Lyapunov spaces (in the natural order of their Lyapunov exponents) if and only if there exist a homeomorphism $h : \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ that maps the finest Morse decomposition of $\mathbb{P}\Phi$ onto the finest Morse decomposition of $\mathbb{P}\Psi$, i.e., h maps Morse sets onto Morse sets and preserves their orders.

5. Let $A \in gl(d, \mathbb{R})$ with associated flows Φ on $\mathbb{R}^d$ and $\mathbb{F}_k\Phi$ on the k-flag.
   (i) For every $k \in \{1, \ldots, d\}$ there exists a unique finest Morse decomposition $\{_k\mathcal{M}_{i_j}\}$ of $\mathbb{F}_k\Phi$, where $i_j \in \{1, \ldots, d\}^k$ is a multi-index, and the number of chain transitive components in $\mathbb{F}_k$ is bounded by $\frac{d!}{(d-k)!}$.
   (ii) Let $\mathcal{M}_i$ with $i \in \{1, \ldots, d\}^k$ be a chain recurrent component in $\mathbb{F}_{k-1}$. Consider the $(d-k+1)$-dimensional vector bundle $\pi : \mathcal{W}(\mathcal{M}_i) \to \mathcal{M}_i$ with fibers

$$\mathcal{W}(\mathcal{M}_i)_{F_{k-1}} = \mathbb{R}^d / V_{k-1} \text{ for } F_k = (V_1, \ldots, V_{k-1}) \in \mathcal{M}_i \subset \mathbb{F}_{k-1}.$$

   Then every chain recurrent component $_{\mathbb{P}}\mathcal{M}_{i_j}$, $j = 1, \ldots, k_i \leq d - k + 1$, of the projective bundle $\mathbb{P}\mathcal{W}(\mathcal{M}_i)$ determines a chain recurrent component $_k\mathcal{M}_{i_j}$ on $\mathbb{F}_k$ via

$$_k\mathcal{M}_{i_j} = \left\{ F_k = (F_{k-1}, V_k) \in \mathbb{F}_k : F_{k-1} \in \mathcal{M}_i \text{ and } \mathbb{P}(V_k / V_{k-1}) \in {}_{\mathbb{P}}\mathcal{M}_{i_j} \right\}.$$

   Every chain recurrent component in $\mathbb{F}_k$ is of this form; this determines the multiindex i_j inductively for $k = 2, \ldots, d$.
6. On every Grassmannian $\mathbb{G}_i$ there exists a finest Morse decomposition of the dynamical system $\mathbb{G}_i\Phi$. Its Morse sets are given by the projection of the chain recurrent components from the complete flag $\mathbb{F}$.
7. Let $A \in gl(d, \mathbb{R})$ be a matrix with flow Φ on $\mathbb{R}^d$. Let L_i, $i = 1, \ldots, l$, be the Lyapunov spaces of A, i.e., their projections $\mathbb{P}L_i = \mathcal{M}_i$ are the finest Morse decomposition of $\mathbb{P}\Phi$ on the projective space. For $k = 1, \ldots, d$ define the index set

$$I(k) = \{(k_1, \ldots, k_m) : k_1 + \ldots + k_m = k \text{ and } 0 \leq k_i \leq d_i = \dim L_i\}.$$

   Then the finest Morse decomposition on the Grassmannian $\mathbb{G}_k$ is given by the sets

$$\mathcal{N}^k_{k_1, \ldots, k_m} = \mathbb{G}_{k_1} L_1 \oplus \ldots.. \oplus \mathbb{G}_{k_m} L_m, \ (k_1, \ldots, k_m) \in I(k).$$

8. For two matrices $A, B \in gl(d, \mathbb{R})$ the vector of the dimensions d_i of the Lyapunov spaces (in the natural order of their Lyapunov exponents) are identical if and only if certain graphs defined on the Grassmannians are isomorphic; see [ACK05].

Examples:

1. For $A \in gl(d, \mathbb{R})$ let Φ be its linear flow in $\mathbb{R}^d$. The flow Φ projects onto a flow $\mathbb{P}\Phi$ on $\mathbb{P}^{d-1}$, given by the differential equation

$$\dot{s} = h(s, A) = (A - s^T A s\ I)\, s, \text{ with } s \in \mathbb{P}^{d-1}.$$

   Consider the matrices

$$A = \mathrm{diag}(-1, -1, 1) \text{ and } B = \mathrm{diag}(-1, 1, 1).$$

   We obtain the following structure for the finest Morse decompositions on the Grassmannians for A:
   $\mathbb{G}_1$: $\mathcal{M}_1 = \{\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2)\}$ and $\mathcal{M}_3 = \{\mathrm{Span}(\mathbf{e}_3)\}$
   $\mathbb{G}_2$: $\mathcal{M}_{1,2} = \{\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2)\}$ and $\mathcal{M}_{1,3} = \{\{\mathrm{Span}(\mathbf{x}, \mathbf{e}_3)\} : \mathbf{x} \in \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2)\}$
   $\mathbb{G}_3$: $\mathcal{M}_{1,2,3} = \{\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\}$
   and for B we have
   $\mathbb{G}_1$: $\mathcal{N}_1 = \{\mathrm{Span}(\mathbf{e}_1)\}$ and $\mathcal{N}_2 = \{\mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)\}$
   $\mathbb{G}_2$: $\mathcal{N}_{1,2} = \{\mathrm{Span}(\mathbf{e}_1, \mathbf{x}) : \mathbf{x} \in \mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)\}$ and $\mathcal{N}_{2,3} = \{\mathrm{Span}(\mathbf{e}_2, \mathbf{e}_3)\}$
   $\mathbb{G}_3$: $\mathcal{N}_{1,2,3} = \{\mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\}$.

On the other hand, the Morse sets in the full flag are given for A and B by

$$\begin{bmatrix} \mathcal{M}_{1,2,3} \\ \mathcal{M}_{1,2} \\ \mathcal{M}_1 \end{bmatrix} \preceq \begin{bmatrix} \mathcal{M}_{1,2,3} \\ \mathcal{M}_{1,3} \\ \mathcal{M}_1 \end{bmatrix} \preceq \begin{bmatrix} \mathcal{M}_{1,2,3} \\ \mathcal{M}_{1,3} \\ \mathcal{M}_3 \end{bmatrix} \text{ and } \begin{bmatrix} \mathcal{N}_{1,2,3} \\ \mathcal{N}_{1,2} \\ \mathcal{N}_1 \end{bmatrix} \preceq \begin{bmatrix} \mathcal{N}_{1,2,3} \\ \mathcal{N}_{1,2} \\ \mathcal{N}_2 \end{bmatrix} \preceq \begin{bmatrix} \mathcal{N}_{1,2,3} \\ \mathcal{N}_{2,3} \\ \mathcal{N}_2 \end{bmatrix},$$

respectively. Thus, in the full flag, the numbers and the orders of the Morse sets coincide, while on the Grassmannians (together with the projection relations between different Grassmannians) one can distinguish also the dimensions of the corresponding Lyapunov spaces. (See [ACK05] for a precise statement.)

