Symmetric submanifolds associated with irreducible symmetric R-spaces

Jürgen Berndt · Jost-Hinrich Eschenburg · Hiroo Naitoh · Kazumi Tsukada

1. Introduction

Symmetric submanifolds are defined analogously to Riemannian symmetric spaces. Riemannian symmetric spaces admit an intrinsic symmetry at each point, whereas symmetric submanifolds admit an extrinsic symmetry at each point. A connected submanifold M of a connected Riemannian manifold $\bar{M}$ is called *symmetric* if at each point p in M there exists an involutive isometry t_p of $\bar{M}$ satisfying $t_p(p) = p$, $t_p(M) = M$, $(t_p)_*X = -X$ for all $X \in T_pM$, and $(t_p)_*\xi = \xi$ for all $\xi \in T_p^\perp M$. Here T_pM and $T_p^\perp M$ denote the tangent and normal space of M at p, respectively. The isometry t_p is called the *extrinsic symmetry* of M at p. For definitions of locally symmetric submanifolds, symmetric immersions and locally symmetric immersions we refer to [10]. The aim of this paper is to classify symmetric submanifolds in Riemannian symmetric spaces.

Ferus ([2]) proved that the symmetric submanifolds of Euclidean spaces are essentially the symmetric orbits of the isotropy representations of semisimple

J. Berndt
Department of Mathematics, University College, Cork, Ireland
(e-mail: `j.berndt@ucc.ie`)

J.-H. Eschenburg
Universität Augsburg, Institut für Mathematik, 86135 Augsburg, Germany
(e-mail: `eschenburg@math.uni-augsburg.de`)

H. Naitoh
Department of Mathematics, Yamaguchi University, 753-8512 Yamaguchi, Japan
(e-mail: `naitoh@po.cc.yamaguchi-u.ac.jp`)

K. Tsukada
Department of Mathematics, Ochanomizu University, 112-8610, Tokyo
Japan (e-mail: `tsukada@math.ocha.ac.jp`)

Riemannian symmetric spaces. The orbits of such representations are known as R-spaces or real flag manifolds. For some symmetric spaces $\bar{M}$ some of these orbits are symmetric spaces, in which case they are symmetric submanifolds in the corresponding tangent space $T_o\bar{M}$ of $\bar{M}$. These symmetric submanifolds are known as symmetric R-spaces or symmetric real flag manifolds. If $\bar{M}$ is noncompact, the projection of these symmetric submanifolds from $T_o\bar{M}$ into $\bar{M}$ via the exponential map at o provides examples of symmetric submanifolds in $\bar{M}$. In this paper we extend these symmetric submanifolds to larger one-parameter families of symmetric submanifolds, and prove that if $\bar{M}$ is irreducible and of rank ≥ 2, then every symmetric submanifold of $\bar{M}$ arises in this way.

We recall some properties of symmetric submanifolds. Since the restriction to M of an extrinsic symmetry t_p gives an intrinsic symmetry of M, each symmetric submanifold is a Riemannian symmetric space. For each point $p \in M$ the tangent space T_pM and the normal space $T_p^\perp M$ are invariant under the curvature tensor $\bar{R}$ of $\bar{M}$, that is,

$$\bar{R}(T_pM, T_pM)T_pM \subset T_pM \quad \text{and} \quad \bar{R}(T_p^\perp M, T_p^\perp M)T_p^\perp M \subset T_p^\perp M\ .$$

Moreover, a symmetric submanifold is equivariant in the following sense. We denote by $I(\bar{M})$ the isometry group of $\bar{M}$ and by $I^o(\bar{M})$ its identity component. Let G_M be the subgroup of $I(\bar{M})$ which is generated by all extrinsic symmetries t_p, $p \in M$. Then $G_M^o = I^o(\bar{M}) \cap G_M$ acts transitively on M. This and further details about the basic theory of symmetric submanifolds can be found in [10].

The above property induces the following classes of submanifolds in the framework of Grassmann geometries that were introduced by Harvey and Lawson ([3]), and which is an essential ingredient of our approach: Let $\bar{M}$ be a Riemannian manifold and $Gr_m(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ consisting of all m-dimensional linear subspaces of the tangent spaces of $\bar{M}$. Let $\mathfrak{O}$ be an arbitrary orbit of the canonical action of $I^o(\bar{M})$ on $Gr_m(T\bar{M})$. An m-dimensional connected submanifold M is called an *$\mathfrak{O}$-submanifold* if all its tangent spaces belong to $\mathfrak{O}$. The collection of all $\mathfrak{O}$-submanifolds forms a class of submanifolds, the so-called *$\mathfrak{O}$-geometry*. If for some, and hence for any, $V \in \mathfrak{O}$ both V and its orthogonal complement $V^\perp$ are invariant under the curvature tensor $\bar{R}$ of $\bar{M}$, the orbit $\mathfrak{O}$ is of *strongly curvature-invariant type* and its $\mathfrak{O}$-geometry is also said to be of strongly curvature-invariant type. From this definition it follows that a symmetric submanifold belongs to some $\mathfrak{O}$-geometry of strongly curvature-invariant type.

If $\bar{M}$ is a Riemannian symmetric space, then the curvature-invariant linear subspaces are also known as Lie triple systems. If $p \in \bar{M}$ and $V \subset T_p\bar{M}$ is a Lie triple system, then there exists a unique connected complete totally geodesic submanifold M of $\bar{M}$ with $p \in M$ and $T_pM = V$. The condition that $V^\perp$ is also a Lie triple system means geometrically that the geodesic reflection of $\bar{M}$ in M is an isometry in some open neighborhood of M, in which case M is called a *reflective* submanifold. The reflective submanifolds are precisely the totally geodesic symmetric submanifolds of $\bar{M}$. The reflective submanifolds of simply connected irreducible Riemannian symmetric spaces were classified by Leung ([7], [8]). We note that this classification is equivalent to the classification of $\mathfrak{O}$-geometries of strongly curvature-invariant type on Riemannian symmetric spaces. The third author obtained in a series of papers ([11], [12], [13], [14]) the classification of such $\mathfrak{O}$-geometries by different methods, and determined all $\mathfrak{O}$-geometries containing non-totally geodesic submanifolds.

Theorem 1.1. *All $\mathfrak{O}$-geometries of strongly curvature-invariant type in simply connected irreducible Riemannian symmetric spaces except the following ones have only totally geodesic submanifolds:*

(1) *the geometry of k-dimensional $(0 < k < n)$ submanifolds of the sphere S^n resp. of the real hyperbolic space $\mathbb{R}H^n$ $(n \geq 2)$;*
(2) *the geometry of k-dimensional $(0 < k < n)$ complex submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*
(3) *the geometry of n-dimensional totally real submanifolds of the complex projective space $\mathbb{C}P^n$ resp. of the complex hyperbolic space $\mathbb{C}H^n$ $(n \geq 2)$;*
(4) *the geometry of $2n$-dimensional totally complex submanifolds of the quaternionic projective space $\mathbb{H}P^n$ resp. of the quaternionic hyperbolic space $\mathbb{H}H^n$ $(n \geq 2)$;*
(5) *the geometries associated with irreducible symmetric R-spaces and their noncompact dual geometries.*

The third author proved Theorem 1.1 for simply connected irreducible Riemannian symmetric spaces of compact type. However, it is easy to see that the proof also holds for the noncompact case. He also obtained a decomposition theorem (Theorem 2.2 in [12]) which shows that it is sufficient to discuss the irreducible case.

The symmetric submanifolds belonging to the geometries of type (1)–(4) in Theorem 1.1 were classified by several authors, we refer to [10], [15] for further details. The symmetric submanifolds belonging to the geometries of type (5) were classified by the third author in [10] for the compact case. In this paper we solve the last remaining case, namely the case of geometries of type (5) for the noncompact case.

The paper is organized as follows. In Section 2 we construct the above mentioned one-parameter families of symmetric submanifolds in Riemannian symmetric spaces of noncompact type. In Section 3 we show that submanifolds belonging to the geometries of type (5) in the noncompact case are locally extrinsically symmetric submanifolds (Theorem 3.3), and that such submanifolds are exhausted by the examples constructed in Section 2 (Theorem 3.5).

2. The new examples of symmetric submanifolds

In this section we construct a one-parameter family of noncongruent symmetric submanifolds $M_c, c \geq 0$, in irreducible Riemannian symmetric spaces of noncompact type. All these submanifolds belong to the $\mathfrak{O}$-geometries of type (5). Any such family contains a family of homothetic symmetric R-spaces and its noncompact dual spaces, and a flat submanifold. Our construction is a noncompact version of the one discussed by the third author in [10], §5, Part II.

