Steepest descent on real flag manifolds

Jost-Hinrich Eschenburg, Augustin-Liviu Mare

Angaben zur Veröffentlichung / Publication details:

Eschenburg, Jost-Hinrich, and Augustin-Liviu Mare. 2006. "Steepest descent on real flag manifolds." *Bulletin of the London Mathematical Society* 38 (2): 323–28. https://doi.org/10.1112/S0024609306018376.

STEEPEST DESCENT ON REAL FLAG MANIFOLDS

J.-H. ESCHENBURG AND A.-L. MARE

1. *Introduction*

Among the compact homogeneous spaces, a very distinguished subclass is formed by the (*generalized*) *real flag manifolds*, which by definition are the orbits of the isotropy representations of Riemannian symmetric spaces (*s-orbits*). This class contains most of the compact symmetric spaces (for example, all hermitian ones), all classical flag manifolds over real, complex and quaternionic vector spaces, all adjoint orbits of compact Lie groups (*generalized complex flag manifolds*), and many others. They form the main examples of isoparametric submanifolds and their focal manifolds (the so-called *constant principal curvature manifolds*); in fact, for most codimensions these are the only such spaces (cf. [**6**–**8**]).

Any real flag manifold M enjoys two very peculiar geometric properties: it carries a transitive action of a *noncompact* Lie group G, and it is embedded in euclidean space as a *taut* submanifold; that is, almost all height or coordinate functions are perfect Morse functions (at least for $\mathbb{Z}/2$-coefficients). The aim of our paper is to link these two properties by the following theorem.

Main Theorem. *The gradient flow of any height function is a one-parameter subgroup of* G*, where the gradient is defined with respect to a suitable homogeneous metric* s *on* M*.*

The explicit construction of the metric s, as well as the proof of the main result, can be found in Section 4. The theorem says that the lines of steepest descent for the height function (gradient flow lines) are obtained by applying a one-parameter subgroup of G. This is an elementary fact when M is a euclidean sphere and G its conformal group: the gradient of any height function is a conformal vector field. We will see in Section 5 that in the case in which M is an adjoint orbit (that is, a (generalized) complex flag manifold), our metric s is a homogeneous Kähler metric. In this case, we recover a fact that was observed earlier by Guest and Ohnita [**4**].

Our more general result can be derived from their theorem, since real flag manifolds are contained in complex flag manifolds as fixed point sets under certain involutions. However, a short, direct proof might be desirable.

2. *Root space decomposition*

Let $P = G/K$ be a symmetric space of noncompact type, where G is a connected noncompact semisimple Lie group and $K \subset G$ is a maximal compact subgroup [**5**]. Let σ be the corresponding involution on G with fixed point set K. Consider the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1}$$

where $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, and $\mathfrak{p}$ denotes the (-1)-eigenspace of (the differential of) σ. The adjoint action of K leaves $\mathfrak{p}$ invariant; this is the isotropy representation of P. As usual, we consider an $\mathrm{Ad}(G)$-invariant indefinite inner product b on $\mathfrak{g}$ with $b > 0$ on $\mathfrak{p}$ and $b < 0$ on $\mathfrak{k}$ (for example, the Killing form), and we define a positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is b on $\mathfrak{p}$ and $-b$ on $\mathfrak{k}$, and for which $\mathfrak{p} \perp \mathfrak{k}$. Then any $\mathrm{ad}(x)$ with $x \in \mathfrak{p}$ is self-adjoint on $\mathfrak{g}$. Hence a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ gives rise to a family of mutually commuting self-adjoint endomorphisms $\mathrm{ad}(x)$ of $\mathfrak{g}$, where $x \in \mathfrak{a}$. These have a common eigenspace decomposition

$$\mathfrak{g} = \sum_{\alpha \in \hat{R}} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{z \in \mathfrak{g};\ [x, z] = \alpha(x)z\ \forall x \in \mathfrak{a}\}, \tag{2}$$

where $\hat{R} \subset \mathfrak{a}^*$ is the set of roots including 0.