56.5 Linear Skew Product Flows

Developing a linear algebra for time varying systems $\dot{\mathbf{x}} = A(t)\mathbf{x}$ means defining appropriate concepts to generalize eigenvalues, linear eigenspaces and their dimensions, and certain normal forms that characterize the behavior of the solutions of a time varying system and that reduce to the constant matrix case if $A(t) \equiv A \in gl(d,\mathbb{R})$. The eigenvalues and eigenspaces of the family $\{A(t),\, t \in \mathbb{R}\}$ do not provide an appropriate generalization; see, e.g., [Hah67], Chapter 62. For certain classes of time varying systems it turns out that the Lyapunov exponents and Lyapunov spaces introduced in section 56.1 capture the key properties of (real parts of) eigenvalues and of the associated subspace decomposition of $\mathbb{R}^d$. These systems are linear skew product flows for which the base is a (nonlinear) system θ_t that enters into the linear dynamics of a differential equation in the form $\dot{\mathbf{x}} = A(\theta_t)\mathbf{x}$. Examples for this type of systems include periodic and almost periodic differential equations, random differential equations, systems over ergodic or chain recurrent bases, linear robust systems, and bilinear control systems. This section concentrates on periodic linear differential equations, random linear dynamical systems, and robust linear systems. It is written to emphasize the correspondences between the linear algebra in Section 56.1, Floquet theory, the multiplicative ergodic theorem, and the Morse spectrum and Selgrade's theorem.
Literature: [Arn98], [BK94], [CK00], [Con97], [Rob98].

Definitions:

A (continuous time) **linear skew-product flow** is a dynamical system with state space $M = \Omega \times \mathbb{R}^d$ and flow $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$, where $\Phi = (\theta,\varphi)$ is defined as follows: $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is a dynamical system, and $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is linear in its $\mathbb{R}^d$-component, i.e., for each $(t,\omega) \in \mathbb{R} \times \Omega$ the map $\varphi(t,\omega,\cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is linear. Skew-product flows are called measurable (continuous, differentiable) if $\Omega = (\theta,\varphi)$ is a measurable space (topological space, differentiable manifold) and Φ is measurable (continuous, differentiable). For the time-t maps, the notation $\theta_t = \theta(t,\cdot) : \Omega \longrightarrow \Omega$ is used again.

Note that the **base component** $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is a dynamical system itself, while the **skew-component** φ is not a dynamical system. The skew-component φ is often called a **co-cycle** over θ.

Let $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$ be a linear skew-product flow. For $\mathbf{x}_0 \in \mathbb{R}^d$, $\mathbf{x}_0 \neq 0$, the **Lyapunov exponent** is defined as $\lambda(\mathbf{x}_0,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t,\omega,\mathbf{x}_0)\|$, where log denotes the natural logarithm and $\|\cdot\|$ is any norm in $\mathbb{R}^d$.

Examples:

1. Time varying linear differential equations: Let $A : \mathbb{R} \longrightarrow gl(d,\mathbb{R})$ be a uniformly continuous function and consider the linear differential equation $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$. The solutions of this differential equation define a dynamical system via $\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \times \mathbb{R}^d$, where $\theta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is given by $\theta(t,\tau) = t+\tau$, and $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined as $\varphi(t,\tau,\mathbf{x}_0) = X(t+\tau,\tau)\mathbf{x}_0$. Here $X(t,\tau)$ is a fundamental matrix of the differential equation $\dot{X}(t) = A(t)X(t)$ in $gl(d,\mathbb{R})$. Note that for $\varphi(t,\tau,\cdot) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}$, we have $\varphi(t+s,\tau) = \varphi(t,\theta(s,\tau)) \circ \varphi(s,\tau)$ and, hence, the

solutions of $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ themselves do not define a flow. The additional component θ "keeps track of time."

2. Metric dynamical systems: Let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e., a set Ω with σ-algebra $\mathcal{F}$ and probability measure P. Let $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ be a measurable flow such that the probability measure P is invariant under θ, i.e., $\theta_t P = P$ for all $t \in \mathbb{R}$, where for all measurable sets $X \in \mathcal{F}$ we define $\theta_t P(X) = P\{\theta_t^{-1}(X)\} = P(X)$. Flows of this form are often called metric dynamical systems.
3. Random linear dynamical systems: A random linear dynamical system is a skew-product flow $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$, where $(\Omega, \mathcal{F}, P, \theta)$ is a metric dynamical system and each $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is linear in its $\mathbb{R}^d$-component. Examples for random linear dynamical systems are given, e.g., by linear stochastic differential equations or linear differential equations with stationary background noise; see [Arn98].
4. Robust linear systems: Consider a linear system with time varying perturbations of the form $\dot{\mathbf{x}} = A(u(t))\mathbf{x} := A_0\mathbf{x} + \sum_{i=1}^{m} u_i(t)A_i\mathbf{x}$, where $A_0, \ldots, A_m \in gl(d, \mathbb{R})$, $u \in \mathcal{U} = \{u : \mathbb{R} \longrightarrow U$, integrable on every bounded interval$\}$, and $U \subset \mathbb{R}^m$ is compact, convex with $0 \in \operatorname{int} U$. A robust linear system defines a linear skew-product flow via the following construction: We endow $\mathcal{U}$ with the weak*-topology of $L^\infty(\mathbb{R}, U)^*$ to make it a compact, metrizable space. The base component is defined as the shift $\theta : \mathbb{R} \times \mathcal{U} \longrightarrow \mathcal{U}$, $\theta(t, u(\cdot)) = u(\cdot + t)$, and the skew-component consists of the solutions $\varphi(t, u(\cdot), \mathbf{x})$, $t \in \mathbb{R}$ of the perturbed differential equation. Then $\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \longrightarrow \mathcal{U} \times \mathbb{R}^d$, $\Phi(t, u, \mathbf{x}) = (\theta(t, u), \varphi(t, u, \mathbf{x}))$ defines a continuous linear skew-product flow. The functions u can also be considered as (open loop) controls.

56.6 Periodic Linear Differential Equations: Floquet Theory

Definitions:

A **periodic linear differential equation** $\dot{\mathbf{x}} = A(\theta_t)\mathbf{x}$ is given by a matrix function $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ that is continuous and periodic (of period $\widehat{t} > 0$). As above, the solutions define a dynamical system via $\Phi : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$, if we identify $\mathbb{R} \bmod \widehat{t}$ with the circle $\mathbb{S}^1$.

Facts:

Literature: [Ama90], [GH83], [Hah67], [Sto92], [Wig96].