We start with recalling some facts about the theory of symmetric R-spaces, for details we refer to Kobayashi and Nagano [5], Nagano [9] and Takeuchi [17]. Let $(\bar{\mathfrak{g}}, \sigma)$ be a positive definite symmetric graded Lie algebra, that is, $\bar{\mathfrak{g}}$ is a real semisimple Lie algebra with a Cartan involution σ satisfying the following properties:

(1) $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1$ (vector space direct sum) and $[\bar{\mathfrak{g}}_p, \bar{\mathfrak{g}}_q] \subset \bar{\mathfrak{g}}_{p+q} \quad (p, q \in \{0, \pm 1\})$;
(2) $\sigma(\bar{\mathfrak{g}}_p) = \bar{\mathfrak{g}}_{-p} \quad (p \in \{0, \pm 1\})$;
(3) $\bar{\mathfrak{g}}_{-1} \neq \{0\}$, and the adjoint action of $\bar{\mathfrak{g}}_0$ on the vector space $\bar{\mathfrak{g}}_{-1}$ is effective.

For the classification of the positive definite symmetric graded Lie algebras see [5], [17], and the table at the end of this paper. We define a linear isomorphism τ of $\bar{\mathfrak{g}}$ by $\tau(X) = (-1)^p X$ for $X \in \bar{\mathfrak{g}}_p$. Then τ is an involutive automorphism of $\bar{\mathfrak{g}}$ with $\sigma\tau = \tau\sigma$. Let $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$ be the Cartan decomposition induced by σ. Then we have $\tau(\bar{\mathfrak{k}}) = \bar{\mathfrak{k}}$ and $\tau(\bar{\mathfrak{p}}) = \bar{\mathfrak{p}}$. Let $\bar{\mathfrak{k}} = \mathfrak{k}_+ + \mathfrak{k}_-$ and $\bar{\mathfrak{p}} = \mathfrak{p}_+ + \mathfrak{p}_-$ be the ± 1-eigenspace decompositions of $\bar{\mathfrak{k}}$ and $\bar{\mathfrak{p}}$ with respect to τ. Obviously, we have

$$\mathfrak{k}_+ = \bar{\mathfrak{k}} \cap \bar{\mathfrak{g}}_0 \ , \ \mathfrak{k}_- = \bar{\mathfrak{k}} \cap (\bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1) \ , \ \mathfrak{p}_+ = \bar{\mathfrak{p}} \cap \bar{\mathfrak{g}}_0 \ , \ \mathfrak{p}_- = \bar{\mathfrak{p}} \cap (\bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1) \ .$$

Since $\bar{\mathfrak{g}}$ is semisimple, there exists a unique element $\nu \in \bar{\mathfrak{g}}_0$ such that

$$\bar{\mathfrak{g}}_p = \{X \in \bar{\mathfrak{g}} \mid \mathrm{ad}(\nu)X = pX\} \text{ for all } p \in \{0, \pm 1\} \ .$$

It is easy to see that $\nu \in \bar{\mathfrak{p}}$ and hence $\nu \in \mathfrak{p}_+$. The following lemma can be proved in a straightforward manner.

Lemma 2.1. *The adjoint transformation* $\mathrm{ad}(\nu)$ *restricted to* $\mathfrak{p}_-$ *is an isomorphism from* $\mathfrak{p}_-$ *onto* $\mathfrak{k}_-$*, and its inverse is* $\mathrm{ad}(\nu)$ *restricted to* $\mathfrak{k}_-$*. Moreover, the following holds:*

(1) $[T, \mathrm{ad}(\nu)X] = \mathrm{ad}(\nu)[T, X]$ *for all* $T \in \mathfrak{k}_+$ *and* $X \in \mathfrak{p}_-$*;*
(2) $[\mathrm{ad}(\nu)X, Y] \in \mathfrak{p}_+$ *and* $[\mathrm{ad}(\nu)X, Y] + [X, \mathrm{ad}(\nu)Y] = 0$ *for all* $X, Y \in \mathfrak{p}_-$*;*
(3) $[\mathrm{ad}(\nu)X, \mathrm{ad}(\nu)Y] = -[X, Y] \in \mathfrak{k}_+$ *for all* $X, Y \in \mathfrak{p}_-$.

The restriction of the Killing form B of $\bar{\mathfrak{g}}$ to $\bar{\mathfrak{p}} \times \bar{\mathfrak{p}}$ is a positive definite inner product on $\bar{\mathfrak{p}}$, which will be denoted by $\langle \cdot, \cdot \rangle$. This inner product is invariant under the adjoint action of $\bar{\mathfrak{k}}$ on $\bar{\mathfrak{p}}$ and under the involution $\tau | \bar{\mathfrak{p}}$. In particular, $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are perpendicular to each other. Let $\bar{G}$ be the simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$ and $\bar{K}$ be the connected Lie subgroup of $\bar{G}$ with Lie algebra $\bar{\mathfrak{k}}$. Let $\bar{\pi} : \bar{G} \to \bar{M} = \bar{G}/\bar{K}$ be the canonical projection and $o = \bar{\pi}(e)$, where e is the identity of $\bar{G}$. The restriction to $\bar{\mathfrak{p}}$ of the differential $\bar{\pi}_{*e} : \bar{\mathfrak{g}} \to T_o\bar{M}$ of $\bar{\pi}$ at e yields a linear isomorphism from $\bar{\mathfrak{p}}$ onto $T_o\bar{M}$. In the following we will always identify $\bar{\mathfrak{p}}$ and $T_o\bar{M}$ via this isomorphism. From the $\mathrm{Ad}(\bar{K})$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\bar{\mathfrak{p}} \cong T_o\bar{M}$ we get a $\bar{G}$-invariant Riemannian metric on $\bar{M}$. Then $\bar{M} = \bar{G}/\bar{K}$ is the simply connected Riemannian symmetric space of noncompact type associated with $(\bar{\mathfrak{g}}, \sigma, \langle \cdot, \cdot \rangle)$.

The Lie algebra of the closed subgroup $K'_+ = \{k \in \bar{K} \mid \mathrm{Ad}(k)\nu = \nu\}$ of $\bar{K}$ is $\mathfrak{k}_+$. The homogeneous space $M' = \bar{K}/K'_+$ is diffeomorphic to the orbits $\mathrm{Ad}(\bar{K}) \cdot \nu \subset \bar{\mathfrak{p}}$ and $\bar{K} \cdot \bar{\pi}(\exp \nu) \subset \bar{M}$, where $\exp : \bar{\mathfrak{g}} \to \bar{G}$ denotes the Lie exponential map from $\bar{\mathfrak{g}}$ into $\bar{G}$. We equip M' with the induced Riemannian metric from $\bar{M}$. Then M' is a compact Riemannian symmetric space associated with the orthogonal symmetric Lie algebra $(\bar{\mathfrak{k}}, \tau|\bar{\mathfrak{k}})$, where $\tau|\bar{\mathfrak{k}}$ is the restriction of τ to $\bar{\mathfrak{k}}$. The symmetric spaces M' arising in this manner are precisely the symmetric R-spaces. If $\bar{\mathfrak{g}}$ is simple, then M' is called an *irreducible symmetric R-space*. Symmetric R-spaces form a class of compact Riemannian symmetric spaces with remarkable properties.

The subspace $\mathfrak{p}_-$ is a Lie triple system in $\bar{\mathfrak{p}} = T_o\bar{M}$ and $[\mathfrak{p}_-, \mathfrak{p}_-] \subset \mathfrak{k}_+$. Thus there exists a connected complete totally geodesic submanifold M of $\bar{M}$ with $o \in M$ and $T_oM = \mathfrak{p}_-$. Moreover, M is a reflective submanifold, as $T_o^\perp M = \mathfrak{p}_+$ is also a Lie triple system. Since M is the image of $\mathfrak{p}_-$ under the exponential map of $\bar{M}$ at o, we see that M is simply connected. We define a subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ by $\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{p}_-$ and denote by G the connected Lie subgroup of $\bar{G}$ with Lie algebra $\mathfrak{g}$. Then, by construction, M is the G-orbit through o. The Lie algebra of the isotropy subgroup K_+ of this action at o is just $\mathfrak{k}_+$. Since $M = G/K_+$ is simply connected, K_+ is connected. The restriction $\tau|\mathfrak{g}$ of τ to $\mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$, and it follows from Lemma 2.1 (1) and (3) that $(\mathfrak{g}, \tau|\mathfrak{g})$ is the orthogonal symmetric Lie algebra dual to $(\bar{\mathfrak{k}}, \tau|\bar{\mathfrak{k}})$. Moreover, M is the Riemannian symmetric space of noncompact type associated with $(\mathfrak{g}, \tau|\mathfrak{g})$.

We now introduce an $\mathfrak{O}$-geometry on $\bar{M}$. We put $\dim \mathfrak{p}_- = m$ and denote by $\mathfrak{O}$ the orbit through $\mathfrak{p}_-$ under the action of $\bar{G}$ on $Gr_m(T\bar{M})$. Since $\mathfrak{p}_+$ is also a Lie triple system, $\mathfrak{O}$ is of strongly curvature-invariant type. This $\mathfrak{O}$-geometry is a geometry of type (5) in Theorem 1.1 for the noncompact case.

