Symplectic homology of Brieskorn manifolds

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Peter Uebele

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University

Gutachter:	1. Prof. Dr. Kai Cieliebak
	2. Prof. Dr. Urs Frauenfelder
	3. Prof. Dr. Alexandru Oancea

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Contents

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Abstract

We discuss various versions of symplectic homology in the context of Brieskorn manifolds and their Stein fillings. Under a certain index condition, some flavors of symplectic homology, namely SH^+ and $\check{SH}$, are independent of the filling. Thus, they can be used to distinguish the contact structures on Brieskorn manifolds. We will examine their relative advantages and disadvantages compared to other invariant such as the formal homotopy class, contact homology and the mean Euler characteristic.

On a concrete example, $\Sigma(2\ell, 2, 2, 2)$, we will show that SH^+ and $\check{SH}$ actually contain more information than contact homology. On the other hand, symplectic homology is always very hard to compute, and we can do so for $\Sigma(2\ell, 2, 2, 2)$ only with the help of a symmetry of the manifold.

For more complicated examples, where the full computation of symplectic homology remains elusive, we will turn our attention to the mean Euler characteristic, which is a quantity derived from positive S^1-equivariant symplectic homology. While it contains significantly less information, its computation is essentially a matter of combinatorics. With this tool, we can prove that there exist infinitely many exotic but homotopically trivial contact structures on S^7, S^{11} and S^{15}, which was previously known only for S^{4m+1}.

Moreover, $\check{SH}$ has the algebraic structure of a unital, graded commutative ring, where multiplication is given by the pair-of-pants product. Again, the full ring structure is extremely hard to compute. However, for a large class of examples, we achieve a partial result in this direction: There is a generator s such that the combination of all products with s gives $\check{SH}$ the structure of a free and finitely generated module over the ring of Laurent polynomials $\mathbb{Z}_2[s, s^{-1}]$. In particular, $\check{SH}$ is finitely generated as a $\mathbb{Z}_2$-algebra, although its vector space dimension is infinite. Similarly, the usual symplectic homology SH of the filling can be given the structure of a finitely generated module over the polynomial ring $\mathbb{Z}_2[s]$.

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1 Introduction

1.1 Motivation and historical background

Brieskorn manifolds, named after the German mathematician Egbert Brieskorn, are defined as the set

$$\Sigma(a) = \Sigma(a_0, \dots, a_n) := \left\{ z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0, \|z\| = 1 \right\}$$

for integers $a_i \geq 2$. They come along with the structure of a smooth submanifold of the ambient space $\mathbb{C}^{n+1}$. Historically, the main motivation of their study was that they provide many examples of exotic spheres, see e.g. [14, 38, 56, 50].

For this goal, people have developed methods to calculate the homology, homeomorphism type and even diffeomorphism type of $\Sigma(a)$ depending on the integers a_i. In many examples, $\Sigma(a)$ turns out to be homeomorphic but not diffeomorphic to the standard sphere S^{2n-1}, thus representing an exotic sphere.

Today, after extensive study, most problems in the area of exotic spheres have been solved. However, Brieskorn manifolds still provide an important resource of examples to many questions in differential and algebraic geometry. In this thesis, we study them mostly from the viewpoint of contact and symplectic topology.

It has been known since [45] that Brieskorn manifolds carry a canonical contact structure, with the explicit contact form

$$\alpha = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\bar{z}_j - \bar{z}_j dz_j). \tag{1.1}$$

Thus, in analogy to the study of exotic spheres, one can use Brieskorn manifolds to find exotic contact structures.

A famous result in this line of thought was discovered by Ustilovsky in [66], which we recall here:

Theorem 1.1.1 (Ustilovsky). *Let $n = 2m + 1$ be odd and $p \equiv \pm 1 \mod 8$. Then, the Brieskorn manifolds $\Sigma(p, \underbrace{2, \dots, 2}_{n})$ with their canonical contact structures are pairwise non-contactomorphic for different values of p.*

On the other hand, by [14], they are all diffeomorphic to $S^{2n-1} = S^{4m+1}$. Hence, they represent infinitely many exotic contact structures on S^{4m+1}. Moreover, infinitely many of these contact structures have the same classical invariants as the standard one.

The main tool used in the proof of this theorem is cylindrical contact homology. Ustilovsky found an explicit non-degenerate perturbation of the contact form (1.1), which means in particular that all closed Reeb orbits are isolated. Then, contact homology is applicable in its original formulation from [23, Section 1.9]. Furthermore, it turns out that the (shifted) Conley–Zehnder indices of all Reeb orbits are even. This implies that the differential of contact homology vanishes trivially, thus making the computation feasible.

As a sequel to this result, new methods were developed to compute the contact homology for other Brieskorn manifolds. In particular, Bourgeois [7] developed a Morse–Bott version of contact homology, which enabled computations without having to perturb the contact form. Building from this work, van Koert [68] was able to compute the contact homology for all Brieskorn manifolds that satisfy a certain index condition (which is necessary for the definition of cylindrical contact homology).

However, one of the drawbacks of contact homology is that it still lacks a solid analytic foundation (although there has been some recent activity aimed at solving this issue, see [55, 5]). Therefore, some people prefer other tools such as Rabinowitz–Floer homology, symplectic homology and their S^1-equivariant versions, whose foundations are well-established. For instance, Fauck [28] has reproven Theorem 1.1.1 using Rabinowitz Floer homology.

Also, it has been known that contact homology is not a complete invariant for contact structures. This means that there exist contact manifolds (even Brieskorn manifolds) that have the same contact homology, yet they are not contactomorphic. This fact brings further motivation to consider other invariants that might distinguish these contact structures.

1.2 Symplectic homology as an invariant

Our main tool for examining Brieskorn manifolds will be symplectic homology, in its various guises. Symplectic homology was introduced in [30, 69] as a generalization of Floer theory to non-compact symplectic manifolds (or compact symplectic manifolds with boundary).

The basic construction of Floer homology goes as follows (see Section 3.1 for some more details). Given a symplectic manifold (W, ω), a compatible almost complex structure J and a Hamiltonian $H \in C^\infty(W)$, one can first define the Hamiltonian vector field X_H by $\iota_{X_H}\omega = -dH$. Under some technical assumptions, one can define a chain complex whose chain groups $SC_*(H)$ are generated by the 1-periodic orbits of X_H. Moreover, they have a grading by Conley–Zehnder indices. Finally, one defines the differential $\partial \colon SC_*(H) \to SC_{*-1}(H)$ by counting cylinders between the orbits that satisfy a particular partial differential equation, known as the Floer equation.

With some analytic work, one can prove that $\partial \circ \partial = 0$ and define the symplectic homology $SH_*(H)$. For a non-compact symplectic manifold, it still depends on the choice of Hamiltonian, but one can get rid of this dependence by a limit process. The resulting symplectic homology $SH_*(W)$ is an invariant of (W, ω). As for coefficients, one can define $SH_*(W)$ with coefficients in any commutative ring, but we will mostly stick to $\mathbb{Z}_2$, unless mentioned otherwise.

In order to apply symplectic homology in the context of Brieskorn manifolds, one first has to choose a symplectic filling for a Brieskorn manifold Σ. Indeed, there exists a Liouville filling W, known as the *Milnor fiber*, coming from a deformation of the singularity. Then, in a process known as symplectic completion, one can attach a cylindrical end to W. Thus, one obtains a non-compact symplectic manifold $\widehat{W}$, to which one can assign its symplectic homology.

However, for the purpose of distinguishing contact manifolds, this construction is only useful if the result is independent of the filling W, which is not the case for the usual symplectic homology. Luckily, there exist other variants, such as the *positive symplectic homology* SH^+ and the $\bigvee$*-shaped symplectic homology* $\check{SH}$ from [17], which is isomorphic to Rabinowitz Floer homology. Under favorable circumstances, these two homology theories are independent of the filling.

A bit more precisely, their chain groups always depend only on Σ and its contact form, and if there is a certain lower bound for the Conley–Zehnder indices of closed Reeb orbits on Σ, so does the differential. This independence is a variation of [17, Theorem 1.14] and can be formulated as follows:

Proposition 1.2.1. *Let $(\Sigma, \xi = \ker \alpha)$ be a $(2n-1)$-dimensional contact manifold with $\pi_1(\Sigma) = 0$, $c_1(\xi) = 0$ and the condition*

$$\mu_{CZ}(c) > 4 - n \qquad \textit{for all closed Reeb orbits } c \tag{1.2}$$

on the Conley–Zehnder indices. Then, $SH_*^+(\Sigma)$ and $\check{SH}_*(\Sigma)$ can be defined in the symplectization $\mathbb{R}_+ \times \Sigma$, without reference to any symplectic filling of Σ. Moreover, if there exists a Liouville filling W such that $c_1(W) = 0$, then $SH_*^+(\Sigma) \cong SH_*^+(W)$ and $\check{SH}_*(\Sigma) \cong \check{SH}_*(W)$.

Besides this index condition, which makes symplectic homology inapplicable in some cases, the other major issue is computational difficulty. Symplectic homology has been known to be very hard to compute explicitly - in most cases even more so than contact homology, because there are more generators. The principal difficulty is that its differential is defined by counting solutions to the Floer equation, which involves, among other things, the choice of a generic almost complex structure, which is even hard to write down.

However, for the cases where a full computation is not feasible, symplectic homology can still provide usual information in the form of a derived quantity, known as the *mean Euler characteristic*. Originally introduced by van Koert in [67], the mean Euler characteristic is an adaptation of the topological Euler characteristic to symplectic homology (more precisely, a variant called *positive* S^1*-equivariant symplectic homology*). As for the topological Euler characteristic, its main advantage is that it can be computed from the chain level, without needing to know the differential. This makes its computation, especially for Brieskorn manifolds, essentially a matter of combinatorics.

On the other hand, as a derived quantity, the mean Euler characteristic does not contain the same amount of information as either symplectic homology or contact homology. Thus, as we will see, there are examples where the mean Euler characteristic cannot distinguish the contact structures. In other cases, though, it can do the job, while the other invariants are impossible to compute.

On the other end of the computability scale, one can define symplectic homology with even more structure. Indeed, as a generalization of Floer homology, it has not only an additive structure (chain groups and a differential), but also other algebraic operations, coming from counting Riemann surfaces with an arbitrary number of positive and negative punctures. Most notably, there is a (commutative and associative) product called pair-of-pants product and a unit, giving symplectic homology the structure of a commutative unital ring.

The same applies to the $\bigvee$-shaped symplectic homology $\check{SH}$, which, in addition, has a chance to be independent of the filling. In fact, we will show that for contact manifolds satisfying a stronger index condition, namely

$$\mu_{CZ}(c) > 3 \qquad \text{for all closed Reeb orbits } c, \tag{1.3}$$

the product can be defined without reference to a filling.

Unfortunately, like the differential, the product structure is extremely hard to compute. For this reason, a complete description of the ring structure of $\check{SH}$ would be desirable, but remains out of reach, and we will not use the ring structure to distinguish contact structures. However, we will find some interesting structural result about the product in the case of Brieskorn manifolds, see Section 1.3.2.

1.3 Summary of the Main Results

1.3.1 The example $\Sigma(2\ell, 2, 2, 2)$

One of the main examples of this text will be the Brieskorn manifolds $\Sigma(2\ell, 2, 2, 2)$ for $\ell \geq 1$. They are all diffeomorphic to $S^2 \times S^3$ (i.e. the unit cotangent bundle of S^3) and, as was pointed out in [68], have the same contact homology for all $\ell \in \mathbb{N}$. The same applies to positive S^1-equivariant symplectic homology, see [41], and therefore to the mean Euler characteristic. Moreover, the underlying almost contact structures coincide, as follows from [34, Proposition 8.1.1] and the fact that their first Chern class vanishes. Thus, the question whether they (or some of them) are contactomorphic was left open.

Using positive symplectic homology, we can answer this question negatively:

Theorem 1.3.1. *The manifolds $\Sigma(2\ell, 2, 2, 2), \ell \geq 1$ with their canonical contact structures are pairwise non-contactomorphic. Hence, there are infinitely many different contact structures on $S^2 \times S^3$.*

At this point, we should mention two other results, by Lerman [46] and Abreu and Macarini [3], respectively, who also find infinitely many contact structures on $S^2 \times S^3$. However, their examples do not overlap with the ones discussed here for the following reasons: The examples of [46] have non-vanishing first Chern class, whereas all Brieskorn manifolds have vanishing first Chern class. The examples of [3] also have vanishing first Chern class, but they can be distinguished from ours by their contact homology. Namely, they all have contact homology in degree 0, whereas the contact homology of $\Sigma(2\ell, 2, 2, 2)$ starts in degree 2 (see [68, Example 3.1.1]).

We prove Theorem 1.3.1 by computing the (positive) symplectic homology of a symplectic filling of $\Sigma(2\ell, 2, 2, 2)$ with $\mathbb{Z}_2$-coefficients. The index condition mentioned above is satisfied for $\Sigma(2\ell, 2, 2, 2)$, so its positive symplectic homology is independent of the filling and can distinguish the contact structures. Along the way, we also compute the positive symplectic homology of $\Sigma(2\ell, \underbrace{2, \dots, 2}_{n})$ for $n \geq 5$ odd, which turns out to be much easier than for $n = 3$.

The result, together with [66] or [28], can also be viewed as a classification of the links of A_k-singularities as contact manifolds. These links can be defined as the Brieskorn manifolds $\Sigma(k + 1, \underbrace{2, \dots, 2}_{n})$ with $k \geq 1$ and $n \geq 2$. If n is even, these manifolds are already distinguished by their singular homology, because $H_{n-1}(\Sigma(k + 1, 2, \dots, 2)) = \mathbb{Z}_{k+1}$ in this case. For n odd and k even, the contact structures are distinguished in [66] and [28]. Note that their results can also be proven using symplectic homology, with computations almost identical to [28]. This leaves the case n odd and k odd, which is treated here.

1.3.2 Product structure on symplectic homology

As mentioned above, the product on symplectic homology is very hard to compute. Therefore, the full ring structure is known only in very few cases, including:

- Subcritical Stein manifolds [15], where symplectic homology vanishes,
- Cotangent bundles [70, 1], where symplectic homology is isomorphic to the homology of the loop space with the Chas–Sullivan product,
- Negative line bundles [59], where symplectic homology is related to Gromov–Witten theory and quantum cohomology.

Here, we attempt to apply the techniques of [59] to the more general case of contact manifolds with periodic Reeb flow. As we want to have a quantity that is independent of the filling, we use $\check{SH}$ instead of SH and assume the conditions of Proposition 1.2.1 and (1.3).

Our main tool, which builds upon ideas from [63, 59], is to study the action of a loop of Hamiltonian diffeomorphisms

$$g\colon S^1 \longrightarrow \mathrm{Ham}(\mathbb{R}_+ \times \Sigma, d(r\alpha)), \qquad t \mapsto g_t$$

on $\check{SH}_*(\Sigma)$. This action is defined by $\gamma(t) \mapsto g_t \cdot \gamma(t)$ on the level of generators, and similarly by $u \mapsto g_t \cdot u$ on the Floer cylinders counted by the differential. In this way, g_t defines an isomorphism

$$S_g\colon \check{SH}_*(\Sigma) \xrightarrow{\cong} \check{SH}_{*+2I(g)}(\Sigma), \tag{1.4}$$

where $I(g)$ is a Maslov index depending only on the loop g_t. For our purposes, we are mainly interested in the example where g_t is given by the Reeb flow on Σ, which is always possible if the Reeb flow is periodic (with the period normalized to one). In most cases, this loop of Hamiltonian diffeomorphisms does not extend to a symplectic filling of Σ, hence the need to work in the symplectization. Note that equation (1.4) already implies that $\check{SH}_*(\Sigma)$ fulfills the periodicity

$$\check{SH}_*(\Sigma) \cong \check{SH}_{*+2I(g)}(\Sigma),$$

which would be hard to verify directly, even in concrete examples.

The isomorphism (1.4) does not preserve the product, but instead satisfies the relation

$$S_g(x \cdot y) = S_g(x) \cdot y.$$

In particular, if we take x to be the unit we get $S_g(y) = s \cdot y$, where $s := S_g(1)$ is the principal orbit of (Σ, α). Furthermore, by taking the loop g in the reverse direction, we get the element s^{-1} inverse to s.

Theorem 1.3.2. *Let (Σ, α) be a contact manifold with periodic Reeb flow satisfying* (1.3) *and $\pi_1(\Sigma) = 0$. Then, $\check{SH}_*(\Sigma)$ is a module over the ring of Laurent polynomials $\mathbb{Z}_2[s, s^{-1}]$, with multiplication given by the pair-of-pants product*

$$(s^k, x) \mapsto S_g^k(x) = s^k \cdot x.$$

If $I(g) \neq 0$ this module is free and finitely generated. By contrast, if $I(g) = 0$ then $\check{SH}_*(\Sigma)$ is a free module (i.e. a vector space) over the field of Laurent series $\mathbb{Z}_2((s^{-1}))$.[1]

In both cases, the dimension of this module is bounded from above by the number of generators (in a Morse–Bott sense) of symplectic homology which correspond to Reeb orbits of length at most one.

To put this result into context, recall that $\check{SH}_*(\Sigma)$ is usually not finitely generated as a $\mathbb{Z}_2$-vector space, so only the product gives a finite algebraic structure. Furthermore, Theorem 1.3.2 gives some product computation that would be very difficult to prove directly. In examples, however, it turns out that there can be further relations between the generators of the module, so Theorem 1.3.2 does not reveal the full ring structure of $\check{SH}_*(\Sigma)$.

Of course, the main examples for Theorem 1.3.2 are Brieskorn manifolds, but we tried to formulate all corresponding results as generally as possible. Also note that, while the index conditions (1.2) and (1.3) are quite restrictive, both of them can be relaxed if Σ admits a Liouville filling W with $c_1(W) = 0$. Then, indeed, (1.2) can be replaced by $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits c. Moreover, if in addition $I(g) \neq 0$, then the conclusion of Theorem 1.3.2 also holds under the weaker assumption that $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits, see Proposition 6.1.21.

[1] The notation $\mathbb{Z}_2((s^{-1}))$ stands for the field of semi-infinite Laurent series of the form $\sum_{j=-\infty}^{N} c_j s^j$, i.e. the powers of s can go to $-\infty$.

Finally, Theorem 1.3.2 can also be used to get some information about the usual symplectic homology $SH_*(W)$ of a Liouville filling W of Σ. The long exact sequence from [17] gives a map

$$f\colon SH_*(W) \longrightarrow \check{SH}_*(\Sigma),$$

whose kernel is a subset of the negative symplectic homology $SH_*^-(W)$.

In fact, f is a ring homomorphism (see Lemma 6.1.23 or [18, Theorem 10.2(e)]), hence $SH_*(W)/\ker(f)$ is a ring and maps injectively to $\check{SH}_*(\Sigma)$. It turns out that, with the right choice of module generators, the image of $SH_*(W)/\ker(f)$ in $\check{SH}_*(\Sigma)$ is the subset of elements with non-negative powers of s.

Corollary 1.3.3. *Let Σ be as in Theorem 1.3.2 and W a Liouville filling of Σ with $c_1(W) = 0$. Then, $SH_*(W)/\ker(f)$ is a free and finitely generated module over $\mathbb{Z}_2[s]$. In particular, $SH_*(W)$ is finitely generated as a $\mathbb{Z}_2$-algebra.*

1.3.3 Exotic contact structures on S^{4m+3}

Going back to Ustilovsky's result, one might wonder whether a similar statement about exotic contact structures on spheres also holds true for S^{4m+3}. These dimensions turn out to be more complicated, mainly because, unlike in dimensions $4m + 1$, there are infinitely many formal homotopy classes of almost contact structures. Hence, it is more difficult to find contact structures representing a given formal homotopy class, e.g. the standard one.

Partial results in this direction were proven in [33], [22] and [44]. In particular, [22] shows the existence of one exotic but homotopically trivial contact structure on S^{4m+3} for every $m \geq 1$, while [44, Corollary 1.5] implies existence of at least two such contact structures on spheres of dimension $2n - 1 \geq 15$.

Here, we treat mainly dimension 7. We can show that there are in fact infinitely many exotic but homotopically trivial contact structures on S^7. Our method is somewhat similar to [66]: We use a class of Brieskorn manifolds, namely $\Sigma(78k+1, 13, 6, 3, 3)$, which we show to be all diffeomorphic to S^7. Moreover, their canonical contact structures all lie in the standard formal homotopy class of S^7. Of course, the numbers were chosen specifically to have this property.[2]

In order to distinguish the contact structures, it would be very difficult to compute any variant of contact homology or symplectic homology, because there are generators in a wide range of degrees and the differential is hard to compute. Also, the index condition is not satisfied, so symplectic homology might actually depend on the filling. However, the mean Euler characteristic is very suitable in this case, being both independent of the filling and rather easy to compute. As it turns out to have different values for different $k \in \mathbb{N}$, this proves that the contact structures are different.

As for higher dimensions, it seems difficult to get a similar example for combinatorial reasons. Instead, some results can be achieved by using a specific Brieskorn

[2] Note that [33, Proposition 19] actually claims that such an example cannot exist. However, its proof contains a mistake, originating from different conventions about Bernoulli numbers.

manifold Σ and the connected sum $\#_k\Sigma$ of k copies of Σ. This simplifies the combinatorics somewhat, making a result possible for dimensions 11 and 15. In principle, a similar construction might even be possible for any dimension, but the computations get increasingly difficult.

Theorem 1.3.4. *There exist infinitely many exotic but homotopically standard contact structures on S^7, S^{11} and S^{15}.*

Nevertheless, it is plausible to conjecture that the same holds true on S^{4m+3} for any $m \geq 1$.

1.4 Outline of the thesis

This thesis is organized as follows. Chapter 2 contains the necessary definitions about Brieskorn manifolds, as well as some standard results about their topology. After that, Chapter 3 takes a more detailed look at symplectic homology. We sketch its standard construction, then move on to its Morse–Bott formulation, $\bigvee$-shaped symplectic homology, product structures, S^1-equivariant symplectic homology and the mean Euler characteristic. Of particular importance is Section 3.5, which deals with the (in)dependence of the filling. Most material in Chapter 3 is well-known to the experts, though some results in Section 3.5 have not been stated in this way before.

Moving towards the application of symplectic homology to Brieskorn manifolds, Chapter 4 contains the necessary computation of the Conley–Zehnder indices. The results of Sections 4.1 and 4.2 have been known before, while Sections 4.3 and 4.4 contain new, albeit not particularly difficult material.

The next three chapters are at the heart of this thesis. They contain the results announced in Sections 1.3.1, 1.3.2 and 1.3.3, respectively, in this order. Also, each of these chapters gives a possible direction for further research: Distinguishing $\Sigma(p\ell, p, 2, 2)$ for $p > 1$ odd, the removal (or weakening) of the index conditions for Theorem 1.3.2, and the question of existence of infinitely many exotic contact structures on S^{4m+3} for $m \geq 4$.

Finally, Chapter 8 introduces a method to compute the volume of certain Brieskorn manifolds. We give an example of two Brieskorn manifolds that all the invariants discussed so far fail to distinguish, yet they are not strictly contactomorphic and conjecturally not even contactomorphic. In the last section, we give some further open questions that came up during the research.

2 Brieskorn manifolds and their topology

2.1 Basic definitions

Brieskorn manifolds are defined as follows: Let n be a natural number and $a = (a_0, \ldots, a_n)$ an $(n+1)$-tuple of integers ≥ 2. Then the singular hypersurface

$$V(a) := \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots + z_n^{a_n} = 0\}$$

is called the *Brieskorn variety* of a, and

$$\Sigma(a) = V(a) \cap S^{2n+1}$$

is called the *Brieskorn manifold*.

Lemma 2.1.1 ([45]). *The one-form*

$$\alpha = \alpha_a = \frac{i}{8} \sum_{j=0}^{n} a_j (z_j d\bar{z}_j - \bar{z}_j dz_j) \tag{2.1}$$

restricts to a contact form on $\Sigma(a)$. Its associated Reeb vector field is given by

$$R_\alpha = R_{\alpha_a} = \left(\frac{4i}{a_0} z_0, \ldots, \frac{4i}{a_n} z_n\right).^{1} \tag{2.2}$$

Proof. Note that

$$\omega_a := d\alpha_a = \frac{i}{8} \sum_{j=0}^{n} a_j dz_j \wedge d\bar{z}_j$$

is a symplectic form on $\mathbb{C}^{n+1}$, which also restricts to a symplectic form on $V(a)$. Define the Liouville vector field X on $V(a)$ by

$$\iota_X \omega|_{V(a)} = \alpha|_{V(a)}.$$

Next, we will show that X is transverse to the regular level sets of $\rho(z) := |z|^2$. This implies that α restricts to contact forms on theses level sets, in particular on $\rho^{-1}(1) = \Sigma(a)$.

[1]By this notation, we actually mean the vector field $\sum_j \left(\frac{4i}{a_j} z_j \partial_{z_j} - \frac{4i}{a_j} \bar{z}_j \partial_{\bar{z}_j}\right)$. In particular, R_α lives in the real tangent space.

Indeed, a short calculation shows that $d\rho = -2 \cdot \iota_{R_\alpha}\omega_a$. Hence,

$$d\rho(X) = 2\omega_a(X, R_\alpha) = \alpha_a(R_\alpha) = \|z\|^2 > 0,$$

which implies that X is transverse to the level sets. The statement that R_α is the Reeb vector field also follows from $\iota_{R_\alpha}(d\alpha) = \iota_{R_\alpha}\omega_a = -\frac{1}{2}d\rho$, whose restriction to $\Sigma(a) = \rho^{-1}(1)$ vanishes, and $\alpha(R_\alpha) = \|z\|^2 = 1$ on $\Sigma(a)$. □

It follows immediately from (2.2) that the Reeb flow is given by

$$\phi_t(z) = \left(e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n\right).$$

Furthermore, it is easy to find exact symplectic fillings of Brieskorn manifolds. Indeed, we can take the deformation

$$V_\epsilon(a) := \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = \epsilon\}$$

of $V(a)$ (with ϵ sufficiently small) and intersect it with the unit ball $B^{2(n+1)}$. The resulting manifold is smooth. Outside the origin, we can undo the deformation again, so that the boundary is just $\Sigma(a)$.

A bit more precisely, we use a smooth, monotone decreasing cutoff function $\phi \in C^\infty(\mathbb{R})$ that fulfills $\phi(x) = 1$ for $x \leq 1/4$ and $\phi(x) = 0$ for $x \geq 3/4$. Then we define

$$W = W_a = \left\{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = \epsilon \cdot \phi(\|z\|)\right\} \cap B^{2n+2}. \tag{2.3}$$

As shown in [28], this is an exact symplectic manifold $(W, \omega = d\theta)$, with boundary $\partial W = \Sigma(a)$ and $\theta|_{\partial W} = \alpha_a$. Alternatively, one could directly take $V_\epsilon(a) \cap B^{2n+2}$ as W and use Gray's stability theorem to see that its boundary is contactomorphic to $\Sigma(a)$.

In Section 4.4 and Chapter 5, we will examine a very special class of Brieskorn manifolds, namely those with $n = 2m+1$ odd and $a = (2\ell, 2, \dots, 2)$. We abbreviate them by

$$\Sigma_\ell := \Sigma(2\ell, 2, \dots, 2).$$

We see immediately that in this case, the formulas for the contact form, the Reeb vector field and its flow simplify to

$$\begin{aligned}
\alpha &= \frac{i\ell}{4}(z_0 d\bar{z}_0 - \bar{z}_0 dz_0) + \frac{i}{4}\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j) \\
R_\alpha &= 2i\left(\ell^{-1} z_0, z_1, \dots, z_n\right) \\
\phi_t(z) &= \left(e^{2it/\ell} z_0, e^{2it} z_1, \dots, e^{2it} z_n\right).
\end{aligned}$$

2.2 Topology of Brieskorn manifolds

The singular homology of Brieskorn manifolds is very well understood. It is a classical fact (see e.g. [50, Theorem 5.2]) that $\Sigma = \Sigma(a)$ is highly-connected, meaning that

$$\pi_1(\Sigma) = \cdots = \pi_{n-2}(\Sigma) = 0$$

Consequently, their homology is concentrated in degrees $0, n-1, n, 2n-1$. Of course, $H_0(\Sigma) \cong H_{2n-1}(\Sigma) \cong \mathbb{Z}$. The homology in the middle dimension can be computed by a combinatorial algorithm from Randell [57] (see also [41, Section 3]).

The algorithm can be described as follows. Denote $I = \{0, 1, \ldots, n\}$ and $I_t = \{i_1, \ldots, i_t\}$ any subset of I with exactly t elements. The rank κ of $H_{n-1}(\Sigma)$ is given by the formula

$$\kappa = \kappa(\Sigma) := \operatorname{rank} H_{n-1}(\Sigma) = \sum_{I_t \subset I} (-1)^{n+1-t} \frac{\prod_{i \in I} a_i}{\operatorname{lcm}_{j \in I_t} a_j}.$$

The torsion part is a bit more complicated. Define the function $C\colon \mathcal{P}(I) \to \mathbb{N}$, where $\mathcal{P}(I)$ denotes the power set of I, recursively by

$$C(\emptyset) = \gcd_{i \in I} a_i$$
$$C(I_s) = \frac{\gcd_{i \in I \setminus I_s} a_i}{\prod_{I_t \subsetneq I_s} C(I_t)},$$

with the convention that $\gcd_{i \in \emptyset} a_i = 1$. Moreover, denote $\kappa(I_t) = \kappa(\Sigma(a_{i_1}, \ldots, a_{i_t}))$ and

$$k(I_t) := \begin{cases} \kappa(I_t) & \text{if } n+1-t \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Then, define

$$d_j := \prod_{\substack{I_s \subset I \\ k(I_s) \geq j}} C(I_s) \quad \text{and} \quad r := \max_{I_s \subset I} k(I_s).$$

Theorem 2.2.1 (Randell). *The homology group $H_{n-1}(\Sigma)$ with coefficients in $\mathbb{Z}$ is given by*

$$H_{n-1}(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^\kappa \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}.$$

By Poincaré duality and the universal coefficient theorem, this implies that $H_n(\Sigma; \mathbb{Z}) = \mathbb{Z}^\kappa$.