1. Consider the periodic linear differential equation $\dot{\mathbf{x}} = A(\theta_t)\mathbf{x}$ with period $\widehat{t} > 0$. A fundamental matrix $X(t)$ of the system is of the form $X(t) = P(t)e^{Rt}$ for $t \in \mathbb{R}$, where $P(\cdot)$ is a nonsingular, differentiable, and $\widehat{t}$-periodic matrix function and $R \in gl(d, \mathbb{C})$.
2. Let $X(\cdot)$ be a fundamental solution with $X(0) = I \in GL(d, \mathbb{R})$. The matrix $X(\widehat{t}) = e^{R\widehat{t}}$ is called the monodromy matrix of the system. Note that R is, in general, not uniquely determined by X, and does not necessarily have real entries. The eigenvalues α_j, $j = 1, \ldots, d$ of $X(\widehat{t})$ are called the characteristic multipliers of the system, and the eigenvalues $\mu_j = \lambda_j + i\nu_j$ of R are the characteristic exponents. It holds that $\mu_j = \frac{1}{\widehat{t}} \log \alpha_j + \frac{2m\pi i}{\widehat{t}}$, $j = 1, \ldots, d$ and $m \in \mathbb{Z}$. This determines uniquely the real parts of the characteristic exponents $\lambda_j = \operatorname{Re} \mu_j = \log \left|\alpha_j\right|$, $j = 1, \ldots, d$. The λ_j are called the Floquet exponents of the system.
3. Let $\Phi = (\theta, \varphi) : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$ be the flow associated with a periodic linear differential equation $\dot{\mathbf{x}} = A(t)\mathbf{x}$. The system has a finite number of Lyapunov exponents λ_j, $j = 1, \ldots, l \leq d$. For each exponent λ_j and each $\tau \in \mathbb{S}^1$ there exists a splitting $\mathbb{R}^d = \bigoplus_{j=1}^{l} L(\lambda_j, \tau)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:

   (a) The subspaces $L(\lambda_j, \tau)$ have the same dimension independent of τ, i.e., for each $j = 1, \ldots, l$ it holds that $\dim L(\lambda_j, \sigma) = \dim L(\lambda_j, \tau) =: d_i$ for all $\sigma, \tau \in \mathbb{S}^1$.

(b) The subspaces $L(\lambda_j, \tau)$ are invariant under the flow Φ, i.e., for each $j = 1, \ldots, l$ it holds that $\varphi(t, \tau)L(\lambda_j, \tau) = L(\lambda_j, \theta(t, \tau)) = L(\lambda_j, t + \tau)$ for all $t \in \mathbb{R}$ and $\tau \in \mathbb{S}^1$.

(c) $\lambda(\mathbf{x}, \tau) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t, \tau, \mathbf{x})\| = \lambda_j$ if and only if $\mathbf{x} \in L(\lambda_j, \tau)\setminus\{0\}$.

4. The Lyapunov exponents of the system are exactly the Floquet exponents. The linear subspaces $L(\lambda_j, \cdot)$ are called the Lyapunov spaces (or sometimes the Floquet spaces) of the periodic matrix function $A(t)$.
5. For each $j = 1, \ldots, l \leq d$ the map $L_j : \mathbb{S}^1 \longrightarrow \mathbb{G}_{d_j}$ defined by $\tau \longmapsto L(\lambda_j, \tau)$ is continuous.
6. These facts show that for periodic matrix functions $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ the Floquet exponents and Floquet spaces replace the real parts of eigenvalues and the Lyapunov spaces, concepts that are so useful in the linear algebra of (constant) matrices $A \in gl(d, \mathbb{R})$. The number of Lyapunov exponents and the dimensions of the Lyapunov spaces are constant for $\tau \in \mathbb{S}^1$, while the Lyapunov spaces themselves depend on the time parameter τ of the periodic matrix function $A(t)$, and they form periodic orbits in the Grassmannians $\mathbb{G}_{d_j}$ and in the corresponding flag.
7. As an application of these results, consider the problem of stability of the zero solution of $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ with period $\widehat{t} > 0$: The stable, center, and unstable subspaces associated with the periodic matrix function $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ are defined as $L^-(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j < 0\}$, $L^0(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j = 0\}$, and $L^+(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j > 0\}$, respectively, for $\tau \in \mathbb{S}^1$. The zero solution $\mathbf{x}(t, 0) \equiv 0$ of the periodic linear differential equation $\dot{\mathbf{x}} = A(t)\mathbf{x}$ is asymptotically stable if and only if it is exponentially stable if and only if all Lyapunov exponents are negative if and only if $L^-(\tau) = \mathbb{R}^d$ for some (and hence for all) $\tau \in \mathbb{S}^1$.
8. Another approach to the study of time-dependent linear differential equations is via transforming an equation with bounded coefficients into an equation of known type, such as equations with constant coefficients. Such transformations are known as Lyapunov transformations; see [Hah67, Secs. 61–63].

Examples:

1. Consider the $\widehat{t}$-periodic differential equation $\dot{\mathbf{x}} = A(t)\mathbf{x}$. This equation has a nontrivial $\widehat{t}$-periodic solution iff the system has a characteristic multiplier equal to 1; see Example 2.3 for the case with constant coefficients ([Ama90, Prop. 20.12]).
2. Let H be a continuous quadratic form in $2d$ variables $x_1, \ldots, x_d, y_1, \ldots, y_d$ and consider the Hamiltonian system

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \dot{y}_i = -\frac{\partial H}{\partial x_i}, i = 1, \ldots, d.$$

Using $\mathbf{z}^T = [\mathbf{x}^T, \mathbf{y}^T]$, we can set $H(\mathbf{x}, \mathbf{y}, t) = \mathbf{z}^T A(t)\mathbf{z}$, where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$ with A_{11} and A_{22} symmetric, and, hence, the equation takes the form

$$\dot{\mathbf{z}} = \begin{bmatrix} A_{12}^T(t) & A_{22}(t) \\ -A_{11}(t) & -A_{12}(t) \end{bmatrix} \mathbf{z} =: P(t)\mathbf{z}.$$

Note that $-P^T(t) = QP(t)Q^{-1}$ with $Q = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, where I is the $d \times d$ identity matrix. Assume that H is $\widehat{t}$-periodic, then the equation for z and its adjoint have the same Floquet exponents and for each exponent λ its negative $-\lambda$ is also a Floquet exponent. Hence, the fixed point $0 \in \mathbb{R}^{2d}$ cannot be exponentially stable ([Hah67, Sec. 60]).
3. Consider the periodic linear oscillator

$$\ddot{y} + q_1(t)\dot{y} + q_2(t)y = 0.$$

Using the substitution $y = z\exp(-\frac{1}{2}\int q_1(u)du)$ one obtains Hill's differential equation

$$\ddot{z} + p(t)z = 0,\ p(t) := q_2(t) - \frac{1}{4}q_1(t)^2 - \frac{1}{2}\dot{q}_1(t).$$

Its characteristic equation is $\lambda^2 - 2a\lambda + 1 = 0$, with a still to be determined. The multipliers satisfy the relations $\alpha_1\alpha_2 = 1$ and $\alpha_1 + \alpha_2 = 2a$. The exponential stability of the system can be analyzed using the parameter a: If $a^2 > 1$, then one of the multipliers has absolute value > 1 and, hence, the system has an unbounded solution. If $a^2 = 1$, then the system has a nontrivial periodic solution according to Example 1. If $a^2 < 1$, then the system is stable. The parameter a can often be expressed in form of a power series; see [Hah67, Sec. 62] for more details. A special case of Hill's equation is the Mathieu equation

$$\ddot{z} + (\beta_1 + \beta_2 \cos 2t)z = 0,$$

with β_1, β_2 real parameters. For this equation numerically computed stability diagrams are available; see [Sto92, Secs. VI. 3 and 4].