After fixing some arbitrary $x \in \mathfrak{a}$, there are three disjoint subsets of $\hat{R}$, formed by the roots α with $\alpha(x) > 0$, $\alpha(x) = 0$ and $\alpha(x) < 0$, respectively. Hence we have the decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{c} \oplus \mathfrak{n}_- \tag{3}$$

with

$$\mathfrak{n}_+ = \sum_{\alpha(x)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{c} = \sum_{\alpha(x)=0} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \sum_{\alpha(x)<0} \mathfrak{g}_\alpha. \tag{4}$$

Since $\sigma = -\mathrm{id}$ on $\mathfrak{a}$, we have

$$(\mathfrak{g}_\alpha)^\sigma = \mathfrak{g}_{-\alpha}, \tag{5}$$

and hence σ interchanges $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Any $a \in \mathfrak{g}$ allows a unique decomposition $a = a_- + a_0 + a_+$ with $a_\pm \in \mathfrak{n}_\pm$ and $a_0 \in \mathfrak{c}$, and if $a \in \mathfrak{k}$ or $a \in \mathfrak{p}$, we have $a_- = a_+^\sigma$ or $a_- = -a_+^\sigma$, respectively.

3. *Generalized real flag manifolds*

By definition, generalized real flag manifolds are the orbits of the isotropy representation of a symmetric space $P = G/K$ of noncompact type. We consider an isotropy orbit $M \subset \mathfrak{p}$. For any fixed $x \in M$ we have $M = \mathrm{Ad}(K)x$. The stabilizer subgroup is

$$S = \{k \in K;\ \mathrm{Ad}(k)x = x\}; \tag{6}$$

thus M may be identified with the coset space K/S by $kS \longmapsto \mathrm{Ad}(k)x$.

Now choose a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ containing x. Such an $\mathfrak{a}$ is uniquely determined up to applying $\mathrm{Ad}(s)$ with $s \in S$; this fact is just the conjugacy of maximal flat subspaces in the symmetric subspace $\tilde{P} = C/S \subset P$, where C (with Lie algebra $\mathfrak{c}$) is the centralizer of x in G. We note that, geometrically, $\tilde{P}$ can be described as the union of all geodesics parallel to the geodesic $\exp(\mathbb{R}x) \subset P$, which is obviously invariant under geodesic reflection; hence it is a symmetric subspace. Note that each $\mathrm{Ad}(s)$, $s \in S$, commutes with $\mathrm{ad}(x)$, and thus preserves the eigenspaces of $\mathrm{ad}(x)$; hence $\mathfrak{n}_- = \sum_{\alpha(x)<0} \mathfrak{g}_\alpha$ is also invariant under $\mathrm{Ad}(s)$.

Next we show that the natural K-action on M can be extended to a G-action. Consider

$$H = \{g \in G;\ \mathrm{Ad}(g)(x + \mathfrak{n}_-) = x + \mathfrak{n}_-\}, \tag{7}$$

which is a closed subgroup of G containing S as a subgroup. One can see that the Lie algebra of H is $\mathfrak{h} = \mathfrak{c} + \mathfrak{n}_-$. Note that H does not depend on the choice of $\mathfrak{a}$. Indeed, if instead of $\mathfrak{a}$ we start our construction with $\mathfrak{a}' = \mathrm{Ad}(s)\mathfrak{a}$, where $s \in S$, then $\mathfrak{n}'_- = \mathrm{Ad}(s)\mathfrak{n}_- = \mathfrak{n}_-$, and we end up with $H' = H$.