If one only wants to know whether $\Sigma(a)$ is a homotopy sphere, there is a simpler criterion, which was already known to Brieskorn (see [14, Satz 1]).

Theorem 2.2.2 (Brieskorn). *The Brieskorn manifold $\Sigma(a_0, \ldots, a_n)$, $n \geq 3$, is a topological sphere if and only if one of the following two conditions holds:*

(i) There are two exponents a_i, a_j which are relatively prime to all the other exponents.

(ii) There is one exponent a_i which is relatively prime to all the other exponents. Additionally, there is a set of exponents $a_{j_1}, \dots, a_{j_r}$, with $r \geq 3$ odd, such that each a_{j_k} is relatively prime to any exponent not in the set, while $\gcd(a_{j_k}, a_{j_\ell}) = 2$ *for all $k \neq \ell$.*

The case $n = 2$ is excluded in this theorem because then, the manifold is no longer simply-connected. For all examples in Section 7.2, condition (i) will be satisfied.

Another classical result concerns the topology of the filling W_a from (2.3). By [50, Theorems 5.1 and 6.5], it is parallelizable and has the homotopy type of a wedge of

$$\mu = \mu(a) = \prod_{i=0}^{n} (a_i - 1)$$

copies of S^n.

Once one knows the topology, one can ask for the diffeomorphism type of a Brieskorn manifold. For the case of topological spheres, see Section 2.3. In other cases, the diffeomorphism type can sometimes be deduced from Wall's classification of highly-connected manifolds [71].

As for Σ_ℓ, Randall's algorithm shows that $H_{n-1}(\Sigma_\ell) \cong \mathbb{Z}$, with no torsion elements. The following result about its diffeomorphism type can be found in [20, Proposition 6.1].

Proposition 2.2.3. *Fix $n \geq 3$ odd. Denote by K the Kervaire sphere of dimension $2n-1$ (which can be defined as the Brieskorn manifold* $\Sigma(3, 2, \dots, 2)$*) and by S^*S^n the unit cotangent bundle of S^n. The diffeomorphism type of Σ_ℓ is given as follows:*

$$\Sigma_\ell \cong \begin{cases} S^{n-1} \times S^n & \textit{if} \quad \ell \equiv 0 \mod 4 \\ S^*S^n & \textit{if} \quad \ell \equiv 1 \mod 4 \\ (S^{n-1} \times S^n) \# K & \textit{if} \quad \ell \equiv 2 \mod 4 \\ S^*S^n \# K & \textit{if} \quad \ell \equiv 3 \mod 4 \end{cases}$$

In dimension 5, the Kervaire sphere is diffeomorphic to the standard sphere [39, Lemma 7.2]. Moreover, the cotangent bundle of S^3 is trivial, so $S^*S^3 \cong S^2 \times S^3$, and we get:

Corollary 2.2.4. $\Sigma(2\ell, 2, 2, 2)$ *is diffeomorphic to $S^2 \times S^3$.*

2.3 Diffeomorphism types of topological spheres

Recall that a manifold M is called *boundary-parallelizable* if there exists a parallelizable manifold with boundary W such that $\partial(W) = M$. For Brieskorn manifolds, this role is played by W_a, hence all Brieskorn manifolds are boundary-parallelizable.

In particular, if $\Sigma(a)$ is a topological sphere, it represents an element of the group bP_{2n} of boundary-parallelizable homotopy spheres of dimension $2n-1$. If n is odd (i.e. in dimensions $4m+1$), this group contains only the standard sphere and the Kervaire sphere, see [39, Theorem 8.5]. For n even, this group can be bigger. In this case, the element represented by $\Sigma(a)$ can be identified from the signature of the filling W_a, as will be explained in the rest of this section.

Let M, M' be boundary-parallelizable homotopy spheres of dimension $4m-1$, $m > 1$. Denote by W, W' their parallelizable fillings and by $\sigma(W), \sigma(W')$ the signatures of their intersection products on $H_{2m}(W), H_{2m}(W')$.

By [39], M is orientation-preserving diffeomorphic to M' if and only if

$$\sigma(W) \equiv \sigma(W') \mod \sigma_m, \tag{2.4}$$

where σ_m is a constant depending only on the dimension. Explicitly,

$$\sigma_m = 2^{2m+1} \cdot (2^{2m-1} - 1) \cdot \text{numerator}\left(\frac{4B_m}{m}\right), \tag{2.5}$$

where B_m is the m-th Bernoulli number, with the convention $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, B_4 = 1/30$ and so on.[2] In particular, for $m = 2$, this formula gives $\sigma_m = 224$.[3]

To apply this result, we need to know the signature of the filling of Brieskorn manifolds. For this, we use [14, Theorem 3]:

Theorem 2.3.1 (Brieskorn). *Assume that $\Sigma = \Sigma(a_0, \dots, a_n)$ is a homotopy sphere, with $n \geq 4$ even. Denote its filling by W_a. Then*

$$\sigma(W_a) = \sigma_a^+ - \sigma_a^-,$$

where

$$\sigma_a^+ = \#\left\{ j = (j_0, \dots, j_n) \mid 0 < j_k < a_k \ \forall k,\ 0 < \sum_{k=0}^{n} \frac{j_k}{a_k} < 1 \mod 2 \right\} \tag{2.6}$$

$$\sigma_a^- = \#\left\{ j = (j_0, \dots, j_n) \mid 0 < j_k < a_k \ \forall k,\ 1 < \sum_{k=0}^{n} \frac{j_k}{a_k} < 2 \mod 2 \right\}. \tag{2.7}$$

By the condition $0 < x < 1 \mod 2$ for a real number x, we mean that x lies

[2] Kervaire and Milnor prove (2.5) for m odd, while for m even, it was left open whether there might be another factor of two in some cases. This uncertainty was removed later, see e.g. [42, Theorem 5.2] and the references therein.

[3] This number is easier to understand by noting that $\sigma(W)$ is divisible by 8, and the number $\sigma(W)/8 \mod 28$ distinguishes the 28 smooth structures on S^7.

in some interval $(2k, 2k+1)$, $k \in \mathbb{Z}$, and similarly for $1 < x < 2 \mod 2$. The numbers σ_a^+ and σ_a^- are precisely the dimensions of the subspaces of $H_n(W_a)$ on which the intersection form is positive and negative, respectively.

As a preparation for Section 7.2, we want to apply Theorem 2.3.1 to $\Sigma(78k + 1, 13, 6, 3, 3)$ for $k \in \mathbb{N}$. Note that $\Sigma(78k + 1, 13, 6, 3, 3)$ is a homotopy sphere by Theorem 2.2.2.

Proposition 2.3.2. *The filling W_k of $\Sigma(78k+1, 13, 6, 3, 3)$ has signature $\sigma(W_k) = 5824k$, with $\sigma_a^+ = 12272k$ and $\sigma_a^- = 6448k$. In particular, $\Sigma(78k+1, 13, 6, 3, 3)$ has the diffeomorphism type of the standard sphere.*

Proof. For a tuple $j = (j_0, \dots, j_4)$ with $0 < j_k < a_k$, denote

$$y_j := \sum_{i=1}^{4} \frac{j_i}{a_i} = \frac{j_1}{13} + \frac{j_2}{6} + \frac{j_3}{3} + \frac{j_4}{3},$$

ignoring j_0 for the moment. We can write $y_j = \frac{p}{78}$ for some positive integer p (relatively prime to 13).

The integer j_0 can take any value from 1 to $78k$. For $0 \leq n < 78$, define

$$I_n := \{nk + 1, nk + 2, \dots, (n+1)k\}.$$

The important point of the proof is that for any $j_0 \in I_n$, we get the inequality

$$\frac{n}{78} < \frac{j_0}{a_0} = \frac{j_0}{78k+1} < \frac{n+1}{78}.$$

Therefore, if we add $\frac{j_0}{a_0}$ to $y_j = \frac{p}{78}$, the result lies in the same integer interval for all $j_0 \in I_n$. It also lies in the same integer interval as $\frac{n+1}{79} + y_j$

For $k = 1$, the proposition is just a trivial computation (most easily done by a computer). However, with the above considerations, we can infer the general case $k > 1$ from $k = 1$. Indeed, we can associate to any tuple $\tilde{j} = (\tilde{j}_0, \dots, \tilde{j}_4)$ from the $(k = 1)$-case (i.e. with $0 < \tilde{j}_0 < 79$) a set of k different tuples $j = (j_0, \dots, j_4)$ such that

$$\frac{\tilde{j}_0}{79} + \sum_{i=1}^{4} \frac{\tilde{j}_i}{a_i} \qquad \text{and} \qquad \frac{j_0}{78k+1} + \sum_{i=1}^{4} \frac{j_i}{a_i}$$

lie in the same integer interval. Explicitly, we set

$$j_i = \tilde{j}_i \qquad \text{and} \qquad j_0 = (\tilde{j}_0 - 1) \cdot k + 1, (\tilde{j}_0 - 1) \cdot k + 2, \dots, \tilde{j}_0 \cdot k.$$

This implies that any tuple $\tilde{j}$ contributing to (2.6) (resp. (2.7)) for $k = 1$ gives k contributions to (2.6) (resp. (2.7)) for $k > 1$, and all tuples j are reached from some $\tilde{j}$ in this way. Thus, σ_a^+ and σ_a^- (and hence $\sigma(W_a)$) both get multiplied by k, giving the result. $\square$

3 Symplectic homology and its variants

3.1 Recap of symplectic homology

In this section, we recall the main notions coming up in symplectic homology and fix some conventions. This is standard material that can be found e.g. in [62, 54, 10]. At some points, the material can be simplified a little because we only consider Liouville domains with vanishing first Chern class and use coefficients in $\mathbb{Z}_2$.

To define symplectic homology, we need to introduce the completion $\widehat{W}$ of a Liouville domain (W, λ). To construct it, denote by Z the Liouville vector field, defined by $\iota_Z \omega = \lambda$. Denote its flow by ϕ_Z^t. A neighborhood U of $\Sigma = \partial W \subset W$ can be parametrized by

$$\psi\colon [-\delta, 0] \times \Sigma \to U, \quad (\rho, x) \mapsto \phi_Z^\rho(x).$$

The symplectic completion is defined as the manifold

$$\widehat{W} := W \cup_\psi (\mathbb{R}_{\geq 0} \times \Sigma),$$

equipped with the symplectic form

$$\hat{\omega} = d\hat{\lambda}, \qquad \hat{\lambda} := \begin{cases} \lambda & \text{on } W \\ e^\rho \lambda & \text{on } \mathbb{R}_{\geq 0} \times \Sigma, \end{cases}$$

where ρ is the coordinate on $\mathbb{R}_{\geq 0}$. Note that $\hat{\lambda}$ is a smooth one-form because $\psi^*(\lambda) = e^\rho \alpha$. We will sometimes abuse the notation by writing λ and ω instead of $\hat{\lambda}$ and $\hat{\omega}$. We will always distinguish between the filling W and its completion $\widehat{W}$, though.

Next, we characterize the Hamiltonians used in the definition of symplectic homology. Given a (possibly time-dependent) Hamiltonian $H\colon S^1 \times \widehat{W} \to \mathbb{R}$, define the Hamiltonian vector field X_H by

$$dH = -\iota_{X_H} \omega.$$

The set $\mathcal{H}$ of *admissible Hamiltonians* consists of all functions $H\colon S^1 \times \widehat{W} \to \mathbb{R}$ satisfying

(i) $H < 0$ on W.

(ii) For $(\rho, x) \in \mathbb{R}_{\geq 0} \times \Sigma$ with ρ sufficiently large,

$$H(t, \rho, x) = ae^\rho + \text{const}$$

for some $a > 0$, $a \notin \operatorname{Spec}(\Sigma, \alpha)$, which is called the *slope* of H.

(iii) All 1-periodic orbits $\gamma\colon S^1 \to \widehat{W}$ of X_H are non-degenerate, i.e. the linearized Hamiltonian flow

$$d\phi_H^1(\gamma(0))\colon T_{\gamma(0)} \to T_{\gamma(0)}$$

has no eigenvalue equal to one.

Moreover, define the set $\mathcal{J}$ of *admissible almost complex structures* as the set of all

$$J\colon S^1 \to \operatorname{End}(T\widehat{W}), \qquad J^2 = -\operatorname{id},$$

which are compatible with $\hat{\omega}$ and convex near infinity, i.e. outside a compact set of $\widehat{W}$,

$$d\rho \circ J_{\mathcal{P}}(s, t, x) = -e^f \lambda,$$

where f is any smooth function.

Given a Hamiltonian H, the symplectic action functional $\mathcal{A}_H\colon C^\infty(S^1, \widehat{W}) \to \mathbb{R}$ is defined as

$$\mathcal{A}_H(\gamma) = \int_{S^1} \gamma^* \hat{\lambda} - \int_{S^1} H(t, \gamma(t))\, dt.$$

Its differential at a loop γ is given by

$$d\mathcal{A}_H(\gamma)\zeta = \int_{S^1} \omega(\dot{\gamma} - X_H(\gamma), \zeta)\, dt,$$

hence the critical points of $\mathcal{A}_H$ are the 1-periodic orbits of X_H.

We want to assign a grading to the 1-periodic Hamiltonian orbits. To do this, we will assume that $c_1(W) = 0$ and $\pi_1(W) = 0$. While it is certainly possible to work without these assumptions, they simplify our task and will be satisfied in the cases we need. Thus, for a 1-periodic Hamiltonian orbit γ, we can choose a disk $\sigma\colon D^2 \to \widehat{W}$ with $\sigma(e^{2\pi i t}) = \gamma(t)$. As D^2 is contractible, the bundle $\sigma^* T\widehat{W}$ has a symplectic trivialization, unique up to homotopy,

$$\Phi\colon D^2 \times \mathbb{R}^{2n} \to \sigma^* T\widehat{W}.$$

Therefore, the linearized Hamiltonian flow $d\phi_{X_H}^t$ defines a path of symplectic matrices

$$\Psi\colon [0,1] \to \operatorname{Sp}(2n), \qquad \Psi(t) := \Phi^{-1} \circ d\phi_{X_H}^t(\gamma(0)) \circ \Phi.$$

By the assumption that γ is non-degenerate, $\Psi(1) \in \operatorname{Sp}(2n)$ has no eigenvalue equal to one. Hence, we can define the *Conley–Zehnder index*

$$\mu(\gamma) := \mu_{CZ}(\Psi) \in \mathbb{Z},$$

where $\mu_{CZ}(\Psi)$ is defined as in [21] (or [60]).

With these preparations, we can define the chains group of symplectic homology $SC_k(H)$ as the $\mathbb{Z}_2$-vector space generated by all 1-periodic Hamiltonian orbits with Conley–Zehnder index k. The next step is to define a differential between these chain groups, so that we can take homology.

Given two 1-periodic Hamiltonian orbits $\overline{\gamma}$, $\underline{\gamma}$, define the moduli space of parametrized Floer cylinders $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ as the set of all maps $u\colon \mathbb{R} \times S^1 \to \widehat{W}$ satisfying the *Floer equation*

$$\partial_s u + J_t(\partial_t u - X_H) = 0 \tag{3.1}$$

and the limits

$$\lim_{s\to\infty} u(s,t) = \overline{\gamma}(t), \quad \lim_{s\to-\infty} u(s,t) = \underline{\gamma}(t), \quad \lim_{s\to\pm\infty} \partial_s u = 0$$

uniformly in t. The equation (3.1) is non-linear because of the point-dependency of J and X_H. By linearizing it in a suitable sense, one obtains the linearized operator

$$\begin{aligned} D_u\colon W^{1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}) &\longrightarrow L^p(\mathbb{R} \times S^1, u^*T\widehat{W}) \\ D_u\zeta &:= \nabla_s\zeta + J\nabla_t\zeta + (\nabla_\zeta J)\partial_t u - \nabla_\zeta(JX_H). \end{aligned}$$

Here, $u \in \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$, $p > 2$ and ∇ is the Levi-Civita connection associated with $\omega(\cdot, J\cdot)$. It can be shown that D_u is a Fredholm operator with Fredholm index

$$\operatorname{ind}(D_u) = \mu(\overline{\gamma}) - \mu(\underline{\gamma})$$

(with an additional term if c_1 is non-zero). We say that $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ is *cut out transversally* if the linearized operator D_u is surjective for all $u \in \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$. It follows from the implicit function theorem for Banach spaces that if $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ is transversally cut out, then it is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J) = \operatorname{ind}(D_u).$$

One of the foundational theorems of Floer theory says that for a generic choice of H and J, all moduli spaces are cut-out transversally:

Theorem 3.1.1. *For any $H \in \mathcal{H}$, there is a comeagre subset $\mathcal{J}_{\mathrm{reg}}(H) \subset \mathcal{J}$ such that for all $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and any 1-periodic orbits $\overline{\gamma}, \underline{\gamma}$ of X_H, the moduli space $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ is cut-out transversally.*

We will assume $J \in \mathcal{J}_{\mathrm{reg}}(H)$ in the following discussion. If $\overline{\gamma} \neq \underline{\gamma}$, there is a free $\mathbb{R}$-action on $\widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ by translations in the domain, $s_0.u(\cdot,\cdot) = u(s_0 + \cdot, \cdot)$. Then, we define the *moduli space of Floer cylinders* as the quotient

$$\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J) := \widehat{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)/\mathbb{R},$$

which is a smooth manifold of dimension

$$\dim \mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J) = \mu(\overline{\gamma}) - \mu(\underline{\gamma}) - 1.$$

For the case $\overline{\gamma} = \underline{\gamma}$, we define $\mathcal{M}(\overline{\gamma}, \overline{\gamma}; H, J) := \widehat{\mathcal{M}}(\overline{\gamma}, \overline{\gamma}; H, J)$, which consists of a single point. Another foundational theorem of Floer theory concerns the compactness of these moduli spaces:

Theorem 3.1.2. *There exists a compactification* $\overline{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ *of* $\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ *whose elements are* broken Floer cylinders*, i.e. tuples*

$$([u_1], \dots, [u_k]) \qquad \textit{where } [u_i] \in \mathcal{M}(\overline{\gamma}_i, \underline{\gamma}_i; H, J)$$

and $\overline{\gamma}_1 = \overline{\gamma}$*,* $\underline{\gamma}_k = \underline{\gamma}$ *and* $\underline{\gamma}_i = \overline{\gamma}_{i+1}$ *for* $i = 1, \dots, k-1$*.*

When such a compactification exist, one usually says that $\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ is *compact up to breaking.* On a more technical note, the topology on $\overline{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J)$ is defined in such a way that a sequence $[u^j]$ in $\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ converges to a tuple $([u_1], \dots, [u_k])$ if and only if there exist $s_i^j \in \mathbb{R}$ such that the reparametrization $u^j(s_i^j + \cdot, \cdot)$ converges to u_i uniformly on compact sets.

If $\mu(\overline{\gamma}) - \mu(\underline{\gamma}) = 1$, then $\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ is zero-dimensional and already compact, meaning $\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ consists of a finite number of points. Thus, we can define the *Floer differential*

$$\partial \colon SC_*(H) \to SC_{*-1}(H)$$

by

$$\partial \overline{\gamma} := \sum_{\substack{\underline{\gamma} \\ \mu(\overline{\gamma}) - \mu(\underline{\gamma}) = 1}} n_{\overline{\gamma}, \underline{\gamma}} \cdot \underline{\gamma},$$

where

$$n_{\overline{\gamma}, \underline{\gamma}} := \#_{\mathbb{Z}_2} \left(\mathcal{M}(\overline{\gamma}, \underline{\gamma}; H, J) \right).$$

Note that, if we were using coefficients other than $\mathbb{Z}_2$, we would have to define orientations on the moduli spaces and define $n_{\overline{\gamma}, \underline{\gamma}}$ as a signed count of elements.

Theorem 3.1.3. $\partial \circ \partial = 0$*. Hence, the symplectic homology of the pair* (H, J) *can be defined as*

$$SH_*(H, J) := H_*(SC_+(H, \partial)).$$

It turns out that $SH_*(H, J)$ is actually independent of the choice of $J \in \mathcal{J}_{\text{reg}}(H)$ (this is a consequence of the continuations maps described below), but it does depend on H. Indeed, choosing a Hamiltonian with a different slope at infinity amounts to including more or less generators in the complex, thus changing its homology drastically. To get an invariant of the Liouville domain W, we use a certain limit process on H.

Given $H_-, H_+ \in \mathcal{H}$, define *admissible homotopy of Hamiltonians* from H_- to H_+ to be a smooth map $H \colon \mathbb{R} \times S^1 \times \widehat{W} \to \mathbb{R}$ such that

(i) $H(s, \cdot, \cdot) = H_-$ for $s \leq -1$ and $H(s, \cdot, \cdot) = H_+$ for $s \geq 1$

(ii) $H < 0$ on W and there are smooth functions $a, b\colon \mathbb{R} \to \mathbb{R}$ such that

$$H(s,\rho,x) = a(s)e^{\rho} + b(s)$$

outside a compact set in $\widehat{W}$.

(iii) $\partial_s H \geq 0$.

Note that (iii) implies in particular that the slopes $a(s)$ are non-decreasing, which will be important for some analytic arguments to work. Similarly, define an *admissible homotopy of almost complex structures* between $J_- \in \mathcal{J}_{\mathrm{reg}}(H_-)$ and $J_+ \in \mathcal{J}_{\mathrm{reg}}(H_+)$ as a smooth map $J\colon \mathbb{R} \to \mathcal{J}$ with $J(s) = J_-$ for $s \leq -1$ and $J(s) = J_+$ for $s \geq 1$.

Given homotopies H and J as well as an orbit $\overline{\gamma}$ of X_{H_+} and an orbit $\underline{\gamma}$ of X_{H_-}, we can define the moduli space $\mathfrak{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ as the set of solutions $u\colon \mathbb{R}\times S^1 \to \widehat{W}$ to the *s-dependent Floer equation*

$$\partial_s u(s,t) + J(s,t,u(s,t))(\partial_t u(s,t) - X_H(s,t,u(s,t))) = 0 \tag{3.2}$$

with the asymptotic conditions

$$\lim_{s\to\infty} u(s,t) = \overline{\gamma}(t), \quad \lim_{s\to-\infty} u(s,t) = \underline{\gamma}(t).$$

Similar to the discussion before, there exist a comeagre subset of all admissible homotopies of almost complex structures for which the linearization of (3.2) is surjective, in which case $\mathfrak{M}(\overline{\gamma}, \underline{\gamma}; H, J)$ is a smooth manifold of dimension $\mu(\overline{\gamma}) - \mu(\underline{\gamma})$. Also, there is a suitable compactness theorem asserting that the zero-dimensional moduli spaces are compact. Thus, we can define the *continuation map*

$$\begin{aligned} \Phi^{H,J}\colon SH_*(H_-, J_-) &\longrightarrow SH_*(H_+, J_+) \\ \overline{\gamma} &\longmapsto \sum_{\substack{\underline{\gamma} \\ \mu(\overline{\gamma})-\mu(\underline{\gamma})=1}} \#_{\mathbb{Z}_2}\left(\mathfrak{M}(\overline{\gamma}, \underline{\gamma}; H, J)\right)\cdot \underline{\gamma}. \end{aligned}$$

Going one step further, there is a “homotopy of homotopies” argument showing that the continuation map is independent of the choice of admissible homotopies. Hence, the set $\{SH_*(H) \mid H \in \mathcal{H}\}$ is a direct system, and we can define the *symplectic homology* of the Liouville domain W as the direct limit

$$SH_*(W) := \varinjlim_{H\in\mathcal{H}} SH_*(H,J). \tag{3.3}$$

Furthermore, $SH_*(W)$ is invariant under isotopies of the Liouville one-form, i.e. $SH_*(W, d\lambda_1) \cong SH_*(W, d\lambda_2)$ if λ_1 and λ_2 are connected by an isotopy. This statement can even be generalized to isotopies of symplectic forms, although we have not defined symplectic homology in the nonexact case.

Remark 3.1.4. While this definition of symplectic homology is the most common, there is an alternative definition that does not use the direct limit. For this approach, we use a Hamiltonian $H^\infty \in C^\infty(S^1 \times \widehat{W})$ that has the form

$$H^\infty(t, \rho, z) = h(t, e^\rho)$$

outside some compact set, but fulfills items (i) and (iii) in the definition of admissibility. Here, the function h should satisfy $\lim_{x\to\infty} \partial_x h(t,x) = \infty$ and $\partial_x^2 h(t,x) > 0$. In particular, this makes sure that the slope of H^∞ goes to infinity as $\rho \to \infty$, which is important for the chain complex associated with H^∞ to contain all the generators appearing in the limit process (3.3).

In fact, it turns out that its associated symplectic homology $SH_*(H^\infty)$ is isomorphic to $SH_*(W)$, see [62, Sections 3]. We will use this definition for the generalization to a Morse–Bott setup in the next section.

The main disadvantage of the second definition is that it is not clear whether $SH_*(H^\infty)$ depends on the choice of H^∞. In fact, the easiest way to prove this independence is probably to show the equivalence to the first definition.

Finally, we also need to define symplectic homology in some action window $[a, b) \subset \mathbb{R}$, denoted by $SH_*^{[a,b)}$. First, define the truncated chain groups $SC_k^{<b}(H)$ as the $\mathbb{Z}_2$ vector space generated by the Hamiltonian orbits γ with $\mu(\gamma) = k$ and action $\mathcal{A}_H(\gamma) < b$.

Lemma 3.1.5. *The differential decreases the action, i.e. for any orbit γ, the differential $\partial(\gamma)$ is a linear combinations of orbits γ' with $\mathcal{A}_H(\gamma') < \mathcal{A}_H(\gamma)$.*

Proof. By the definition of the differential, $\partial(\gamma)$ is a linear combination of orbits γ' with $\mu(\gamma') = \mu(\gamma) - 1$ for which the moduli space $\widehat{\mathcal{M}}(\gamma, \gamma'; H, J)$ is nonempty. Moreover, the elements of $\widehat{\mathcal{M}}(\gamma, \gamma'; H, J)$ can be viewed negative gradient flow lines of the action functional $\mathcal{A}_H$ with respect to the L^2-metric on $C^\infty(S^1, \widehat{W})$ associated with $g = \omega(\cdot, J\cdot)$. Explicitly, we can compute

$$\begin{aligned}
\langle \nabla^{L^2} \mathcal{A}_H(\gamma), \zeta \rangle_{L^2} &= d\mathcal{A}_H(\gamma)\zeta \\
&= \int_{S^1} \omega(\dot\gamma - X_H, \zeta)\, dt \\
&= \langle \dot\gamma - X_H, J\zeta \rangle_{L^2} \\
&= -\langle J(\dot\gamma - X_H), \zeta \rangle_{L^2}.
\end{aligned}$$

Hence,

$$\partial_s \mathcal{A}_H(u(s,\cdot)) = \langle \nabla^{L^2} \mathcal{A}_H, \partial_s u \rangle_{L^2} = -\langle J(\partial_t u - X_H), \partial_s u \rangle_{L^2} = \|\partial_s u\|_{L^2}^2$$

is strictly positive for any non-trivial cylinder u, which implies $\mathcal{A}_H(\gamma') < \mathcal{A}_H(\gamma)$. ∎

By Lemma 3.1.5, $SC_*^{<b}(H)$ is a subcomplex of $SC_*(H)$, and $SH_*^{<b}(H)$ is defined to be its homology. More generally, for $-\infty \le a < b \le \infty$, we can define the quotient complex

$$SC_*^{[a,b)}(H) := SC_*^{<b}(H)/SC_*^{<a}(H)$$

and define $SH_*^{[a,b)}(H)$ as its homology. Here, $SC_*^{<-\infty}(H)$ is understood to be empty, while $SC_*^{<\infty}(H)$ is understood to be $SC_*(H)$. By taking the direct limit as in (3.3), we define $SH_*^{[a,b)}(W)$, which we refer to as the symplectic homology of (W,ω) in the action window $[a,b)$. It shoud be noted, however, that these truncated symplectic homology are not invariant under isotopies of the Liouville one-form.

Note that the chain groups of $SH_*^{<b}(H)$ are the same for any admissible H with slope greater or equal to b. It turns out that the differential is also the same. Thus, for $b \notin \operatorname{Spec}(\Sigma,\alpha)$,

$$SH_*^{<b}(W) = SH_*^{<b}(W) := SH_*^{(-\infty,b)}(W) = SH_*(H_b),$$

where H_b is an admissible Hamiltonian of slope b.

As special cases of truncated symplectic homology, define

$$SH_*^-(W) := SH_*^{(-\infty,\delta)}(W) \qquad \text{and} \qquad SH_*^+(W) := SH_*^{[\delta,\infty)}(W),$$

where $0 < \delta < \min(\operatorname{Spec}(\Sigma,\alpha)$, which is called *negative* and *positive* symplectic homology, respectively. In fact, $SH_*^-(W)$ is isomorphic to to the singular homology of W relative to Σ,

$$SH_*^-(W) \cong H_{n+*}(W,\Sigma) \overset{PD}{\cong} H^{n-*}(W).$$

The reason is fairly intuitive: Take a cofinal family of admissible Hamiltonians H such that $H|_W$ is negative but C^2-small and time-independent. For such an H, the only 1-periodic orbits with action less than δ are constant, i.e. critical points of H in W. Further, the Floer cylinders between these critical points are also time-independent, satisfying $\partial_s u = JX_H$. With respect to the metric $g = \omega(\cdot, J\cdot)$, the right hand side JX_H is exactly the negative gradient of H. Hence, $SH_*^-(W)$ is isomorphic to Morse homology, which in turn is isomorphic to singular homology.

3.2 Morse–Bott setup of symplectic homology

The definition of SH from the previous section is very difficult to use for explicit computations. Apart from the difficulty in computing the differential (which is hard to avoid), it is already difficult to write down the chain groups, because any admissible Hamiltonian must be time-dependent. Otherwise, if H is time-independent and γ an orbit of X_H, $\gamma(t_0+\cdot)$ is another orbit for any $t_0 \in S^1$. Hence, the Hamiltonian orbit come at least in S^1-families. In this case, the linearized Hamiltonian flow would have one as an eigenvalue, so the non-degeneracy (condition (iii) in the definition of admissibility) would be violated.