We will now construct a one-parameter family of symmetric submanifolds of $\bar{M}$ consisting of $\mathfrak{O}$-submanifolds and containing the reflective submanifold M and the symmetric R-space M'. For each $c \in \mathbb{R}$ we define a linear subspace $\mathfrak{p}_c$ of $\mathfrak{p}_- + \mathfrak{k}_- = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1$ by

$$\mathfrak{p}_c = \{X + c\,\mathrm{ad}(\nu)X \mid X \in \mathfrak{p}_-\}\,.$$

Both $\mathfrak{p}_1 = \bar{\mathfrak{g}}_1$ and $\mathfrak{p}_{-1} = \bar{\mathfrak{g}}_{-1}$ are Abelian subalgebras of $\bar{\mathfrak{g}}$. From Lemma 2.1 it follows that $\mathfrak{g}_c = \mathfrak{k}_+ + \mathfrak{p}_c$ is a subalgebra of $\bar{\mathfrak{g}}$. The adjoint action of $\mathfrak{k}_+$ on

$\mathfrak{p}_c$ is effective. In fact, suppose that $T \in \mathfrak{k}_+$ satisfies $[T, X + c\,\mathrm{ad}(\nu)X] = 0$ for all $X \in \mathfrak{p}_-$. Since $[T, X] \in \mathfrak{p}_-$ and $[T, \mathrm{ad}(\nu)X] \in \mathfrak{k}_-$, this implies $[T, X] = 0$ and $[T, \mathrm{ad}(\nu)X] = 0$ for all $X \in \mathfrak{p}_-$. This implies that $0 = [T, \mathfrak{p}_- + \mathfrak{k}_-] = [T, \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1]$. Since the adjoint action of $\bar{\mathfrak{g}}_0$ on $\bar{\mathfrak{g}}_{-1}$ is effective, this gives $T = 0$. On $\mathfrak{k}_+$ we have $\tau = I$ (the identity map), and since $\mathfrak{p}_c \subset \mathfrak{p}_- + \mathfrak{k}_- = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_1$, we get $\tau(\mathfrak{p}_c) = \mathfrak{p}_c$ and $\tau = -I$ on $\mathfrak{p}_c$. Therefore $\mathfrak{g}_c$ is invariant under τ and $(\mathfrak{g}_c, \tau|\mathfrak{g}_c)$ is an orthogonal symmetric Lie algebra. We denote by G_c the connected Lie subgroup of $\bar{G}$ with Lie algebra $\mathfrak{g}_c$ and by M_c the G_c-orbit through o in $\bar{M}$.

Proposition 2.2. *For each $c \in \mathbb{R}$ the orbit $M_c = G_c \cdot o$ is a symmetric submanifold of $\bar{M}$ belonging to the $\mathfrak{O}$-geometry of $\bar{M}$. The submanifolds M_c and M_{-c} are congruent via the geodesic symmetry s_o of $\bar{M}$ at o.*

Proof. We denote by $\bar{\pi}$ the canonical projection from $\bar{\mathfrak{g}}$ onto $\bar{\mathfrak{p}}$ with respect to the Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$. Using our identification of $\bar{\mathfrak{p}}$ with $T_o\bar{M}$, we have $T_oM_c = \bar{\pi}(\mathfrak{g}_c) = \mathfrak{p}_-$, and hence $T_o^\perp M_c = \mathfrak{p}_+$. Since M_c is a G_c-equivariant submanifold, we see that M_c is an $\mathfrak{O}$-submanifold. The Lie group automorphism of $\bar{G}$ whose differential at e is the Lie algebra automorphism τ will also be denoted by τ. The Lie group automorphism τ induces an involutive isometry t_o of $\bar{M}$. Since $\tau(\mathfrak{g}_c) = \mathfrak{g}_c$, we have $\tau(G_c) = G_c$, and therefore $t_o(M_c) = M_c$. Moreover, by construction, t_o satisfies $(t_o)_*X = -X$ for all $X \in T_oM_c = \mathfrak{p}_-$ and $(t_o)_*\xi = \xi$ for all $\xi \in T_o^\perp M_c = \mathfrak{p}_+$. Thus t_o is an extrinsic symmetry of M_c at o, and the G_c-equivariance of M_c implies that M is a symmetric submanifold of $\bar{M}$. Finally, the involution σ of $\bar{\mathfrak{g}}$ satisfies $\sigma(\mathfrak{p}_c) = \mathfrak{p}_{-c}$ and hence $\sigma(\mathfrak{g}_c) = \mathfrak{g}_{-c}$. Therefore M_{-c} is congruent to M_c via the geodesic symmetry s_o of $\bar{M}$ at o induced from σ. □

The third author proved in [10], Lemma 4.3, an analogon of Proposition 2.2 for the compact case. Our next aim is to study the geometry of the submanifolds M_c $(c \geq 0)$ in more detail.

Theorem 2.3. *The submanifolds M_c, $0 \leq c < 1$, form a family of noncompact symmetric submanifolds which are homothetic to the reflective submanifold M. The submanifolds M_c, $1 < c < \infty$, form a family of compact symmetric submanifolds which are homothetic to the symmetric R-space M'. The submanifold M_1 is a flat symmetric space which is isometric to a Euclidean space. The second fundamental form α_c of M_c is given by*

$$\alpha_c(X, Y) = c[\mathrm{ad}(\nu)X, Y] \in \mathfrak{p}_+ = T_o^\perp M_c\ ,\ \ X, Y \in \mathfrak{p}_- = T_oM_c\ .$$

In particular, all submanifolds M_c, $0 \leq c < \infty$, are pairwise noncongruent.

Proof. We use the following lemma, which can be proved by a straightforward calculation.

Lemma 2.4. *For all $t \in \mathbb{R}$ we have*

(1) $\mathrm{Ad}(\exp t\nu)T = T$ *for all* $T \in \mathfrak{k}_+$*;*
(2) $\mathrm{Ad}(\exp t\nu)X = \cosh(t)X + \sinh(t)\mathrm{ad}(\nu)X$ *for all* $X \in \mathfrak{p}_- + \mathfrak{k}_-$.

For each $t \in \mathbb{R}$ we define an inner automorphism ρ_t of $\bar{G}$ by $\rho_t(g) = (\exp t\nu)g(\exp(-t\nu))$ for all $g \in \bar{G}$. The induced Lie algebra automorphism of $\bar{\mathfrak{g}}$ is $\mathrm{Ad}(\exp t\nu)$. From Lemma 2.4 it follows that $\mathrm{Ad}(\exp t\nu)(\mathfrak{g}) = \mathfrak{g}_{\tanh(t)}$ and $\mathrm{Ad}(\exp t\nu)(\bar{\mathfrak{k}}) = \mathfrak{g}_{\coth(t)}$ $(t > 0)$, which implies that $\rho_t(G) = G_{\tanh(t)}$ and $\rho_t(\bar{K}) = G_{\coth(t)}$. Thus $M_{\tanh(t)}$ is congruent to the orbit $G \cdot \exp(-t\nu)o \subset \bar{M}$ and $M_{\coth(t)}$ is congruent to the orbit $\bar{K} \cdot \exp(-t\nu)o \subset \bar{M}$. Note that, since $\nu \in \bar{\mathfrak{p}}$, the curve $t \mapsto \exp(-t\nu)o$ is a geodesic in $\bar{M}$. The fact that the orbits of G through the points on this geodesic are symmetric submanifolds in $\bar{M}$ was already discovered by Osipova in [16].

The case $0 \le c < 1$: For $c = 0$ we have $\mathfrak{g}_0 = \mathfrak{k}_+ + \mathfrak{p}_-$, and hence M_0 coincides with the reflective submanifold M. Let t be the nonnegative real number given by $\tanh(t) = c$. Since K_+ is connected, Lemma 2.4 (1) implies $\rho_t(K_+) = K_+$. We define a G-equivariant map f_t from $M = G/K_+$ into $\bar{M} = \bar{G}/\bar{K}$ by $f_t(gK_+) = \rho_t(g)(o)$ for all $g \in G$. Then we obviously have $f_t(M) = M_{\tanh(t)} = M_c$. Using our identification of T_oM with $\mathfrak{p}_-$ and of $T_o\bar{M}$ with $\bar{\mathfrak{p}}$, the differential f_{t*o} of f_t at o is given by

$$f_{t*o}X = \bar{\pi}(\mathrm{Ad}(\exp t\nu)X) = \cosh(t)X\ ,\ \ X \in \mathfrak{p}_-\ ,$$

where $\bar{\pi}$ denotes the canonical projection from $\bar{\mathfrak{g}}$ onto $\bar{\mathfrak{p}}$. We denote by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_t$ the induced Riemannian metrics on M from the totally geodesic imbedding and from f_t, respectively. Then we have $\langle\cdot,\cdot\rangle_t = \cosh^2(t)\langle\cdot,\cdot\rangle$. Therefore f_t is a homothetic immersion from M into $\bar{M}$ and hence a covering map onto M_c. We will show that f_t is a diffeomorphism from M onto M_c. The normal exponential map $\mathrm{Exp} : T^\perp M \to \bar{M}$ from the normal bundle $T^\perp M$ onto $\bar{M}$ is a diffeomorphism. Since

$$f_t(gK_+) = (\exp t\nu)g(\exp(-t\nu))(o) = (\exp t\nu)(\mathrm{Exp}_{g(o)}(-tg_{*o}\nu))\ ,$$

we have $f_t(gK_+) = f_t(K_+)$ if and only if $g \in K_+$. Therefore f_t is an imbedding from M into $\bar{M}$ and hence a diffeomorphism from M onto M_c.