Lemma 3.1. *K acts transitively on the coset space G/H with stabilizer S. Hence G/H can be identified with $K/S = M$.*

Proof. Let $M' \subset G/H$ be the orbit of $eH \in G/H$ under the subgroup $K \subset G$. We show first that it is open in G/H. To see this, it suffices to show that $\mathfrak{k} + \mathfrak{h} = \mathfrak{g}$; that is, one needs to show that $\mathfrak{g}_\alpha \subset \mathfrak{k} + \mathfrak{h}$ for each $\alpha \in R$ with $\alpha(x) > 0$. In fact, take $z \in \mathfrak{g}_\alpha$, decompose it as $z = v + u$ with $v \in \mathfrak{p}$ and $u \in \mathfrak{k}$, and notice that $z^\sigma = -v + u \in \mathfrak{g}_{-\alpha} \subset \mathfrak{h}$. Thus the K-orbit M' is open. However, it is also closed in G/H since K is compact. So M' coincides with G/H.

The K-stabilizer of $eH \in G/H$ is $K \cap H$; we have to show that $K \cap H = S$. Clearly, $S \subset K \cap H$. Conversely, if $k \in K \cap H$, then

$$\mathrm{Ad}(k)x - x \in \mathfrak{p} \cap \mathfrak{n}_-.$$

But $\mathfrak{p} \cap \mathfrak{n}_- = 0$, because if z belongs to this intersection, then $z^\sigma = -z \in \mathfrak{n}_+$, and hence $z \in \mathfrak{n}_- \cap \mathfrak{n}_+ = 0$. We deduce that $\mathrm{Ad}(k)x = x$, which means that $k \in S$. □

Thus the action of G on $G/H = K/S = M$ is an extension of the K-action. We will denote this action $G \times M \longrightarrow M$ by $(g, x) \longmapsto g.x$. When restricted to $k \in K \subset G$, we have $k.x = \mathrm{Ad}(k)x$. Similarly, the infinitesimal action $\mathfrak{g} \times M \longrightarrow TM$ will be denoted by

$$(a, x) \longmapsto a.x := \frac{d}{dt}\Big|_{t=0} \exp(ta).x,$$

where $a \in \mathfrak{g}$, and $a.x = [a, x]$ whenever $a \in \mathfrak{k}$.

We take the opportunity here to give a more geometric description of the action of G on M, although it will not be used in our paper. Consider $\mathfrak{p}$ as the tangent space of $P = G/K$ at some base point $o \in P$. We may project any nonzero $x \in T_0P$ to the boundary at infinity $P(\infty)$ (bearing in mind that P is a simply connected space of nonpositive curvature) by the map $\pi_\infty(x) = \gamma_x(-\infty)$ where γ_x is the geodesic in P starting at o with initial vector x. The isometry group G acts on $P(\infty)$ and leaves $\pi_\infty(M)$ invariant; this is the G-action. See [**1**] for details.

4. *Steepest descent*

Theorem 4.1. *Let G/K be a symmetric space of noncompact type, let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, and let $M \subset \mathfrak{p}$ be a real flag manifold (isotropy orbit). Let $q \in \mathfrak{p}$ and $f : M \longrightarrow \mathbb{R}$, $f(x) = \langle q, x\rangle$. Then there is a K-invariant Riemannian metric $s : TM \longrightarrow T^*M$ on M such that the flow lines $x(t)$ of the s-gradient $\nabla^s f = s^{-1}df$ satisfy*

$$x(t) = \exp(-tq).x(0). \tag{8}$$

Proof. Choose an arbitrary $x \in M$, and consider the decomposition (3) corresponding to x. We have to show that $\nabla^s f(x) = -q.x$. To compute $q.x$, we look for $r \in \mathfrak{k}$ with $q.x = r.x$; that is, $q - r \in \mathfrak{h}$. We have $q = q_0 + \sum_{\alpha(x)>0} q_\alpha$ with $q_0 \in \mathfrak{c}$ and $q_\alpha = z_\alpha - z_\alpha^\sigma$ for some $z_\alpha \in \mathfrak{g}_\alpha$. We may assume that $q_0 = 0$, since $q_0.x = 0$. Then we put $r = \sum_+(z_\alpha + z_\alpha^\sigma) \in \mathfrak{k}$, and hence $r - q = 2\sum_+ z_\alpha^\sigma \in \mathfrak{h}$, where $\sum_+$ always denotes $\sum_{\alpha(x)>0}$. Now

$$\begin{aligned} r.x = [r, x] &= -\operatorname{ad}(x)\textstyle\sum_+(z_\alpha + z_\alpha^\sigma) \\ &= -\textstyle\sum_+\alpha(x)(z_\alpha - z_\alpha^\sigma) = -\textstyle\sum_+\alpha(x)q_\alpha. \end{aligned}$$