However, there is a Morse–Bott approach to symplectic homology, developed in [10], which deals with Hamiltonians with degenerate orbits. While [10] considers only the case of Hamiltonians for which the 1-periodic orbits are transversally non-degenerate (i.e. appear only in S^1-families), it turns out that analogous statements are true for more general Hamiltonians.

For our purposes, it is convenient to use a Hamiltonian H on $\widehat{W}$ which is C^2-small and negative on W and has the form

$$H|_{\mathbb{R}_{\geq 0}\times\partial W} = h(e^\rho)$$

on the cylindrical end, where h is some strictly increasing function satisfying $\lim_{x\to\infty} h'(x) = \infty$ and $h'' > 0$. Thus, H is similar to H^∞ from Remark 3.1.4, except for the time-independence of h. The Hamiltonian vector field is now explicitly given by

$$X_H(\rho, x) = h'(e^\rho)R_\alpha(\rho, x)$$

for any point $(\rho, x) \in \mathbb{R}_{\geq 0} \times \Sigma$, where R_α denotes the Reeb vector field of $\alpha = \lambda|_\Sigma$. Hence, the 1-periodic orbits of X_H are either

- critical points of H in W, or
- 1-periodic orbits on the level sets $\{\rho\} \times \Sigma$, which are in one-to-one correspondence with closed Reeb orbits of period $h'(e^\rho)$ on Σ.

As explained at the end of the previous section, the orbits of the first kind give the negative part of symplectic homology, $SH_*^-(W) \cong H_{n+*}(W, \Sigma)$. Thus we will now focus on the positive part of symplectic homology $SH_*^+(W)$, generated by the closed Reeb orbits in Σ.

The relevant conditions for this approach to work are:

- The space
$$\mathcal{N}_T := \{z \in \Sigma \mid \phi_T(z) = z\}$$
consisting of T-periodic orbits is a closed submanifold for any $T \in \mathbb{R}_{\geq 0}$, such that the rank $d\alpha|_{\mathcal{N}_T}$ is locally constant and $T_p\mathcal{N}_T = \ker(d_p\phi_T - \mathrm{id})$.
- The set $\{T \geq 0 \mid \mathcal{N}_T \neq \emptyset\}$ is discrete.[1]

These conditions guarantee that we have a Morse–Bott setting for symplectic homology, with the critical submanifolds $\mathcal{N}_T$. Moreover, to have a well-defined grading, assume that

- $c_1(W) = 0$ and
- all closed Reeb orbits of Σ are contractible in Σ.

[1] In fact, recent work by Fauck [29, Section 2.1] shows that the second condition follows from the first. We chose to leave it as a separate condition for the sake of explicitness, since it is obvious for Brieskorn manifolds anyway.

The second assumption makes sure that the grading of generators of $SH^+(W)$ is independent of the filling W (provided that $c_1(W) = 0$). For Brieskorn manifolds of dimension at least five, both conditions are clearly satisfied.

Next, we choose a Morse function f_T on each (non-empty) $\mathcal{N}_T$. Then, the generators of the complex $SC^+(W)$ are given by pairs (T, η), where η is a critical point of f_T. Its grading is given by (see [7, Lemma 2.4])

$$\mu(T, \eta) = \mu_{\mathrm{RS}}(\mathcal{N}_T) + \mathrm{ind}(\eta) - \frac{1}{2}(\dim \mathcal{N}_T - 1), \tag{3.4}$$

where $\mathrm{ind}(\eta)$ is the Morse index of η as a critical point of $f_T \colon \mathcal{N}_T \to \mathbb{R}$, and $\mu_{\mathrm{RS}}(\mathcal{N}_T)$ is the Robbin–Salamon index of the path of symplectic matrices induced by an orbit in $\mathcal{N}_T$, as described in [60]. (Some authors call μ_{RS} the Maslov index, but this terminology is somewhat ambiguous.)

Remark 3.2.1. In some examples, one would like to use a *perfect* Morse function for f_T (i.e. a Morse function such that the differential of Morse homology vanishes), but it is not clear whether such a function exists. However, by the work of Fauck [29, Section 4.3], at least with field coefficients, one can formally work with a chain complex as if one had perfect Morse functions. Roughly, the argument is that the generators for other Morse functions fit into a Morse–Bott spectral sequence whose first page consists of the homology of the critical submanifolds (with appropriate degree shifts). Hence, one can use the total complex of the first page for the chain complex.

It remains to define the differential for this chain complex (and to prove that its homology is isomorphic to $SH_*(W)$). The idea is that instead of counting Floer cylinders as before, we count *cascades of Floer and Morse trajectories*, i.e. trajectories that switch between Floer cylinders between critical submanifolds and negative gradient flow lines of the chosen Morse functions. The details, including the equivalence to the previous definition for a time-dependent perturbation of H, are described in [10].

However, the differential is still very hard to compute. Therefore, in most applications, including Chapter 4, the strategy is rather to get as much information as possible without knowing the differential. We will calculate the differential only for the examples in Chapter 5, so we postpone its definition to that point.

3.3 $\vee$-shaped symplectic homology

For some of the purposes of this text, notably Chapter 6, it will be important to have a variant of symplectic homology that is defined on the symplectization of a contact manifold, without reference to a symplectic filling. This is definitely not possible for the usual SH, as even some of its generators live in the filling (indeed, its negative part SH^- is isomorphic to the singular cohomology of the filling).

Its positive part SH^+ is, under favorable conditions, independent of the filling. However, while SH has as product with unit, SH^+ does not (see Section 3.4, in particular Remark 3.4.2). This ring structure will be essential for Chapter 6.

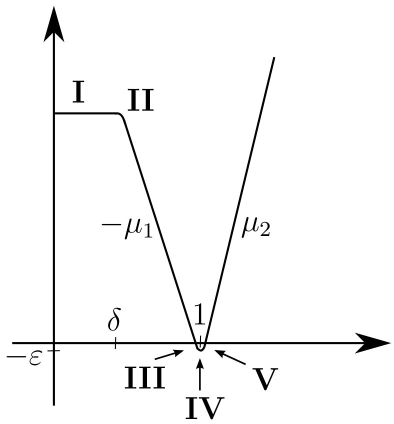

Figure 3.1: A Hamiltonian used to define $\check{SH}$

The solution to this is to use the V-shaped symplectic homology $\check{SH}$ of [17]. Let us quickly recall how this homology theory is constructed: Take a Hamiltonian as in Figure 3.1, where the coordinate of the horizontal axis is $r = e^\rho$. Assume that $\mu_1, \mu_2 \notin \mathrm{Spec}(\Sigma, \alpha)$. (In [17], $\mu_1 = \mu_2$, but it causes no problems to have different values.) This Hamiltonian can be either time-independent, in which case we use methods as in Section 3.2, or we can add a small time-dependent perturbation.

The 1-periodic orbits of H are concentrated in the areas (I) to (V). However, as explained in [17, Proposition 2.9], the orbits in (I) and (II) are excluded by their action, by taking a suitable quotient complex. Indeed, given an action window (a, b), one can choose the constants μ_1, μ_2, δ and ε such that all generators with action in (a, b) are of the following types:

- Nonconstant orbits in (III), coming from negatively parametrized Reeb orbits with action greater than $a > -\mu_1$.
- Constant orbits in (IV), coming from the singular cohomology of Σ.
- Nonconstant orbits in (V), coming from positively parametrized Reeb orbits with action less than $b < \mu_2$.

Then, define

$$\check{SH}_k^{(a,b)}(\widehat{W}) := \varinjlim_{H} HF_k^{(a,b)}(H) \tag{3.5}$$

as the direct limit as $\mu_1, \mu_2 \to \infty$, and define

$$\check{SH}_k(\widehat{W}) := \varinjlim_{b} \varprojlim_{a} \check{SH}_k^{(a,b)}(\widehat{W}), \tag{3.6}$$

where the limits mean $b \to \infty$ and $a \to -\infty$, respectively.

By [17, Theorem 1.5], $\check{SH}(\widehat{W})$ is isomorphic to the Rabinowitz Floer homology of W. Moreover, the positive part $\check{SH}^{(0,\infty)}(\widehat{W})$ is isomorphic to the usual positive symplectic homology $SH^+(\widehat{W})$, while $\check{SH}^{(-\epsilon,\epsilon)}(\widehat{W})$ (for $\epsilon > 0$ sufficiently small) is isomorphic to the singular cohomology of Σ.

3.4 Product structures

The product in Hamiltonian Floer theory always involves a count of pairs-of-pants between three Hamiltonian orbits, although the precise definition varies slightly in the literature. Here, we will follow the approach from [2]. Let $\mathcal{P} := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Riemann sphere with three punctures, two of which are called *positive* (or inputs) and one is called *negative* (or the output). Fix parametrizations $[0, \infty) \times S^1$ near the positive punctures and $(-\infty, 0] \times S^1$ near the negative puncture, called *cylindrical ends.*

Given admissible Hamiltonians $H_0, H_1, H_2 \in \mathcal{C}^\infty(\widehat{W})$, almost complex structure J_0, J_1, J_2 and 1-periodic orbits $\gamma_0, \gamma_1, \gamma_2$ of the Hamiltonians, respectively, we want the define the product

$$HF(H_1, J_1) \times HF(H_2, J_2) \to HF(H_0, J_0). \tag{3.7}$$

To define this product, we need the following data:

- A Hamiltonian $H_\mathcal{P}$, parametrized by the pair-of-pants surface $\mathcal{P}$, such that $H_\mathcal{P}(s, t, x) = H_i(t, x)$ in the parametrization near the puncture z_i.
- An almost complex structure $J_\mathcal{P}$, parametrized by $\mathcal{P}$, such that $J_\mathcal{P}(s, t, x) = J_i(t, x)$ in the parametrization near the puncture z_i.
- A one-form $\beta \in \Omega^1(\mathcal{P})$ which restricts to dt in the parametrizations near the punctures.

Assume that $J_\mathcal{P}$ is convex near infinity. Moreover, assume that the Hamiltonians H_0, H_1, H_2 are linear at infinity with slopes $b_0, b_1, b_2 \geq 0$ and $H_\mathcal{P}$ is linear at infinity with slope function $b_\mathcal{P} \colon \mathcal{P} \to \mathbb{R}_+$. Then we require (for compactness of the moduli spaces below) that

$$d(b_\mathcal{P}\beta) \leq 0. \tag{3.8}$$

By [2, Exercise 2.3.4], it is possible the choose β and $H_\mathcal{P}$ such that (3.8) is satisfied if and only if $b_0 \geq b_1 + b_2$. Now, we define the moduli space of pairs-of-pants

$$\mathcal{M}(\gamma_1, \gamma_2, \gamma_0; \beta, H_\mathcal{P}, J_\mathcal{P})$$

as the set of smooth maps $u \colon \mathcal{P} \to \widehat{W}$ which converge to γ_1, γ_2 at the positive punctures and to γ_0 at the negative puncture and satisfy the Floer equation

$$(du - X_{H_\mathcal{P}} \otimes \beta)^{0,1} = \frac{1}{2}\left((du - X_{H_\mathcal{P}} \otimes \beta) + J \circ (du - X_{H_\mathcal{P}} \otimes \beta) \circ j\right) = 0. \tag{3.9}$$

For a generic choice of $H_\mathcal{P}$ and $J_\mathcal{P}$, this moduli space is a smooth manifold of dimension

$$\dim(\mathcal{M}(\gamma_1, \gamma_2, \gamma_0; \beta, H_\mathcal{P}, J_\mathcal{P})) = \mu(\gamma_1) + \mu(\gamma_2) - \mu(\gamma_0) - n,$$

where $\mu = \mu_{CZ}$ denotes the Conley–Zehnder index. Moreover, there is a suitable compactification by adding lower-dimensional strata. In particular, for $\mu(\gamma_0) =$

$\mu(\gamma_1) + \mu(\gamma_2) + n$, the moduli space is a finite set of points. Hence, we can define the product of γ_1 and γ_2 as

$$\gamma_1 \cdot \gamma_2 = \sum_{\substack{\gamma_0 \\ \mu(\gamma_0)=\mu(\gamma_1)+\mu(\gamma_2)-n}} \#_2[\mathcal{M}(\gamma_1, \gamma_2, \gamma_0; \beta, H_{\mathcal{P}}, J_{\mathcal{P}})]\, \gamma_0,$$

giving the definition of (3.7). By [2, Section 2.3.6], this product behaves well with respect to continuation maps. Hence, taking direct limits on the Hamiltonians, it induces a product

$$SH_k(W) \times SH_\ell(W) \to SH_{k+\ell-n}(W).$$

It turns out that this product is associative and graded commutative, although this is not obvious from the definition. Also, there is an element of SH acting as a unit of this product, namely the image of the generator of $H^0(W)$ under the map $H^*(W) \cong SH^-_{*-n}(W) \to SH_{*-n}(W)$. Hence, it gives SH the structure of a unital, graded-commutative ring.

As explained in [18], this product structure can also be defined on $\check{SH}(\widehat{W})$, with similar properties. In particular, $\check{SH}(\widehat{W})$ is also a graded ring with unit, where the unit comes from the generator of $H^0(\Sigma)$.

Remark 3.4.1 (Product Grading)**.** By definition, the pair-of-pants product has degree n in the usual grading. In order to have a product of degree zero, it can be convenient to simply shift the grading by n and define the "product grading"

$$\mu_{\text{product}} := \mu - n.$$

Remark 3.4.2. Although we will not need this, let us recall how the pair-of-pants product on symplectic homology respects the action filtration. For this purpose, it is convenient to use a slightly different definition of the product, in which the Hamiltonians H_1, H_2 and H_0 are positive multiples of a common Hamiltonian H, see e.g. [58]. (The induced product on SH is still the same.) Then, by [58, Section 16.3], it holds that

$$\mathcal{A}_{H_0}(\gamma_1 \cdot \gamma_2) \leq \mathcal{A}_{H_1}(\gamma_1) + \mathcal{A}_{H_2}(\gamma_2),$$

As a consequence, the product restricts to a map

$$\cdot \colon SH^{[a,b)} \times SH^{[a',b')} \to SH^{[\max\{a+b',a'+b\},b+b')},$$

where on the right hand side, it is necessary to divide out all generators with action less than $\max\{a + b', a' + b\}$ to make the map well-defined. For example, this does not give a product on the whole positive symplectic homology, but one can define maps

$$SH^{[\delta,b)} \times SH^{[\delta,b)} \to SH^{[b+\delta,2b)} \tag{3.10}$$

that contain a part of the information of the product on SH.

3.5 Independence of the filling

In this section, we will explore under which circumstances $\check{SH}$ (or SH^+) can be defined in the symplectization of Σ, as opposed to the completion $\widehat{W}$ of a Liouville filling W. We take the model $(\mathbb{R}_+ \times \Sigma, \omega = d(r\alpha))$ for the symplectization. An ω-compatible almost complex structure J_t is called *SFT-like* if it satisfies

- $J_t(r\partial_r) = R_\alpha$, where R_α denotes the Reeb vector field.
- J_t preserves the contact distribution $\xi = \ker(\alpha)$.
- J_t is invariant under translations $r \mapsto e^c r$ for $c \in \mathbb{R}$.

Now, fix a Hamiltonian $H = H_{\mu_1,\mu_2}$ as in Figure 3.1 and an ω-compatible almost complex structure J_t which is SFT-like near the negative end of the symplectization.

Lemma 3.5.1. *Assume $c_1(\Sigma) = 0$ and $\mu_{CZ}(c) > 3 - n$ for all contractible Reeb orbits c. Let γ_+, γ_- be two Hamiltonian orbits in the part where H is convex with $\mu(\gamma_+) - \mu(\gamma_-) = 1$. Then, the moduli space $\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(\gamma_+, \gamma_-; H, J)$ is compact, i.e. the Floer cylinders do not escape to the negative end of the symplectization.*

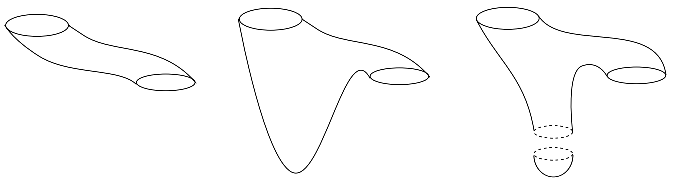

Figure 3.2: Possible breaking of cylinders. Hamiltonian orbits are represented by continuous lines, Reeb orbits by dashed lines.

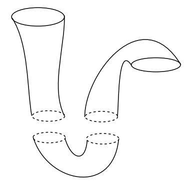

Figure 3.3: Such a breaking cannot occur, due to the maximum principle

Proof. Assume that there exists a sequence $u_j \in \mathcal{M}^{\mathbb{R}_+ \times \Sigma}(\gamma_+, \gamma_-; H, J)$ with $\lim_{j\to\infty} \inf(\pi_{\mathbb{R}_+}(u_j)) = 0$. By the usual SFT-compactness, and since H is constant on the negative end, they converge to a broken cylinder (see Figure 3.2). Its top level component is a Floer cylinder with punctures, at which it is asymptotic to contractible Reeb orbits $c_1, \dots, c_k$. As was shown in [9, Section 5.2], the domain of the top component is connected. (The reason is that the $\mathbb{R}_+$-component of the Floer cylinder approaches the orbit γ_- from above, hence a breaking as in Figure 3.3 is prevented by the maximum principle.) The moduli space of such punctured Floer cylinders has virtual dimension

$$\mu(\gamma_+) - \mu(\gamma_-) - \sum_{j=1}^{k} (\mu(c_j) + n - 3) - 1 = -\sum_{j=1}^{k} (\mu(c_j) + n - 3), \tag{3.11}$$

where the -1 comes from dividing out the free $\mathbb{R}$-action by shifts in the domain (see [9, Section 5.2]). By the assumption on the indices of contractible Reeb orbits, this dimension is negative. Hence, by transversality (assuming J_t was chosen sufficiently generic), this space is empty, giving a contradiction. $\square$

In the same way, one can show that the moduli spaces for continuation maps are compact. In this case, there is no $\mathbb{R}$-action divided out, so the virtual dimension is bigger by one compared to (3.11). However, the difference of Conley–Zehnder indices $\mu(\gamma_+) - \mu(\gamma_-)$ is zero, hence one gets the same contradiction.

Corollary 3.5.2. *Assume that $c_1(\Sigma) = 0$ and either*

(i) $\mu_{CZ}(c) > 4 - n$ for all contractible Reeb orbits c, or

(ii) Σ admits a Liouville filling W with $c_1(W) = 0$ and $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits c which are contractible in W.[2]

Then, $\check{SH}$ can be defined by counting Floer cylinders on the symplectization $\mathbb{R}_+ \times \Sigma$ instead of a filling.

Proof. In addition to the compactness of the moduli spaces for the differential and the continuation maps, we have to show that $\partial \circ \partial = 0$. As usual, this is done by examining the moduli spaces $\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(\gamma_+, \gamma_-; H, J)$ for $\mu(\gamma_+) - \mu(\gamma_-) = 2$. We have to prove again that its elements do not escape to the negative end of the symplectization, so that the moduli space has the usual compactification by products of one-dimensional moduli spaces.

If $\mu_{CZ}(c) > 4 - n$ for all contractible Reeb orbits c, we can use the same proof as for Lemma 3.5.1. Indeed, the virtual dimension of the top component is

$$\mu(\gamma_+) - \mu(\gamma_-) - \sum_{j=1}^{k} (\mu(c_j) + n - 3) - 1 = 1 - \sum_{j=1}^{k} (\mu(c_j) + n - 3),$$

which is again negative by the stronger index assumption.

If, on the other hand, we only know $\mu_{CZ}(c) > 3 - n$, this strategy does not work, since the virtual dimension might just be zero. Instead, if (*ii*) holds, the strategy is to show that the differential defined by Lemma 3.5.1 and the differential defined by the filling coincide. We have to show that, for any orbits γ_+, γ_- with $\mu(\gamma_+) - \mu(\gamma_-) = 1$, the moduli spaces

$$\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(\gamma_+, \gamma_-; H, J) \quad \text{and} \quad \mathcal{M}^{\widehat{W}}(\gamma_+, \gamma_-; H, J) \tag{3.12}$$

are in bijective correspondence. We use the "neck-stretching" operation as in [9, Section 5.2]. This basically means that we insert a piece of the symplectization with constant Hamiltonian near $\partial W \cong \{1\} \times \Sigma \subset \widehat{W}$ and make this piece larger and larger. Under this operation, the elements of $\mathcal{M}^{\widehat{W}}(\gamma_+, \gamma_-; H, J)$ which are not contained in $\mathbb{R}_{\geq 1} \times \Sigma \subset \widehat{W}$ converge to broken cylinders as in the right of Figure 3.2. However, by the same index calculation as in Lemma 3.5.1, such a breaking is not possible. Hence, this neck-stretching operation gives the correspondence (3.12). □

[2]For a Reeb orbit c that is contractible in W but not in Σ, we have to use the grading $\mu_{CZ}(c)$ coming from a filling disk in W.

Remark 3.5.3. In case *(ii)* of Corollary 3.5.2, one can wonder whether $\check{SH}$ is independent of the choice of filling W. Indeed, the only place where the choice of W still plays a role is the grading. For a Reeb orbit c which is not contractible in Σ, the grading generally depends on the choice of a "reference loop" in the free homotopy class of c. If c is contractible in W, however, W gives a canonical choice of grading. This grading might differ for different Liouville fillings with $c_1(W) = 0$.

Apart from this grading ambiguity, $\check{SH}$ is independent of W. In particular, this is the case if $\pi_1(\Sigma) = 0$, or more generally if the induced map $\pi_1(\Sigma) \to \pi_1(W)$ is injective.

Once product structures are taken into account, the grading issue becomes more complicated. Then, the reference loops for different free homotopy classes can no longer be chosen independently from each other, and it is not clear what choices one has in general for the grading of non-contractible orbits. One possible way to go is to split symplectic homology into different homology classes in $H_1(W)$, as opposed to free homotopy classes. If $H_1(W)$ is free, one can assign gradings consistently as in [23]. However, if $H_1(W)$ has torsion, one runs into the same problems as in [23, Section 2.9.1].

To avoid these issues, we assume from now on that $\pi_1(\Sigma) = 0$. The only exception in this text will be the example of A_k-surface singularities in Section 6.2.2, but these have an explicit Liouville filling with $c_1(W) = 0$ and $\pi_1(W)$ which can be used to define the grading.

Alternatively, one can consider the subring $\check{SH}^{\text{contractible}} \subset \check{SH}$ generated by contractible Reeb orbits, for which the grading is always well-defined.

Remark 3.5.4. The statements of Corollary 3.5.2 and Remark 3.5.3 hold equally true for SH^+ instead of $\check{SH}$. In particular, if Σ is simply-connected and fulfills $c_1(\Sigma) = 0$, $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits c and admits a Liouville filling W with $c_1(W) = 0$, then $SH^+(W)$ is independent of the choice of W.

Definition 3.5.5. We call a contact manifold (Σ, ξ) *index-positive* if there exists a contact form α with $\xi = \ker(\alpha)$ such that the assumption of Corollary 3.5.2 is satisfied.

In the following, we will always assume that Σ is index-positive. In view of Corollary 3.5.2, we will also write $\check{SH}(\Sigma)$ instead of $\check{SH}(W)$.

We would like to have statements analogous to Lemma 3.5.1 and Corollary 3.5.2 also for moduli spaces of pairs-of-pants. However, there is an additional complication: While the top component of a broken Floer cylinder was always connected, a pair-of-pants can also break as in Figure 3.4. We must exclude this by another index condition.

As as preparation, the next lemma gives the general dimension formula for the moduli spaces of broken Floer curves that appear in the limit process. As always in this section, we assume that $c_1(\Sigma) = 0$.

Let $\Gamma^+ = (\gamma_1^+, \dots, \gamma_{k_+}^+)$ and $\Gamma^- = (\gamma_1^-, \dots, \gamma_{k_-}^-)$ be collections of Hamiltonian orbits in $\mathbb{R}_+ \times \Sigma$ and $C = (c_1, \dots, c_\ell)$ be a collection of contractible Reeb orbits of Σ. Further, let H, J, β be Floer data as in Section 3.4 (with the straightforward

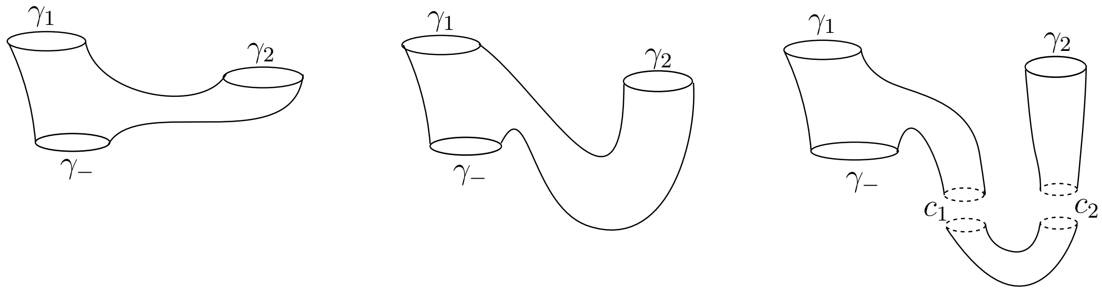

Figure 3.4: Possible breaking of pairs-of-pants. Hamiltonian orbits are represented by continuous lines, Reeb orbits by dashed lines.

generalization to any number of positive and negative punctures). Denote by $\mathcal{M}(\Gamma^+, \Gamma^-, C; \beta, H, J)$ the moduli space of maps

$$u\colon \mathbb{CP}^1 \setminus \{z_1^+, \dots, z_{k_+}^+, z_1^-, \dots, z_{k_-}^-, \tilde{z}_1, \dots, \tilde{z}_\ell\} \longrightarrow \mathbb{R}_+ \times \Sigma$$

which fulfill Floer's equation (3.9), converge to $\gamma_i^\pm$ as $z \to z_i^\pm$ in the sense of Floer theory and converge to $\{0\} \times c_j$ at $\tilde{z}_j$ in the sense of SFT. The conformal structure on $\mathbb{CP}^1 \setminus \{z_1^+, \dots, z_{k_+}^+, z_1^-, \dots, z_{k_-}^-\}$ is understood to be fixed, while the points $\tilde{z}_1, \dots, \tilde{z}_\ell$ can vary freely.

Lemma 3.5.6. *The virtual dimension of this moduli space is*

$$\dim \mathcal{M}(\Gamma^+, \Gamma^-, C; \beta, H, J) = \\ = \sum_{i=1}^{k_+} \mu(\gamma_i^+) - \sum_{i=1}^{k_-} \mu(\gamma_i^-) - \sum_{j=1}^{\ell} (\mu(c_j) + n - 3) + n(2 - |\Gamma^+| - |\Gamma^-|).$$

Proof. For $C = \emptyset$, the formula is fairly standard (see e.g. [65, Theorem 3.3.11]). The general case can be deduced by gluing J-holomorphic discs to the orbits c_j. By [9, Section 3], the dimension of the moduli space of J-holomorphic discs asymptotic to a Reeb orbit c_j is

$$\mu(c_j) + n - 3.$$

As the dimension formula is additive under gluing, the result follows. □

Remark 3.5.7. For $|\Gamma^+| = |\Gamma^-| = 1$, this is the moduli space of punctured holomorphic cylinders. For this case, the dimension was already computed in [9], and we applied the result in the proof of Lemma 3.5.1 above. In the following lemma, we need the cases $|\Gamma^+| = |\Gamma^-| = 1$ and $|\Gamma^+| = 1$, $|\Gamma^-| = 0$, as these cases appear in Figure 3.4.

Lemma 3.5.8. *Fix Hamiltonian orbits $\gamma_1, \gamma_2, \gamma_-$ with*

$$\mu(\gamma_1) + \mu(\gamma_2) - \mu(\gamma_-) - n = 0. \tag{3.13}$$

Assume that Σ, in addition to being index-positive, satisfies

$$\mu(c) > \max\{3 - |\mu(\gamma_1)|, 3 - |\mu(\gamma_2)|\} \tag{3.14}$$

for all Reeb orbits c. Then, the moduli space $\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(\gamma_1, \gamma_2, \gamma_-; \beta, H, J)$ is compact.

Proof. We need to rule out the breaking as in Figure 3.4 (and similarly with γ_1 and γ_2 exchanged). Then, the rest of the proof works as in Lemma 3.5.1.

For the top level on the right of Figure 3.4 to have positive dimension, by Lemma 3.5.6, we would need

$$\mu(\gamma_1) - \mu(\gamma_-) - \mu(c_1) - n + 3 \geq 0$$

and

$$\mu(\gamma_2) - \mu(c_2) + 3 \geq 0.$$

Using (3.13), these conditions simplify to

$$\mu(c_1) \leq 3 - \mu(\gamma_2) \quad \text{and} \quad \mu(c_2) \leq 3 + \mu(\gamma_2).$$

By the assumption (3.14), these two equations lead to

$$3 - |\mu(\gamma_2)| < 3 - \mu(\gamma_2) \quad \text{and} \quad 3 - |\mu(\gamma_2)| < 3 + \mu(\gamma_2).$$

The first equation implies $\mu(\gamma_2) < 0$ while the second equation implies $\mu(\gamma_2) > 0$, giving a contradiction. □

Remark 3.5.9. The cylinder in the bottom level on the right of Figure 3.4 is a holomorphic curve of the kind studied in SFT. As such, it lives in a moduli space of virtual dimension $\mu(c_1) + \mu(c_2)$ (which might not be cut out transversally). Thus, it seems that the virtual dimensions appearing in Figure 3.4 are not additive under gluing. The reason for the mismatch is that upon gluing, one does in general not recover the conformal structure that was fixed in the left part of Figure 3.4.