Next, we compute the second fundamental form α_c of M_c. We denote by $\bar{\nabla}$ the Riemannian connection of $\bar{M}$ and by Z^* the Killing vector field of $\bar{M}$ generated by $Z \in \bar{\mathfrak{g}}$. At the origin o of $\bar{M}$ we have $\bar{\nabla}_X Z^* = [Z_{\bar{\mathfrak{k}}}, X]$ for all $X \in T_o\bar{M} = \bar{\mathfrak{p}}$, where $Z_{\bar{\mathfrak{k}}}$ denotes the $\bar{\mathfrak{k}}$-component of Z with respect to the Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$. Therefore we have

$$\begin{aligned}(\bar{\nabla}_{f_{t*}U} f_{t*}V^*)_o &= (\bar{\nabla}_{f_{t*}U}(\mathrm{Ad}(\exp t\nu)V)^*)_o \\ &= [(\mathrm{Ad}(\exp t\nu)V)_{\bar{\mathfrak{k}}}, f_{t*}U] \\ &= \sinh(t)\cosh(t)[\mathrm{ad}(\nu)V, U] \\ &= \sinh(t)\cosh(t)[\mathrm{ad}(\nu)U, V] \quad \in \mathfrak{p}_+ = T_oM_c^\perp\end{aligned}$$

for all $U, V \in \mathfrak{p}_-$. For $X, Y \in \mathfrak{p}_-$ we define $U = X/\cosh(t)$ and $V = Y/\cosh(t)$. Then we have $f_{t*}U = X$ and $f_{t*}V = Y$ at o, and the above equation implies $\alpha_c(X, Y) = c\,[\mathrm{ad}(\nu)X, Y]$.

The case $1 < c < \infty$: Let t be the positive real number defined by $\coth(t) = c$. For $t = 1$, $M_{\coth(1)}$ is congruent to the symmetric R-space $M' = \bar{K}/K'_+$. Since $\mathrm{Ad}(k)\nu = \nu$ for all $k \in K'_+$, we have $k(\exp t\nu)k^{-1} = \exp t\nu$ and hence $\rho_t(k) = k$. Moreover, $\rho_t(g)(o) = o$ holds for $g \in \bar{K}$ if and only if $g \in K'_+$. In fact, for $g \in \bar{K}$ we get

$$\begin{aligned}\rho_t(g)(o) = o &\iff (\exp t\nu)g(\exp(-t\nu))\bar{\pi}(e) = \bar{\pi}(e)\\ &\iff g\bar{\pi}(\exp(-t\nu)) = \bar{\pi}(\exp(-t\nu))\\ &\iff \bar{\pi}(\exp(-t\mathrm{Ad}(g)\nu)) = \bar{\pi}(\exp(-t\nu))\\ &\iff \mathrm{Ad}(g)\nu = \nu\ .\end{aligned}$$

Here we note that $\bar{\pi} \circ \exp|\bar{\mathfrak{p}} : \bar{\mathfrak{p}} \to \bar{M}$ is a diffeomorphism. Therefore the map h_t from $M' = \bar{K}/K'_+$ into $\bar{M}$ defined by $h_t(gK'_+) = \rho_t(g)(o)$ for all $g \in \bar{K}$ is a diffeomorphism from M' onto $M_{\coth(t)} = M_c$. The differential $h_{t*eK'_+}$ of h_t at eK'_+ is given by

$$h_{t*eK'_+}X = \bar{\pi}(\mathrm{Ad}(\exp t\nu)X) = \sinh(t)\mathrm{ad}(\nu)X\ ,\ \ X \in \mathfrak{k}_-\ ,$$

where we identify $T_{eK'_+}M'$ and $T_o\bar{M}$ with $\mathfrak{k}_-$ and $\bar{\mathfrak{p}}$, respectively. We denote by $\langle\cdot,\cdot\rangle_t$ the inner product on $\mathfrak{k}_-$ that corresponds to the induced Riemannian metric on M' by h_t. Then we have $\langle X, Y\rangle_t = \sinh^2(t)\langle\mathrm{ad}(\nu)X, \mathrm{ad}(\nu)Y\rangle$ for all $X, Y \in \mathfrak{k}_-$, which shows that h_t is a homothetic diffeomorphism from M' onto M_c. By a computation similar to the case $0 \le c < 1$ we see that the second fundamental form α_c of M_c, $1 < c < \infty$, is given by $\alpha_c(X, Y) = c\,[\mathrm{ad}(\nu)X, Y]$ for all $X, Y \in \mathfrak{p}_-$, where we use the identification of T_oM_c with $\mathfrak{p}_-$ and of $T_o^\perp M_c$ with $\mathfrak{p}_+$.

The case $c = 1$: The submanifold M_1 arises as the limit of the two families $M_{\tanh(t)}$ and $M_{\coth(t)}$ for $t \to \infty$. Since $\mathfrak{p}_1 = \bar{\mathfrak{g}}_1$ is an Abelian ideal in $\mathfrak{g}_1 = \mathfrak{k}_+ + \mathfrak{p}_1 = \mathfrak{k}_+ + \bar{\mathfrak{g}}_1$, $(\mathfrak{g}_1, \tau|\mathfrak{g}_1)$ is an orthogonal symmetric Lie algebra of Euclidean type. Therefore M_1 is a flat symmetric space. We shall prove now that M_1 is isometric to a Euclidean space. Let $\bar{G}_1$ be the connected Lie subgroup of $\bar{G}$ with Lie algebra $\bar{\mathfrak{g}}_1$. Then $\bar{G}_1$ is Abelian and we have $\exp(\bar{\mathfrak{g}}_1) = \bar{G}_1$. Moreover, the orbit $\bar{G}_1 \cdot o$ coincides with $G_1 \cdot o = M_1$. There exists an Iwasawa decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{a}} + \bar{\mathfrak{n}}$ of $\bar{\mathfrak{g}}$ such that the maximal Abelian subspace $\bar{\mathfrak{a}}$ of $\bar{\mathfrak{p}}$ contains the element ν and the nilpotent subalgebra $\bar{\mathfrak{n}}$ contains $\bar{\mathfrak{g}}_1$ (cf. [17], Chapter I, § 4). Let $\bar{G} = \bar{K} \cdot \bar{A} \cdot \bar{N}$ be the Iwasawa decomposition of $\bar{G}$ that corresponds to the above Iwasawa decomposition of $\bar{\mathfrak{g}}$. Since the exponential map $\exp : \bar{\mathfrak{n}} \to \bar{N}$ is a diffeomorphism (cf. [4], Chapter VI, § 5), $\bar{G}_1$ is a closed subgroup of $\bar{N}$ that is diffeomorphic to $\bar{\mathfrak{g}}_1$. Since $\bar{G}_1 \cap \bar{K} = \{e\}$, the orbit $M_1 = \bar{G}_1 \cdot o$ is isometric to the simply connected Abelian Lie group $\bar{G}_1$ with a suitable left-invariant Riemannian metric. By a computation similar to the case $0 \le c < 1$ we see that the second fundamental form α_1 of M_1 is given by $\alpha_1(X, Y) = [\mathrm{ad}(\nu)X, Y]$ for all $X, Y \in \mathfrak{p}_-$. $\square$

Recall that a submanifold M of a Riemannian manifold $\bar{M}$ is *pseudo-umbilical* if the second fundamental form α in direction of the mean curvature vector field ξ of M is a multiple of the induced Riemannian metric on M, that is, $\langle \alpha(X, Y), \xi\rangle = \lambda\langle X, Y\rangle$ for all vector fields X, Y tangent to M and some smooth function λ on M.

Corollary 2.5. *The mean curvature vector ξ_c of M_c at o is given by $\xi_c = (c/2m)\nu \in \mathfrak{p}_+ = T_o^\perp M_c$, where $m = \dim M_c = \dim \mathfrak{p}_-$, and M_c is a pseudo-umbilical submanifold of $\bar{M}$.*