56.7 Random Linear Dynamical Systems

Definitions:

Let $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ be a metric dynamical system on the probability space $(\Omega, \mathcal{F}, P)$. A set $\Delta \in \mathcal{F}$ is called **P-invariant under** θ if $P[(\theta^{-1}(t,\Delta) \setminus \Delta) \cup (\Delta \setminus \theta^{-1}(t,\Delta))] = 0$ for all $t \in \mathbb{R}$. The flow θ is called **ergodic**, if each invariant set $\Delta \in \mathcal{F}$ has P-measure 0 or 1.

Facts:

Literature: [Arn98], [Con97].

1. (Oseledets Theorem, Multiplicative Ergodic Theorem) Consider a random linear dynamical system $\Phi = (\theta, \varphi) : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$ and assume

$$\sup_{0 \le t \le 1} \log^+ \|\varphi(t,\omega)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \text{ and } \sup_{0 \le t \le 1} \log^+ \left\|\varphi(t,\omega)^{-1}\right\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}),$$

where $\|\cdot\|$ is any norm on $GL(d,\mathbb{R})$, $\mathcal{L}^1$ is the space of integrable functions, and $\log^+$ denotes the positive part of log, i.e.,

$$\log^+(x) = \begin{cases} \log(x) & \text{for } \log(x) > 0 \\ 0 & \text{for } \log(x) \le 0. \end{cases}$$

Then there exists a set $\widehat{\Omega} \subset \Omega$ of full P-measure, invariant under the flow $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$, such that for each $\omega \in \widehat{\Omega}$ there is a splitting $\mathbb{R}^d = \bigoplus_{j=1}^{l(\omega)} L_j(\omega)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:

(a) The number of subspaces is θ-invariant, i.e., $l(\theta(t,\omega)) = l(\omega)$ for all $t \in \mathbb{R}$, and the dimensions of the subspaces are θ-invariant, i.e., $\dim L_j(\theta(t,\omega)) = \dim L_j(\omega) =: d_j(\omega)$ for all $t \in \mathbb{R}$.

(b) The subspaces are invariant under the flow Φ, i.e., $\varphi(t,\omega)L_j(\omega) \subset L_j(\theta(t,\omega))$ for all $j = 1, \ldots, l(\omega)$.

(c) There exist finitely many numbers $\lambda_1(\omega) < \ldots < \lambda_{l(\omega)}(\omega)$ in $\mathbb{R}$ (with possibly $\lambda_1(\omega) = -\infty$), such that for each $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent $\lambda(\mathbf{x},\omega)$ exists as a limit and

$\lambda(\mathbf{x},\omega) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t,\tau,\mathbf{x})\| = \lambda_j(\omega)$ if and only if $\mathbf{x} \in L_j(\omega)\backslash\{0\}$. The subspaces $L_j(\omega)$ are called the Lyapunov (or sometimes the Oseledets) spaces of the system Φ.

2. The following maps are measurable: $l : \Omega \longrightarrow \{1,\ldots,d\}$ with the discrete σ-algebra, and for each $j = 1,\ldots,l(\omega)$ the maps $L_j : \Omega \longrightarrow \mathbb{G}_{d_j}$ with the Borel σ-algebra, $d_j : \Omega \longrightarrow \{1,\ldots,d\}$ with the discrete σ-algebra, and $\lambda_j : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ with the (extended) Borel σ-algebra.
3. If the base flow $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ is ergodic, then the maps l, d_j, and λ_j are constant on $\widehat{\Omega}$, but the Lyapunov spaces $L_j(\omega)$ still depend (in a measurable way) on $\omega \in \widehat{\Omega}$.
4. As an application of these results, we consider random linear differential equations: Let $(\Gamma,\mathcal{E},Q)$ be a probability space and $\xi : \mathbb{R} \times \Gamma \longrightarrow \mathbb{R}^m$ a stochastic process with continuous trajectories, i.e., the functions $\xi(\cdot,\gamma) : \mathbb{R} \longrightarrow \mathbb{R}^m$ are continuous for all $\gamma \in \Gamma$. The process ξ can be written as a measurable dynamical system in the following way: Define $\Omega = \mathcal{C}(\mathbb{R},\mathbb{R}^m)$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^m$. We denote by $\mathcal{F}$ the σ-algebra on Ω generated by the cylinder sets, i.e., by sets of the form $Z = \{\omega \in \Omega,\ \omega(t_1) \in F_1,\ldots,\omega(t_n) \in F_n,\ n \in \mathbb{N},\ F_i$ Borel sets in $\mathbb{R}^m\}$. The process ξ induces a probability measure P on $(\Omega,\mathcal{F})$ via $P(Z) = Q\{\gamma \in \Gamma, \xi(t_i,\gamma) \in F_i$ for $i = 1,\ldots,n\}$. Define the shift $\theta : \mathbb{R} \times \Omega \longrightarrow \mathbb{R} \times \Omega$ as $\theta(t,\omega(\cdot)) = \omega(t+\cdot)$. Then $(\Omega,\mathcal{F},P,\theta)$ is a measurable dynamical system. If ξ is stationary, i.e., if for all $n \in \mathbb{N}$, and $t, t_1,\ldots,t_n \in \mathbb{R}$, and all Borel sets $F_1,\ldots,F_n$ in $\mathbb{R}^m$, it holds that $Q\{\gamma \in \Gamma, \xi(t_i,\gamma) \in F_i$ for $i = 1,\ldots,n\} = Q\{\gamma \in \Gamma$, $\xi(t_i+t,\gamma) \in F_i$ for $i = 1,\ldots,n\}$, then the shift θ on Ω is P-invariant, and $(\Omega,\mathcal{F},P,\theta)$ is a metric dynamical system.
5. Let $A : \Omega \longrightarrow gl(d,\mathbb{R})$ be measurable with $A \in \mathcal{L}^1$. Consider the random linear differential equation $\dot{\mathbf{x}}(t) = A(\theta(t,\omega))\mathbf{x}(t)$, where $(\Omega,\mathcal{F},P,\theta)$ is a metric dynamical system as described before. We understand the solutions of this equation to be ω-wise. Then the solutions define a random linear dynamical system. Since we assume that $A \in \mathcal{L}^1$, this system satisfies the integrability conditions of the Multiplicative Ergodic Theorem.
6. Hence, for random linear differential equations $\dot{\mathbf{x}}(t) = A(\theta(t,\omega))\mathbf{x}(t)$ the Lyapunov exponents and the associated Oseledets spaces replace the real parts of eigenvalues and the Lyapunov spaces of constant matrices $A \in gl(d,\mathbb{R})$. If the "background" process ξ is ergodic, then all the quantities in the Multiplicative Ergodic Theorem are constant, except for the Lyapunov spaces that do, in general, depend on chance.
7. The problem of stability of the zero solution of $\dot{\mathbf{x}}(t) = A(\theta(t,\omega))\mathbf{x}(t)$ can now be analyzed in analogy to the case of a constant matrix or a periodic matrix function: The stable, center, and unstable subspaces associated with the random matrix process $A(\theta(t,\omega))$ are defined as $L^-(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) < 0\}$, $L^0(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) = 0\}$, and $L^+(\omega) = \bigoplus\{L_j(\omega), \lambda_j(\omega) > 0\}$, respectively for $\omega \in \widehat{\Omega}$. We obtain the following characterization of stability: The zero solution $\mathbf{x}(t,\omega,0) \equiv 0$ of the random linear differential equation $\dot{\mathbf{x}}(t) = A(\theta(t,\omega))\mathbf{x}(t)$ is P-almost surely exponentially stable if and only if P-almost surely all Lyapunov exponents are negative if and only if $P\{\omega \in \Omega$, $L^-(\omega) = \mathbb{R}^d\} = 1$.