To have the chain level product of any orbits well-defined in the symplectization, Lemma 3.5.8 implies that the condition

$$\mu(c) > 3 \quad \text{for all closed Reeb orbits } c \tag{3.15}$$

is sufficient. In order for the product to descend to homology, one also need compactness of the one-dimensional moduli spaces. However, a quick calculation (as in the proof of Lemma 3.5.8) shows that (3.15) is sufficient for this as well.

Definition 3.5.10. We call a contact manifold (Σ, ξ) with $c_1(\Sigma) = 0$ and $\pi_1(\Sigma) = 0$ *product-index-positive* if there exist a contact form α with $\xi = \ker(\alpha)$ such that (3.15) holds.

As $\dim(\Sigma) = 2n - 1$, we have $n \geq 1$, so product-index-positivity implies index-positivity.

Corollary 3.5.11. *For a product-index-positive contact manifold* Σ, $\check{SH}$ *and its product structure can be defined by counting Floer cylinders and pairs-of-pants in the symplectization* $\mathbb{R}_+ \times \Sigma$.

3.6 S^1-equivariant symplectic homology

On the loop space $C^\infty(S^1, \widehat{W})$, where the action is defined, there exists the obvious S^1-action by reparametrization. As Floer homology can be heuristically thought of as the Morse homology of this action functional, it seems plausible that there exists an S^1-equivariant version. This is the starting point for S^1-equivariant symplectic homology, first developed by Viterbo in [69], with the standard references now being [11, 12]. We will only need it for the definition of the mean Euler characteristic in Section 3.7, so we just give a rough sketch of the definition. In the context of S^1-equivariant symplectic homology, we use coefficients in $\mathbb{Q}$, since the theory would be quite different (and more complicated) with coefficients in a finite field.

Recall that in the Borel construction in algebraic topology, the S^1-equivariant homology of a space X with an S^1-action is defined as

$$H_*^{S^1}(X) = H_*(X \times_{S^1} S^\infty). \tag{3.16}$$

Here, we take $\mathbb{CP}^\infty = S^\infty/S^1$ as a model for the classifying space BS^1, and $X \times_{S^1} S^\infty$ means the quotient of $X \times S^\infty$ by the diagonal action. We try to mimic the Morse theoretic version of this construction in symplectic homology.

We approximate S^∞ by S^{2N+1}, with the usual S^1-action by complex multiplication, and let $N \to \infty$ in the last step. Take a Hamiltonian

$$H\colon S^1 \times \widehat{W} \times S^{2N+1} \longrightarrow \mathbb{R}$$

which is S^1-invariant, i.e. satisfies $H(t+\tau, x, \tau z) = H(t,x,z)$ for all $t, \tau \in S^1$, $x \in \widehat{W}$ and $z \in S^{2N+1}$. The action functional $\mathcal{A}_H$ is defined as before, with $z \in S^{2N+1}$ as another variable. Its critical points are tuples (γ, z), where γ is a 1-periodic orbit of X_H and $z \in S^{2N+1}$ that satisfy the equation

$$\int_{S^1} \partial_z H(t, \gamma(t), z)\, dt = 0.$$

The set of critical points is clearly S^1-invariant, and we define the chain group $SC_*^{S^1,N}(H)$ as the $\mathbb{Q}$-span of the S^1-orbits of critical points. The grading of $[(\gamma, z)]$ is defined as $\mu(\gamma) - N$.

To define the differential, we also need an S^1-invariant family of almost complex structures $J\colon S^1 \times S^{2N+1} \to \operatorname{End}(\widehat{W})$ and an S^1-invariant Riemannian metric g on S^{2N+1}. Then,

$$\partial^{S^1}\colon SC_*^{S^1,N}(H) \longrightarrow SC_{*-1}^{S^1;N}(H)$$

is defined by counting tuples (u, z), where u satisfies a parametrized version of the Floer equation and $z\colon \mathbb{R} \to S^{2N+1}$ is a gradient flow of the left hand side of (3.16).

Of course, a precise definition would require some further restrictions on H and J as well as some amount of analysis, which we won't go into here. Also, since we use rational coefficients, one would really have to introduce orientations on the moduli spaces involved.

After doing this work, one can prove that $\partial^{S^1} \circ \partial^{S^1} = 0$, so one can define the corresponding homology $SH_*^{S^1,N}(H)$. Standard arguments show that this group does not depend on the choices of J and g, and that there exist continuation maps between different Hamiltonians. Thus, one can define

$$SH_*^{S^1,N}(W) := \varinjlim_{H} SH_*^{S^1,N}(H).$$

Finally, the S^1*-equivariant symplectic homology groups* are defined as

$$SH_*^{S^1}(W) := \varinjlim_{N} SH_*^{S^1,N}(W),$$

where the maps in the direct system are induced by the embeddings $S^{2N+1} \hookrightarrow S^{2N+3}$.

As with the usual symplectic homology, there is an action filtration on $SH_*^{S^1}(W)$. In particular, one can define $SH_*^{+,S^1}(W)$, the *positive* S^1*-equivariant symplectic homology* of W. As shown in [11], this group is isomorphic to the linearized contact homology of Σ with the filling W, assuming the latter is well-defined.

3.7 Mean Euler characteristic

Like in the non-equivariant version, the differential of $SH_*^{S^1}(W)$ is very hard to compute. However, there is a derived quantity, called the mean Euler characteristic, which can be computed on chain level. It was originally introduced in [67] in the context of contact homology and has been transferred to S^1-equivariant symplectic homology in [32].

Assume, as always in this chapter, that $(W, \omega = d\lambda)$ be a Liouville domain with boundary $\Sigma = \partial W$ and $c_1(W)|_{\pi_2(W)} = 0$. For simplicity, assume also that both Σ and W are simply-connected. Define the i-th Betti number of the positive S^1-equivariant symplectic homology as

$$b_i(W) := \dim\left(SH_i^{+,S^1}(W;\mathbb{Q})\right).$$

Now, suppose further that there exists a chain complex for positive S^1-equivariant symplectic homology for which the ranks of the chain groups are uniformly bounded in all degrees. This chain complex can either come from a contact form with non-degenerate Reeb orbits or from a suitable Morse–Bott setup. Then, we can define the *mean Euler characteristic* as

$$\chi_m(W) := \frac{1}{2}\left(\liminf_{N\to\infty} \frac{1}{N}\sum_{i=-N}^{N}(-1)^i b_i(W) + \limsup_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{N}(-1)^i b_i(W)\right)$$

In all the cases considered in this paper, the limit actually exists, so the formula reduces to

$$\chi_m(W) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i b_i(W).$$

Remark 3.7.1. The assumption of having a chain complex for $SH_i^{+,S^1}(W)$ whose ranks are uniformly bounded is fairly strong. However, it can be weakened to the condition that the contact manifold (Σ, ξ) has *convenient dynamics*, see [41, Definition 5.14]. Under this condition, χ_m is also well-defined and satisfies all properties listed below. The notion of convenient dynamics is useful in particular because it is preserved under subcritical surgery, see [41, Appendix C].

By [32, Corollary 2.2] and with the assumption made above, the mean Euler characteristic depends only on Σ and its contact structure, i.e. it is independent of the filling W. Therefore, we will also write $\chi_m(\Sigma)$ instead of $\chi_m(W)$.

The next proposition gives an explicit formula for the mean Euler characteristic in the case of a contact form with periodic Reeb flow. So, let $(\Sigma, \xi = \ker\alpha)$ be a contact manifold with a Morse–Bott contact form α and assume that the Reeb vector field induces an S^1-action with finitely many orbit spaces. Denote the periods, in increasing order, by $T_1 < T_2 < \cdots < T_k$ and the orbit spaces by Σ_{T_i}. So T_k is the period of the principal orbit and all T_i divide T_k. Define the *frequency*

$$\phi_{T_i;T_{i+1},\ldots,T_\ell} = \#\{a \in \mathbb{N} \mid aT_i < T_\ell \text{ and } aT_i \notin T_j\mathbb{N} \text{ for any } j = i+1,\ldots,\ell\}.$$

By convention, $\phi_{T_k;\emptyset} = 1$.

Proposition 3.7.2 ([32], [41])**.** *Let $(\Sigma, \xi = \ker\alpha)$ be a simply-connected contact manifold with periodic Reeb flow as above. Assume the following conditions:*

- *There exists an exact symplectic filling $(W, d\lambda)$ of Σ such that $c_1(W) = 0$ and $\pi_1(W) = 0$.*
- *The restriction of the tangent bundle of the symplectization to Σ, $T(\mathbb{R}\times\Sigma)|_\Sigma$ is symplectically trivial.*
- *For any periodic Reeb orbit γ, the linearized Reeb flow is complex linear in some unitary trivialization of ξ along γ.*
- *The Robbin–Salamon index of the principal orbit $\mu_P := \mu(\Sigma_{T_k})$ does not vanish.*

Then the mean Euler characteristic exists, is independent of the filling W and can be computed by the formula

$$\chi_m(\Sigma) = \frac{\sum_{i=1}^{\ell} (-1)^{\mu(\Sigma_{T_i}) - \frac{1}{2}\dim(\Sigma_{T_i}/S^1)} \phi_{T_i;T_{i+1},\ldots,T_\ell} \cdot \chi^{S^1}(\Sigma_{T_i})}{|\mu_P|}. \tag{3.17}$$

Let us briefly explain how this formula arises. The main idea is to use a Morse–Bott spectral sequence converging to $SH^{+,S^1}(W;\mathbb{Q})$, whose first page is given by

$$E^1_{p,q} = \bigoplus_{\substack{T \text{ such that} \\ \mu(\Sigma_T) - \frac{1}{2}\dim(\Sigma_T/S^1) = p}} H_q^{S^1}(\Sigma_T;\mathcal{L}). \tag{3.18}$$

Here, $\mathcal{L}$ is a real line bundle, meaning that homology with local coefficients is used. However, the third assumption in Proposition 3.7.2 guarantees that this bundle is trivial, so one has coefficients in $\mathbb{Q}$. Then, adding all the contributions from (3.18) to χ_m over one period of the S^1-action with the correct signs result in the formula (3.17).

We will see in Section 4.2 that Proposition 3.7.2 can be applied to a Brieskorn manifold $\Sigma(a)$ if $\sum_j \frac{1}{a_j} \neq 1$. Moreover, Section 7.2 contains an explicit sample computation.

4 Index computations for Brieskorn manifolds

4.1 General formula for the Robbin–Salamon index

Recall from Section 2.1 the formula

$$\phi_t(z) = \left(e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n\right) \tag{4.1}$$

for the Reeb flow of the standard contact form on a Brieskorn manifold $\Sigma(a)$. From this, we see that the Morse–Bott submanifold $\mathcal{N}_T$ is empty unless $T = L \cdot \frac{\pi}{2}$ for some integer L which is divisible by at least two of the exponents a_i.

The first step in the computation of any variant of symplectic homology for $\Sigma(a)$ is to compute the Robbin–Salamon indices of the critical submanifolds $\mathcal{N}_{L\pi/2}$. The following formulas been computed by Fauck [28], and in a slightly different notation earlier by van Koert [68]. For the special case $a = (p, 2, \dots, 2)$, the computation was already done by Ustilovsky [66].

Proposition 4.1.1 ([68, 28])**.** *For a Brieskorn manifold $\Sigma(a)$, the Robbin–Salamon index of the critical submanifold $\mathcal{N}_{L\pi/2}$ is*

$$\mu_{\mathrm{RS}}(\mathcal{N}_{L\pi/2}) = \sum_{j=0}^{n} \left(\left\lfloor \frac{L}{a_j} \right\rfloor + \left\lceil \frac{L}{a_j} \right\rceil \right) - 2L. \tag{4.2}$$

Proof. Regard ϕ_t (with the formula from (4.1)) as a map from $\mathbb{C}^{n+1}$ to itself. Similarly, $\omega := d\alpha$ can be regarded as a symplectic form on $\mathbb{C}^{n+1}$. Thus, the ambient tangent space

$$T_z\mathbb{C}_{n+1} = \xi \oplus \xi^\omega$$

is the direct sum of the contact distribution ξ and its symplectic complement ξ^ω. The latter has an $\mathbb{R}$-basis

$$\begin{aligned} X_1 &= (\bar{z}_0^{a_0-1}, \dots, \bar{z}_n^{a_n-1}), & Y_1 &= iX_1, \\ X_2 &= -2i\left(\frac{z_0}{a_0}, \dots, \frac{z_n}{a_n}\right), & Y_2 &= (z_0, \dots, z_n). \end{aligned}$$

Here, X_1 and Y_1 are a basis of the symplectic complement of $T_zV(a)$, while X_2 and Y_2 span the symplectic complement of ξ inside $T_zV(a)$. By a symplectic analog of the Gram–Schmidt process, we can modify this basis to a symplectic basis

$$\begin{aligned}
\tilde{X}_1 &= \frac{X_1}{\omega(X_1, Y_1)}, & \tilde{Y}_1 &= \frac{Y_1}{\omega(X_1, Y_1)}, \\
\tilde{X}_2 &= X_2, & \tilde{Y}_2 &= Y_2 - \frac{\omega(X_1, Y_2)Y_1 - \omega(Y_1, Y_2)X_1}{\omega(X_1, Y_1)} = Y_2 - \frac{\sum_j a_j z^{a_j}}{2\omega(X_1, Y_1)}.
\end{aligned}$$

See [28] for a more thorough derivation of these formulas. Furthermore, a straightforward computation shows that the linearized Reeb flow on ξ^ω satisfies

$$\begin{aligned}
d\phi_t(\tilde{X}_1(z)) &= e^{4it}\tilde{X}_1(\phi_t(z)), & d\phi_t(\tilde{Y}_1(z)) &= e^{4it}\tilde{Y}_1(\phi_t(z)) \\
d\phi_t(\tilde{X}_2(z)) &= \tilde{X}_2(\phi_t(z)), & d\phi_t(\tilde{Y}_2(z)) &= \tilde{Y}_2(\phi_t(z)),
\end{aligned}$$

so it is represented by the matrix $\Phi_{\xi^\omega} := \operatorname{diag}(e^{4it}, 1)$. Meanwhile, the linearized Reeb flow on the ambient space $T\mathbb{C}^{n+1}|_\Sigma$ is represented in the standard basis by

$$\Phi_{\mathbb{C}^{n+1}} := \operatorname{diag}(e^{4it/a_0}, \dots, e^{4it/a_n}).$$

Finally, denote by Φ_ξ the linearized Reeb flow on the bundle ξ. It follows from the additivity under direct sums of the Robbin–Salamon index under direct sums (see [60, Section 4]) that

$$\mu_{\mathrm{RS}}(\mathcal{N}_T) = \mu_{\mathrm{RS}}(\Phi_\xi) = \mu_{\mathrm{RS}}(\Phi_{\mathbb{C}^{n+1}}) - \mu_{\mathrm{RS}}(\Phi_{\xi^\omega}), \tag{4.3}$$

where all paths of symplectic matrices are understood to go from $t = 0$ to $t = T$. To compute $\mu_{\mathrm{RS}}(\Phi_{\mathbb{C}^{n+1}})$ and $\mu_{\mathrm{RS}}(\Phi_{\xi^\omega})$, we use again additivity and the fact that the Robbin–Salamon index of the path

$$\Phi(t) = e^{it} \in \mathrm{Sp}(2), \qquad 0 \le t \le T,$$

is given by

$$\mu_{\mathrm{RS}}(\Phi) = \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil = \begin{cases} \frac{T}{\pi} & \text{if } T \in 2\pi\mathbb{Z} \\ 2\left\lfloor \frac{T}{2\pi} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Hence, taking the reparametrizations into account and setting $T = L\frac{\pi}{2}$,

$$\mu_{\mathrm{RS}}(\Phi_{\mathbb{C}^{n+1}}) = \sum_{j=0}^{n} \left(\left\lfloor \frac{L}{a_j} \right\rfloor + \left\lceil \frac{L}{a_j} \right\rceil \right)$$

and

$$\mu_{\mathrm{RS}}(\Phi_{\mathbb{C}^{n+1}}) = (\lfloor L \rfloor + \lceil L \rceil) = 2L.$$

The result now follows from (4.3). $\square$

As a by-product, this proof shows that ξ^ω and $T\mathbb{C}^{n+1}|_\Sigma$ are symplectically trivial, hence their first Chern class vanishes. Thus, $c_1(\xi)$ vanishes as well.

4.2 The Mean Euler characteristic for Brieskorn manifolds

The mean Euler characteristic, as described in Section 3.7, is a very useful invariant for the contact structures of Brieskorn manifold.

Lemma 4.2.1 ([41])**.** *For a Brieskorn manifold $\Sigma = \Sigma(a_0, \ldots, a_n)$ with*

$$\sum_{j=0}^{n} \frac{1}{a_j} \neq 1,$$

the mean Euler characteristic $\chi_m(\Sigma)$ is a well-defined invariant of the contact structure and can be computed by (3.17)*.*

Proof. We check that the conditions of Proposition 3.7.2 are satisfied. Indeed, the only thing not obvious for Brieskorn manifolds is that Robbin–Salamon index $\mu_P = \mu(\Sigma_{T_k})$ of the principal orbit is nonzero. By (4.2), this index is given by

$$\begin{aligned} \mu_P &= \sum_{j=0}^{n} \left(\left\lfloor \frac{\operatorname{lcm}(a_i)}{a_j} \right\rfloor + \left\lceil \frac{\operatorname{lcm}(a_i)}{a_j} \right\rceil \right) - 2 \operatorname{lcm}(a_i) \\ &= 2 \operatorname{lcm}(a_i) \cdot \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1 \right), \end{aligned} \tag{4.4}$$

which is nonzero if and only if $\sum_{j=0}^{n} \frac{1}{a_j} \neq 1$. □

The assumption that $\sum_{j=0}^{n} \frac{1}{a_j} \neq 1$ is necessary: Otherwise, if $\sum_{j=0}^{n} \frac{1}{a_j} = 1$, some of the Betti numbers of positive S^1-equivariant symplectic homology are infinite (similarly as we will see in Section 4.3 for non-equivariant symplectic homology), so that the mean Euler characteristic is not well-defined.

We will compute the mean Euler characteristic of some explicit examples in Section 7.2.

4.3 Examples of infinite-dimensional $SH_k(W)$

As a first application of the index formulas, we note that for some Brieskorn manifolds, SH_* is infinite-dimensional in certain degrees. The following theorem has been found independently by Fauck in [29].

Theorem 4.3.1. *Let $\Sigma = \Sigma(a_0, \ldots, a_n)$ be a Brieskorn manifold with $n \geq 3$ and*

$$\sum_{j=0}^{n} \frac{1}{a_j} = 1.$$

Then, the symplectic homology $SH_k(W)$ of the filling W (with field coefficients) is infinite-dimensional in the degrees $k = -n + 1$ and $k = n$.

Proof. By (3.4), the index of a generator η of SH^+ lying on the critical submanifold $\mathcal{N}_T$ is given by

$$\mu(T,\eta) = \mu_{\mathrm{RS}}(\mathcal{N}_T) + \mathrm{ind}(\eta) - \frac{1}{2}(\dim \mathcal{N}_T - 1),$$

where the first summand is the Robbin–Salamon index of $\mathcal{N}_T$ and the second summand is the Morse index of a critical point. For Brieskorn manifolds, the period T is a multiple of $\pi/2$, and the Robbin Salamon index can be computed by (4.2). If we take $T = L\pi/2$ with L a multiple of $\mathrm{lcm}(a_j)$ (so that $\mathcal{N}_T = \Sigma$) and the minimum of a Morse function on $\mathcal{N}_{L\pi/2}$, the index is given by

$$\begin{aligned}
\mu &= \sum_{j=0}^{n} \left(\left\lfloor \frac{L}{a_j} \right\rfloor + \left\lceil \frac{L}{a_j} \right\rceil \right) - 2L - (n-1) \\
&= 2\sum_{j=0}^{n} \frac{L}{a_j} - 2L - n + 1 \\
&= -n + 1.
\end{aligned}$$

So this degree appears infinitely many times on chain level. Next, we show that all other generators of SH^+ have degree at least $-n+3$.

For generators (T,η) with $\mathcal{N}_T = \Sigma$, this follows from $n \geq 3$. Namely, since Brieskorn manifolds are highly connected, we can use Remark 3.2.1 to see that all critical points other than the minimum have Morse-index at least $n-1 \geq 2$.

So take L not a multiple of $\mathrm{lcm}(a_j)$. Say, possibly after reordering, that L divides $a_0, \dots, a_{j_0}$, but does not divide $a_{j_0+1}, \dots, a_n$. Then, the index of the minimum of the Morse function on $\mathcal{N}_{L\pi/2}$ is

$$\begin{aligned}
\mu &= 2\sum_{j=0}^{j_0} \frac{L}{a_j} + \sum_{j=j_0+1}^{n} \left(2\left\lceil \frac{L}{a_j} \right\rceil - 1 \right) - 2L - (j_0 - 1) \\
&= 2\left(\underbrace{\sum_{j=0}^{j_0} \frac{L}{a_j} + \sum_{j=j_0+1}^{n} \left\lceil \frac{L}{a_j} \right\rceil}_{=:x} \right) - (n - j_0) - 2L - j_0 + 1
\end{aligned}$$

By assumption, $j_0 < n$, so we have $x > L$ strictly. As $x, L \in \mathbb{Z}$, this implies $x \geq L + 1$, hence

$$\mu \geq 2(L+1) - (n - j_0) - 2L - j_0 + 1 = -n + 3.$$

Thus, the generators in degree $-n+1$ cannot be killed by the differential (except possibly finitely many of them by SH^-, but even this turns out not to be the case, as the elements of SH^- live in degrees 0 and n).

For the degree $k = n$, the argument is similar: For L a multiple of $\mathrm{lcm}(a_j)$, the maximum of a Morse function on $\mathcal{N}_{L\pi/2}$ has degree n. Further, by the same reasoning as above, all other generators have degree at most $n-2$. $\square$

4.4 $\Sigma(2\ell, 2, \ldots, 2)$ for $n \geq 5$

In this section, we use the setup from Section 3.2 and the result of Proposition 4.1.1 to compute the positive symplectic homology SH^+ of the filling of Σ_ℓ for $n \geq 5$ odd. The chain groups are valid also for $n = 3$.

If we start at a point $z \in \Sigma_\ell$, minimal period of the Reeb flow is π if $z_0 = 0$ and $\ell\pi$ otherwise. Hence we get the critical submanifolds

$$\mathcal{N}_T = \begin{cases} \Sigma_\ell & \text{if } T = N\pi, N \in \mathbb{N}, \ell \mid N \\ \{z \in \Sigma_\ell \mid z_0 = 0\} & \text{if } T = N\pi, N \in \mathbb{N}, \ell \nmid N \\ \emptyset & \text{else.} \end{cases}$$

(In our convention, $\mathbb{N} = \{1, 2, \ldots\}$.) In particular, all periods are in fact multiples of π. So for $T = N\pi$ (i.e. $L = 2N$) and $a = (2\ell, 2\ldots, 2)$, (4.2) specializes to

$$\begin{aligned} \mu_{\mathrm{RS}}(\mathcal{N}_{N\pi}) &= \left\lfloor \frac{N}{\ell} \right\rfloor + \left\lceil \frac{N}{\ell} \right\rceil + 2N(n-2) \\ &= \begin{cases} 2\frac{N}{\ell} + 2N(n-2) & \text{if } \ell \mid N \\ 2\left\lfloor \frac{N}{\ell} \right\rfloor + 2N(n-2) + 1 & \text{if } \ell \nmid N. \end{cases} \end{aligned}$$

As for the Morse functions, first note that

$$\{z \in \Sigma_\ell \mid z_0 = 0\} = \left\{ z \in \mathbb{C}^{n+1} \mid z_0 = 0, \sum_{j=1}^{n} z_k^2 = 0, |z|^2 = 1 \right\}$$

is diffeomorphic to the unit cotangent bundle S^*S^{n-1} of S^{n-1} [28, Section 3.2]. As shown in the appendix of [28], there exists a perfect Morse function (for $\mathbb{Z}_2$-coefficients) on S^*S^{n-1}, i.e. a Morse function that has only four critical points with indices $0, n-2, n-1, 2n-3$. If $n = 3$, Σ_ℓ is diffeomorphic to S^*S^3 (see Corollary 2.2.4), so we can use the same Morse function on Σ_ℓ.

For $n > 3$, the diffeomorphism type of Σ_ℓ is given by Proposition 2.2.3. In the cases $\ell \equiv 2, 3 \mod 4$, the existence of a perfect Morse function on Σ_ℓ is not obvious. However, for $\mathbb{Z}_2$-coefficients, we can use Remark 3.2.1 and pretend to have a Morse function on Σ_ℓ whose only four critical points have indices $0, n-1, n, 2n-3$.

When plugging the Robbin–Salamon and Morse indices into (3.4), the chain groups of the positive part of symplectic homology (with coefficients in $\mathbb{Z}_2$) take the form

$$SC_k^+(W) \quad \cong \bigoplus_{\substack{N\in\mathbb{N},\ \ell\mid N \\ d\in\{-n+1,0,1,n\}}} (\mathbb{Z}_2)_{2N/\ell+2N(n-2)+d} \quad \oplus \bigoplus_{\substack{N\in\mathbb{N},\ \ell\nmid N \\ d\in\{-n+3,1,2,n\}}} (\mathbb{Z}_2)_{2\lfloor N/\ell\rfloor+2N(n-2)+d} \tag{4.5}$$

where $(\mathbb{Z}_2)_k$ denotes the $\mathbb{Z}_2$-vector space on one generator of degree k.

As the differential decreases the degree by one, many differentials can already be excluded by degree reasons. Note also that there cannot be a differential between

$2N/\ell + 2N(n-2)+1$ and $2N/\ell + 2N(n-2)$ or between $2\lfloor N/\ell \rfloor + 2N(n-2)+2$ and $2\lfloor N/\ell \rfloor + 2N(n-2)+1$, as the corresponding generators lie on the same critical manifold and the Morse–Bott differential is zero (as we had a perfect Morse function). Thus, we only have to check whether there can be non-zero differentials between indices with different values for N.

Now, we assume $n \geq 5$. (The case $n = 3$ will be examined in Chapter 5.) Then, an easy computation shows that each degree occurs at most once and we have precisely one pair of generators in consecutive degrees for any pair $(N, N+1)$. However, there is an easy way to see that the differential vanishes. This is because the index of the generator with period $(N+1)$ is lower by one than the index of the generator with period N, and the differential cannot increase the period. To see this, remember that the generators of SH^+ are critical points of the action functional

$$\mathcal{A}_H \colon \mathcal{C}^\infty(S^1, \widehat{W}) \to \mathbb{R}, \qquad \mathcal{A}_H(\gamma) = \int_{S^1} \gamma^* \hat{\theta} - \int_{S^1} H(t, \gamma(t))\, dt.$$

Its value at a critical orbit γ is $\mathcal{A}_H(\gamma) = e^r h'(e^r) - h(e^r)$. Note that $\partial_r(e^r h'(e^r) - h(e^r)) = e^{2r} h''(e^r) > 0$, so Hamiltonian orbits with larger action correspond to larger values of r, and thus to Reeb orbits with larger period. By Lemma 3.1.5, ∂ decreases the action, hence it decreases the periods of the corresponding Reeb orbits.

This concludes that the positive symplectic homology is generated by the chains above:

Theorem 4.4.1. *For $n \geq 5$, the positive symplectic homology of the filling of Σ_ℓ is given by*

$$SH_k^+(W) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 2N/\ell + 2N(n-2) - n + 1 \\ & \text{or } k = 2N/\ell + 2N(n-2) \\ & \text{or } k = 2N/\ell + 2N(n-2) + 1 \\ & \text{or } k = 2N/\ell + 2N(n-2) + n \qquad \text{for some } N \in \mathbb{N}, \ell \mid N \\ \mathbb{Z}_2 & \text{if } k = 2\lfloor N/\ell \rfloor + 2N(n-2) - n + 3 \\ & \text{or } k = 2\lfloor N/\ell \rfloor + 2N(n-2) + 1 \\ & \text{or } k = 2\lfloor N/\ell \rfloor + 2N(n-2) + 2 \\ & \text{or } k = 2\lfloor N/\ell \rfloor + 2N(n-2) + n \qquad \text{for some } N \in \mathbb{N}, \ell \nmid N \\ 0 & \text{else.} \end{cases}$$

By Remark 3.5.4, we can use this theorem to distinguish the contact structures on Σ_ℓ. Alternatively, we could argue that Σ_ℓ and $\Sigma_{\ell'}$ are non-contactomorphic because there is a degree $k \geq n+2$ in which the filling Σ_ℓ has non-vanishing symplectic homology, while, for a suitable non-degenerate contact form, there is no Reeb orbit with Conley–Zehnder index k on $\Sigma_{\ell'}$.

Corollary 4.4.2. *For $n \geq 5$, the manifolds $\Sigma_\ell = \Sigma(2\ell, 2, \ldots, 2)$ are pairwise non-contactomorphic.*

Remark 4.4.3. In this section, one could have chosen coefficients in some other field instead of $\mathbb{Z}_2$. The only change would be in the homology of the critical submanifolds, and accordingly in $SH_*^+(W)$. Presumably, there is also a similar theorem for integer coefficients. However, this raises some difficulties because the critical submanifolds may not admit perfect Morse functions (e.g. S^*S^{n-1} does not), and the spectral sequence argument from Remark 3.2.1 does not work over the integers.

Remark 4.4.4. From here, one can easily compute the full symplectic homology of the filling W of Σ_ℓ. Indeed, the singular relative homology of the pair (W, Σ_ℓ) is

$$H_k(W, \Sigma_\ell; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{2\ell-1} & \text{if } k = n \\ \mathbb{Z} & \text{if } k = 2n \\ 0 & \text{else,} \end{cases}$$

as can be seen from the statements at the beginning of Section 2.2. Then, one uses $SH_k^-(W) \cong H_{k+n}(W, \Sigma_\ell)$ and the long exact sequence

$$\cdots \longrightarrow SH_*^-(W) \longrightarrow SH_*(W) \longrightarrow SH_*^+(W) \longrightarrow SH_{*-1}^-(W) \longrightarrow \cdots$$

coming from the tautological exact sequence $0 \to SC_*^- \to SC_* \to SC_*^+ \to 0$. One can read off $SH_*(W)$ directly from this sequence, without having to know any of the maps.