Proof. Let $e_1, \dots, e_m$ be an orthonormal basis of $T_o M_c = \mathfrak{p}_-$. Then the vectors $\mathrm{ad}(\nu)e_1, \dots, \mathrm{ad}(\nu)e_m$ form an orthonormal basis of $\mathfrak{k}_-$ with respect to the negative definite inner product $B|(\mathfrak{k}_- \times \mathfrak{k}_-)$, and

$$\sum_{i=1}^m B(\mathrm{ad}(\eta)\mathrm{ad}(\nu)\mathrm{ad}(\nu)e_i, \mathrm{ad}(\nu)e_i) = -\sum_{i=1}^m \langle \mathrm{ad}(\eta)\mathrm{ad}(\nu)e_i, e_i\rangle\ ,$$

which implies

$$B(\eta, \nu) = \mathrm{tr}(\mathrm{ad}(\eta)\mathrm{ad}(\nu)) = 2\sum_{i=1}^m \langle \mathrm{ad}(\eta)\mathrm{ad}(\nu)e_i, e_i\rangle$$

for all $\eta \in T_o^\perp M_c = \mathfrak{p}_+$. Thus

$$\begin{aligned}
\langle \xi_c, \eta\rangle &= \frac{1}{m}\sum_{i=1}^m \langle \alpha_c(e_i, e_i), \eta\rangle \ = \ \frac{c}{m}\sum_{i=1}^m \langle [\mathrm{ad}(\nu)e_i, e_i], \eta\rangle \\
&= -\frac{c}{m}\sum_{i=1}^m B(\mathrm{ad}(\nu)e_i, \mathrm{ad}(\eta)e_i) \ = \ \frac{c}{m}\sum_{i=1}^m B(\mathrm{ad}(\eta)\mathrm{ad}(\nu)e_i, e_i) \\
&= \frac{c}{2m}B(\eta, \nu) = \langle \frac{c}{2m}\nu, \eta\rangle\ ,
\end{aligned}$$

which implies $\xi_c = (c/2m)\nu$ and

$$\langle \alpha_c(X, Y), \xi_c\rangle = \frac{c^2}{2m}\langle [\mathrm{ad}(\nu)X, Y], \nu\rangle = \frac{c^2}{2m}\langle \mathrm{ad}(\nu)^2 X, Y\rangle = \frac{c^2}{2m}\langle X, Y\rangle$$

for all $X, Y \in \mathfrak{p}_- = T_o M_c$. $\square$

Remark 2.6. In the case of Table, No.13, i=1, $\bar{M}$ is a real hyperbolic space $\mathbb{R}H^n$, and the family of symmetric submanifolds M_c constructed in this section is the family of complete totally umbilical hypersurfaces. The totally geodesic submanifold M is a totally geodesic hyperplane $\mathbb{R}H^{n-1}$, and the submanifold $M_{\tanh(t)}$, $t > 0$, is congruent to the equidistant hypersurface at distance t from M. The hypersurface M_1 is a flat Euclidean space $\mathbb{R}^{n-1}$ which is imbedded in $\mathbb{R}H^n$ as a horosphere. The submanifold $M_{\coth(t)}$, $t > 0$, is congruent to a distance sphere with radius t in $\mathbb{R}H^n$. So our construction of symmetric submanifolds can be viewed as a generalization of this well-known family of symmetric submanifolds in a real hyperbolic space.

3. The classification of symmetric submanifolds

In this section we will prove that for each $\mathfrak{O}$-geometry as introduced in Section 2 every $\mathfrak{O}$-submanifold is locally extrinsically symmetric, and that such submanifolds are exhausted by the examples constructed in Section 2. We continue to use the notations that we introduced in Section 2. Let $\bar{M} = \bar{G}/\bar{K}$ be a simply connected Riemannian symmetric space of noncompact type and $\mathfrak{O}$ a Grassmann geometry of strongly curvature-invariant type on $\bar{M}$.

We first study the submanifold geometry of an $\mathfrak{O}$-submanifold (M, f) following Chapter VII in Kobayashi and Nomizu [6]. We denote by u_o the restriction to $\bar{\mathfrak{p}}$ of the differential $\bar{\pi}_{*e} : \bar{\mathfrak{g}} \to T_o\bar{M}$. Then u_o is a linear isometry from $\bar{\mathfrak{p}}$ onto $T_o\bar{M}$ and induces an orthonormal frame for $T_o\bar{M}$. We define a smooth map ϕ from $\bar{G}$ into the orthonormal frame bundle $O(\bar{M})$ by $\phi(g) = g_{*o}u_o$. Then ϕ is a $\bar{K}$-bundle homomorphism that corresponds to the Lie group homomorphism $\mathrm{Ad}_{\bar{\mathfrak{p}}} : \bar{K} \to O(\bar{\mathfrak{p}})$. We denote by $\bar{\omega}_{\bar{\mathfrak{k}}}$ the $\bar{\mathfrak{k}}$-component of the Maurer-Cartan form $\bar{\omega}$ on $\bar{G}$ with respect to the Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$. General theory of Riemannian symmetric spaces implies that ϕ maps the canonical connection of $\bar{G}$ with connection form $\bar{\omega}_{\bar{\mathfrak{k}}}$ to the Riemannian connection of $O(\bar{M})$.

The orbit $\mathfrak{O}$ can be identified with the fibre bundle $\bar{G} \times_{\bar{K}} (\bar{K}/\bar{K}_+)$ that is associated with the principal fibre bundle $\bar{\pi} : \bar{G} \to \bar{M}$, where $\bar{K}_+ = \{k \in \bar{K} \mid \mathrm{Ad}_{\bar{\mathfrak{p}}}(k)(\mathfrak{p}_-) = \mathfrak{p}_-\}$. It is easy to see that the Lie algebra of $\bar{K}_+$ is just $\mathfrak{k}_+$. Let (M, f) be an $\mathfrak{O}$-submanifold, that is, f is an immersion from an m-dimensional smooth manifold M into $\bar{M}$ such that $f_{*p}T_pM \in \mathfrak{O}$ for all $p \in M$. We denote by

$$P = \{(p, g) \in M \times \bar{G} \mid f(p) = \bar{\pi}(g)\ ,\ \phi(g)\mathfrak{p}_- = f_{*p}T_pM\}$$

the bundle of the adapted frames of (M, f), and by $\pi : P \to M$ and $\tilde{f} : P \to \bar{G}$ the restrictions to P of the canonical projection from $M \times \bar{G}$ onto the first factor M and the second factor $\bar{G}$, respectively. Then $\pi : P \to M$ is a principal fibre bundle with structure group $\bar{K}_+$, and $\tilde{f}$ is the bundle homomorphism that corresponds to the Lie group homomorphism $\bar{K}_+ \to \bar{K}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & \bar{G} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow[f]{} & \bar{M} \end{array} .$$

We denote by $\rho' : \bar{K}_+ \to O(\mathfrak{p}_-)$ and $\rho'' : \bar{K}_+ \to O(\mathfrak{p}_+)$ the representations of $\bar{K}_+$ that are obtained by restricting $\mathrm{Ad}_{\bar{\mathfrak{p}}}(\bar{K}_+)$ to $\mathfrak{p}_-$ and $\mathfrak{p}_+$, respectively. Let $O(M)$ be the orthornormal frame bundle of M with respect to the induced Riemannian metric and $O(T^\perp M)$ be the orthonormal frame bundle of the normal bundle $T^\perp M$. For each $u \in P$ we define linear isomorphisms $\phi'(u) : \mathfrak{p}_- \to T_{\pi(u)}M$ and $\phi''(u) : \mathfrak{p}_+ \to T^\perp_{\pi(u)}M$ by restricting $\phi(\tilde{f}(u))$ to $\mathfrak{p}_-$ and $\mathfrak{p}_+$, respectively. Then $\phi' : P \to O(M)$ and $\phi'' : P \to O(T^\perp M)$ are bundle homomorphisms with corresponding Lie group homomorphism ρ' and ρ'', respectively.

The decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}} = \mathfrak{k}_+ + \mathfrak{k}_- + \mathfrak{p}_- + \mathfrak{p}_+$ induces a decomposition of the pull back $\omega = \tilde{f}^*\bar{\omega}$ of the Maurer-Cartan form $\bar{\omega}$ on $\bar{G}$ to P into

$$\omega = \omega_{\mathfrak{k}_+} + \omega_{\mathfrak{k}_-} + \omega_{\mathfrak{p}_-} ,$$

since $\omega_{\mathfrak{p}_+}$ vanishes. We denote by $\bar{\nabla}$ and ∇ the Riemannian connections of $\bar{M}$ and M, respectively, and by α the second fundamental form of (M, f). For each vector field Y on M we define a smooth function $\tilde{Y} : P \to \mathfrak{p}_-$ by

$$\tilde{Y}(u) = \phi(\tilde{f}(u))^{-1}(Y_{\pi(u)}) , \ u \in P .$$

For each vector field X on M we denote by X^* a lift of X to P with respect to $\pi : P \to M$. Then, by definition of $\bar{\nabla}$, we have

$$\begin{aligned}\phi(\tilde{f}(u))^{-1}(\bar{\nabla}_X Y) &= X^*_u(\tilde{Y}) + \mathrm{ad}_{\bar{\mathfrak{p}}}(\omega_{\bar{\mathfrak{k}}}(X^*))(\tilde{Y}(u)) \\ &= X^*_u(\tilde{Y}) + \mathrm{ad}_{\bar{\mathfrak{p}}}(\omega_{\mathfrak{k}_+}(X^*) + \omega_{\mathfrak{k}_-}(X^*))(\tilde{Y}(u)) \\ &= X^*_u(\tilde{Y}) + \rho'(\omega_{\mathfrak{k}_+}(X^*))(\tilde{Y}(u)) + [\omega_{\mathfrak{k}_-}(X^*), \tilde{Y}(u)]\end{aligned}$$

for all $u \in P$. On the other hand, we have

$$\phi(\tilde{f}(u))^{-1}(\nabla_X Y + \alpha(X, Y)) = \phi'(u)^{-1}(\nabla_X Y) + \phi''(u)^{-1}(\alpha(X, Y)) .$$