Any $v \in T_xM$ has a decomposition $v = \sum_+ v_\alpha$ with $v_\alpha \in (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$ for $\alpha(x) > 0$, and our metric s on T_xM will be of the form

$$\langle v, w\rangle_s = \textstyle\sum_+ s_\alpha \langle v_\alpha, w_\alpha\rangle$$

for certain numbers $s_\alpha > 0$. We have to choose s such that for all $v \in T_xM$,

$$\langle \nabla^s f(x), v\rangle_s = -\langle r.x, v\rangle_s.$$

The left-hand side is

$$\langle \nabla^s f(x), v\rangle_s = df_x v = \langle q, v\rangle = \textstyle\sum_+\langle q_\alpha, v_\alpha\rangle,$$

while the right-hand side is

$$-\langle r.x, v\rangle_s = \textstyle\sum_+ s_\alpha \alpha(x)\langle q_\alpha, v_\alpha\rangle.$$

We put

$$s_\alpha = 1/\alpha(x), \tag{9}$$

and the result follows. □

5. *Extrinsically symmetric spaces*

An *extrinsically symmetric space* is a submanifold M of euclidian space such that M is preserved by the reflections at all of its (affine) normal spaces. By a result of Ferus [**3**] (see also [**2**]), after splitting off euclidean factors, M has precisely the form of an s-orbit $M = \operatorname{Ad}(K)x_0 \subset \mathfrak{p}$, where $\mathfrak{p}$ corresponds to a symmetric space $P = G/K$ and where $x_0 \in \mathfrak{p}$ satisfies

$$\alpha(x_0) \in \{-1, 0, 1\},$$

for all $\alpha \in R$. In this case the metric s of Theorem 4.1 agrees with the given inner product $\langle\ ,\ \rangle$; cf. (9). By applying Theorem 4.1, one obtains the following theorem.

THEOREM 5.1. *If $M = \mathrm{Ad}(K)x_0 \subset \mathfrak{p}$ is extrinsically symmetric, then the gradient lines of the height function $h(x) = \langle q, x\rangle$ with respect to the metric $\langle\ ,\ \rangle$ are of the form*

$$x(t) = \exp(-tq).x(0).$$

6. *Adjoint orbits*

In the particular case of *complex* flag manifolds (that is, adjoint orbits of compact Lie groups), we will establish relations between Theorem 4.1 and previously known results.

Let K be a compact semisimple Lie group of Lie algebra $\mathfrak{k}$, and let $T \subset K$ be a maximal torus of Lie algebra $\mathfrak{t}$. Consider the adjoint orbit $M = \mathrm{Ad}(K)x$ for $x \in \mathfrak{k}$. If $G = K^{\mathbb{C}}$ is the complexification of K, then G/K is a non-compact symmetric space and

$$\mathfrak{g} = \mathfrak{k} + i\,\mathfrak{k}$$

is a Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{k} \otimes \mathbb{C}$ (the involution σ is just the complex conjugation). Since M is (up to a multiple of i) an isotropy orbit of G/K, the results of the previous section can be applied here, too. The goal of this section is to point out that the metric on M for which the lines of steepest descent are orbits of one-parameter subgroups of G is well known: it is the Kähler metric (cf. [**3**, **4**]).