Furthermore, one can also compute the Rabinowitz Floer homology $RFH_*(W) \cong \check{SH}(W)$, either directly with the Morse–Bott methods used here (and in [28]) or from $SH_*(W)$, using the long exact sequence from [17].

5 The role of symmetries

The goal of this chapter is to compute SH_*^+ for $\Sigma_\ell = \Sigma(2\ell, 2, 2, 2)$ and ultimately prove Theorem 1.3.1. The chain complex (4.5) for $n = 3$ contains many generators in consecutive degrees, so one needs to worry about possible differentials. We will manage to compute these differentials with the help of a $\mathbb{Z}_2$-symmetry of Σ_ℓ and its filling.

5.1 Chain groups for $SH_*^+(\Sigma(2\ell, 2, 2, 2))$

It turns out to be convenient to leave the full Morse–Bott formalism and work instead in a perturbed setup. We will use the same perturbation as Ustilovsky in [66]. In fact, we are still in a Morse–Bott situation after the perturbation, but with all critical manifolds being S^1. This is exactly the setup used in [10].

To start, Ustilovsky changes the coordinates by the following unitary transformation:

$$w_0 = z_0, \qquad w_1 = z_1, \qquad \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}.$$

In these coordinates,

$$\Sigma_\ell = \left\{ w \in \mathbb{C}^4 \mid w_0^{2\ell} + w_1^2 + 2w_2w_3 = 0, \|w\|^2 = 1 \right\}.$$

Next, Ustilovsky introduces a new contact form $\alpha' := K^{-1}\alpha$, where

$$K(w) := \|w\|^2 + \epsilon \left(|w_2|^2 - |w_3|^2 \right)$$

and $\epsilon > 0$ is a sufficiently small, irrational number.[1] He then shows that the corresponding Reeb vector field is

$$R_{\alpha'} = \left(\frac{2i}{\ell} w_0, 2iw_1, 2i(1+\epsilon)w_2, 2i(1-\epsilon)w_3 \right).$$

[1] For the sake of simplicity, we have, as Ustilovsky, perturbed the contact form. In fact, it is possible to get the same outcome by perturbing the Hamiltonian, as is more common in symplectic homology, although it cannot be written down as nicely. To perturb the Hamiltonian orbits on the level set $\{r_0\} \times \Sigma_\ell$, one has to add $H_{\mathrm{pert},r_0} := \epsilon h'(e^{r_0})e^r(|w_2|^2 - |w_3|^2)$ to the Hamiltonian. Doing this for all level sets containing critical submanifolds (with suitable cutoff functions), one gets that the Hamiltonian vector field equals $h'(e^r) \cdot R_{\alpha'}$ near the critical submanifolds, hence the perturbed orbits are the same.

Hence, the only simple (i.e. not multiply covered) periodic Reeb orbits are

$$
\begin{aligned}
\gamma^{0,+}(t) &= \left(re^{2it/\ell}, ir^\ell e^{2it}, 0, 0\right), && r > 0, r^{2\ell} + r^2 = 1, \quad 0 \le t \le \ell\pi, && (5.1)\\
\gamma^{0,-}(t) &= \left(re^{2it/\ell}, -ir^\ell e^{2it}, 0, 0\right), && r > 0, r^{2\ell} + r^2 = 1, \quad 0 \le t \le \ell\pi, && (5.2)\\
\gamma^{+}(t) &= \left(0, 0, e^{2it(1+\epsilon)}, 0\right), && 0 \le t \le \frac{\pi}{1+\epsilon}, && (5.3)\\
\gamma^{-}(t) &= \left(0, 0, 0, e^{2it(1-\epsilon)}\right), && 0 \le t \le \frac{\pi}{1-\epsilon}, && (5.4)
\end{aligned}
$$

and their multiples, all of which are transversally non-degenerate. Furthermore, for ϵ sufficiently small, the Conley–Zehnder indices of these orbits and their multiple covers (denoted by $N\gamma$ for a simple orbit γ) are given by

$$
\begin{aligned}
\mu_{\mathrm{CZ}}(N\gamma^{0,\pm}) &= 2N + 2N\ell \quad \stackrel{N' :\equiv N\ell}{=} \quad 2\frac{N'}{\ell} + 2N' && (5.5)\\
\mu_{\mathrm{CZ}}(N\gamma^{+}) &= 2\left\lceil \frac{N}{\ell} \right\rceil + 2N - 2 && (5.6)\\
\mu_{\mathrm{CZ}}(N\gamma^{-}) &= 2\left\lfloor \frac{N}{\ell} \right\rfloor + 2N + 2, && (5.7)
\end{aligned}
$$

by a computation analogous to [66, Lemma 4.2].

Applying the Morse–Bott formalism of Section 3.2 to this situation, we get two generators for each of the orbits above. We denote these by γ_m and γ_M, with $\mu(\gamma_M) = \mu(\gamma_m) + 1$.[2] Hence we get the generators of $SC^+(W)$ as in Table 5.1.

Degree	2	3	4	5	6	$\cdots$	$2\ell+1$	$2\ell+2$	$2\ell+3$	$2\ell+4$	$\cdots$
Generators	γ_m^+	γ_M^+	$2\gamma_m^+$	$2\gamma_M^+$	$3\gamma_m^+$	$\cdots$	$\ell\gamma_M^+$	$\gamma_m^{0,+}$	$\gamma_M^{0,+}$	$(\ell+1)\gamma_m^+$	$\cdots$
			γ_m^-	γ_M^-	$2\gamma_m^-$	$\cdots$	$(\ell-1)\gamma_M^-$	$\gamma_m^{0,-}$	$\gamma_M^{0,-}$	$\ell\gamma_m^-$	$\cdots$

Table 5.1: The generators of $SC^+(W)$ for $n = 3$, in the perturbed Morse–Bott setup.

At this point, let us recall from [10] how the differential is defined in the Morse–Bott formalism. Denote by S_γ the S^1-family of orbits with geometric image $\mathrm{im}(\gamma)$. Given a Hamiltonian H as in Section 3.2 and an ω-compatible, time-dependent almost complex structure J, the set $\widehat{\mathcal{M}}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ is defined as the space of cylinders $u\colon \mathbb{R} \times S^1 \to \widehat{W}$ satisfying the Floer equation

$$\partial_s u + J_{(t,u)}(\partial_t u - X_H(u)) = 0, \tag{5.8}$$

which converge asymptotically to some orbits in $S_{\overline{\gamma}}$ and $S_{\underline{\gamma}}$. The last part means

[2] Due to our use of *negative* gradient flow lines, the indices of minimum and maximum are interchanged compared to [10].

that there exist orbits $\overline{\gamma} \in S_{\overline{\gamma}}$, $\underline{\gamma} \in S_{\underline{\gamma}}$ such that

$$\lim_{s\to-\infty} u(s,t) = \underline{\gamma}(t), \qquad \lim_{s\to\infty} u(s,t) = \overline{\gamma}(t), \qquad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0,$$

uniformly in t.

If $S_{\overline{\gamma}} \neq S_{\underline{\gamma}}$, there is a free $\mathbb{R}$-action on $\widehat{\mathcal{M}}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$, defined by $s_0 \cdot u(\cdot,\cdot) = u(s_0 + \cdot, \cdot)$. Define the moduli space as $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) := \widehat{\mathcal{M}}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)/\mathbb{R}$.

These moduli spaces come along with evaluation maps $\overline{\text{ev}}, \underline{\text{ev}}$, defined by $u \mapsto \overline{\gamma}(0)$ and $u \mapsto \underline{\gamma}(0)$, respectively. For J in a comeagre set, the moduli spaces $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ are transversally cut out (see [10, Theorem 3.5]). In this case, we can choose perfect Morse functions f_γ on the spaces S_γ such that their stable and unstable manifolds (denoted by W^s and W^u, respectively) are transverse to $\overline{\text{ev}}$ and $\underline{\text{ev}}$ (see [10, Lemma 3.6]). The minima and maxima of these Morse functions give the generators in Table 5.1.

For two generators $\overline{\gamma}_p, \underline{\gamma}_q$ with $S_{\overline{\gamma}} \neq S_{\underline{\gamma}}$, the fibered product

$$\mathcal{M}(\overline{\gamma}_p, \underline{\gamma}_q; H, J) := W^s(p) \times_{\overline{\text{ev}}} \mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)_{\underline{\text{ev}}} \times W^u(q). \tag{5.9}$$

is a smooth, compact manifold of dimension $\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) - 1$. In particular, if $\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) = 1$, it is a finite set. The coefficient $\langle \partial(\overline{\gamma}_p), \underline{\gamma}_q \rangle$ of the differential is then defined as the count (modulo 2) of its elements.

If $S_{\overline{\gamma}} = S_{\underline{\gamma}}$, the coefficient $\langle \partial(\overline{\gamma}_p), \underline{\gamma}_q \rangle$ of the differential is simply the corresponding coefficient of the Morse differential. In our case, since all critical manifolds are circles and we use $\mathbb{Z}_2$-coefficients, all these differentials vanish. (For integer or rational coefficients, this would be true only for *good* Reeb orbits, see Section 3.7 for the definition. For our choice of contact form on Σ_ℓ, all Reeb orbits are actually good, as can be checked from equations (5.5) to (5.7).)

The next goal is to collect as much information as possible about the differential between the generators in Table 5.1. First, it follows from [9, Proposition 2], that there is no differential between γ_M and γ_m for any γ, so that we only need the definition above for the differential. Another important observation is given by the next lemma.

Lemma 5.1.1. *In the degrees $2N(\ell+1)+j$, where $N \in \mathbb{N}$ and $j \in \{-1,0,1,2\}$, the rank of $SH^+(W)$ is at most one. In particular, there are some non-trivial differentials in Table 5.1 in these degrees.*

Proof. The chain complex from the full Morse–Bott setup of Section 3.2 will give the same symplectic homology groups. Hence, the ranks of the chain groups give upper bounds. Checking the ranks in (4.5), one sees that this upper bound is one in the degrees $2N(\ell+1)+j$. □

We claim that away from these degrees, all differentials vanish. The proof of this uses a $\mathbb{Z}_2$-symmetry of Σ_ℓ (and W) and will occupy the next three sections.

Remark 5.1.2. Although we will not need this, let us point out a few cases where the vanishing of the differential also follows from other reasons. For instance, by [9, Proposition 3 and Remark 14], there cannot be a non-zero differential between γ_M and $\tilde{\gamma}_m$ for any orbits $\gamma \neq \tilde{\gamma}$ with $\mu(\gamma) = \mu(\tilde{\gamma})$, at least if we assume that transversality as in [9, Remark 9] holds for Σ_ℓ. Checking the degrees in Table 5.1, this means that there is no differential from an odd degree to an even degree.

One can also argue that there is no differential from $N\gamma_m^-$ to $N\gamma_M^+$. This is because in the full Morse–Bott picture from Section 3.2, the generators corresponding to $N\gamma_m^-$ and $N\gamma_M^+$ belonged to the same critical manifold. They can be viewed as originating from a perturbation thereof. If there were a differential between them, it would have shown up in Section 3.2 as a Morse differential on this critical manifold, which it did not, as the Morse function on each critical submanifold was perfect. See e.g. [6, Theorem 5.2.2] for the correspondence between the Morse–Bott formalism and the perturbed version.

5.2 Idea of symmetries

Define the involutive isomorphism

$$\psi\colon \mathbb{C}^4 \to \mathbb{C}^4, \qquad \psi(w_0, w_1, w_2, w_3) = (-w_0, -w_1, w_2, w_3).$$

From the definition of Σ_ℓ, one sees immediately that ψ leaves Σ_ℓ invariant (because the exponents $a_0 = 2\ell, a_1 = 2$ are even). For the same reason, the filling W, as defined in (2.3), is left invariant, as well as its completion. Moreover, ψ preserves the contact (resp. symplectic) form on Σ_ℓ (resp. $\widehat{W}$).

We can view ψ as the generator of a $\mathbb{Z}_2$ symmetry. Denote by $\mathcal{J}_{\mathrm{symm}}$ the set of all time-dependent, ω-almost complex structures that are symmetric under ψ, i.e. $\psi_* J = J$. The idea behind this definition is that we apply ψ to the Floer cylinders that appear in the differential. The hope is that these cylinders always come in pairs $u, \psi \circ u$, so that the differential with coefficients in $\mathbb{Z}_2$ vanishes.

As a first step, the next lemma lets us assume that the fixed point set $\widehat{W}_{\mathrm{fixed}}$ of ψ does not contain any Floer cylinders.

Lemma 5.2.1. *Let $\overline{\gamma}_p$ and $\underline{\gamma}_q$ be two generators with $S_{\overline{\gamma}} \neq S_{\underline{\gamma}}$. There are no elements of* (5.9) *whose Floer cylinders are contained in the fixed point set $\widehat{W}_{\mathrm{fixed}}$ of ψ.*

Proof. If one of the underlying orbits $\overline{\gamma}$ or $\underline{\gamma}$ lies outside of $\widehat{W}_{\mathrm{fixed}}$, there is nothing to show. So assume that both are multiples of γ^+ or γ^-. For this case, we are going to show that any two of these orbits live in distinct homotopy classes in $\widehat{W}_{\mathrm{fixed}}$.

The fixed point set

$$\widehat{W}_{\mathrm{fixed}} = \{(w_2, w_3) \in \mathbb{C}^2 \mid 2w_2 w_3 = \epsilon\phi(\|w\|)\}$$

is diffeomorphic to $\mathbb{C}^* \cong \mathbb{R} \times S^1$. Explicitly, this isomorphism can be taken to be the composition

$$\widehat{W}_{\text{fixed}} \longrightarrow \left\{ w_2 w_3 = \phi\left(\sqrt{\epsilon/2}\|w\|\right) \right\} \longrightarrow \{w_2 w_3 = 1\} \longrightarrow \mathbb{C}^*,$$

where the first map is scaling by $\sqrt{2/\epsilon}$, the second is

$$(w_2, w_3) \mapsto \begin{cases} \left(w_2, \frac{1}{w_2}\right) & \text{if} \quad |w_2| \geq |w_3| \\ \left(\frac{1}{w_3}, w_3\right) & \text{if} \quad |w_2| \leq |w_3|, \end{cases}$$

and the third is the inverse of $z \mapsto (z, 1/z)$. Therefore, the orbit $N\gamma^{\pm}$ is mapped to a loop homotopic to

$$[0,1] \to \mathbb{C}^*, \qquad t \mapsto e^{\pm 2\pi N i t}.$$

For different values of $N \in \mathbb{N}$ and $\pm$, these loops all represent different elements of $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$, hence there can be no cylinder in between. $\square$

Remark 5.2.2. This simple proof was suggested by the referee. In the first version of this article, an alternative argument was given, based on the fact that the orbits $N\gamma^{\pm}$ have two different kinds of Conley–Zehnder indices: one with $\widehat{W}$ as the ambient manifold and one with $\widehat{W}_{\text{fixed}}$ as the ambient manifold. It turns out that for some orbits, the difference of the latter indices is smaller than the difference of the former indices. Thus, for generic $J \in \mathcal{J}_{\text{symm}}$, the moduli space of Floer cylinders in $\widehat{W}_{\text{fixed}}$ has negative dimension.

Proposition 5.2.3. *Let $J \in \mathcal{J}_{\text{symm}}$ and $u \colon \mathbb{R} \times S^1 \to \widehat{W}$ be a J-Floer cylinder between orbits $\overline{\gamma} \neq \underline{\gamma}$ which are contained in $\widehat{W}_{\text{fixed}}$. Then $\psi \circ u$ is again a Floer cylinder between the same orbits. Moreover, there is no constant s_0 such that $\psi \circ u(s,t) = u(s + s_0, t)$ for all $(s,t) \in \mathbb{R} \times S^1$.*

The second claim ensures that $\psi \circ u$ and u are counted separately in the moduli space for the differential.

Proof. By assumption, both J and X_H are equivariant with respect to ψ. Thus, if u satisfies the Floer equation (5.8), we can apply ψ_* to both sides and get

$$\begin{aligned} 0 &= \psi_* \left(\partial_s u + J_{(t,u)}(\partial_t u - X_H(u))\right) \\ &= \partial_s(\psi \circ u) + J_{(t,\psi\circ u)}\left(\partial_t(\psi \circ u) - X_H(\psi \circ u)\right), \end{aligned}$$

establishing that $\psi \circ u$ is a Floer cylinder. Since $\overline{\gamma}$ and $\underline{\gamma}$ lie in $\widehat{W}_{\text{fixed}}$, the asymptotics of $\psi \circ u$ and u are the same.

If there were such a constant s_0, we could use $\psi \circ \psi = \text{id}$ to get

$$u(s + 2s_0, t) = \psi \circ u(s + s_0, t) = \psi \circ \psi \circ u(s,t) = u(s,t).$$

So the function $s \mapsto u(s,t)$ would be periodic with period $2s_0$, but it also has a limit as $s \to \infty$. Since it is not constant, this implies $s_0 = 0$ and hence $\psi \circ u = u$. But u cannot lie in $\widehat{W}_{\text{fixed}}$ by Lemma 5.2.1, which gives a contradiction. $\square$

We want to apply this proposition to show that certain Floer cylinders contributing to the differential of symplectic homology come in pairs, so that the differential vanishes in $\mathbb{Z}_2$. Before, though, we must show that there are almost complex structures in $\mathcal{J}_{\text{symm}}$ such that the relevant moduli spaces are cut out transversally.

5.3 Transversality

Proposition 5.3.1. *Given H as in Section 3.2 (time-independent), there exists a comeagre set $\mathcal{J}_{\text{symm,reg}} \subset \mathcal{J}_{\text{symm}}$ for which all moduli spaces $\mathcal{M}(\overline{\gamma}_p, \underline{\gamma}_q; H, J)$ with $\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) = 1$ and $S_{\overline{\gamma}} \neq S_{\underline{\gamma}}$ are transversally cut out.*

Proof. Fix two generators $\overline{\gamma}_p$ and $\underline{\gamma}_q$. For the most part, we have to prove the existence of a comeagre set $\mathcal{J}_{\text{symm,reg}}$ such that the moduli space $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ appearing in the fibered product (5.9) is transversally cut out. Then, the statement follows from a generic choice of Morse functions as in [10, Lemma 3.6].

To prove this, much of the proof of [10, Proposition 3.5 (ii)] can be followed very closely. We will only point out the parts that are different. The most important difference is that for all sets of almost complex structures (like $\mathcal{J}^\ell$, $\mathcal{J}^\ell(H)$, etc.), we additionally demand that $J \in J_{\text{symm}}$. We then denote the corresponding sets by $\mathcal{J}^\ell_{\text{symm}}$, $\mathcal{J}^\ell_{\text{symm}}(H)$, etc.

So we take the universal moduli space

$$\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}, H, \mathcal{J}^\ell_{\text{symm}}(H)) = \left\{(u,J) \;\middle|\; J \in \mathcal{J}^\ell_{\text{symm}}(H), u \in \mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}, H, J)\right\}.$$

We want to prove that this space is transversally cut out. Then we define $\mathcal{J}_{\text{symm,reg}}$ as the set of regular values of the projection to the second factor.

As usual, $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}, H, \mathcal{J}^\ell_{\text{symm}}(H))$ can be written as the preimage $\bar{\partial}_H^{-1}(0)$ under the section $\bar{\partial}_H$ of a Banach vector bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{J}^\ell_{\text{symm}}(H)$. We do not write the details here, as this part is entirely analogous to [10].

It remains to show that the vertical differential

$$\begin{aligned} D\bar{\partial}_H(u,J)\colon T_u\mathcal{B} \times T_J\mathcal{J}^\ell_{\text{symm}}(H) &\longrightarrow \mathcal{E}_{(u,J)} \\ (\zeta, Y) &\longmapsto D_u\zeta + Y_t(u)(\partial_t u - X_H(u)) \end{aligned}$$

is surjective. Again as in [10], D_u (the linearization of the Cauchy–Riemann operator) is Fredholm, so the range of $D\bar{\partial}_H(u,J)$ is closed. We have to show that it is also dense, and this is where some differences to [10] appear.

Let η be in the cokernel of $D\bar{\partial}_H(u,J)$, which means

$$\int_{\mathbb{R}\times S^1} \langle \eta, D_u\zeta\rangle e^{d|s|} ds\, dt = 0, \qquad \int_{\mathbb{R}\times S^1} \langle \eta, Y_t(u)(\partial_t u - X_H(u))\rangle e^{d|s|} ds\, dt = 0 \tag{5.10}$$

for all ζ, Y, where $d > 0$ is some exponential weight. The first equation is still the same as in [10]. It implies that, assuming $\eta \not\equiv 0$, the set $\{(s,t) \mid \eta(s,t) \neq 0\}$ is open and dense. Also, by [31, Lemma 4.5], the set of *regular points*

$$\begin{aligned} R(u) := \Big\{(s,t) \in \mathbb{R}\times S^1 \mid \partial_s u(s,t) \neq 0, u(s,t) \neq \overline{\gamma}(t), \underline{\gamma}(t), \\ u(s,t) \notin u(\mathbb{R}\setminus\{s\}, t)\Big\} \end{aligned}$$

is open and dense for any $u \in \bar{\partial}^{-1}(0)$.

Furthermore, we claim that the set

$$S(u) := \Big\{(s,t) \in \mathbb{R}\times S^1 \mid \partial_s u(s,t) \neq 0, u(s,t) \neq \overline{\gamma}(t), \underline{\gamma}(t), \psi(u(s,t)) \notin u(\mathbb{R},t)\Big\}$$

is open and dense. Indeed, this can be proven in exactly the same way as [31, Lemma 4.5], one just has to replace u by $\psi \circ u$ at the right places and use that $\operatorname{im}(u) \not\subset \widehat{W}_{\text{fixed}}$ by Lemma 5.2.1. The upshot is that we can find a point $(s_0,t_0) \in R(u) \cap S(u)$ with $\eta(s_0,t_0) \neq 0$.

Now, it is always possible to choose a matrix $Y_{t_0}(u(s_0,t_0)) \in T_{J(u(s_0,t_0))}\mathcal{J}^\ell(H)$ which maps the vector $J(u(s_0,t_0))(\partial_t u - X_H(u))$ to $\eta(u(s_0,t_0))$ (see e.g. [49, Lemma 3.2.2]). Letting $\rho\colon S^1 \times \widehat{W} \to [0,1]$ be a time-dependent cutoff function supported near $(t_0, u(s_0,t_0))$, we define $Y := \rho \cdot Y_{t_0}(u(s_0,t_0))$. Then

$$\int_{\mathbb{R}\times S^1} \langle \eta, Y_t(u)(\partial_t u - X_H(u))\rangle e^{d|s|} ds\, dt \neq 0. \tag{5.11}$$

However, Y is a priori not contained in the tangent space to $J^\ell_{\text{symm}}(H)$. For this, we have to make it symmetric under ψ. Hence we define $Y^{\text{symm}} := Y + \psi_*(Y)$. We want to show that (5.11) is still true with Y replaced by Y^{symm}.

By construction, Y^{symm} is supported near the two point $(t_0, u(s_0,t_0))$ and $(t_0, \psi(u(s_0,t_0))$. If $\psi(u(s_0,t_0)) \notin \operatorname{im}(u)$, we are done, because $\psi_*(Y)$ does not affect (5.11) (provided the cutoff function ρ was chosen well). Otherwise, since $(s_0,t_0) \in R(u) \cap S(u)$, we know that $\psi(u(s_0,t_0)) = u(s_1,t_1)$ for some $t_1 \neq t_0$. But since ρ is time-dependent and localized near t_0, $\psi_*(Y)$ still does not affect (5.11). Thus, (5.11) is indeed true with Y replaced by $Y^{\text{symm}} \in T_J\mathcal{J}^\ell_{\text{symm}}(H)$, which contradicts (5.10).

This shows that $D\bar{\partial}_H(u,J)$ is surjective, hence the universal moduli space is cut out transversally. Define the set $\mathcal{J}_{\text{symm,reg}}$ as the set of regular values under its projection to the second factor. By Sard's theorem, this set is comeagre and by construction, $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ is cut out transversally for $J \in \mathcal{J}_{\text{symm,reg}}$. $\square$

5.4 Conclusion

Corollary 5.4.1. *Let $\overline{\gamma}_p, \underline{\gamma}_q$ be two generators with $\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) = 1$, such that the underlying orbits lie in $\widehat{W}_{\text{fixed}}$. Then, the differential satisfies $\left\langle \partial(\overline{\gamma}_p), \underline{\gamma}_q \right\rangle = 0$.*

Proof. By Proposition 5.3.1, we can assume that J is symmetric under ψ. Then, Lemma 5.2.1 and Proposition 5.2.3 tell us that the elements in (5.9) come in pairs, namely by replacing the Floer cylinder $u \in \mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ with $\psi \circ u$. Hence the algebraic count is an even number, and thus vanishes for $\mathbb{Z}_2$-coefficients. □

Looking again at Table 5.1, this shows that all differentials involving only the orbits $N\gamma^+$ and $N\gamma^-$ vanish. This proves the claim made at the end of Section 5.1. Hence, the rank of symplectic homology in degrees $4, \dots, 2\ell$ is two (and again in degrees $2\ell+5, \dots, 4\ell+2$, etc).

Up to here, we already know enough to distinguish the contact structures of Σ_ℓ for different ℓ, but we can get another observation almost for free:

Lemma 5.4.2. *For $N \in \mathbb{N}$ and $j \in \{-1, 0, 1, 2\}$, the groups $SH^+_{(2\ell+2)N+j}(W)$ are isomorphic to $\mathbb{Z}_2$.*

Proof. We already know from Lemma 5.1.1 that these groups can have at most rank 1. To see that they do not vanish, define the map

$$\tilde{\psi}\colon \mathbb{C}^4 \longrightarrow \mathbb{C}^4, \qquad \tilde{\psi}(w_0, w_1, w_2, w_3) = (w_0, -w_1, w_2, w_3).$$

As with ψ, this map descends to a $\mathbb{Z}_2$-symmetry on $\widehat{W}$. Furthermore, it exchanges the orbits $N\gamma^{0,+}$ and $N\gamma^{0,-}$, while leaving all other orbits fixed. In analogy to Proposition 5.3.1, we can find an almost complex structure $\tilde{J}$ with $\tilde{J} = \tilde{\psi}_*(\tilde{J})$ such that the moduli spaces $\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, \tilde{J})$ are regular if at least one of the orbits $\overline{\gamma}, \underline{\gamma}$ lies outside of the fixed point set of $\tilde{\psi}$ (so that Floer cylinders in the fixed point set are excluded).

To justify the switch to a different almost complex structure, consider the continuation homomorphism Φ from (H, J) to $(H, \tilde{J})$. As we change only the almost complex structure, not the Hamiltonian, Φ can be represented in each degree by an invertible matrix. Moreover, Φ intertwines the differential, i.e. $\partial \circ \Phi = \Phi \circ \partial$. Hence, the new differential is a conjugation of the old one, and as such has the same rank. For this lemma, only the rank is of interest, so we can indeed switch to another (regular) almost complex structure.

With $\tilde{J}$ (and suitable Morse functions on $S_{N\gamma^{0,\pm}}$), we get that

$$\left\langle \partial N\gamma_m^{0,+}, N\ell\gamma_M^+ \right\rangle = \left\langle \partial N\gamma_m^{0,-}, N\ell\gamma_M^+ \right\rangle,$$

as $\tilde{\psi}$ interchanges all contributing cylinders. Similarly,

$$\left\langle \partial(N\gamma_m^{0,+}), N(\ell-1)\gamma_M^- \right\rangle = \left\langle \partial(N\gamma_m^{0,-}), N(\ell-1)\gamma_M^- \right\rangle,$$

hence the map

$$\partial\colon SC^+_{N(2\ell+2)} \cong (\mathbb{Z}_2)^2 \longrightarrow SC^+_{N(2\ell+2)-1} \cong (\mathbb{Z}_2)^2$$

is represented by a matrix of the form $\left(\begin{smallmatrix} a & a \\ b & b \end{smallmatrix}\right) \in \mathbb{Z}_2^{2\times 2}$. This matrix has rank at most one, but it cannot have rank zero, as this would contradict Lemma 5.1.1.

Now, note that we cannot have any differential from $N\gamma_M^{0,+}$ to $N\gamma_m^{0,-}$ or from $N\gamma_M^{0,-}$ to $N\gamma_m^{0,+}$. The easiest way to see this is that the underlying orbits have exactly the same period, hence exactly the same action, while the differential strictly decreases the action.

This proves the lemma for $j = -1, 0$. For $j = 1, 2$, note that, for the same reasons as above,

$$\partial\colon SC^+_{N(2\ell+2)+2} \cong (\mathbb{Z}_2)^2 \longrightarrow SC^+_{N(2\ell+2)+1} \cong (\mathbb{Z}_2)^2$$

is represented by a matrix of the form $\left(\begin{smallmatrix} a & b \\ a & b \end{smallmatrix}\right) \in \mathbb{Z}_2^{2\times 2}$. As this matrix also has rank at most one and rank zero would again contradict Lemma 5.1.1, the claim follows. $\square$

Summing up, we have proven:

Theorem 5.4.3. *Let W be a Liouville filling of $\Sigma(2\ell, 2, 2, 2)$, $\ell \geq 2$ with $c_1(W) = 0$. The positive part of symplectic homology of W with coefficients in $\mathbb{Z}_2$ is given by*

$$SH_k^+(W) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 2,3 \text{ or } k = (2\ell+2)N + j \\ & \text{for any } N \in \mathbb{N},\ j \in \{-1,0,1,2\} \\ (\mathbb{Z}_2)^2 & \text{if } k \geq 4, \text{ unless } k \text{ is as above} \\ 0 & \text{if } k \leq 1. \end{cases}$$

The case $\ell = 1$ is even easier and can be read off directly from (4.5). For a Liouville filling W of $\Sigma(2,2,2,2)$ with $c_1(W) = 0$, we get

$$SH_k^+(W) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 2 \text{ or } k \geq 4 \\ 0 & \text{else.} \end{cases}$$

Together with Remark 3.5.4, Theorem 5.4.3 implies:

Corollary 5.4.4. *The Brieskorn manifolds $\Sigma(2\ell, 2, 2, 2) \cong S^2 \times S^3$ with their natural contact structure are pairwise non-contactomorphic.*

5.5 A generalization: $\Sigma(\ell p, p, 2, 2)$

The methods of Section 5.1 to 5.4 can also be applied to $\Sigma(\ell p, p, 2, 2)$, $p \geq 2$, at least for p even. These manifolds (suggested to me by Otto van Koert) provide further examples for contact manifolds that have the same contact homology (for p fixed), but for which symplectic homology can distinguish the contact structures for different values of ℓ. From this point of view, the work above was the special case $p = 2$. We sketch the main points for the general case:

- Application of Randell's algorithm shows that

  $$H_2(\Sigma(\ell p, p, 2, 2); \mathbb{Z}) \cong \mathbb{Z}^{p-1}.$$

  Using this result and the classification of simply-connected spin 5-manifolds [64], we get that $\Sigma(\ell p, p, 2, 2)$ is diffeomorphic to a connected sum of $(p-1)$ copies of $S^2 \times S^3$.