Comparing the $\mathfrak{p}_-$- and $\mathfrak{p}_+$-components of both equations, we obtain

$$\begin{aligned}\phi'(u)^{-1}(\nabla_X Y) = X^*_u(\tilde{Y}) + \rho'(\omega_{\mathfrak{k}_+}(X^*))(\tilde{Y}(u)) \text{ and } \phi''(u)^{-1}(\alpha(X, Y)) \\ = [\omega_{\mathfrak{k}_-}(X^*), \tilde{Y}(u)] .\end{aligned}$$

Analogously, for a normal vector field ξ and a tangent vector field X of M we get

$$\begin{aligned}&\phi'(u)^{-1}(A_\xi X) = -[\omega_{\mathfrak{k}_-}(X^*), \tilde{\xi}(u)] \text{ and} \\ &\phi''(u)^{-1}(\nabla^\perp_X \xi) = X^*_u(\tilde{\xi}) + \rho''(\omega_{\mathfrak{k}_+}(X^*))(\tilde{\xi}(u)) ,\end{aligned}$$

where A and $\nabla^\perp$ denote the shape operator and the normal connection of M, respectively. From these equations we get

Lemma 3.1. *The maps ϕ' and ϕ'' map the connection in P with connection form $\omega_{\mathfrak{k}_+}$ to the Riemannian connection in $O(M)$ and the normal connection in $O(T^{\perp}M)$, respectively. Moreover, for each $u \in P$ there exists a linear map $\hat{\alpha}_u : \mathfrak{p}_- \to \mathfrak{k}_-$ such that*

$$\phi''(u)^{-1}(\alpha(\phi'(u)X, \phi'(u)Y)) = [\hat{\alpha}_u X, Y]$$

for all $X, Y \in \mathfrak{p}_-$.

We define a $\mathfrak{k}_+$-invariant $\mathfrak{p}_+$-valued symmetric bilinear form $\tilde{\alpha}_1$ on $\mathfrak{p}_-$ by

$$\tilde{\alpha}_1(X, Y) = [\mathrm{ad}(\nu)X, Y]\ ,\ \ X, Y \in \mathfrak{p}_-\ .$$

We recall that the subgroup K_+ of $\bar{K}$ is the identity component of $\bar{K}_+$ and its Lie algebra is $\mathfrak{k}_+$. Since $\tilde{\alpha}_1$ is invariant under the action of $\mathfrak{k}_+$, it is also invariant under the action of K_+. From now on we assume that M is simply connected. Let Q be the principal subbundle of P over M with structure group K_+. We define a $T^{\perp}M$-valued symmetric bilinear form α_1 on M by

$$\alpha_1(X, Y) = \phi''(u)(\tilde{\alpha}_1(\phi'(u)^{-1}X, \phi'(u)^{-1}Y)) \tag{3.1}$$

for all $X, Y \in T_pM$, $p \in M$, $u \in Q$ with $\pi(u) = p$. Since both the Riemannian connection on M and the normal connection on $T^{\perp}M$ are reduced to Q by Lemma 3.1, α_1 is a parallel tensor field. The following lemma is a key point for our classification result.

Lemma 3.2. *Let $\bar{M}$ be an irreducible Riemannian symmetric space of noncompact type as in the Table, except No. 13 for $i = 1$, and let (M, f) be an $\mathfrak{O}$-submanifold of $\bar{M}$. Then there exists a smooth function c on M such that the second fundamental form α of (M, f) satisfies $\alpha = c\alpha_1$.*

Proof. We will reduce the proof to arguments of Theorem 5.7 in [10] and adapt these arguments to our context. By Lemma 3.1, for each $u \in Q$ there exists a linear map $\hat{\alpha}_u : \mathfrak{p}_- \to \mathfrak{k}_-$ such that

$$\phi''(u)^{-1}(\alpha(\phi'(u)X, \phi'(u)Y)) = [\hat{\alpha}_u X, Y]\ ,\ \ X, Y \in \mathfrak{p}_-\ .$$

The symmetry of the second fundamental form α yields $[\hat{\alpha}_u X, Y] = [\hat{\alpha}_u Y, X]$, and the definition of α_1 implies $\hat{\alpha}_1 = \mathrm{ad}(\nu)$. We will prove that $\hat{\alpha}_u = c\hat{\alpha}_1$ for some $c \in \mathbb{R}$.

In order to be able to apply the arguments in the proof of Theorem 5.7 in [10] we consider the compact dual of $(\bar{\mathfrak{g}}, \sigma)$. We put $\hat{\mathfrak{g}} = \bar{\mathfrak{k}} + \sqrt{-1}\bar{\mathfrak{p}}$ and define involutive automorphisms $\hat{\sigma}$ and $\hat{\tau}$ of $\hat{\mathfrak{g}}$ by $\hat{\sigma}(T + \sqrt{-1}X) = T - \sqrt{-1}X$ and $\hat{\tau}(T + \sqrt{-1}X) = \tau(T) + \sqrt{-1}\tau(X)$ for all $T \in \bar{\mathfrak{k}}$ and $X \in \bar{\mathfrak{p}}$. Then $(\hat{\mathfrak{g}}, \hat{\sigma})$ is an orthogonal symmetric Lie algebra of compact type and the negative of the Killing form of $\hat{\mathfrak{g}}$ defines a positive definite inner product $\langle \cdot, \cdot \rangle$ on $\hat{\mathfrak{g}}$. For the linear maps $\lambda_1, \lambda_u : \sqrt{-1}\mathfrak{p}_- \to \mathfrak{k}_-$ given by $\lambda_1 X = -\hat{\alpha}_1(\sqrt{-1}X)$ and $\lambda_u X = -\hat{\alpha}_u(\sqrt{-1}X)$ we have

$$[\lambda_u X, Y] = [\lambda_u Y, X] \text{ and } \lambda_1 X = \mathrm{ad}(-\sqrt{-1}\nu)X$$

for all $X, Y \in \sqrt{-1}\mathfrak{p}_-$. Note that $-\sqrt{-1}\nu \in \sqrt{-1}\mathfrak{p}_+$ and that the map $J = \mathrm{ad}(-\sqrt{-1}\nu)$ on $\mathfrak{p}^* = \sqrt{-1}\mathfrak{p}_- + \mathfrak{k}_-$ interchanges $\sqrt{-1}\mathfrak{p}_-$ and $\mathfrak{k}_-$ and satisfies $J^2 = -I_{\mathfrak{p}^*}$.

The previous equation for λ_u implies

$$\langle \mathrm{ad}(\xi)\lambda_u X, Y\rangle = \langle \xi, [\lambda_u X, Y]\rangle = -\langle \xi, [X, \lambda_u Y]\rangle = -\langle \lambda_u^t \mathrm{ad}(\xi)X, Y\rangle$$

for all $X, Y \in \sqrt{-1}\mathfrak{p}_-$ and $\xi \in \sqrt{-1}\mathfrak{p}_+$, which implies

$$\mathrm{ad}(\xi)\lambda_u = -\lambda_u^t \mathrm{ad}(\xi)|_{\sqrt{-1}\mathfrak{p}_-}\ ,\ \xi \in \sqrt{-1}\mathfrak{p}_+\ . \tag{3.2}$$

Here, $\lambda_u^t : \mathfrak{k}_- \to \sqrt{-1}\mathfrak{p}_-$ denotes the transpose map of λ_u with respect to $\langle\cdot,\cdot\rangle$.

From (3.2) we will now deduce the equation

$$\lambda_u \mathrm{ad}(\xi)|_{\mathfrak{k}_-} = -\mathrm{ad}(\xi)\lambda_u^t\ ,\ \xi \in \sqrt{-1}\mathfrak{p}_+\ . \tag{3.3}$$

The equation (3.2) applied to $\xi = -\sqrt{-1}\nu$ implies that $J\lambda_u = -\lambda_u^t J$. Since each $\xi \in \sqrt{-1}\mathfrak{p}_+$ commutes with $-\sqrt{-1}\nu$, we get $[\mathrm{ad}(-\sqrt{-1}\nu), \mathrm{ad}(\xi)] = \mathrm{ad}([-\sqrt{-1}\nu, \xi]) = 0$. Hence, by conjugating the left-hand side of (3.3) with $J = \mathrm{ad}(-\sqrt{-1}\nu)$, we see that $J\lambda_u \mathrm{ad}(\xi)J^{-1} = -\lambda_u^t J\mathrm{ad}(\xi)J^{-1} = -\lambda_u^t \mathrm{ad}(\xi)$, which is the right-hand side of (3.2). Analogously, the left-hand side of (3.2) is obtained by conjugating the right-hand side of (3.3) with J.

Now we extend the linear map $\lambda_u : \sqrt{-1}\mathfrak{p}_- \to \mathfrak{k}_-$ to a skewsymmetric endomorphism A_u of $\mathfrak{p}^* = \sqrt{-1}\mathfrak{p}_- + \mathfrak{k}_-$ by putting $A_u = \lambda_u$ on $\sqrt{-1}\mathfrak{p}_-$ and $A_u = -\lambda_u^t$ on $\mathfrak{k}_-$. From (3.2) and (3.3) we obtain $[\mathrm{ad}(\xi), A_u] = 0$ for all $\xi \in \sqrt{-1}\mathfrak{p}_+$, and then $[\mathrm{ad}(T), A_u] = 0$ for all $T \in \mathfrak{h}_+ + \sqrt{-1}\mathfrak{p}_+$, where $\mathfrak{h}_+$ is the subalgebra of $\mathfrak{k}_+$ given by $\mathfrak{h}_+ = [\sqrt{-1}\mathfrak{p}_+, \sqrt{-1}\mathfrak{p}_+]$.