Examples:

1. The case of constant matrices: Let $A \in gl(d,\mathbb{R})$ and consider the dynamical system $\varphi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ generated by the solutions of the linear differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. The flow φ can be considered as the skew-component of a random linear dynamical system over the base flow given by $\Omega = \{0\}$, $\mathcal{F}$ the trivial σ-algebra, P the Dirac measure at $\{0\}$, and $\theta : \mathbb{R} \times \Omega \longrightarrow \Omega$ defined as the constant map $\theta(t,\omega) = \omega$ for all $t \in \mathbb{R}$. Since the flow is ergodic and satisfies the integrability condition, we can recover all the results on Lyapunov exponents and Lyapunov spaces for φ from the Multiplicative Ergodic Theorem.
2. Weak Floquet theory: Let $A : \mathbb{R} \longrightarrow gl(d,\mathbb{R})$ be a continuous, periodic matrix function. Define the base flow as follows: $\Omega = \mathbb{S}^1$, $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{S}^1$, P is the uniform distribution on $\mathbb{S}^1$, and θ is the shift $\theta(t,\tau) = t+\tau$. Then $(\Omega,\mathcal{F},P,\theta)$ is an ergodic metric dynamical system. The solutions $\varphi(\cdot,\tau,\mathbf{x})$ of $\dot{\mathbf{x}} = A(t)\mathbf{x}$ define a random linear dynamical system $\Phi : \mathbb{R} \times \Omega \times \mathbb{R}^d \longrightarrow \Omega \times \mathbb{R}^d$ via

$\Phi(t,\omega,\mathbf{x}) = (\theta(t,\omega),\varphi(t,\omega,\mathbf{x}))$. With this set-up, the Multiplicative Ergodic Theorem recovers the results of Floquet Theory with P-probability 1.

3. Average Lyapunov exponent: In general, Lyapunov exponents for random linear systems are difficult to compute explicitly — numerical methods are usually the way to go. In the ergodic case, the average Lyapunov exponent $\overline{\lambda} := \frac{1}{d}\sum d_j\lambda_j$ is given by $\overline{\lambda} = \frac{1}{d} tr\, E(A \mid \mathcal{I})$, where $A : \Omega \longrightarrow gl(d,\mathbb{R})$ is the random matrix of the system, and $E(\cdot,\mathcal{I})$ is the conditional expectation of the probability measure P given the σ-algebra $\mathcal{I}$ of invariant sets on Ω. As an example, consider the linear oscillator with random restoring force

$$\ddot{y}(t) + 2\beta\dot{y}(t) + (1+\sigma f(\theta(t,\omega)))y(t) = 0,$$

where $\beta,\sigma \in \mathbb{R}$ are positive constants and $f : \Omega \to \mathbb{R}$ is in $\mathcal{L}^1$. We assume that the background process is ergodic. Using the notation $x_1 = y$ and $x_2 = \dot{y}$ we can write the equation as

$$\dot{\mathbf{x}}(t) = A(\theta(t,\omega)\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -1-\sigma f(\theta(t,\omega)) & -2\beta \end{bmatrix} \mathbf{x}(t).$$

For this system we obtain $\overline{\lambda} = -\beta$ ([Arn98, Remark 3.3.12]).

56.8 Robust Linear Systems

Definitions:

Let $\Phi : \mathbb{R}\times\mathcal{U}\times\mathbb{R}^d \longrightarrow \mathcal{U}\times\mathbb{R}^d$ be a linear skew-product flow with continuous base flow $\theta : \mathbb{R}\times\mathcal{U} \longrightarrow \mathcal{U}$. Throughout this section, $\mathcal{U}$ is compact and θ is chain recurrent on $\mathcal{U}$. Denote by $\mathcal{U}\times\mathbb{P}^{d-1}$ the projective bundle and recall that Φ induces a dynamical system $\mathbb{P}\Phi : \mathbb{R}\times\mathcal{U}\times\mathbb{P}^{d-1} \longrightarrow \mathcal{U}\times\mathbb{P}^{d-1}$. For $\varepsilon, T > 0$ an (ε, T)-chain ζ of $\mathbb{P}\Phi$ is given by $n \in \mathbb{N}$, $T_0,\ldots,T_n \geq T$, and $(u_0,p_0),\ldots,(u_n,p_n) \in \mathcal{U}\times\mathbb{P}^{d-1}$ with $d(\mathbb{P}\Phi(T_i,u_i,p_i),(u_{i+1},p_{i+1})) < \varepsilon$ for $i = 0,\ldots,n-1$.

Define the **finite time exponential growth rate** of such a chain ζ (or **chain exponent**) by

$$\lambda(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} (\log\|\varphi(T_i,x_i,u_i)\| - \log\|x_i\|),$$

where $x_i \in \mathbb{P}^{-1}(p_i)$.

Let $\mathcal{M} \subset \mathcal{U}\times\mathbb{P}^{d-1}$ be a chain recurrent component of the flow $\mathbb{P}\Phi$. Define the **Morse spectrum over** $\mathcal{M}$ as

$$\Sigma_{Mo}(\mathcal{M}) = \left\{ \begin{array}{c} \lambda \in \mathbb{R}, \text{there exist sequences } \varepsilon_n \to 0,\ T_n \to \infty \text{ and} \\ (\varepsilon_n, T_n)\text{-chains } \zeta_n \text{ in } \mathcal{M} \text{ such that } \lim \lambda(\zeta_n) = \lambda \end{array} \right\}$$

and the **Morse spectrum of the flow** as

$$\Sigma_{Mo}(\Phi) = \left\{ \begin{array}{c} \lambda \in \mathbb{R}, \text{there exist sequences } \varepsilon_n \to 0,\ T_n \to \infty \text{ and } (\varepsilon_n, T_n)\text{-} \\ \text{chains } \zeta_n \text{ in the chain recurrent set of } \mathbb{P}\Phi \text{ such that } \lim \lambda(\zeta_n) = \lambda \end{array} \right\}.$$

Define the **Lyapunov spectrum over** $\mathcal{M}$ as

$$\Sigma_{Ly}(\mathcal{M}) = \{\lambda(u,x),\ (u,x) \in \mathcal{M},\ x \neq 0\}$$

and the **Lyapunov spectrum of the flow** Φ as

$$\Sigma_{Ly}(\Phi) = \{\lambda(u,\mathbf{x}),\ (u,\mathbf{x}) \in \mathcal{U}\times\mathbb{R}^d,\ \mathbf{x} \neq 0\}.$$

Facts:

Literature: [CK00], [Gru96], [HP05].