- Analogous to Section 5.1, we use the coordinate change

  $$\Sigma(\ell p, p, 2, 2) \cong \{w \in \mathbb{C}^4 \mid w_0^{\ell p} + w_1^p + 2w_2w_3 = 0\}$$

  and perturb the contact form. The resulting simple closed Reeb orbits are

  $$\gamma^+(t) = \left(0, 0, e^{2it(1+\epsilon)}, 0\right), \qquad 0 \leq t \leq \frac{\pi}{1+\epsilon}, \tag{5.12}$$
  $$\gamma^-(t) = \left(0, 0, 0, e^{2it(1-\epsilon)}\right), \qquad 0 \leq t \leq \frac{\pi}{1-\epsilon} \tag{5.13}$$

  and

  $$\gamma^{0,k}(t) = \left(re^{4it/\ell p}, \zeta^{2k+1} r^\ell e^{4it/p}, 0, 0\right), \qquad 0 \leq t \leq \ell p \frac{\pi}{2}, \tag{5.14}$$

  where $r > 0$ is the constant satisfying $r^{2\ell} + r^2 = 1$, $\zeta = e^{\pi i/p}$ is a primitive $2p$-th root of unity and $k = 0, \ldots, p-1$. The main difference from Section 5.1 is that we get p different simple orbits living in the first two coordinates. The Conley–Zehnder indices of these orbits (and their multiple covers) are similar to (5.5) to (5.7), namely

  $$\mu_{\mathrm{CZ}}(N\gamma^{0,k}) = 2N + 2N\ell \overset{N' := N\ell}{=} 2\frac{N'}{\ell} + 2N' \tag{5.15}$$
  $$\mu_{\mathrm{CZ}}(N\gamma^+) = 2\left\lceil \frac{2N}{\ell p} \right\rceil + 2\left\lceil \frac{2N}{p} \right\rceil - 2 \tag{5.16}$$
  $$\mu_{\mathrm{CZ}}(N\gamma^-) = 2\left\lfloor \frac{2N}{\ell p} \right\rfloor + 2\left\lfloor \frac{2N}{p} \right\rfloor + 2. \tag{5.17}$$

  At this point, one sees that contact homology cannot distinguish different values of ℓ. Indeed, checking the indices gives

$$CH_k(\Sigma(\ell p, p, 2, 2); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}^{p-1} & \text{for } k = 2 \\ \mathbb{Q}^p & \text{for } k \geq 4 \text{ even} \\ 0 & \text{else.} \end{cases}$$

- Each of these orbits gives two generators for SH_*^+, corresponding to minimum and maximum of a Morse function on S^1. Putting them in a table analogous to Table 5.1, we see that there are $p-1$ generators in degrees 2 and 3 and p generators in all higher degrees. Moreover, the generators in degrees $2N(\ell+1)$ and $2N(\ell+1)+1$ (for $N \in \mathbb{N}$) come exclusively from the orbits $N\gamma^{0,k}$, while in other degrees, they come from $N\gamma^+$ and $N\gamma^-$.

- Now, assume that p is even and that we use $\mathbb{Z}_2$-coefficients. Again, we get an essential ingredient from the full Morse–Bott setup of Section 3.2. After working through this setup (which now involves three critical submanifolds), one sees that there are only $p-1$ generators in degrees $2N(\ell+1)+j$ for $N \in \mathbb{N}$ and $j \in \{-1, 0, 1, 2\}$. Hence, there has to be a non-zero differential involving the generators from $N\gamma^{0,k}$.

- As in Sections 5.2 and 5.3, one can show that the Floer cylinders between orbits $N\gamma^+$ and $N\gamma^-$ come in pairs. Hence, there is no differential between these orbits over $\mathbb{Z}_2$-coefficients. (Here, the assumption that p is even is essential, otherwise, there is no $\mathbb{Z}_2$-symmetry.) As a consequence, $\operatorname{rank}(SH_k^+(W)) = p$ for $k = 4, 5, \ldots, 2\ell$ but $\operatorname{rank}(SH_k^+(W)) = p-1$ for $k = 2\ell+1, \ldots, 2\ell+4$ (and, by an analog of Lemma 5.4.2, equality hold in the latter identity, but this is not needed). By Remark 3.5.4, we get

Theorem 5.5.1. *For p even, the manifolds $\Sigma(\ell p, p, 2, 2), \ell \geq 1$ with their canonical contact structures are all diffeomorphic to $\#_{p-1} S^2 \times S^3$ and have the same contact homology, yet they are pairwise non-contactomorphic.*

A natural question is whether the same is true for p odd. The obvious thing to try is to apply the same strategy for the $\mathbb{Z}_p$ symmetry generated by

$$\psi(w_0, w_1, w_2, w_3) = (e^{2\pi i/p} w_0, e^{2\pi i/p} w_1, w_2, w_3)$$

and $\mathbb{Z}_p$ coefficients. On the face of it, everything seems to work fine. However, more checks need to be done, in particular about the orientations of contributing Floer cylinders. This could be the subject of future work.

6 Structural results and products

6.1 S^1-actions by Hamiltonian diffeomorphisms

In the previous two chapters, we studied only the additive structure of symplectic homology, not the pair-of-pants product from Section 3.4. This product structure will be the topic of this chapter. The main method for the computation will be to study the action of a loop of Hamiltonian diffeomorphisms on symplectic homology. Thus, we will generalize the results of [63, 59] to the setup of contact manifolds with periodic Reeb flow. Although Brieskorn manifolds are the main examples, the results of Section 6.1 hold in a more general context. For instance, links of other isolated hypersurface singularities would provide further examples.

6.1.1 Recollections from the closed case

We start by recalling some facts from [63] about the action of a loop of Hamiltonian diffeomorphisms on Floer homology on a closed symplectic manifold (M, ω). Let

$$g\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathrm{Ham}(M, \omega), \qquad t \mapsto g_t$$

be a loop of Hamiltonian diffeomorphisms based at $g_0 = \mathrm{id}$. Denote by

$$K^g\colon S^1 \times M \to \mathbb{R}$$

a Hamiltonian function that generates g, i.e. $\partial_t(g_t\cdot) = X_{K^g}(t, g_t\cdot)$.

In this text, we will only work with manifolds (M, ω) that satisfy $c_1(M)|_{\pi_2(M)} = 0$ and $\omega|_{\pi_2(M)} = 0$ (actually, in the non-closed case, ω will be an exact form). Therefore, the grading and the action functional will be well-defined and we do not need any cover of the loop space or Novikov coefficients (see [59, Section 2.4]).

The loop g acts on the loop space $C^\infty(S^1, M)$ by

$$(g \cdot \gamma)(t) = g_t(\gamma(t)).$$

Define the pullback (g^*H, g^*J) of a pair of Hamiltonian H and almost complex structure J as

$$\begin{aligned}(g^*H_t)(x) &= H_t(g_t(x)) - K^g_t(g_t(x)),\\ g^*J_t &= dg_t^{-1} \circ J_t \circ dg_t.\end{aligned}$$

Similarly, define the pushforward (g_*H, g_*J) as

$$(g_*H_t)(x) = \left((g^{-1})^*H_t\right)(x) = H_t(g_t^{-1}(x)) + K_t^g(g_t^{-1}(x)),$$
$$g_*J_t = (g^{-1})^*J_t = dg_t \circ J_t \circ dg_t^{-1}.$$

Lemma 6.1.1. *The action of g has the following properties:*

1. $g^*(d\mathcal{A}_H) = d\mathcal{A}_{g^*H}$*, where* $\mathcal{A}_H(\gamma) = \int_{D^2} \bar{\gamma}^*(\omega) - \int_{S^1} H_t(\gamma(t))dt$ *is the usual symplectic action functional. Equivalently,* $\mathcal{A}_{g^*H} = g^*\mathcal{A}_H$ *up to a constant (depending on the choice of additive constant for* K^g*).*

2. 1*-periodic orbits of* H *correspond bijectively to* 1*-periodic orbits of* g_*H *via* $x \mapsto g \cdot x$

3. *Floer trajectories satisfy the bijective correspondence*

$$\mathcal{M}(\gamma_+, \gamma_-; H, J) \xrightarrow{\cong} \mathcal{M}(g \cdot \gamma_+, g \cdot \gamma_-; g_*H, g_*J), \qquad u \mapsto g \cdot u,$$

and similarly for the moduli spaces appearing in the continuation maps.

See [63, Section 4] for the proof of Lemma 6.1.1. As for the grading, the Maslov index $I(g) \in \mathbb{Z}$ is defined as follows. For any contractible loop $\gamma \in C^\infty(S^1, M)$, choose a filling disk, which induces a symplectic trivialization

$$\tau_\gamma \colon \gamma^*(TM) \to S^1 \times (\mathbb{R}^{2n}, \omega_0)$$

of the pullback bundle $\gamma^*(TM)$. By [63, Lemma 2.2], $g \cdot \gamma$ is also contractible. Thus, $g(t)$ induces a loop of symplectomorphisms $\ell(t) \in \mathrm{Sp}(2n, \mathbb{R})$ by

$$\ell(t) = \tau_{g\gamma}(t) \circ dg_t(\gamma(t)) \circ \tau_\gamma(t)^{-1}.$$

Define the Maslov index $I(g) := \deg(\ell)$, where $\deg \colon H_1(\mathrm{Sp}(2n, \mathbb{R})) \to \mathbb{Z}$ is the isomorphism induced by the determinant on $U(n) \subset Sp(2n, \mathbb{R})$. By the assumption that $c_1(M)|_{\pi_2(M)} = 0$, this index is independent of the choice of filling disks. In fact, it is also independent of γ and only depends on the homotopy class of g_t in $\pi_0(\mathrm{Ham}(M, \omega))$. So

$$\mu(g \cdot \gamma) = \mu(\gamma) + 2I(g),$$

by one of the axioms of the Conley–Zehnder index.

Corollary 6.1.2. *The loop g_t induces a map on Floer homology*

$$S_g \colon HF_*(H) \to HF_{*+2I(g)}(g_*H).$$

As g^{-1} gives the inverse map, S_g is in fact an isomorphism.

The following proposition gives two further properties, whose proofs are a bit more involved (see [63, Sections 5 and 6]):

Proposition 6.1.3. *1. If g_t and $\tilde{g}_t$ are homotopic through a homotopy of loops of Hamiltonian diffeomorphisms g_t^r with $g_0^r = \mathrm{id}$ for all r, then*

$$S_g = S_{\tilde{g}} \colon HF_*(M,\omega) \to HF_{*+2I(g)}(M,\omega).$$

2. The isomorphism S_g and the pair-of-pants product $\cdot$ fulfill the relation

$$S_g(x \cdot y) = S_g(x) \cdot y.$$

6.1.2 S^1-actions by Hamiltonian loops on $\widehat{W}$

All of the statements of Section 6.1.1, including Proposition 6.1.3, admit a rather straightforward generalization to symplectic homology, provided that the filling W admits a Hamiltonian S^1-action. This generalization has been worked out by Ritter in [59]. Unfortunately, in many examples, one has a suitable S^1-action (e.g. by the Reeb flow) only on the contact manifold (and hence on its symplectization), but not on the filling. Indeed, the following lemma shows that in many cases, the S^1-action cannot be extended to a Liouville filling.

Lemma 6.1.4. *Let Σ be a contact manifold with periodic Reeb flow and W a Liouville filling of Σ (with arbitrary first Chern class) such that $SH(W)$ has infinite rank. Then the S^1-action on Σ by the Reeb flow does not extend to an S^1-action of Hamiltonian diffeomorphisms on W.*

Proof. This lemma is closely related to [59, Sections 1.6 and 1.7]. Assume that there is an S^1-action of Hamiltonian diffeomorphisms g_t on $\widehat{W}$ extending the Reeb flow. The corresponding Hamiltonian K_g has constant slope one on the symplectization part $\mathbb{R}_+ \times \Sigma \subset \widehat{W}$. Hence, with H_0 a generic Hamiltonian of slope $\epsilon > 0$ sufficiently small, we can successively define $H_i := g_*(H_{i-1})$, which gives a generic Hamiltonian of slope $i + \epsilon$.

Application of the S^1-action g_t to Floer homology gives isomorphisms

$$S_g \colon HF(H_i) \cong SH^{<i+\epsilon}(W) \xrightarrow{\cong} HF(H_{i+1}) \cong SH^{<i+\epsilon+1}(W). \tag{6.1}$$

In particular, this means that $SH^{<n+\epsilon}(W)$ has the same vector space dimension for any value of $n \in \mathbb{N}$. As $SH^{<\epsilon}(W)$ is isomorphic to the singular cohomology of W, this means

$$SH(W) = \varinjlim_i SH^{<i+\epsilon}(W)$$

has finite rank, giving a contradiction. □

Corollary 6.1.5. *Let Σ be a contact manifold with periodic Reeb flow and W a Liouville filling of Σ such that $c_1(W) = 0$ and $SH(W) \neq 0$. Assume that the index $I(g) \neq 0$. Then the S^1-action on Σ by the Reeb flow does not extend to an S^1-action of Hamiltonian diffeomorphisms on $\widehat{W}$.*

Proof. As $c_1(W) = 0$, symplectic homology has a well-defined integer grading. With the gradings made explicit, (6.1) becomes

$$S_g\colon SH_*^{<i+\epsilon}(W) \xrightarrow{\cong} SH_{*+2I(g)}^{<i+\epsilon+1}(W).$$

Hence, after taking direct limits, we get $SH_*(W) \cong SH_{*+2I(g)}(W)$. With $I(g) \neq 0$, this implies that $SH(W)$ is either zero or infinite-dimensional (and zero is excluded by assumption). The result now follows from Lemma 6.1.4. $\square$

Note that the assumption $SH \neq 0$ is necessary, since otherwise, the ball in $\mathbb{C}^n$ would provide a counterexample.

Lemma 6.1.4 and Corollary 6.1.5 can be applied directly to Brieskorn manifolds. For a Brieskorn manifold $\Sigma(a)$ with $\sum_j \frac{1}{a_j} \neq 1$, the index shift $I(g)$ is non-zero (see Section 6.2 for a formula for $I(g)$ for Brieskorn manifolds). As the standard filling W fulfills $c_1(W) = 0$ and $SH(W) \neq 0$ by [41, Theorem 6.3] (provided that $a_j \geq 2$ for all j), Corollary 6.1.5 tells us that the S^1-action g_t does not extend to W.

It is instructive to consider the example $\Sigma(2, \ldots, 2)$, which is contactomorphic to the unit cotangent bundle S^*S^n of S^n. The S^1-action by the Reeb flow agrees with the geodesic flow for the standard Riemannian metric on S^n. While the geodesic flow extends to the filling D^*S^n, the period varies, so this does not give an S^1-action. On the other hand, the normalized geodesic flow is an S^1-action, but it does not extend across the zero-section in D^*S^n.

For a Brieskorn manifold $\Sigma(a)$ with $\sum_j \frac{1}{a_j} = 1$, the index shift $I(g)$ is zero. However, if $\dim(\Sigma(a)) \geq 5$, Theorem 4.3.1 tells us that $\dim SH_*(W) = \infty$ in certain degrees (for any filling W). Hence, by Lemma 6.1.4, no S^1-equivariant Liouville filling can exist.

Because of this non-existence, the only way one can hope to apply the results of Section 6.1.1 to Brieskorn manifolds is to use a version of symplectic homology that can be defined purely on the symplectization. As seen in Section 3.5, this is possible for $\check{SH}$ in many cases.

6.1.3 S^1-actions by Hamiltonian loops on $\mathbb{R}_+ \times \Sigma$

Let Σ be any contact manifold for which the Reeb flow is periodic. After normalizing the period to one, the Reeb flow defines an S^1-action, which we denote by $e^{2\pi it}.z$, with $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Using this, we can define a loop of Hamiltonian diffeomorphisms

$$g_t\colon \mathbb{R}_+ \times \Sigma \to \mathbb{R}_+ \times \Sigma, \qquad g_t(r,z) = (r, e^{2\pi i\varphi(t)}.z). \tag{6.2}$$

on the symplectization. Here, $\varphi\colon [0,1] \to \mathbb{R}$ is any map with $\varphi(0) = 0$ and $\varphi(1) \in \mathbb{Z}$ (e.g. the identity map, though we will also need others below). The corresponding Hamiltonian function K_t^g on $\mathbb{R}_+ \times \Sigma$ is (up to a possibly time-dependent constant)

$$K_t^g(t,r,z) = \varphi'(t) \cdot r.$$

The following lemma gives a characterization of the Hamiltonians that can be written as g_*H for H constant and g as in (6.2).

Lemma 6.1.6. *A linear Hamiltonian G on $(\mathbb{R}_{>0} \times \Sigma, d(r\alpha))$ can be written as g_*H for $H \equiv$ constant and g as in* (6.2) *if and only if its slope $\sigma(t)$ depends only on t and fulfills $\int_0^1 \sigma(t)\, dt \in \mathbb{Z}$.*

Proof. For a loop of Hamiltonian diffeomorphisms $g_t(x, r)$ and $H \equiv$ constant,

$$g_*H_t = H_t + K_t^g = \text{constant} + \varphi'(t) \cdot r$$

has slope $\sigma(t) = \varphi'(t)$. The integral

$$\int_0^1 \sigma(t, r)\, dt = \int_0^1 \varphi'(t)\, dt$$

is the winding number of the loop $\varphi\colon S^1 \to S^1$, hence it has values in $\mathbb{Z}$.

Conversely, assume the slope $\sigma(t)$ of G fulfills $\int_0^1 \sigma(t)\, dt \in \mathbb{Z}$. Then, define

$$\varphi(t) := \int_0^t \sigma(\tau)\, d\tau,$$

which fulfills $\varphi(1) \in \mathbb{Z}$ and thus descends to a loop on S^1. The corresponding loop of Hamiltonian diffeomorphisms $g_t(x, r) = (e^{2\pi i \varphi(t)} \,.\, x, r)$ is associated with the Hamiltonian $K_t^g = \sigma(t) r$, which coincides (up to a constant) with G. $\square$

Note that for g_*H, with g_t as in (6.2), Lemma 3.5.1 cannot be applied directly, because g_*H is not constant on the negative end. However, the bijection of moduli spaces from Lemma 6.1.1 still holds, so the compactness of the moduli space $\mathcal{M}(\gamma_+, \gamma_-; H, J)$ induces compactness of the moduli space $\mathcal{M}(g\cdot\gamma_+, g\cdot\gamma_-; g_*H, g_*J)$. This gives a possible definition of $HF_*(g_*H)$, basically as the image of $HF_*(H)$ under S_g.

A problem with this definition is that one has to worry about compactness again for the continuation maps. We deal with this compactness issue in three steps:

- Given a continuation map $\Phi^{H\tilde{H}}$ between two Hamiltonians $H, \tilde{H}$ as in Figure 3.1, we get a continuation map between g_*H and $g_*\tilde{H}$ by using the fact that g gives a bijection of the moduli spaces involved. This means that we can define continuation maps for Hamiltonians within the family g_*H for a fixed g.

- In Lemma 6.1.7, we show that if g_1 is homotopic to g_2, we can define continuation maps between $(g_1)_*H$ and $(g_2)_*\tilde{H}$.

- In Proposition 6.1.8, we show that we get the same Floer homology as for g_*H if we make the Hamiltonian constant near the negative end of the symplectization. Therefore, this Floer homology can be used in the limit process to $\check{SH}(\Sigma)$.

Lemma 6.1.7. *Let g_1 and g_2 be homotopic through loops of Hamiltonian diffeomorphisms. Then, for $H, \tilde{H}$ two Hamiltonians as in Figure 3.1 (with $\tilde{H}$ steeper at ∞ than H) and $J, \tilde{J}$ regular almost complex structures, there exists a continuation map from $((g_1)_*H, (g_1)_*J)$ to $((g_2)_*\tilde{H}, (g_2)_*\tilde{J})$.*

Proof. By concatenation with g_2^{-1}, we can reduce the general case to the case $g_2 = \mathrm{id}$. Denote by $g_{s,t}$, $s \in \mathbb{R}$, the homotopy from g_t to id, and arrange it such that $g_{s,t} = \mathrm{id}$ for $s \geq 1$ and $g_{s,t} = g_t$ for $s \leq -1$. By the assumption on the slopes, there is a homotopy $(H_{s,t}, J_{s,t})$ from (H, J) to $(\tilde{H}, \tilde{J})$ that defines a continuation map. In particular, the moduli spaces

$$\mathfrak{M}(\gamma, \tilde{\gamma}; H_{s,t}, J_{s,t})$$

are compact for all H-orbits γ and $\tilde{H}$-orbits $\tilde{\gamma}$ with $\mu(\tilde{\gamma}) - \mu(\gamma) = 0$. Now, we can apply $g_{s,t}$ to its elements. As in Lemma 6.1.1 (and because $g_{s,t} = \mathrm{id}$ for $s \geq 1$), this gives a bijective correspondence between the moduli space above and

$$\mathfrak{M}\left(g_t \cdot \gamma, \tilde{\gamma}; (g_{s,t})_* H_{s,t}, (g_{s,t})_* J_{s,t}\right).$$

Hence, these moduli spaces are also compact and define a continuation map from (g_*H, g_*J) to $(\tilde{H}, \tilde{J})$. □

Proposition 6.1.8. *Denote by $[g_*H]_0$ the Hamiltonian which, up to a smoothing, equals g_*H on $(e^{-T}, \infty) \times \Sigma$ and is constant on $(0, e^{-T}) \times \Sigma$. Then, for T is sufficiently large (dependent on g_*H), there is a bijection between the zero-dimensional moduli spaces*

$$\mathcal{M}(\gamma_+, \gamma_-; g_*H, g_*J) \cong \mathcal{M}(\gamma_+, \gamma_-; [g_*H]_0, g_*J).$$

Proof. Denote by $u_1, \ldots, u_n$ the elements of the moduli space $\mathcal{M}(\gamma_+, \gamma_-; g_*H, g_*J)$. By compactness, they live in a compact region $[e^{-T}, e^T] \times \Sigma$ of the symplectization. We choose this value for T. Then, in this region, $g_*H = [g_*H]_0$, hence $u_1, \ldots, u_n$ are also elements of $\mathcal{M}(\gamma_+, \gamma_-; [g_*H]_0, g_*J)$.

Assume that the latter moduli space has some further element u'. By applying g_t^{-1}, this gives an element

$$g^{-1}u' \in \mathcal{M}(g^{-1}\gamma_+, g^{-1}\gamma_-; g^*[g_*H]_0, J).$$

Since $g^*[g_*H]_0 = H = \text{constant on } (e^{-T}, \delta) \times \Sigma$, we can use a neck-stretching operation there, as in the proof of Corollary 3.5.2. So we insert a piece of the symplectization near $\{e^{-T}\} \times \Sigma$. Under this operation, the Floer cylinder $g^{-1}u'$ converges to a broken cylinder as in Figure 3.2. However, as in the proof of Lemma 3.5.1, the index condition on the Reeb orbits makes sure that the cylinder is in fact unbroken. This implies that $g^{-1}u'$ was in fact a Floer cylinder for the original Hamiltonian H, hence u' was a Floer cylinder for the Hamiltonian g_*H. This contradicts the assumption that u' was not among the elements $u_1, \ldots, u_n$.

Thus, all elements of the moduli space $\mathcal{M}(\gamma_+, \gamma_-; [g_*H]_0, g_*J)$ are already contained in $\mathcal{M}(\gamma_+, \gamma_-; g_*H, g_*J)$, which gives the bijection. □

Together, Lemma 6.1.7 and Proposition 6.1.8 show that Hamiltonians g_*H (with H as in Figure 3.1) can be used in the definition of $\check{SH}_*(\Sigma)$. Indeed, by Lemma 6.1.7, we can arrange that the slope of g_*H is time-independent. Further, we can use continuation maps from g_*H to $g_*\tilde{H}$ such that the slopes μ_1, μ_2 grow arbitrarily large, while δ remains small and the slope of $g_*\tilde{H}$ at the negative end of the symplectization stays constant. This makes sure any orbits created in the transition from $g_*\tilde{H}$ to $[g_*\tilde{H}]_0$ have action outside of the fixed action window (a, b). Hence, the generators of $HF^{(a,b)}([g_*\tilde{H}]_0)$ are the same as those of $HF^{(a,b)}(g_*\tilde{H})$, and Proposition 6.1.8 shows that the differential agrees as well. As $[g_*\tilde{H}]_0$ is constant at the negative end, it is clear that it can be used in to define $\check{SH}_*(\Sigma)$.

The statements of Lemma 6.1.1 and Corollary 6.1.2 hold as in the closed case.

Example 6.1.9. Take the specific loop of Hamiltonian diffeomorphisms

$$g_t(r, z) = (r, e^{2\pi i t}.z), \tag{6.3}$$

i.e. the case $\varphi = \mathrm{id}_{[0,1]}$, and normalize the corresponding Hamiltonian to

$$K_t^g(t, r, z) = r - 1.$$

Then, for H_t as in Figure 3.1 (only dependent on the radial coordinate r), the Hamiltonian

$$g_*H(t, r, z) = H_t(r) + K_t^g(t, r, z) = H_t(r) + (r - 1)$$

is again normalized such that $g_*H = -\varepsilon$ at $r = 1$. Thus, except for the non-zero slope at the negative end (which equals one), g_*H looks as in Figure 3.1, but with μ_1 decreased and μ_2 increased by one, respectively. As for the action, first note that because of the chain rule

$$\frac{d}{dt}(g_t\gamma(t)) = (g_t)_*\gamma'(t) + \frac{d}{d\tau}\Big|_{\tau=t} g_\tau(\gamma(t))$$

and $g_t^*\alpha = \alpha$, we get that

$$\int_{S^1} (g_t\gamma)^*\alpha = \int_{S^1} \gamma^*\alpha + 1.$$

For the second term,

$$-\int_{S^1} (g_*H)(g_t\gamma(t))\,dt = -\int_{S^1} H(\gamma(t))\,dt - \underbrace{\int_{S^1} K_t^g(\gamma(t))\,dt}_{\approx 0},$$

where the second summand vanishes up to an arbitrary small error due to the the smoothing of H. Hence, except for this small error,

$$\mathcal{A}_{g_*H}(g \cdot \gamma) = \mathcal{A}_H(\gamma) + 1, \tag{6.4}$$

which gives an isomorphism

$$S_g\colon HF^{(a,b)}(H_{\mu_1,\mu_2}) \xrightarrow{\cong} HF^{(a+1,b+1)}(H_{\mu_1-1,\mu_2+1}).$$

Taking the direct limits $\mu_1, \mu_2 \to \infty$, this induces an isomorphism

$$S_g\colon S\check{H}^{(a,b)}(\Sigma) \xrightarrow{\cong} S\check{H}^{(a+1,b+1)}(\Sigma), \tag{6.5}$$

and, after taking the additional limit from (3.6), an isomorphism on $S\check{H}(\Sigma)$, which we still denote by S_g.

Lemma 6.1.10. *Assume that $S\check{H}(\Sigma) \neq 0$. Then, the elements $S_g^k(1)$ for $k \in \mathbb{Z}$ are linearly independent. More generally, if $\gamma \neq 0 \in S\check{H}(\Sigma)$, then the elements $S_g^k(\gamma)$ for $k \in \mathbb{Z}$ are linearly independent.*

Proof. As $S\check{H}(\Sigma) \neq 0$, the unit $1 \in S\check{H}(\Sigma)$ is non-zero, hence the first claim is a special case of the second. For $\gamma \neq 0$, it follows from the fact that S_g is an isomorphism that all the elements $S_g^k(\gamma)$ for $k \in \mathbb{Z}$ are non-zero.

If $I(g) \neq 0$, linear independence follows immediately from the fact that these elements all have different degrees. For the case $I(g) = 0$, we have to use a different argument involving the action filtration. Namely, for any element $x \in S\check{H}(\Sigma)$, define the quantity

$$a(x) := \inf\left\{a \in \mathbb{R} \mid x \in \operatorname{im}\left(\iota\colon S\check{H}^{(-\infty,a)}(\Sigma) \to S\check{H}(\Sigma)\right)\right\} \in [-\infty, \infty).$$

Note that $a(x) = -\infty$ only for $x = 0 \in S\check{H}(\Sigma)$. For the unit, $a(1) = 0$, and

$$a(x+y) \leq \max\{a(x), a(y)\} \tag{6.6}$$

for any $x, y \in S\check{H}(\Sigma)$. Moreover, by (6.4),

$$a(S_g(\gamma)) = a(\gamma) + 1,$$

hence $a(S_g^k(\gamma)) = a(\gamma) + k$. Together with (6.6), this implies linear independence. □

If $S\check{H}(\Sigma) \neq 0$, Lemma 6.1.10 implies that the ring of Laurent polynomials $\mathbb{Z}_2[t, t^{-1}]$ injects into $S\check{H}(\Sigma)$. Moreover, the multiplication

$$\mathbb{Z}_2[t, t^{-1}] \times S\check{H}(\Sigma) \to S\check{H}(\Sigma), \qquad (t^k, \gamma) \mapsto S_g^k(\gamma) \tag{6.7}$$

gives $S\check{H}(\Sigma)$ the structure of a module over the ring $\mathbb{Z}_2[t, t^{-1}]$. Lemma 6.1.10 implies that this module is torsion-free.