Assume that $(\bar{\mathfrak{g}}, \sigma)$ is as in the Table, No. $7 - 18$. Then $(\hat{\mathfrak{g}}, \hat{\tau})$ is an orthogonal symmetric Lie algebra that corresponds to an irreducible Hermitian symmetric space with isotropy algebra $\mathfrak{k}^* = \mathfrak{k}_+ + \sqrt{-1}\mathfrak{p}_+$. Moreover, it satisfies $\mathfrak{k}_+ = [\sqrt{-1}\mathfrak{p}_+, \sqrt{-1}\mathfrak{p}_+]$ unless we are in the case of No. 13 for $i = 1$. Since the above skewsymmetric endomorphism A_u of $\mathfrak{p}^* = \sqrt{-1}\mathfrak{p}_- + \mathfrak{k}_-$ satisfies $[\mathrm{ad}(T), A_u] = 0$ for all $T \in \mathfrak{k}^*$, A_u is a real multiple of J. This implies $\lambda_u = c\,\mathrm{ad}(-\sqrt{-1}\nu) = c\lambda_1$ on $\sqrt{-1}\mathfrak{p}_-$ for some $c \in \mathbb{R}$ and hence $\hat{\alpha}_u = c\hat{\alpha}_1$ on $\mathfrak{p}_-$. The same holds in the case of No. $1 - 6$. For more details we refer to the proof of Theorem 5.7 in [10]. □

Theorem 3.3. *Let $\bar{M}$ be an irreducible Riemannian symmetric space as in the Table, except No.* 13 *for $i = 1$. Then every $\mathfrak{O}$-submanifold (M, f) of $\bar{M}$ is a locally extrinsically symmetric submanifold.*

Proof. Let $p \in M$ and U be a simply connected open neighborhood of p in M. On U we consider the $T^{\perp}M$-valued symmetric bilinear form α_1 as defined in

(3.1). By Lemma 3.2, there exists a smooth function c on U such that the second fundamental form α of (M, f) satisfies $\alpha = c\alpha_1$. Since α_1 is parallel, we have $(\bar{\nabla}_X \alpha)(Y, Z) = dc(X)\alpha_1(Y, Z)$ on U. Assume that $dc_q \neq 0$ at some point $q \in U$. Since $\dim M \geq 2$, there exist nonzero vectors $X, Y \in T_q M$ such that $dc_q(X) \neq 0$ and $dc_q(Y) = 0$. Since the tangent space $T_q M$ is invariant under the curvature tensor $\bar{R}$ of $\bar{M}$, the Codazzi equation is of the form $(\bar{\nabla}_X \alpha)(Y, Z) = (\bar{\nabla}_Y \alpha)(X, Z)$. From this equation we get

$$dc_q(X)\alpha_1(Y, Y) = (\bar{\nabla}_X \alpha)(Y, Y) = (\bar{\nabla}_Y \alpha)(X, Y) = dc_q(Y)\alpha_1(X, Y) = 0.$$

It follows from the proof of Corollary 2.5 that $\alpha_1(Y, Y) \neq 0$, and thus we have $dc_q(X) = 0$, which contradicts our assumption $dc_q(X) \neq 0$. Thus dc vanishes on U, which implies that α is parallel. From Theorem 1.3 in [10] we now conclude that M is a locally extrinsically symmetric submanifold of $\bar{M}$. □

The compact dual $\bar{M}^*$ of an irreducible Riemannian symmetric space $\bar{M}$ as above is associated with an orthogonal symmetric Lie algebra $(\hat{\mathfrak{g}}, \hat{\sigma}, \langle\cdot,\cdot\rangle)$ as in the proof of Lemma 3.2. If we consider on $\bar{M}^*$ the $\mathfrak{O}$-geometry induced from the Lie triple system $\sqrt{-1}\mathfrak{p}_-$, the arguments in Lemma 3.2 and Theorem 3.3 are still valid. Therefore we have

Theorem 3.4. *Let $\bar{M}^*$ be the compact dual to an irreducible Riemannian symmetric space $\bar{M}$ as in the Table, except No.* 13 *for $i = 1$. Then every $\mathfrak{O}$-submanifold M of $\bar{M}^*$ is a locally extrinsically symmetric submanifold.*

The second author proved this result in [1] under the two additional assumptions that the reflective submanifold tangent to $\sqrt{-1}\mathfrak{p}_-$ is irreducible and of rank ≥ 2. His approach is different from ours and uses the Gauss equation.

Eventually, we can now prove the final classification result.

Theorem 3.5. *Let $\bar{M}$ be an irreducible Riemannian symmetric space of noncompact type as in the Table, except No.* 13 *for $i = 1$, and consider an $\mathfrak{O}$-geometry of type (5) on $\bar{M}$. Then every symmetric submanifold M of $\bar{M}$ which belongs to this $\mathfrak{O}$-geometry is congruent to some submanifold M_c as constructed in Section 2.*

Proof. Without loss of generality we may assume that M contains the origin o of $\bar{M}$ and that $T_o M = \mathfrak{p}_-$, using our identification of $T_o\bar{M}$ with $\bar{\mathfrak{p}}$. Let α be the second fundamental form of M. By Lemma 3.2, there exists a real number c such that $\alpha = c\alpha_1$ at o. By Theorem 2.3, we have $\alpha = \alpha_c$ at o, where α_c is the second fundamental form of M_c. By a result of Naitoh and Takeuchi [15] it follows that $M = M_c$. □

Remark 3.6. As we have seen in Remark 2.6, in the case of No. 13 for $i = 1$, $\bar{M}$ is a real hyperbolic space $\mathbb{R}H^n$ and its $\mathfrak{O}$-geometry is formed by hypersurfaces. Symmetric hypersurfaces in $\mathbb{R}H^n$ have already been classified by Takeuchi in [18]. In fact, they are totally umbilical or the product Riemannian manifolds $S^k \times \mathbb{R}H^{n-k-1}$.

Table This table is a modification of Table II in [10]. The notation for real semisimple Lie algebras is as in [4].
$(\bar{\mathfrak{g}}, \sigma)$ $\bar{M}$: an irreducible Riemannian symmetric space $\bar{M}$ associated with a positive definite symmetric graded Lie algebra $(\bar{\mathfrak{g}}, \sigma)$
$\mathfrak{p}_-$ (M): a totally geodesic submanifold M tangent to $\mathfrak{p}_-$
$\mathfrak{k}_-$ (M'): a symmetric R-space
$(\hat{\mathfrak{g}}, \hat{\tau})$ M^*: an irreducible Hermitian symmetric space M^* associated with $(\hat{\mathfrak{g}}, \hat{\tau})$
$\mathfrak{p}_+$ $(M^{\perp})$: a totally geodesic submanifold $M^{\perp}$ tangent to $\mathfrak{p}_+$