1. The projected flow $\mathbb{P}\Phi$ has a finite number of chain-recurrent components $\mathcal{M}_1, \ldots, \mathcal{M}_l$, $l \leq d$. These components form the finest Morse decomposition for $\mathbb{P}\Phi$, and they are linearly ordered $\mathcal{M}_1 \prec \ldots \prec \mathcal{M}_l$. Their lifts $\mathbb{P}^{-1}\mathcal{M}_i \subset \mathcal{U} \times \mathbb{R}^d$ form a continuous subbundle decomposition of $\mathcal{U} \times \mathbb{R}^d = \bigoplus_{i=1}^{l} \mathbb{P}^{-1}\mathcal{M}_i$.
2. The Lyapunov spectrum and the Morse spectrum are defined on the Morse sets, i.e., $\Sigma_{Ly}(\Phi) = \bigcup_{i=1}^{l} \Sigma_{Ly}(\mathcal{M}_i)$ and $\Sigma_{Mo}(\Phi) = \bigcup_{i=1}^{l} \Sigma_{Mo}(\mathcal{M}_i)$.
3. For each Morse set $\mathcal{M}_i$ the Lyapunov spectrum is contained in the Morse spectrum, i.e., $\Sigma_{Ly}(\mathcal{M}_i) \subset \Sigma_{Mo}(\mathcal{M}_i)$ for $i = 1, \ldots, l$.
4. For each Morse set, its Morse spectrum is a closed, bounded interval $\Sigma_{Mo}(\mathcal{M}_i) = [\kappa_i^*, \kappa_i]$, and $\kappa_i^*, \kappa_i \in \Sigma_{Ly}(\mathcal{M})$ for $i = 1, \ldots, l$.
5. The intervals of the Morse spectrum are ordered according to the order of the Morse sets, i.e., $\mathcal{M}_i \prec \mathcal{M}_j$ is equivalent to $\kappa_i^* < \kappa_j^*$ and $\kappa_i < \kappa_j$.
6. As an application of these results, consider robust linear systems of the form $\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \longrightarrow \mathcal{U} \times \mathbb{R}^d$, given by a perturbed linear differential equation $\dot{\mathbf{x}} = A(u(t))\mathbf{x} := A_0\mathbf{x} + \sum_{i=1}^{m} u_i(t) A_i \mathbf{x}$, with $A_0, \ldots, A_m \in gl(d, \mathbb{R})$, $u \in \mathcal{U} = \{u : \mathbb{R} \longrightarrow U$, integrable on every bounded interval$\}$ and $U \subset \mathbb{R}^m$ is compact, convex with $0 \in intU$. Explicit equations for the induced perturbed system on the projective space $\mathbb{P}^{d-1}$ can be obtained as follows: Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere embedded into $\mathbb{R}^d$. The projected system on $\mathbb{S}^{d-1}$ is given by

$$\dot{s}(t) = h(u(t), s(t)),\ u \in \mathcal{U},\ s \in \mathbb{S}^{d-1}$$

where

$$h(u, s) = h_0(s) + \sum_{i=1}^{m} u_i h_i(s) \text{ with } h_i(s) = \left(A_i - s^T A_i s \cdot I\right) s,\ i = 0, 1, \ldots, m.$$

Define an equivalence relation on $\mathbb{S}^{d-1}$ via $s_1 \sim s_2$ if $s_1 = -s_2$, identifying opposite points. Then the projective space can be identified as $\mathbb{P}^{d-1} = \mathbb{S}^{d-1}/\sim$. Since $h(u, s) = -h(u, -s)$, the differential equation also describes the projected system on $\mathbb{P}^{d-1}$. For the Lyapunov exponents one obtains in the same way

$$\lambda(u, \mathbf{x}) = \limsup_{t \to \infty} \frac{1}{t} \log \|\mathbf{x}(t)\| = \limsup_{t \to \infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau))\, d\tau$$

with

$$q(u, s) = q_0(s) + \sum_{i=1}^{m} u_i q_i(s) \text{ with } q_i(s) = \left(A_i - s^T A_i s \cdot I\right) s,\ i = 0, 1, \ldots, m.$$

For a constant perturbation $u(t) \equiv u \in \mathbb{R}$ for all $t \in \mathbb{R}$ the corresponding Lyapunov exponents $\lambda(u, \mathbf{x})$ of the flow Φ are the real parts of the eigenvalues of the matrix $A(u)$ and the corresponding Lyapunov spaces are contained in the subbundles $\mathbb{P}^{-1}\mathcal{M}_i$. Similarly, if a perturbation $u \in \mathcal{U}$ is periodic, the Floquet exponents of $\dot{\mathbf{x}} = A(u(\cdot))\mathbf{x}$ are part of the Lyapunov (and, hence, of the Morse) spectrum of the flow Φ, and the Floquet spaces are contained in $\mathbb{P}^{-1}\mathcal{M}_i$. The systems treated in this example can also be considered as "bilinear control systems" and studied relative to their control behavior and (exponential) stabilizability — this is the point of view taken in [CK00].
7. For robust linear systems "generically" the Lyapunov spectrum and the Morse spectrum agree see [CK00] for a precise definition of "generic" in this context.
8. Of particular interest is the upper spectral interval $\Sigma_{Mo}(\mathcal{M}_l) = [\kappa_l^*, \kappa_l]$, as it determines the robust stability of $\dot{\mathbf{x}} = A(u(t))\mathbf{x}$ (and stabilizability of the system if the set $\mathcal{U}$ is interpreted as a

set of admissible control functions; see [Gru96]). The stable, center, and unstable subbundles of $\mathcal{U} \times \mathbb{R}^d$ associated with the perturbed linear system $\dot{\mathbf{x}} = A(u(t))\mathbf{x}$ are defined as $L^- = \bigoplus\{\mathbb{P}^{-1}\mathcal{M}_j, \kappa_j < 0\}$, $L^0 = \bigoplus\{\mathbb{P}^{-1}\mathcal{M}_j, 0 \in [\kappa_j^*, \kappa_j]\}$, and $L^+ = \bigoplus\{\mathbb{P}^{-1}\mathcal{M}_j, \kappa_j^* > 0\}$, respectively. The zero solution of $\dot{\mathbf{x}} = A(u(t))\mathbf{x}$ is exponentially stable for all perturbations $u \in \mathcal{U}$ if and only if $\kappa_l < 0$ if and only if $L^- = \mathcal{U} \times \mathbb{R}^d$.

Examples:

1. In general, it is not possible to compute the Morse spectrum and the associated subbundle decompositions explicitly, even for relatively simple systems, and one has to revert to numerical algorithms; compare [CK00, App. D]. Let us consider, e.g., the linear oscillator with uncertain restoring force

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$. Figure 56.1 shows the spectral intervals for this system depending on $\rho \geq 0$.