Morally speaking, we should think of this as a free module. However, there is a subtle issue coming from the distinction between Laurent polynomials and Laurent series. Consider first the case $I(g) = 0$. Then, the elements $S_g^k(\gamma)$ for $k \in \mathbb{Z}$ all live in the same degree. Hence, because of the inverse limit in (3.6), infinite sums of the form

$$\sum_{k=-\infty}^{N} \lambda_k S_g^k(\gamma), \qquad \lambda_k \in \mathbb{Z}_2$$

are included in $\check{SH}_*(\Sigma)$. Thus, $\check{SH}(\Sigma)$ is also a module over the ring $\Bbbk := \mathbb{Z}_2((t^{-1}))$ of semi-infinite Laurent series. In fact, $\Bbbk$ is a field, so $\check{SH}(\Sigma)$ is a free module (i.e. a vector space) over $\Bbbk$, but not over $\mathbb{Z}_2[t, t^{-1}]$.

By contrast, for $I(g) \neq 0$, we will see that $\check{SH}(\Sigma)$ is in fact a free module over the ring $\mathbb{Z}_2[t, t^{-1}]$. We start with a simple chain-level observation.

The chain complex

$$\check{SC}(\Sigma) := \bigoplus_{k \in \mathbb{Z}} \check{SC}_k(\Sigma)$$

is defined analogous to (3.5) and (3.6), just without taking homology. By [16, Proposition 3.4], the homology of $\check{SC}(\Sigma)$ is isomorphic to $\check{SH}(\Sigma)$, i.e. taking homology commutes with taking the limits.

Lemma 6.1.11. *For $I(g) \neq 0$, the $\mathbb{Z}_2[t, t^{-1}]$-module $\check{SC}(\Sigma)$ (with the multiplication* (6.7)*) is finitely generated.*

Proof. Let $\gamma_0 = 1, \gamma_1, \dots, \gamma_N$ be all generators in the action window $(-\epsilon, 1-\epsilon)$ with ϵ sufficiently small (i.e. constant orbits and positive Reeb orbits of length < 1). By the discreteness of $\operatorname{Spec}(\Sigma)$, there are only finitely many of them. As the chain complex is periodic and S_g maps the generators of one period to the next one, all generators of $\check{SC}(\Sigma)$ are of the form

$$S_g^j(\gamma_i) \quad \text{for some} \quad j \in \mathbb{N},\ i \in \{0, \dots, N\}.$$

Moreover, since $I(g) \neq 0$, there is at most one j such that $S_g^j(\gamma_i)$ has degree k. By definition, elements of $\check{SC}(\Sigma) = \oplus_{k \in \mathbb{Z}} \check{SC}_k(\Sigma)$ are supported only in finitely many degrees. Hence, any $x \in \check{SC}(\Sigma)$ can be written as

$$x = \sum_{\text{finite}} S_g^j(\gamma_i),$$

meaning that $\check{SC}(\Sigma)$ is a module over $\mathbb{Z}_2[t, t^{-1}]$ with generators $\gamma_0, \dots, \gamma_N$. $\square$

Remark 6.1.12. For $I(g) = 0$, a similar proof shows that $\check{SC}(\Sigma)$ (and thus $\check{SH}(\Sigma)$) is a finite-dimensional vector space over $\Bbbk = \mathbb{Z}_2((t^{-1}))$.

To go further with the case $I(g) \neq 0$, we can make use of some facts from algebra. First, as a localization of the principal ideal domain $\mathbb{Z}_2[t]$, the ring $\mathbb{Z}_2[t, t^{-1}]$ is itself a principal ideal domain ([43, Exercise II.4]). Over such rings, any submodule of a finitely generated module is itself finitely generated (see [43, Corollary III.7.2]). Hence, $\ker(\partial) \subset \check{SC}(\Sigma)$ is a finitely generated $\mathbb{Z}_2[t, t^{-1}]$-module. The same is (trivially) true for quotients, thus $\check{SH}(\Sigma)$ is in fact a finitely generated $\mathbb{Z}_2[t, t^{-1}]$-module.

It follows from the structure theorem for finitely generated modules over a principal ideal domain (see e.g. [61, Theorem 9.3] for the version we need) that any finitely generated, torsion-free module over a principal ideal domain is free. Hence $\check{SH}(\Sigma)$ is a free and finitely generated $\mathbb{Z}_2[t, t^{-1}]$-module. Even better, the dimension (i.e. the number of generators) of $\check{SH}(\Sigma)$ is bounded by the dimension of $\check{SC}(\Sigma)$, which is given by the number of generators in the action window $(-\epsilon, 1-\epsilon)$. Indeed, by [43, Theorem III.7.1], the dimension can only decrease when taking submodules, and by the proof of [61, Theorem 9.3], the same is true for quotients.

We sum up this discussion in the following theorem:

Theorem 6.1.13. *Assume that Σ has periodic Reeb flow and is index-positive. If $I(g) \neq 0$ for g_t as in* (6.3)*, then $\check{SH}(\Sigma)$ is a free and finitely generated module over $\mathbb{Z}_2[t, t^{-1}]$, with the module structure from* (6.7)*. If $I(g) = 0$, then $\check{SH}(\Sigma)$ is a finite-dimensional vector space over $\Bbbk = \mathbb{Z}_2((t^{-1}))$.*

In both cases, the dimension is bounded by the number of generators of $\check{SC}^{(-\epsilon,1-\epsilon)}$.

Remark 6.1.14. If one prefers to work over the ring of Laurent series for the case $I(g) \neq 0$, one can define a variant of $\check{SH}(\Sigma)$, namely

$$\widetilde{SH}(\Sigma) := \varinjlim_{b} \varprojlim_{a} \check{SH}^{(a,b)}(\Sigma).$$

The difference with (3.6) is that here, we do not fix the grading, so we allow for any infinite sum of terms whose actions go to $-\infty$. Then, similarly to the case $I(g) = 0$, $\widetilde{SH}(\Sigma)$ is a finite-dimensional vector space over $\Bbbk = \mathbb{Z}_2((t^{-1}))$.

6.1.4 Homotopy invariance

This section and the next one are devoted to stating, proving and using the statements of Proposition 6.1.3 in the current setup. Σ is assumed to be index-positive.

Proposition 6.1.15. *Let g_t and $\tilde{g}_t$ be homotopic through a homotopy of loops of Hamiltonian diffeomorphisms $g_{t,r}$ with $g_{0,r} = \mathrm{id}$ for all r. Then, the isomorphisms*

$$S_g, S_{\tilde{g}} \colon \check{SH}(\Sigma) \xrightarrow{\cong} \check{SH}(\Sigma)$$

coincide.

Proof. The proof follows the lines of [63, Section 5] and is a variation of the standard "homotopy of homotopies" argument, which is used in Floer homology to show that continuation maps do not depend on the chosen homotopy (H_s, J_s). We omit some of the details that do not differ from the closed case.

First, note that S_g satisfies the concatenation property

$$S_{g^1} \circ S_{g^2} = S_{g^1 \# g^2}$$

for two loops g^1_t, g^2_t of Hamiltonian diffeomorphisms. Therefore, it suffices to prove the proposition in the special case $g_{t,0} = \tilde{g}_t \equiv \mathrm{id}$.

Denote by $\tilde{H}$ a Hamiltonian as in Figure 3.1 such that the slopes of $(g_r)_*\tilde{H}$ at infinity are steeper than those of H for all r. Further, let (H', J') be a regular homotopy from (H, J) to $((g_1)_*\tilde{H}, (g_1)_*J)$ and (H'', J'') a regular homotopy from (H, J) to $(\tilde{H}, J)$.

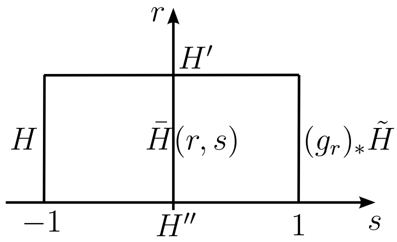

Figure 6.1: Visualization of a deformation of homotopies

Definition 6.1.16. A *deformation of homotopies* is pair of a function $\bar{H} \in C^\infty([0,1] \times \mathbb{R} \times S^1 \times \Sigma, \mathbb{R})$ and a family of ω-compatible almost complex structures $(\bar{J}_{r,s,t})$ parametrized by $(r, s, t) \in [0,1] \times \mathbb{R} \times S^1$ such that

$$
\begin{aligned}
&\bar{H}(r,s,t,x) = H(t,x), && \bar{J}_{r,s,t} = J_t && \text{for} \quad s \leq -1,\\
&\bar{H}(r,s,t,x) = \left((g_r)_*\tilde{H}\right)(t,x), && \bar{J}_{r,s,t} = (g_r)_*J_t && \text{for} \quad s \geq 1,\\
&\bar{H}(0,s,t,x) = H''(s,t,x), && \bar{J}_{0,s,t} = J''_t && \text{and}\\
&\bar{H}(1,s,t,x) = H'(s,t,x), && \bar{J}_{1,s,t} = J'_t. &&
\end{aligned}
$$

See Figure 6.1 for a visualization. By Lemma 6.1.6, we can choose a deformation of homotopies $(\bar{H}, \bar{J})$ such that on the negative end of the symplectization, $\bar{H}$ is of the form g_*H for some g as in (6.2) and H constant. This makes sure that Floer cylinders for $\bar{H}$ do not escape to the negative end of the symplectization, as in Lemma 3.5.1.

For γ_- an H-orbit and γ_+ an $\tilde{H}$-orbit, define the moduli space

$$\mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})$$

as the set of pairs $(r, u) \in [0,1] \times C^\infty(\mathbb{R} \times S^1, \mathbb{R}_+ \times \Sigma)$ satisfying

$$\partial_s u + \bar{J}_{r,s,t}(u(s,t))\left(\partial_t u - X_{\bar{H}}(r,s,t,u(s,t))\right) = 0 \tag{6.8}$$

and the asymptotic conditions

$$\lim_{s \to -\infty} u(s) = \gamma_-, \quad \lim_{s \to \infty} u(s) = g_r(\gamma_+).$$

For a sufficiently generic choice of $(\bar{H}, \bar{J})$, this is a smooth manifold of dimension

$$\dim \mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J}) = \mu(\gamma_+) - \mu(\gamma_-) + 1.$$

Its boundary consists of solutions of (6.8) with $r = 0$ or $r = 1$. In these cases, equation (6.8) becomes

$$\partial_s u + \bar{J}''_{s,t}(u(s,t))\,(\partial_t u - X_{H''}(s,t,u(s,t))) = 0 \tag{6.9}$$

and

$$\partial_s u + \bar{J}'_{s,t}(u(s,t))\,(\partial_t u - X_{H'}(s,t,u(s,t))) = 0, \tag{6.10}$$

respectively. These are precisely the equations for the continuation maps corresponding to (H'', J'') and (H', J') respectively.

Lemma 6.1.17. *(i) If $\mu(\gamma_+) = \mu(\gamma_-) - 1$, the moduli space $\mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})$ is a finite set.*

(ii) If $\mu(\gamma_+) = \mu(\gamma_-) = k$, $\dim \mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J}) = 1$, and there is a smooth compactification $\overline{\mathcal{M}}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})$ whose boundary consists, in addition to $\partial\mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})$, of elements of

$$\mathcal{M}^h(\gamma_+, \gamma; \bar{H}, \bar{J}) \times (\mathcal{M}(\gamma, \gamma_-; \tilde{H}, J)/\mathbb{R}) \tag{6.11}$$

for γ an $\tilde{H}$-orbit of index $\mu(\gamma) = k+1$ and

$$(\mathcal{M}(\gamma_+, \gamma'; H, J)/\mathbb{R}) \times \mathcal{M}^h(\gamma', \gamma_-; \bar{H}, \bar{J}) \tag{6.12}$$

for γ' an H-orbit of index $\mu(\gamma) = k-1$.

See [63] and its references for the proof of Lemma 6.1.17. By this compactness result, it makes sense to define a map

$$\begin{aligned} h_k^{\bar{H},\bar{J}} \colon CF_k(H) &\longrightarrow CF_{k+1}(\tilde{H}) \\ \gamma_+ &\longmapsto \sum_{\substack{\gamma_- \\ \mu(\gamma_-)=k+1}} \#\mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})\,\gamma_-. \end{aligned}$$

Lemma 6.1.18. *For all k,*

$$\partial_{k+1}^{\tilde{H},J} \circ h_k^{\bar{H},\bar{J}} + h_{k-1}^{\bar{H},\bar{J}} \circ \partial_k^{H,J} = \Phi_k^{H',J'} \circ S_{g_{t,1}} - \Phi_k^{H'',J''},$$

where, as before, Φ_k denotes the continuation map.

Proof. By definition,

$$\begin{aligned} \Phi_k^{H',J'} \circ S_{g_{t,1}}(\gamma_+) &= \sum_{\gamma_-} \#\left(\mathfrak{M}^\Phi(g_1(\gamma_+), \gamma_-; H', J')\right)\gamma_- \\ &\overset{(6.10)}{=} \sum_{\gamma_-} \#\left\{(1,u) \in \mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})\right\}\gamma_-. \end{aligned}$$

As (6.9) is the Floer equation for the continuation map $\Phi^{H'',J''}$, this implies

$$\left(\Phi_k^{H',J'} \circ S_{g_{t,1}} - \Phi_k^{H'',J''}\right)(\gamma_+) = \sum_{\gamma_-} \#\left(\partial\mathcal{M}^h(\gamma_+, \gamma_-; \bar{H}, \bar{J})\right)\gamma_-.$$

Moreover, since $\overline{\mathcal{M}}^h(\gamma_+,\gamma_-;\bar{H},\bar{J})$ is a compact 1-dimensional manifold with boundary, its boundary has an even number of points. Hence, for $\mathbb{Z}_2$-coefficients, we can replace $\#\left(\partial\mathcal{M}^h(\gamma_+,\gamma_-;\bar{H},\bar{J})\right)$ with the contributions from (6.11) and (6.12). These equations count contributions from the composition of h_k with the differential, thus giving

$$\left(\Phi_k^{H',J'}\circ S_{g_{t,1}}-\Phi_k^{H'',J''}\right)(\gamma_+)=\left(\partial_{k+1}^{\tilde{H},J}\circ h_k^{\bar{H},\bar{J}}+h_{k-1}^{\bar{H},\bar{J}}\circ\partial_k^{H,J}\right)(\gamma_+),$$

which proves the lemma. □

The statement of Lemma 6.1.18 means that $\Phi_k^{H',J'}\circ S_{g_{t,1}}$ is chain homotopic to a continuation map. Thus, up to continuation maps, $S_{g_{t,1}}$ is the identity map on Floer homology. □

6.1.5 Application to product computations

Proposition 6.1.19. *Assume that Σ is product-index-positive. The isomorphism $S_g\colon \check{SH}_*(\Sigma)\to\check{SH}_{*+2I(g)}(\Sigma)$ satisfies the relation*

$$S_g(x\cdot y)=S_g(x)\cdot y \tag{6.13}$$

with the product on $\check{SH}(\Sigma)$.

Proof. Having established Proposition 6.1.15, the proof is essentially the same as in [59, Theorem 23] and [63, Proposition 6.3]. Namely, by Proposition 6.1.15, we can homotope g_t to another loop of Hamiltonian diffeomorphisms satisfying $g_t=\mathrm{id}$ for $t\in(-\epsilon,\epsilon)$ for some $0<\epsilon<1/4$.

For the domain of the pair-of-pants, we take the specific surface $\mathbb{R}\times S^1\setminus\{(0,0)\}$. Choose a cylindrical parametrization (s,t) near $\{0,0\}$, e.g.

$$e(s,t)=\left(\frac{1}{4}e^{-2\pi s}\cos(2\pi t),\frac{1}{4}e^{-2\pi s}\sin(2\pi t)\right)$$

with $s\in(-\infty,0)$. Let $\gamma_+,\gamma_0,\gamma_-$ be 1-periodic orbits of H_+,H_0,H_-, respectively, and choose $\beta,H_{\mathcal{P}}$ and $J_{\mathcal{P}}$ as in Section 3.4. Then, the product counts maps

$$u\colon\mathbb{R}\times S^1\to\mathbb{R}_+\times\Sigma$$

satisfying

$$(du-X_H\otimes\beta)^{0,1}=0,$$

with the asymptotic conditions

$$\lim_{s\to\pm\infty}u(s,t)=\gamma_\pm(t),$$

at the punctures $\pm\infty$ and

$$\lim_{s\to\infty}u\circ e(s,t)=\gamma_0(t)$$

at the puncture $(0,0)$. Since $g_t = \mathrm{id}$ in a neighborhood of $t = 0$, we note that $g \cdot u$ satisfies the asymptotic conditions

$$\lim_{s\to\pm\infty}(g\cdot u)(s,t) = (g\cdot\gamma_\pm)(t) \quad \text{and} \quad \lim_{s\to\infty}(g\cdot u)\circ e(s,t) = \gamma_0(t).$$

Hence, the assignment $u \mapsto g \cdot u$ gives a bijection of moduli spaces

$$\mathcal{M}(\gamma_+,\gamma_0,\gamma_-;\beta,H_{\mathcal{P}},J_{\mathcal{P}}) \cong \mathcal{M}(g\cdot\gamma_+,\gamma_0,g\cdot\gamma_-;\beta,g_*H_{\mathcal{P}},g_*J_{\mathcal{P}}).$$

By an analog of Proposition 6.1.8 for pairs-of-pants (which holds by the same proof), the moduli space on the right-hand side does not change if we cut off $g_*H_{\mathcal{P}}$ to a constant near the negative end of the symplectization. Therefore, the elements of the right-hand side are counted by the product

$$HF_*(g_*H_+) \times HF_*(H_0) \to HF_*(g_*H_-)$$

of the elements $S_g(\gamma_+) = g\cdot\gamma_+$, γ_0 and $S_g(\gamma_-) = g\cdot\gamma_-$, while the elements of the left-hand side are counted by the product

$$HF_*(H_+) \times HF_*(H_0) \to HF_*(H_-)$$

of the elements γ_+, γ_0 and γ_-. Hence, when taking direct limits to pass to $\check{SH}_*(\mathbb{R}_+\times\Sigma)$, this bijection of moduli spaces gives

$$\langle S_g(\gamma_+)\cdot\gamma_0,\ S_g(\gamma_-)\rangle = \langle\gamma_+\cdot\gamma_0,\ \gamma_-\rangle.$$

Since the right-hand side is the same as $\langle S_g(\gamma_+\cdot\gamma_0), S_g(\gamma_-)\rangle$, this implies (6.13). $\square$

As mentioned in Section 3.3, the ring structure on $\check{SH}(\Sigma)$ has a unit, coming from the generator of $H^0(\Sigma)$. Hence, we can use (6.13) with $x = 1$ being the unit and $y = \gamma$ some other generator, getting

$$S_g(\gamma) = S_g(1\cdot\gamma) = S_g(1)\cdot\gamma. \tag{6.14}$$

Specifically, choose g_t to be the simple loop (6.3) from Example 6.1.9. For this case, define

$$s := S_g(1) \in \check{SH}_{n+2I(g)}(\Sigma).$$

Hence,

$$S_g(\gamma) = s\cdot\gamma, \tag{6.15}$$

and similarly $S_g^{-1}(\gamma) = s^{-1}\cdot\gamma$, where s^{-1} is the inverse of s in the ring $\check{SH}(\Sigma)$.

Corollary 6.1.20. *The isomorphism S_g is simply (left-) multiplication by the element $s \in \check{SH}(\Sigma)$. In particular, the structure of $\check{SH}(\Sigma)$ as a module over the* *ring of Laurent polynomials from* (6.7) *is given by*[1]

$$\mathbb{Z}_2[s, s^{-1}] \times \check{SH}(\Sigma) \to \check{SH}(\Sigma), \qquad (s^k, \gamma) \mapsto s^k \cdot \gamma.$$

While the proof given above, specifically Proposition 6.1.19, was given under the assumption that Σ is product-index-positive, it turns out that, at least if $I(g) \neq 0$, a weaker assumption suffices:

Proposition 6.1.21. *Assume that Σ is simply-connected,*[2] *admits a Liouville filling W with $c_1(W) = 0$ and fulfills $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits c (i.e. it fulfills condition (i) in the definition of index-positivity). Assume further that $I(g) \neq 0$ (for g as in* (6.3)*). Then, although one needs the filling W to the define the product structure, equation* (6.15) *(and hence Corollary 6.1.20) holds as before.*

Proof. As Σ is index-positive, both S_g and $s := S_g(1)$ are still well-defined. By Lemma 3.5.8, the product $\gamma_1 \cdot \gamma_2$ can be computed in the symplectization if (3.14) holds. As $\mu(c) > 3 - n$ for all Reeb orbits c, this is guaranteed if $|\mu(\gamma_1)| \geq n$ and $|\mu(\gamma_2)| \geq n$.

Therefore, the proof of (6.13) goes through as before if

$$|\mu(x)| \geq n, \quad |\mu(y)| \geq n \quad \text{ and } \quad |\mu(S_g(x))| \geq n.$$

Recall that the unit has degree n, so we can use it for x or y. Without loss of generality, assume that $I(g) > 0$ (otherwise replace g by its inverse). Then, $\mu(s^k) \geq n$ for all $k \geq 0$, so we can use (6.13) inductively to get

$$S_g^k(1) = s^k \qquad \forall k \geq 0. \tag{6.16}$$

The next step is to see that s^N is invertible, at least for N sufficiently large. Denote by g^{-N} the $(-N)$-fold cover of g and define $x := S_{g^{-N}}(1)$. For N sufficiently large, $\mu(x) \leq -n$, so we can use (6.13) to get

$$x \cdot s^N = S_{g^{-N}}(1) \cdot s^N = S_{g^{-N}}(1 \cdot s^N) = S_g^{-N}(s^N) = 1,$$

where the last step follows from (6.16). Hence, $x = (s^N)^{-1}$. Now, for any generator $\gamma \in \check{SH}(\Sigma)$, choose N sufficiently large so that $\mu((s^N)^{-1} \cdot \gamma) < -n$. Then, we can calculate

$$\begin{aligned} S_g(\gamma) &= S_g\left(s^N \cdot (s^N)^{-1} \cdot \gamma\right) \\ &= S_g(s^N) \cdot \left((s^N)^{-1} \cdot \gamma\right) \\ &\overset{(6.16)}{=} s^{N+1} \cdot (s^N)^{-1} \cdot \gamma \\ &= s \cdot \gamma \end{aligned}$$

[1] We renamed the variable of the Laurent polynomials from t to s to emphasize that s is itself an element and $\mathbb{Z}_2[s, s^{-1}]$ is a subset of $\check{SH}(\Sigma)$.

[2] Again, the assumption $\pi_1(\Sigma) = 0$ is used only to have a grading of $\check{SH}$ compatible with the product structure and the broken curve in Figure 3.4, see Remark 3.5.3.

which finishes the proof. □

To better understand the structure of $\check{SH}(\Sigma)$, let us use the chain complex from the Morse–Bott setup. S_g maps the whole critical submanifold $\mathcal{N}_T$ (of Reeb orbits of length T) to $\mathcal{N}_{T+1}$. Putting the same Morse function on these manifolds, we see that each generator from $\mathcal{N}_T$ gets mapped under S_g to the corresponding generator on $\mathcal{N}_{T+1}$. Equation (6.15) tells us that this mapping is done by the pair-of-pants product with s. In formulas, this means

$$s \cdot [\mathcal{N}_T, \eta] = S_g([\mathcal{N}_T, \eta]) = [\mathcal{N}_{T+1}, \eta], \tag{6.17}$$

where η is a critical point of a Morse function on $\mathcal{N}_T \cong \mathcal{N}_{T+1}$ and $[\mathcal{N}_T, \eta]$ denotes the homology class represented by $(\mathcal{N}_T, \eta)$. As the unit of $\check{SH}(\Sigma)$ (which corresponds to the unit of $H^*(\Sigma)$ under the isomorphism $\check{SH}^{(-\epsilon,\epsilon)} \cong H^*(\Sigma)$) is given by the maximum[3] on $\mathcal{N}_0 \cong \Sigma$, equation (6.17) says in particular that

$$s = S_g([\mathcal{N}_0, \max]) = [\mathcal{N}_1, \max]. \tag{6.18}$$

The following theorem summarizes the results of this section:

Theorem 6.1.22. *Assume that Σ has periodic Reeb flow and satisfies one on the following:*

- *Σ is product-index-positive, or*
- *Σ fulfills $\pi_1(\Sigma) = 0$, $\mu_{CZ}(c) > 3 - n$ for all Reeb orbits c and admits a Liouville filling W with $c_1(W) = 0$.*

Let g_t be defined as in (6.3) *and assume $I(g) \neq 0$. Then, the multiplication*

$$\mathbb{Z}_2[s, s^{-1}] \times \check{SH}(\Sigma) \to \check{SH}(\Sigma), \qquad (s^k, x) \mapsto s^k \cdot x,$$

where $s = S_g(1)$ and $\cdot$ denotes the pair-of-pants product, gives $\check{SH}(\Sigma) \cong RFH(W)$ the structure of a free and finitely generated module over $\mathbb{Z}_2[s, s^{-1}]$. The generators of this module are the unit and possibly other finite linear combinations of Reeb orbits. In particular, $\check{SH}(\Sigma)$ is finitely generated as an algebra.

If $I(g) = 0$ and Σ is product-index-positive, the same holds true if we replace $\mathbb{Z}_2[s, s^{-1}]$ by $\mathbb{Z}_2((s^{-1}))$.

This theorem also includes the (uninteresting) case when $\check{SH}(\Sigma) = 0$, as e.g. for the standard contact sphere. Note that by [58, Theorem 13.3], $\check{SH}(\Sigma) \cong RFH(W;\mathbb{Z}_2) \neq 0$ is equivalent to $SH(W;\mathbb{Z}_2) \neq 0$.

Unfortunately, this theorem does not necessarily give the complete product structure of $\check{SH}(\Sigma)$. Indeed, the module generators might not be algebraically independent (one might be the product of two others), or even the generator s might be the square (or some higher power) of some other generator.

[3]Whether it is the minimum or the maximum is a matter of convention.

6.1.6 Back to usual symplectic homology

Finally, we can use Theorem 6.1.22 to gain some information about the usual symplectic homology of some Liouville filling W of Σ with $c_1(W) = 0$. The long exact sequence constructed in [17] gives in particular a map

$$f\colon SH(W) \to \check{SH}(\Sigma). \tag{6.19}$$

This map is constructed as follows: The Floer homology of a Hamiltonian on W as in Figure 3.1 with the action window $(-\infty, b)$ is isomorphic to $SH^{(-\infty,b)}(W)$. The Floer homology $HF^{(a,b)}(H)$ used in the definition of $\check{SH}$ arises from dividing out the chains of action less than a (provided that μ_1 is sufficiently large). Thus, the map (6.19) is just the quotient map $HF^{(-\infty,b)}(H) \to HF^{(a,b)}(H)$ after taking the appropriate limits.

The next lemma is a special case of [18, Theorem 10.2(e)].

Lemma 6.1.23. *The maps f respects the product structures,*

$$f(x \cdot y) = f(x) \cdot f(y).$$

Proof. The product on $\check{SH}$ is constructed by applying the limits (3.5) and (3.6) (in the correct order) to the product

$$HF^{[a,b)}(H) \times HF^{[a,b)}(H) \to HF^{[a+b,2b)}(2H). \tag{6.20}$$

But (6.20) also defines the product on $SH(W)$ of any elements that survive the quotient map $HF^{(-\infty,b)}(H) \to HF^{(a,b)}(H)$. □

One should think of the map f as dividing out a part of the negative symplectic homology $SH^-_*(W) \cong H^{n-*}(W)$. This can be seen most easily from the long exact sequence

$$\cdots \longrightarrow SH^{-k} \xrightarrow{h} SH_k \xrightarrow{f} \check{SH}_k \longrightarrow SH^{-(k-1)} \longrightarrow \cdots \tag{6.21}$$

where the map $h\colon SH^{-k} \to SH_k$ factors by [17, Proposition 1.3] as

$$SH^{-k}(W) \to H^{-k+n}(W, \partial W) \xrightarrow{PD} H_{k+n}(W) \xrightarrow{\text{incl}_*} H_{k+n}(W, \partial W) \to SH_k(W). \tag{6.22}$$

By exactness, the induced map $\bar{f}\colon SH(W)/\operatorname{im}(h) \to \check{SH}(\Sigma)$ is injective, and $\operatorname{im}(h)$ is a subset of the image of $SH^-(W) \to SH(W)$.

Furthermore, for reasons similar to Lemma 6.1.23, f maps the unit of SH to the unit of $\check{SH}$. Indeed, both units have the same definition in terms of orbits of H, and it can be checked from (6.22) that $\operatorname{im}(h)$ has no elements of degree n. Hence, the generators defining the unit are not divided out by f.

Corollary 6.1.24. $SH(W)/\operatorname{im}(h)$ *is a commutative ring with unit.*

Proof. As the kernel of the ring homomorphism f, $\operatorname{im}(h) \subset SH(W)$ is an ideal, hence the quotient is a ring. □

Theorem 6.1.25. *For Σ and W as in Theorem 6.1.22, $SH(W)/\operatorname{im}(h)$ is a free and finitely generated module over the polynomial ring $\mathbb{Z}_2[s]$. In particular, $SH(W)$ is finitely generated as a $\mathbb{Z}_2$-algebra.*

Proof. Denote by $\gamma_0 = 1, \gamma_1, \dots, \gamma_N$ all generators of $\check{SC}^{(-\epsilon,1-\epsilon)}(\Sigma)$. By the proof of Lemma 6.1.11, this set generates $\check{SC}(\Sigma)$ as a $\mathbb{Z}_2[s, s^{-1}]$-module (resp. $\mathbb{Z}_2((s^{-1}))$-module if $I(g) = 0$). Further, by the proof of [43, Theorem 7.1], $\ker(\partial)$ (and hence $\check{SH}(\Sigma)$) can be generated by a finite number of linear combinations of $\gamma_0, \dots, \gamma_N$, which we denote by $g_0, \dots, g_M$.