No.	$(\bar{\mathfrak{g}}, \sigma)$ $\bar{M}$	$\mathfrak{p}_-$ (M)	$\mathfrak{k}_-$ (M')
1	$\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$\mathfrak{su}(i,n-i)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$	$\mathfrak{su}(n)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$
2	$\mathfrak{so}(2n;\mathbb{C})/\mathfrak{so}(2n)$	$\mathfrak{so}^*(2n)/\mathfrak{u}(n)$	$\mathfrak{so}(2n)/\mathfrak{u}(n)$
3	$\mathfrak{so}(n;\mathbb{C})/\mathfrak{so}(n)$	$\mathfrak{so}(n-2,2)/\mathfrak{so}(n-2)+\mathbb{T}$	$\mathfrak{so}(n)/\mathfrak{so}(n-2)+\mathbb{T}$
4	$\mathfrak{sp}(n;\mathbb{C})/\mathfrak{sp}(n)$	$\mathfrak{sp}(n;\mathbb{R})/\mathfrak{u}(n)$	$\mathfrak{sp}(n)/\mathfrak{u}(n)$
5	$E_6^{\mathbb{C}}/E_6$	$E_6^{-14}/\mathfrak{so}(10)+\mathbb{T}$	$E_6/\mathfrak{so}(10)+\mathbb{T}$
6	$E_7^{\mathbb{C}}/E_7$	$E_7^{-25}/E_6+\mathbb{T}$	$E_7/E_6+\mathbb{T}$
7	$\mathfrak{su}(n,n)/\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(n))$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$\mathbb{T}+\mathfrak{su}(n)+\mathfrak{su}(n)/\mathfrak{su}(n)$
8	$\mathfrak{so}^*(4n)/\mathfrak{u}(2n)$	$\mathbb{R}+\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$\mathbb{T}+\mathfrak{su}(2n)/\mathfrak{sp}(n)$
9	$\mathfrak{sp}(n;\mathbb{R})/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$\mathbb{T}+\mathfrak{su}(n)/\mathfrak{so}(n)$
10	$E_7^{-25}/E_6+\mathbb{T}$	$\mathbb{R}+E_6^{-26}/F_4$	$\mathbb{T}+E_6/F_4$
11	$\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$\mathfrak{so}(i,n-i)/\mathfrak{so}(i)+\mathfrak{so}(n-i)$	$\mathfrak{so}(n)/\mathfrak{so}(i)+\mathfrak{so}(n-i)$
12	$\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$\mathfrak{sp}(i,n-i)/\mathfrak{sp}(i)+\mathfrak{sp}(n-i)$	$\mathfrak{sp}(n)/\mathfrak{sp}(i)+\mathfrak{sp}(n-i)$
13	$\mathfrak{so}(i,n-i)/$ $\mathfrak{so}(i)+\mathfrak{so}(n-i)$	$\mathfrak{so}(i-1,1)/\mathfrak{so}(i-1)$ $+\mathfrak{so}(n-i-1,1)/\mathfrak{so}(n-i-1)$	$\mathfrak{so}(i)/\mathfrak{so}(i-1)$ $+\mathfrak{so}(n-i)/\mathfrak{so}(n-i-1)$
14	$\mathfrak{so}(n,n)/\mathfrak{so}(n)+\mathfrak{so}(n)$	$\mathfrak{so}(n;\mathbb{C})/\mathfrak{so}(n)$	$\mathfrak{so}(n)+\mathfrak{so}(n)/\mathfrak{so}(n)$
15	$\mathfrak{sp}(n,n)/\mathfrak{sp}(n)+\mathfrak{sp}(n)$	$\mathfrak{sp}(n;\mathbb{C})/\mathfrak{sp}(n)$	$\mathfrak{sp}(n)+\mathfrak{sp}(n)/\mathfrak{sp}(n)$
16	$E_6^6/\mathfrak{sp}(4)$	$\mathfrak{sp}(2,2)/\mathfrak{sp}(2)+\mathfrak{sp}(2)$	$\mathfrak{sp}(4)/\mathfrak{sp}(2)+\mathfrak{sp}(2)$
17	E_6^{-26}/F_4	$F_4^{-20}/\mathfrak{so}(9)$	$F_4/\mathfrak{so}(9)$
18	$E_7^7/\mathfrak{su}(8)$	$\mathfrak{su}^*(8)/\mathfrak{sp}(4)$	$\mathfrak{su}(8)/\mathfrak{sp}(4)$

Table (Continued)

No.	$(\hat{\mathfrak{g}}, \hat{\tau})$ M^*	$\mathfrak{p}_+$ $(M^\perp)$	Remark
1	$\mathfrak{su}(n)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$ $+\mathfrak{su}(n)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$	$\mathbb{R}+\mathfrak{sl}(i;\mathbb{C})/\mathfrak{su}(i)$ $+\mathfrak{sl}(n-i;\mathbb{C})/\mathfrak{su}(n-i)$	$1 \leq i \leq n-i, n \geq 3$
2	$\mathfrak{so}(2n)/\mathfrak{u}(n)$ $+\mathfrak{so}(2n)/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$n \geq 4$
3	$\mathfrak{so}(n)/\mathfrak{so}(n-2)+\mathbb{T}$ $+\mathfrak{so}(n)/\mathfrak{so}(n-2)+\mathbb{T}$	$\mathbb{R}+\mathfrak{so}(n-2;\mathbb{C})/\mathfrak{so}(n-2)$	$n \geq 5, n \neq 6$
4	$\mathfrak{sp}(n)/\mathfrak{u}(n)$ $+\mathfrak{sp}(n)/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$n \geq 3$
5	$E_6/\mathfrak{so}(10)+\mathbb{T}$ $+E_6/\mathfrak{so}(10)+\mathbb{T}$	$\mathbb{R}+\mathfrak{so}(10;\mathbb{C})/\mathfrak{so}(10)$	
6	$E_7/E_6+\mathbb{T}$ $+E_7/E_6+\mathbb{T}$	$\mathbb{R}+E_6^{\mathbb{C}}/E_6$	
7	$\mathfrak{su}(2n)/\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(n))$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{C})/\mathfrak{su}(n)$	$n \geq 2$
8	$\mathfrak{so}(4n)/\mathfrak{u}(2n)$	$\mathbb{R}+\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$n \geq 2$
9	$\mathfrak{sp}(n)/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$n \geq 2$
10	$E_7/E_6+\mathbb{T}$	$\mathbb{R}+E_6^{-26}/F_4$	
11	$\mathfrak{su}(n)/\mathfrak{s}(\mathfrak{u}(i)+\mathfrak{u}(n-i))$	$\mathbb{R}+\mathfrak{sl}(i;\mathbb{R})/\mathfrak{so}(i)$ $+\mathfrak{sl}(n-i;\mathbb{R})/\mathfrak{so}(n-i)$	$1 \leq i \leq n-i, n \geq 3$
12	$\mathfrak{su}(2n)/\mathfrak{s}(\mathfrak{u}(2i)+\mathfrak{u}(2n-2i))$	$\mathbb{R}+\mathfrak{su}^*(2i)/\mathfrak{sp}(i)$ $+\mathfrak{su}^*(2(n-i))/\mathfrak{sp}(n-i)$	$1 \leq i \leq n-i, n \geq 3$
13	$\mathfrak{so}(n)/\mathfrak{so}(n-2)+\mathbb{T}$	$\mathbb{R}+\mathfrak{so}(i-1,n-i-1)/$ $\mathfrak{so}(i-1)+\mathfrak{so}(n-i-1)$	$i=1: n \geq 3, i=2: n \geq 7,$ $i=3: n \geq 7, i=4: n \geq 9,$ $5 \leq i \leq n-i$
14	$\mathfrak{so}(2n)/\mathfrak{u}(n)$	$\mathbb{R}+\mathfrak{sl}(n;\mathbb{R})/\mathfrak{so}(n)$	$n \geq 4$
15	$\mathfrak{sp}(2n)/\mathfrak{u}(2n)$	$\mathbb{R}+\mathfrak{su}^*(2n)/\mathfrak{sp}(n)$	$n \geq 2$
16	$E_6/\mathfrak{so}(10)+\mathbb{T}$	$\mathbb{R}+\mathfrak{so}(5,5)/\mathfrak{so}(5)+\mathfrak{so}(5)$	
17	$E_6/\mathfrak{so}(10)+\mathbb{T}$	$\mathbb{R}+\mathfrak{so}(1,9)/\mathfrak{so}(9)$	
18	$E_7/E_6+\mathbb{T}$	$\mathbb{R}+E_6^6/\mathfrak{sp}(4)$	

References

1. Eschenburg, J.H.: Higher rank curved Lie triples. J. Math. Soc. Japan **54**, 551–564 (2002)
2. Ferus, D.: Symmetric submanifolds of Euclidean space. Math. Ann. **247**, 81–93 (1980)
3. Harvey, R., Lawson, H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)
4. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. Academic Press, San Diego, 1978
5. Kobayashi, S., Nagano, T.: On filtered Lie algebras and geometric structures I. J. Math. Mech. **13**, 875–907 (1964)
6. Kobayashi, S., Nomizu, K.: Foundations of differential geometry II, Wiley, New York, 1969
7. Leung, D.S.P.: On the classification of reflective submanifolds of Riemannian symmetric spaces, Indiana Univ. Math. J. **24**, 327–339 (1974); Errata: ibid. **24**, 1199 (1975)
8. Leung, D.S.P.: Reflective submanifolds. III. Congruency of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds. J. Differential Geom. **14**, 167–177 (1979)
9. Nagano, T.: Transformation groups on compact symmetric spaces. Trans. Am. Math. Soc. **118**, 428–453 (1965)
10. Naitoh, H.: Symmetric submanifolds of compact symmetric spaces. Tsukuba J. Math. **10**, 215–242 (1986)
11. Naitoh, H.: Compact simple Lie algebras with two involutions and submanifolds of compact symmetric spaces I, II. Osaka J. Math. **30**, 653–690, 691–732 (1993)
12. Naitoh, H.: Grassmann geometries on compact symmetric spaces of general type. J. Math. Soc. Japan **50**, 557–592 (1998)
13. Naitoh, H.: Grassmann geometries on compact symmetric spaces of exceptional type. Japanese J. Math. **26**, 157–206 (2000)
14. Naitoh, H.: Grassmann geometries on compact symmetric spaces of classical type. Japanese J. Math. **26**, 219–319 (2000)
15. Naitoh, H., Takeuchi, M.: Symmetric submanifolds of symmetric spaces. Sugaku **2**, 157–188 (1989)
16. Osipova, D.: Symmetric submanifolds in symmetric spaces. Diff. Geom. Appl. **16**, 199–211 (2002)
17. Takeuchi, M.: Cell decompositions and Morse equalities on certain symmetric spaces. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **12**, 81–192 (1965)
18. Takeuchi, M.: Parallel submanifolds of space forms, in: Manifolds and Lie groups, Pap. in Honor of Y. Matsushima, Prog. Math. **14**, 429–447 (1981)