2. We consider robust linear systems as described in Fact 6, with varying perturbation range by introducing the family $U^\rho = \rho U$ for $\rho \geq 0$. The resulting family of systems is

$$\dot{\mathbf{x}}^\rho = A(u^\rho(t))\mathbf{x}^\rho := A_0\mathbf{x}^\rho + \sum_{i=1}^{m} u_i^\rho(t) A_i \mathbf{x}^\rho,$$

with $u^\rho \in \mathcal{U}^\rho = \{u : \mathbb{R} \longrightarrow U^\rho$, integrable on every bounded interval$\}$. The corresponding maximal spectral value $\kappa_l(\rho)$ is continuous in ρ and we define the (asymptotic) stability radius of this family as $r = \inf\{\rho \geq 0$, there exists $u_0 \in \mathcal{U}^\rho$ such that $\dot{x}^\rho = A(u_0(t))x^\rho$ is not exponentially stable$\}$. This stability radius is based on asymptotic stability under all time varying perturbations. Similarly one can introduce stability radii based on time invariant perturbations (with values in $\mathbb{R}^m$ or $\mathbb{C}^m$) or on quadratic Lyapunov functions ([CK00], Chapter 11 and [HP05]).

3. Linear oscillator with uncertain damping: Consider the oscillator

$$\ddot{y} + 2(b + u(t))\dot{y} + (1 + c)y = 0$$

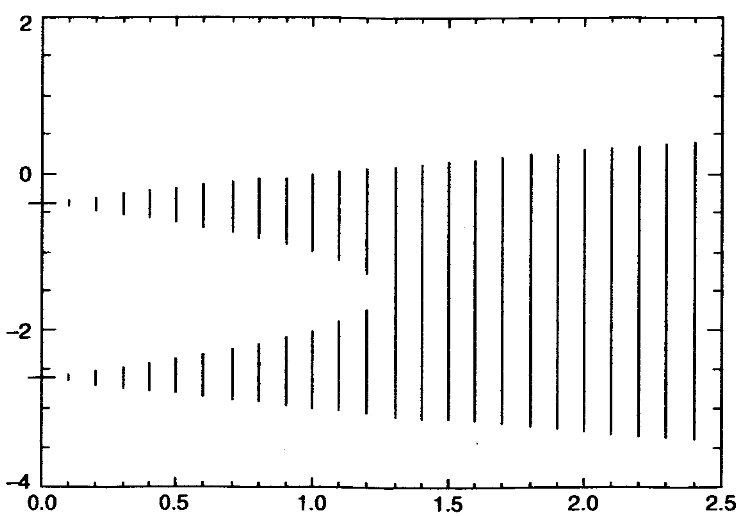

FIGURE 56.1 Spectral intervals depending on $\rho \geq 0$ for the system in Example 1.

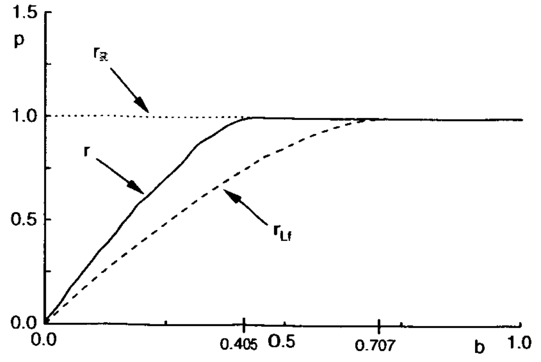

FIGURE 56.2 Stability radii for the system in Example 4.

with $u(t) \in [-\rho, \rho]$ and $c \in \mathbb{R}$. In equivalent first-order form the system reads

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-c & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Clearly, the system is not exponentially stable for $c \leq -1$ with $\rho = 0$, and for $c > -1$ with $\rho \geq b$. It turns out that the stability radius for this system is

$$r(c) = \begin{cases} 0 & \text{for} \quad c \leq -1 \\ b & \text{for} \quad c > -1. \end{cases}$$

4. Linear oscillator with uncertain restoring force: Here we look again at a system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$. (For $b \leq 0$ the system is unstable even for constant perturbations.) A closed form expression of the stability radius for this system is not available and one has to use numerical methods for the computation of (maximal) Lyapunov exponents (or maxima of the Morse spectrum); compare [CK00, App. D]. Figure 56.2 shows the (asymptotic) stability radius r, the stability radius under constant real perturbations $r_{\mathbb{R}}$, and the stability radius based on quadratic Lyapunov functions r_{Lf}, all in dependence on $b > 0$; see [CK00, Ex. 11.1.12].

56.9 Linearization

The local behavior of the dynamical system induced by a nonlinear differential equation can be studied via the linearization of the flow. At a fixed point of the nonlinear system the linearization is just a linear differential equation as studied in Sections 56.1 to 56.4. If the linearized system is hyperbolic, then the theorem of Hartman and Grobman states that the nonlinear flow is topologically conjugate to the linear flow. The invariant manifold theorem deals with those solutions of the nonlinear equation that are asymptotically attracted to (or repelled from) a fixed point. Basically these solutions live on manifolds that are described by nonlinear changes of coordinates of the linear stable (and unstable) subspaces.

Fact 4 below describes the simplest form of the invariant manifold theorem at a fixed point. It can be extended to include a "center manifold" (corresponding to the Lyapunov space with exponent 0). Furthermore, (local) invariant manifolds can be defined not just for the stable and unstable subspace, but for all Lyapunov spaces; see [BK94], [CK00], and [Rob98] for the necessary techniques and precise statements.

Both the Grobman–Hartman theorem as well as the invariant manifold theorem can be extended to time varying systems, i.e., to linear skew product flows as described in Sections 56.5 to 56.8. The general situation is discussed in [BK94], the case of linearization at a periodic solution is covered in [Rob98], random dynamical systems are treated in [Arn98], and robust systems (control systems) are the topic of [CK00].

Definitions:

A (nonlinear) **differential equation** in $\mathbb{R}^d$ is of the form $\dot{\mathbf{y}} = f(\mathbf{y})$, where f is a vector field on $\mathbb{R}^d$. Assume that f is at least of class C^1 and that for all $\mathbf{y}_0 \in \mathbb{R}^d$ the solutions $\mathbf{y}(t, \mathbf{y}_0)$ of the initial value problem $\mathbf{y}(0, \mathbf{y}_0) = \mathbf{y}_0$ exist for all $t \in \mathbb{R}$.

A point $\mathbf{p} \in \mathbb{R}^d$ is a **fixed point** of the differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ if $\mathbf{y}(t, \mathbf{p}) = \mathbf{p}$ for all $t \in \mathbb{R}$.

The **linearization** of the equation $\dot{\mathbf{y}} = f(\mathbf{y})$ at a fixed point $\mathbf{p} \in \mathbb{R}^d$ is given by $\dot{\mathbf{x}} = D_{\mathbf{y}} f(\mathbf{p})\mathbf{x}$, where $D_{\mathbf{y}} f(\mathbf{p})$ is the Jacobian (matrix of partial derivatives) of f at the point $\mathbf{p}$.

A fixed point $\mathbf{p} \in \mathbb{R}^d$ of the differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ is called **hyperbolic** if $D_{\mathbf{y}} f(\mathbf{p})$ has no eigenvalues on the imaginary axis, i.e., if the matrix $D_{\mathbf{y}} f(\mathbf{p})$ is hyperbolic.