It is well known that any adjoint orbit $M = \mathrm{Ad}(K)x$ is a complex manifold. In fact, in the language of Section 4 we have $M = G/H$ but here G and H are *complex* Lie groups, and hence M is a complex manifold. The corresponding complex structure J on T_xM can be described as follows. Choose a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{k}$ with $x \in \mathfrak{t}$. The corresponding roots are considered as real linear forms $\alpha \in \mathfrak{t}^*$, while the eigenvalues of $\mathrm{ad}(x)$ are purely imaginary, $i\,\alpha(x)$. Let $\mathfrak{k}_\alpha \subset \mathfrak{k} \otimes \mathbb{C}$ be the root spaces. Then

$$T_xM = \sum_{\alpha(x)>0} \mathfrak{k}_\alpha^r,$$

where $\mathfrak{k}_\alpha^r = (\mathfrak{k}_\alpha + \mathfrak{k}_{-\alpha}) \cap \mathfrak{k}$ is the *real root space*. Now J leaves invariant each $\mathfrak{k}_\alpha^r$ and on $\mathfrak{k}_\alpha^r$ it is a multiple of $\mathrm{ad}(x)$:

$$\mathrm{ad}(x) = \alpha(x)J. \tag{10}$$

The second ingredient for the Kähler metric is the Kähler form ω, which is defined as follows: If $v = \mathrm{ad}(a)x$ and $w = \mathrm{ad}(b)x$ are tangent vectors of M at the point x, then

$$\omega_x(v, w) := \langle x, [a, b]\rangle = \langle [x, a], b\rangle = \langle v, b\rangle = \langle v, \mathrm{ad}(x)^{-1}w\rangle \tag{11}$$

where $\langle\ ,\ \rangle$ denotes an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$.

Now the Kähler metric $(\ ,\ )$ on T_xM is defined as follows. For $v, w \in T_xM$ we have

$$(v, w) = \omega_x(v, Jw). \tag{12}$$

Hence from (10) and (11) we obtain

$$(v, w) = \omega_x(v, Jw) = \langle v, \mathrm{ad}(x)^{-1}Jw\rangle = \frac{1}{\alpha(x)}\langle v, w\rangle = \langle v, w\rangle_s; \tag{13}$$

cf. (9).

Since $\mathfrak{p} = i\,\mathfrak{k}$ in the present case, we obtain the following result from Theorem 4.1.

THEOREM 6.1 (cf. [**3**, **4**]). *Let K be a compact Lie group and $M = \mathrm{Ad}(K)x_0 \subset \mathfrak{k}$ an adjoint orbit, equipped with its Kähler metric $(\ ,\ )$ as in (12) and acted on by the complexified group $G = K^{\mathbb{C}}$ as described above. Let $\langle\ ,\ \rangle$ be the corresponding $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. Then, for any $q \in \mathfrak{k}$, the gradient lines $x(t)$ of the function $f : M \longrightarrow \mathbb{R}$, $f(x) = \langle q, x\rangle$, are orbits of a 1-parameter subgroup of $G = K^{\mathbb{C}}$, namely*

$$x(t) = \exp(i\,tq).x(0).$$

Acknowledgements. We would like to thank Martin Guest and Peter Quast for hints and discussions.

References

1. W. BALLMANN, M. GROMOV and V. SCHROEDER, *Manifolds of nonpositive curvature* (Birkhäuser, Basel, 1985).

2. J.-H. ESCHENBURG and E. HEINTZE, 'Extrinsic symmetric spaces and orbits of *s*-representations', *Manuscripta Math.* 88 (1995) 517–524.

3. D. FERUS, 'Symmetric submanifolds of euclidean space', *Math. Ann.* 247 (1980) 81–93.

4. M. A. GUEST and Y. OHNITA, 'Group actions and deformations for harmonic maps', *J. Math. Soc. Japan* 45 (1993) 671–704.

5. S. HELGASON, *Differential geometry, Lie groups and symmetric spaces* (Academic Press, 1978).

6. E. HEINTZE, C. OLMOS and G. THORBERGSSON, 'Submanifolds with constant principal curvatures and normal holonomy group', *Int. J. Math.* 2 (1991) 167–185.

7. C. OLMOS, 'Isoparametric submanifolds and their homogeneous structure', *J. Differential Geom.* 38 (1993) 225–234.

8. G. THORBERGSSON, 'Isoparametric foliations and their buildings', *Ann. of Math.* 133 (1991) 429–446.