Consider a differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ in $\mathbb{R}^d$ with flow $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, hyperbolic fixed point $\mathbf{p}$ and neighborhood $U(\mathbf{p})$. In this situation the **local stable manifold** and the **local unstable manifold** are defined as

$$W^s_{loc}(\mathbf{p}) = \{\mathbf{q} \in U\colon \lim_{t\to\infty} \Phi(t, \mathbf{q}) = \mathbf{p}\} \text{ and } W^u_{loc}(\mathbf{p}) = \{\mathbf{q} \in U\colon \lim_{t\to-\infty} \Phi(t, \mathbf{q}) = \mathbf{p}\},$$

respectively.

The local stable (and unstable) manifolds can be extended to **global invariant manifolds** by following the trajectories, i.e.,

$$W^s(\mathbf{p}) = \bigcup\nolimits_{t\geq 0} \Phi(-t, W^s_{loc}(\mathbf{p})) \text{ and } W^u(\mathbf{p}) = \bigcup\nolimits_{t\geq 0} \Phi(t, W^u_{loc}(\mathbf{p})).$$

Facts:

Literature: [Arn98], [AP90], [BK94], [CK00], [Rob98].

See Facts 3 and 4 in Section 56.2 for dynamical systems induced by differential equations and their fixed points.

1. (Hartman–Gobman) Consider a differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ in $\mathbb{R}^d$ with flow $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Assume that the equation has a hyperbolic fixed point $\mathbf{p}$ and denote the flow of the linearized equation $\dot{\mathbf{x}} = D_{\mathbf{y}} f(\mathbf{p})\mathbf{x}$ by $\Psi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Then there exist neighborhoods $U(\mathbf{p})$ of $\mathbf{p}$ and $V(\mathbf{0})$ of the origin in $\mathbb{R}^d$, and a homeomorphism $h : U(\mathbf{p}) \longrightarrow V(\mathbf{0})$ such that the flows $\Phi\,|_{U(\mathbf{p})}$ and $\Psi\,|_{V(\mathbf{0})}$ are (locally) C^0-conjugate, i.e., $h(\Phi(t, \mathbf{y})) = \Psi(t, h(\mathbf{y}))$ for all $\mathbf{y} \in U(\mathbf{p})$ and $t \in \mathbb{R}$ as long as the solutions stay within the respective neighborhoods.
2. Consider two differential equations $\dot{\mathbf{y}} = f_i(\mathbf{y})$ in $\mathbb{R}^d$ with flows $\Phi_i : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ for $i = 1, 2$. Assume that Φ_i has a hyperbolic fixed point $\mathbf{p}_i$ and the flows are C^k-conjugate for some $k \geq 1$ in neighborhoods of the $\mathbf{p}_i$. Then $\sigma(D_{\mathbf{y}} f_1(\mathbf{p}_1)) = \sigma(D_{\mathbf{y}} f_2(\mathbf{p}_2))$, i.e., the eigenvalues of the linearizations agree; compare Facts 5 and 6 in Section 56.2 for the linear situation.
3. Consider two differential equations $\dot{\mathbf{y}} = f_i(\mathbf{y})$ in $\mathbb{R}^d$ with flows $\Phi_i : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ for $i = 1, 2$. Assume that Φ_i has a hyperbolic fixed point $\mathbf{p}_i$ and the number of negative (or positive) Lyapunov exponents of $D_{\mathbf{y}} f_i(\mathbf{p}_i)$ agrees. Then the flows Φ_i are locally C^0-conjugate around the fixed points.
4. (Invariant Manifold Theorem) Consider a differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ in $\mathbb{R}^d$ with flow $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Assume that the equation has a hyperbolic fixed point p and denote the linearized equation by $\dot{\mathbf{x}} = D_{\mathbf{y}} f(p)\mathbf{x}$.

   (i) There exists a neighborhood $U(\mathbf{p})$ in which the flow Φ has a local stable manifold $W^s_{loc}(\mathbf{p})$ and a local unstable manifold $W^u_{loc}(\mathbf{p})$.

(ii) Denote by L^- (and L^+) the stable (and unstable, respectively) subspace of $D_y f(\mathbf{p})$; compare the definitions in Section 56.1. The dimensions of L^- (as a linear subspace of $\mathbb{R}^d$) and of $W^s_{loc}(\mathbf{p})$ (as a topological manifold) agree, similarly for L^+ and $W^u_{loc}(\mathbf{p})$.

(iii) The stable manifold $W^s_{loc}(\mathbf{p})$ is tangent to the stable subspace L^- at the fixed point $\mathbf{p}$, similarly for $W^u_{loc}(\mathbf{p})$ and L^+.

5. Consider a differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$ in $\mathbb{R}^d$ with flow $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. Assume that the equation has a hyperbolic fixed point $\mathbf{p}$. Then there exists a neighborhood $U(\mathbf{p})$ on which Φ is C^0-equivalent to the flow of a linear differential equation of the type

$$\dot{\mathbf{x}}_s = -\mathbf{x}_s,\ \mathbf{x}_s \in \mathbb{R}^{d_s},$$

$$\dot{\mathbf{x}}_u = \mathbf{x}_u,\ \mathbf{x}_u \in \mathbb{R}^{d_u},$$

where d_s and d_u are the dimensions of the stable and the unstable subspace of $D_y f(\mathbf{p})$, respectively, with $d_s + d_u = d$.

Examples:

1. Consider the nonlinear differential equation in $\mathbb{R}$ given by $\ddot{z} + z - z^3 = 0$, or in first-order form in $\mathbb{R}^2$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1 + y_1^3 \end{bmatrix} = f(\mathbf{y}).$$

The fixed points of this system are $\mathbf{p}_1 = [0,0]^T$, $\mathbf{p}_2 = [1,0]^T$, $\mathbf{p}_3 = [-1,0]^T$. Computation of the linearization yields

$$D_y f = \begin{bmatrix} 0 & 1 \\ -1 + 3y_1^2 & 0 \end{bmatrix}.$$

Hence, the fixed point $\mathbf{p}_1$ is not hyperbolic, while $\mathbf{p}_2$ and $\mathbf{p}_3$ have this property.

2. Consider the nonlinear differential equation in $\mathbb{R}$ given by $\ddot{z} + \sin(z) + \dot{z} = 0$, or in first-order form in $\mathbb{R}^2$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -\sin(y_1) - y_2 \end{bmatrix} = f(\mathbf{y}).$$

The fixed points of the system are $\mathbf{p}_n = [n\pi, 0]^T$ for $n \in \mathbb{Z}$. Computation of the linearization yields

$$D_y f = \begin{bmatrix} 0 & 1 \\ -\cos(y_1) & -1 \end{bmatrix}.$$

Hence, for the fixed points $\mathbf{p}_n$ with n even the eigenvalues are $\mu_1, \mu_2 = -\frac{1}{2} \pm i\sqrt{\frac{3}{4}}$ with negative real part (or Lyapunov exponent), while at the fixed points $\mathbf{p}_n$ with n odd one obtains as eigenvalues $\nu_1, \nu_2 = -\frac{1}{2} \pm \sqrt{\frac{5}{4}}$, resulting in one positive and one negative eigenvalue. Hence, the flow of the differential equation is locally C^0-conjugate around all fixed points with even n, and around all fixed points with odd n, while the flows around, e.g., $\mathbf{p}_0$ and $\mathbf{p}_1$ are not conjugate.

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