It follows from the construction of the map f in [17] that the image $\operatorname{im}(f)$ consists of all elements of $\check{SH}(\Sigma)$ that are represented by orbits in the regions (IV) and (V) of Figure 3.1. Thus, $g_0, \dots, g_M$ lie in the image of f, and so does any positive power of s multiplied to some g_j. In contrast, any negative power of s has action less than $-\epsilon$, hence it can only be represented with orbits in region (III) and does not lie in $\operatorname{im}(f)$. In total, $g_0, \dots, g_M$ generate $\operatorname{im}(f) \cong SH(W)/\operatorname{im}(h)$ as a module over $\mathbb{Z}_2[s]$. By Lemma 6.1.10, this module is torsion-free, hence it is free by [61, Theorem 9.3] (since $\mathbb{Z}_2[s]$ is a principal ideal domain). $\square$

Remark 6.1.26. There is no obvious $\mathbb{Z}_2[s]$-module structure on the full $SH(W)$. One possible definition would be to use a non-canonical isomorphism

$$SH(W) \cong \operatorname{im}(h) \oplus SH(W)/\operatorname{im}(h)$$

and extend the module structure from $SH(W)/\operatorname{im}(h)$ to $SH(W)$, e.g. by $(s^k, x) \mapsto 0$ for $x \in \operatorname{im}(h)$. However, any such module cannot be torsion-free, simply because in many examples (e.g. many Brieskorn manifolds)

$$\dim_{\mathbb{Z}_2}(SH_0(W)) > \dim_{\mathbb{Z}_2}(SH_{2I(g)}(W)).$$

6.2 Brieskorn manifolds as examples

For a Brieskorn manifold $\Sigma = \Sigma(a_0, \dots, a_n)$ with the canonical contact structure $\xi = \ker(\alpha)$, the Reeb flow

$$\phi_t(z) = \left(e^{4it/a_0} z_0, \dots, e^{4it/a_n} z_n\right),$$

is periodic with period $T_P := L_P \cdot \frac{\pi}{2}$, with the abbreviation $L_P := \operatorname{lcm}_j(a_j)$. So we define the S^1-action of Hamiltonian diffeomorphisms on $\mathbb{R}_+ \times \Sigma$ as

$$g_t(r, z) := (r, \phi_{t \cdot T_P}(z)) = \left(r, \left(e^{2\pi i t L_P/a_0} z_0, \dots, e^{2\pi i t L_P/a_n} z_n\right)\right).$$

To compute the Maslov index $I(g)$, first note that the linearization

$$dg_t \colon T(\mathbb{R}_+ \times \Sigma) \to T(\mathbb{R}_+ \times \Sigma)$$

is the identity on $\operatorname{span}(R_\alpha, \partial_r)$. Hence, we can use a trivialization of the bundle ξ instead of $T(\mathbb{R}_+ \times \Sigma)$. Also, the Maslov index is additive under direct sums, so we can compute it similarly to Section 4.1: Take the decomposition $T_z\mathbb{C}^{n+1} = \xi \oplus \xi^\omega$ of the ambient tangent space $T_z\mathbb{C}^{n+1}$ of a point $z \in \Sigma$ into the contact distribution ξ and its symplectic complement ξ^ω. With the obvious extension of the Reeb flow to $\mathbb{C}^{n+1}$, the linearization dg_t on the ambient tangent space is given by

$$dg_t = \operatorname{diag}\left(e^{2\pi i L_P t/a_0}, \ldots, e^{2\pi i L_P t/a_n}\right).$$

So its determinant is $\det(dg_t) = e^{2\pi i L_P \sum_j 1/a_j}$ and the degree is $L_P \cdot \sum_j \frac{1}{a_j}$.

On ξ^ω, we can use the basis from Section 4.1, in which dg_t is given by

$$dg_t|_{\xi^\omega} = \begin{pmatrix} e^{2\pi i t L_P} & 0 \\ 0 & 1 \end{pmatrix},$$

so the degree is L_P. By taking the difference, we see that

$$I(g) = L_P \cdot \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1\right). \tag{6.23}$$

6.2.1 Computing the degrees

As a consistency check, let us verify that all the degrees in $\mathbb{Z}_2[s, s^{-1}]$ actually appear in the chain complex. We use the grading

$$\mu_{\text{product}} = \mu - n$$

from Remark 3.4.1, which is preserved by the product. In this grading, the generator s has degree $2I(g)$. So the degrees appearing in $\check{SH}(\Sigma)$ are a finite collection of integers, together with all shifts by multiples of $2I(g)$.

By (6.18), $s = [\mathcal{N}_{T_P}, \max]$, i.e. the maximum of a Morse function on the critical submanifold $\mathcal{N}_{T_P} \cong \Sigma$. To see that the degrees coincide, we compute

$$\begin{aligned}
\mu_P &:= \mu_{\text{product}}([\mathcal{N}_{T_P}, \max]) = \mu_{\text{RS}}(\mathcal{N}_{T_P}) + \dim(N_{T_P}) - \frac{1}{2}(\dim(\mathcal{N}_{T_P}) - 1) - n \\
&= \sum_{j=0}^{n} \left(\left\lfloor \frac{L_P}{a_j} \right\rfloor + \left\lceil \frac{L_P}{a_j} \right\rceil \right) - 2L_P + (2n-1) - (n-1) - n \\
&= 2\sum_{j=0}^{n} \frac{L_P}{a_j} - 2L_P \\
&= 2L_P \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1 \right) \\
&= 2I(g),
\end{aligned}$$

which, as expected, equals the degree of s.

Furthermore, let $[\mathcal{N}_T, \eta]$ be any generator of SH, i.e. η is a critical point of a Morse function on $\mathcal{N}_T$. As $\mathcal{N}_{T+T_P} \cong \mathcal{N}_T$, we can use the same Morse function on $\mathcal{N}_{T+T_P}$ and get a corresponding generator $[\mathcal{N}_{T+T_P}, \eta]$. According to (6.17), the degrees of $[\mathcal{N}_{T+T_P}, \eta]$ and $s \cdot [\mathcal{N}_T, \eta]$ should match, i.e.

$$\mu_{\text{product}}([\mathcal{N}_{T+T_P}, \eta]) = \mu_{\text{product}}([\mathcal{N}_T, \eta]) + \mu_P. \tag{6.24}$$

To see this, note that the period of any Reeb orbit of Σ is a multiple of $\frac{\pi}{2}$, so we can write $T = L \cdot \frac{\pi}{2}$. Then, we can compute

$$\begin{aligned}\mu_{\text{RS}}(\mathcal{N}_{T+T_P}) &= \sum_{j=0}^{n} \left(\left\lfloor \frac{L+L_P}{a_j} \right\rfloor + \left\lceil \frac{L+L_P}{a_j} \right\rceil \right) - 2(L+L_P) \\ &= \sum_{j=0}^{n} \left(\left\lfloor \frac{L}{a_j} \right\rfloor + \left\lceil \frac{L}{a_j} \right\rceil \right) - 2L + 2\sum_{j=0}^{n} \frac{L_P}{a_j} - 2L_P \\ &= \mu_{\text{RS}}(\mathcal{N}_T) + \mu_P.\end{aligned}$$

The other terms in the degree formula are the same for $[\mathcal{N}_T, \eta]$ and $[\mathcal{N}_{T+T_P}, \eta]$, thus (6.24) is verified.

Example 6.2.1. In Chapter 5, we computed the symplectic homology for the specific example

$$\Sigma_\ell := \Sigma(2\ell, 2, 2, 2), \quad \ell \geq 1.$$

While the focus in that chapter was on positive symplectic homology SH^+, the same methods work for computing $\check{SH}(\Sigma_\ell)$. The result can be stated as

$$\check{SH}(\Sigma_\ell) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = (2\ell+2)N + j \text{ for any } N \in \mathbb{Z}, j \in \{-1,0,1,2\} \\ (\mathbb{Z}_2)^2 & \text{else.} \end{cases}$$

Note also that Σ_ℓ is index-positive, hence Theorem 6.1.22 can be applied. The index shift is

$$2I(g) = 4\ell \cdot \left(\frac{1}{2\ell} + \frac{3}{2} - 1 \right) = 2\ell + 2,$$

which matches the periodicity of $\check{SH}(\Sigma_\ell)$. Thus, counting the number of generators in one period, we see that $\check{SH}(\Sigma_\ell)$ is a $\mathbb{Z}_2[s, s^{-1}]$-module of dimension

$$\dim_{\mathbb{Z}_2[s,s^{-1}]} \left(\check{SH}(\Sigma_\ell) \right) = 4\ell.$$

Remark 6.2.2. It is tempting to think that this dimension (or the degree of the principal orbit) can distinguish the contact structures of Brieskorn manifolds with different exponents. After all, by Corollary 3.5.11, $\check{SH}$ and its product structure depend only on the contact manifold Σ (at least under the assumption that Σ is product-index-positive, but by Proposition 6.1.21, the statements about the module structure hold more generally). In this way, one might for instance try to distinguish the contact structures on $\Sigma(\ell p, p, 2, 2)$ for fixed $p \in \mathbb{N}$ and different values of ℓ, see Section 5.5.

However, there is a fundamental difficulty: Since the principal orbit might be itself a power of another generator, the module structure is not uniquely determined. Hence, to distinguish the contact manifolds Σ and Σ' whose principal orbits have degrees μ_P and μ'_P, respectively, one would have to exclude the possibility that $\check{SH}(\Sigma)$ is a free module over the Laurent polynomials in a variable s whose degree is a common divisor of μ_P and μ'_P (e.g. by seeing that $\check{SH}(\Sigma)$ does not have this periodicity). For the example $\Sigma(\ell p, p, 2, 2)$, this is probably not possible without explicitly computing some differentials.

6.2.2 Comparison with known examples

Cotangent bundles of spheres

The $(2n-1)$-dimensional Brieskorn manifold $\Sigma(2, \dots, 2)$ is contactomorphic to the unit cotangent bundle S^*S^n of S^n, and its standard filling W is symplectomorphic to D^*S^n. Hence, by a famous theorem first proven by Viterbo [70], its symplectic homology is isomorphic to the homology of the free loop space LS^n of S^n,

$$SH_*(D^*S^n; \mathbb{Z}) \cong H_*(LS^n; \mathbb{Z}). \tag{6.25}$$

Moreover, by [1], the pair-of-pants product on $SH_*(D^*S^n)$ corresponds to the Chas–Sullivan product on $H_*(LS^n)$. (Note that since S^n is spin, a later correction to this theorem from [40] does not apply here.) The right-hand side of (6.25) was computed in [19]. Making the degree shift

$$\mathbb{H}_*(LM; \mathbb{Z}) := H_{*+n}(LM; \mathbb{Z})$$

in order for the product to have degree zero, their results can be stated as follows. For n even,

$$\mathbb{H}_*(LS^n; \mathbb{Z}) = \Lambda[b] \otimes \mathbb{Z}[a, v]/(a^2, ab, 2av), \tag{6.26}$$

where $\Lambda[b]$ denotes the exterior algebra and the degrees of the variables are $|b| = -1$, $|a| = -n$ and $|v| = 2n - 2$. For $n > 1$ odd,

$$\mathbb{H}_*(LS^n; \mathbb{Z}) = \Lambda[a] \otimes \mathbb{Z}[u], \tag{6.27}$$

where $|a| = -n$ and $|u| = n - 1$. However, if we take $\mathbb{Z}_2$-coefficients, it follows easily from the proof given in [19] that for any $n \geq 0$ (even or odd),

$$\mathbb{H}_*(LS^n; \mathbb{Z}_2) = \mathbb{Z}_2[a, u]/(a^2), \tag{6.28}$$

with $|a| = -n$ and $|u| = n - 1$.

To compare with Theorem 6.1.22, we need to apply the map f from (6.19).

Claim 6.2.3. *For $(W, \Sigma) = (D^*S^n, S^*S^n)$ and with $\mathbb{Z}_2$-coefficients, the map $f\colon SH(W) \to \check{SH}(\Sigma)$ is injective.*

Proof. By exactness of the sequence (6.21), it suffices to show that the map h from

(6.22) vanishes. For this, in turn, it suffices to show that the map

$$\mathrm{incl}_*\colon H_k(D^*S^n) \longrightarrow H_k(D^*S^n, S^*S^n)$$

vanishes in all degrees. As $D^*S^n \simeq S^n$, $H_k(D^*S^n)$ vanishes for $k \neq 0, n$, and $H_k(D^*S^n, S^*S^n)$ vanishes for $k = 0$. Thus, the only non-trivial degree is $k = n$, for which it follows from the long exact sequence of the pair (D^*S^n, S^*S^n) with $\mathbb{Z}_2$-coefficients

$$\cdots \to H_n(D^*S^n) \longrightarrow H_n(D^*S^n, S^*S^n) \longrightarrow H_{n-1}(S^*S^n) \longrightarrow H_{n-1}(D^*S^n) \to \cdots$$

and $H_{n-1}(S^*S^n) \cong \mathbb{Z}_2$, $H_{n-1}(D^*S^n) = 0$ that

$$\mathrm{incl}_*\colon H_n(D^*S^n) \cong \mathbb{Z}_2 \longrightarrow H_n(D^*S^n, S^*S^n) \cong \mathbb{Z}_2$$

is the zero map. □

Remark 6.2.4. For n odd, Claim 6.2.3 is also true for $\mathbb{Z}$-coefficients, while for n even, the last step in the proof only works over $\mathbb{Z}_2$.

Now, we compare with $\check{SH}(\Sigma(2,\dots,2))$. Note that all critical manifolds are of the form $\mathcal{N}_{N\pi}$ for $N \in \mathbb{Z}$, hence they are diffeomorphic to $\Sigma = \Sigma(2,\dots,2)$. The degree of a generator $[\mathcal{N}_{N\pi}, \eta]$, in the product grading, can be computed as

$$\begin{aligned}
\mu_{\text{product}}([\mathcal{N}_{N\pi}, \eta]) &= \mu_{\text{RS}}(\mathcal{N}_{N\pi}) - \frac{1}{2}(\dim(\Sigma) - 1) + \mathrm{ind}_{\text{Morse}}(\eta) - n \\
&= \sum_{j=0}^{n} (\lfloor N \rfloor + \lceil N \rceil) - 4N - (n-1) + \mathrm{ind}_{\text{Morse}}(\eta) - n \\
&= 2N(n-1) - 2n + 1 + \mathrm{ind}_{\text{Morse}}(\eta),
\end{aligned}$$

and if we choose a perfect Morse function on $\Sigma \cong S^*S^n$, $\mathrm{ind}_{\text{Morse}}(\eta)$ lies in the set $\{0, n-1, n, 2n-1\}$. Also note that all generators with $N > 0$, corresponding to positively oriented Reeb orbits, have Conley–Zehnder index at least $n-1$, from which it follows that Σ is index-positive for $n \geq 3$. As for the differential, it turns out that, at least for $n \geq 3$, all differentials of this chain complex vanish. For $n \geq 4$, this follows immediately for degree and action reasons, while for $n = 3$, it is a special case of the computations done in Chapter 5.

Hence, as a $\mathbb{Z}_2$-vector space, the $\bigvee$-shaped symplectic homology of Σ is given by

$$\check{SH}(\Sigma) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 2N(n-1) \\ & \text{or } k = 2N(n-1) - n + 1 \\ & \text{or } k = 2N(n-1) - n \\ & \text{or } k = 2N(n-1) - 2n + 1 \quad \text{for some } N \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases} \tag{6.29}$$

It can easily be checked that these degrees with $N \geq 0$ match those in (6.28), in accordance with (6.25) and Claim 6.2.3. Moreover, the generator $s = S_g(1) =$

$[\mathcal{N}_1, \max]$ appears in the first line of (6.29) with $N = 1$.

Now, the main point in the comparison concerns the product structure. Theorem 6.1.22 says that $\check{SH}(\Sigma)$ is a free module over $\mathbb{Z}_2[s, s^{-1}]$, with the module structure given by the pair-of-pants product. This matches with (6.28), where s corresponds to u^2.

However, Theorem 6.1.22 does not see that s has a square root. Instead, we only see that $\check{SH}(\Sigma)$ is a four-dimensional free module over $\mathbb{Z}_2[s, s^{-1}]$, with the first four lines in (6.29) each giving a generator. This implies that as an algebra, $\check{SH}(\Sigma)$ can be generated by at most four elements, while (6.25) and (6.28) show that two generators suffice.

As an interesting side note, Theorem 6.1.22 in combination with (6.28) and Lemma 6.1.23 reveals the full ring structure on $\check{SH}(\Sigma)$:

Theorem 6.2.5. *The ring structure of $\check{SH}(S^*S^n)$ for $n \geq 3$ is given by*

$$\check{SH}(S^*S^n) = \mathbb{Z}_2[a, u, u^{-1}]/(a^2), \tag{6.30}$$

where $|a| = -n$ and $|u| = n - 1$.

Proof. As a $\mathbb{Z}_2$-vector space, this follows from (6.29). So it remains to show that the product matches, i.e. that the expressions $\langle x \cdot y, z\rangle$ are what (6.30) predicts.

To see this, note that for any $x, y \in \check{SH}(\Sigma)$, we can find an $N \geq 0$ such that $s^N \cdot x, s^N \cdot y \in \operatorname{im}(f)$. (Here it is important that we use $\check{SH}$ and not $\widetilde{SH}$.) Now we can compute

$$\langle x \cdot y, z\rangle = \langle S_g^{2N}(x \cdot y), S_g^{2N}(z)\rangle = \langle (s^N \cdot x) \cdot (s^N \cdot y), S_g^{2N}(z)\rangle,$$

and the right hand side only involves terms in $\operatorname{im}(f)$. For those, we already know from (6.28) that the product structure is the one predicted by (6.30). $\square$

Note that by [17, Theorem 1.10], there is an isomorphism

$$\check{SH}_k(D^*M) \cong H^{-k+n+1}(LM) \qquad \text{for } k < n \tag{6.31}$$

between $\check{SH}$ of the cotangent bundle and the cohomology of the free loop space of M in sufficiently negative degrees (in the product grading). On this part, the pair-of-pants product is conjectured to be related to the Goresky–Hingston product on $H^*(LM, L_0M)$ (the cohomology of the free loop space, relative to constant loops). Indeed, if we restrict the degrees further to the range where $H^*(LM, L_0M) \cong H^*(LM)$ (i.e. $* > n+1$), these products might actually coincide.

For spheres, the Goresky–Hingston product has been computed in [35]. With $\mathbb{Z}_2$-coefficients and up to a grading shift, the result is

$$H^*(LS^n, L_0S^n) \cong \Lambda(U) \otimes \mathbb{Z}_2[T]_{\geq 2},$$

where $\deg(T) = n-1$, $\deg(U) = 1$ and $\mathbb{Z}_2[T]_{\geq 2}$ denotes the ideal in $\mathbb{Z}_2[T]$ generated by T^2. Thus, this example supports the conjecture that the product coincides with the pair-of-pants product on (6.30), with the identification $T \mapsto u^{-1}$ and $U \mapsto au$.

A_k-surface singularities

Besides cotangent bundles, the only example of Brieskorn manifolds for which the product structure on symplectic homology has been computed are the A_k-surface singularities. They are by definition the fillings of the Brieskorn manifolds

$$\Sigma(k+1,2,2) \cong L(k+1,k)$$

for $k > 1$, which are contactomorphic to the lens spaces $L(k,k+1)$. The symplectic homology of their canonical filling, along with its ring structure has been computed in [26] (although it should be mentioned that their methods rely on theorems from [8], which are note yet proven in full rigor). The following theorem specializes the results of [26, Theorem 40] to $\mathbb{Z}_2$-coefficients.

Theorem 6.2.6 ([26])**.** *Denote by W_k the canonical filling of $\Sigma(k+1,2,2)$. For k even, its symplectic homology is given by*

$$SH(W_k) = \mathbb{Z}_2[s_1,\dots,s_k,t_1,t_0,t_{-2}] \Big/ (s_i s_j = 0,\ s_i t_j = 0,\ t_1^2 = 0,\ t_0^k = 0), \quad (6.32)$$

where the degrees are $|s_i| = -2$, $|t_1| = -1$, $|t_0| = 0$ and $t_{-2} = 2$. For k odd,

$$\begin{aligned} SH(W_k) = \mathbb{Z}_2[s_1,\dots,s_k,t_1,t_0,u_{-1},t_{-2}] \Big/ (&s_i s_j = 0,\ s_i t_1 = 0,\ s_i t_0 = 0,\ t_1^2 = 0,\\ &s_i u_{-1} = t_1 t_0^{k-1},\ s_i t_{-2} = t_0^k,\\ &t_0 u_{-1} = t_1 t_{-2},\ t_1 u_{-1} = \alpha t_0^k,\\ &u_{-1}^2 = \beta t_0^{n-1}), \end{aligned} \quad (6.33)$$

where the degrees are $|s_i| = -2$, $|t_1| = -1$, $|t_0| = 0$, $u_{-1} = 1$ and $t_{-2} = 2$, and $\alpha = \beta = 1$ if $4|(k+1)$, otherwise $\alpha = \beta = 0$.

Here, the gradings are defined via filling disks in W_k, which is simply-connected. Note that, due to different conventions, our grading differs from [26] by a minus sign.

Unfortunately, $\Sigma(k+1,2,2)$ is not index-positive, because there are Reeb orbits with Conley–Zehnder index one (and which represent non-trivial classes in contact homology, so taking another contact form does not help). Thus, Theorem 6.1.22 cannot really be applied. However, as far as one can infer from $SH(W_k)$, its conclusion still seems to holds. For k even, the grading shift is

$$\mu_P = 4(k+1)\left(\sum_j \frac{1}{a_j} - 1\right) = 4,$$

so it suffices to see that there is a generator of degree four whose products make $\check{SH}(\Sigma)$ periodic. Indeed, $f(t_{-2})^2$ has degree four. Moreover, it follows from (6.22) and exactness of (6.21) that all s_i get divided out by f. Hence, in $\operatorname{im}(f) \subset \check{SH}(\Sigma)$, there is no relation involving $f(t_{-2})$ (or its square), thus periodicity holds.

For k odd, the grading shift is

$$\mu_P = 2(k+1)\left(\sum_j \frac{1}{a_j} - 1\right) = 2,$$

so the generator corresponding to the principle orbit could be $f(t_2)$ directly. The ring structure is more complicated in this case, but it still turns out that none of the relations in (6.33) destroys the periodicity coming from multiplication by $f(t_{-2})$.

Remark 6.2.7. In light of this result, it seems reasonable to conjecture that the conclusion of Theorem 6.1.22 holds for Brieskorn manifolds in general, even if they are not index-positive.

This conjecture can be heuristically justified as follows: If Σ is not index-positive, we can still use neck-stretching. By [9, Section 5.2], Floer cylinders can then break into more complicated buildings, see Figure 3.2. The top level is a punctured Floer cylinder

$$u\colon \mathbb{R} \times S^1 \setminus \{z_1, \dots, z_k\} \longrightarrow \mathbb{R}_+ \times \Sigma$$

in the symplectization. At the interior punctures $z_i = (s_i, t_i)$, it converges to some Reeb orbits at the negative end of the symplectization.

We want to apply g_t from (6.3) to this top level, while leaving the lower levels unchanged. This does not work directly because it would move the punctures. However, we can homotope g_t to a loop that is constant around t_i, similarly as we did for the product structure. Then, $g \cdot u$ is another Floer cylinder with the same punctures, which can be glued to the lower levels of the original broken cylinder.

If one works in some finite action window, there are only finitely many broken Floer cylinders, each of which has only finitely many punctures. Thus, one can avoid all of them and define S_g as before. The expectation is that it still has the same properties.

To make this rigorous, however, more work would be necessary. First, one would have to argue that counting broken Floer cylinders is fine even though transversality in the lower levels might be hard to achieve. Even more crucially, there is probably no way around working with Hamiltonians that have a non-zero slope at the negative end of the symplectization. Further, in order for S_g to be independent of the chosen homotopy, one would have to reprove the homotopy invariance from Proposition 6.1.15 in the context of broken cylinders.

Still, these difficulties seem mostly technical in nature, so it seems reasonable to expect that the module structure of Theorem 6.1.13 holds. For statements concerning the product, though, the breaking of pairs-of-pants as in Figure 3.4 poses a more fundamental difficulty. Proposition 6.1.21 probably cannot help, because its proof relies on the assumption of index-positivity in an essential way. Thus, one would either have to exclude these broken pairs-of-pants or find a way to work with them.

7 Exotic contact structures on spheres

The main goal of this Chapter is to prove Theorem 1.3.4 about the existence of infinitely many exotic but homotopically trivial contact structures on S^7, S^{11} and S^{15}. The methods are mainly combinatorial: While Sections 2.2 and 2.3 already provide combinatorial descriptions for the topology of Brieskorn manifolds, Section 7.1 will provide a similar description for the almost contact structure. Finally, the contact structures are distinguished by the mean Euler characteristic, which also has a the combinatorial description from Proposition 3.7.2.

7.1 Classical invariants

Definition 7.1.1. An *almost contact structure* on a manifold M of dimension $2n-1$ is a reduction of the structure group from $SO(2n-1)$ to $U(n-1)\times \mathrm{id}$. Equivalently, if $f\colon M \to BSO(2n-1)$ denotes the classifying map of the tangent bundle, an almost contact structure is a lift $\bar{f}\colon M \to B(U(n-1)\times\mathrm{id})$, i.e. a map $\bar{f}$ such that the diagram

$$\begin{tikzcd} & B(U(n-1)\times\mathrm{id}) \arrow[d] \\ M \arrow[ur, "\bar{f}"] \arrow[r, "f"] & BSO(2n-1) \end{tikzcd}$$

commutes.

A (cooriented) contact structure $\xi = \ker(\alpha)$ induces an almost contact structure by the splitting $TM = \xi \oplus \langle R_\alpha\rangle$. The almost contact structure of a contact structure is also called its formal homotopy class.

The map $B(U(n-1)\times\mathrm{id}) \to BSO(2n-1)$ is a fibration whose fibers are $SO(2n-1)/U(n-1)$. Moreover, the inclusion $SO(2n-1)\hookrightarrow SO(2n)$ induces a diffeomorphism of the homogeneous spaces (see e.g. [36, Corollary 3.1.3])

$$SO(2n-1)/U(n-1) \cong SO(2n)/U(n).$$

Hence, if M is stably parallelizable, the almost contact structures on M are in one-to-one correspondence with homotopy classes of maps from M to $SO(2n)/U(n)$. In particular, if Σ is a (topological) sphere, almost complex structures on Σ are classified by $\pi_{2n-1}(SO(2n)/U(n))$, with $0 \in \pi_{2n-1}(SO(2n)/U(n))$ corresponding to the trivial almost contact structure. By a classical result from Massey [48],

$$\pi_{2n-1}(SO(2n)/U(n)) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n \equiv 0 \mod 4 \\ \mathbb{Z}_{(n-1)!} & \text{for } n \equiv 1 \mod 4 \\ \mathbb{Z} & \text{for } n \equiv 2 \mod 4 \\ \mathbb{Z}_{\frac{(n-1)!}{2}} & \text{for } n \equiv 3 \mod 4 \end{cases}$$

For Brieskorn manifolds diffeomorphic to standard spheres, Morita [53] gives an explicit formula for the almost contact structure in terms of the exponents a_j. Denote by ξ_a the canonical contact structure of $\Sigma(a)$ and by ac the map sending its underlying almost contact structure to the groups above. Further, abbreviate

$$S_m := \frac{2^{2m}(2^{2m-1}-1)B_m}{(2m)!},$$

where B_m denotes the m-th Bernoulli number, with the same convention as in (2.5). Then, Morita's result states that

$$ac(\Sigma(a), \xi_a) = \begin{cases} \left(\frac{1}{4S_m}\sigma(W_a) - \frac{1}{2}\mu(a), 0\right) & \text{for } n \equiv 0 \mod 4 \\ \frac{1}{2}\mu(a) & \text{for } n \equiv 1 \mod 4 \\ -\frac{1}{4S_m}\sigma(W_a) - \frac{1}{2}\mu(a) & \text{for } n \equiv 2 \mod 4 \\ \frac{1}{2}\mu(a) & \text{for } n \equiv 3 \mod 4 \end{cases} \tag{7.1}$$

Here, $\mu(a) = \prod_{j=0}^{n}(a_j - 1)$ is the rank of $H_n(W_a)$ and $m = n/2$ in the first and third line. In dimension 7, where $n = 4$ and $m = 2$, this formula gives

$$ac(\Sigma(a), \xi_a) = \left(\frac{45}{28}\sigma(W_a) - \frac{1}{2}\mu(a), 0\right).$$

The standard almost contact structure on S^7 is represented by $(0, 0)$. Thus, we want

$$\frac{45}{28}\sigma(W_a) = \frac{1}{2}\mu(a),$$

or, expressed in the dimensions of the positive and negative eigenspaces of the intersection form (with $\sigma(W_a) = \sigma_a^+ - \sigma_a^-$ and $\mu(a) = \sigma_a^+ + \sigma_a^-$),

$$31\sigma_a^+ = 59\sigma_a^-.$$

By Proposition 2.3.2, this condition is satisfied for $\Sigma(78k+1, 13, 6, 3, 3)$. Hence:

Theorem 7.1.2. *For any $k \in \mathbb{N}$, the Brieskorn manifold $\Sigma(78k+1, 13, 6, 3, 3)$ is diffeomorphic to S^7. Moreover, its canonical contact structure is homotopically standard, i.e. its underlying almost contact structure is homotopic to that of S^7.*

At this point, one could already use [25, Theorem 6.1] to see that the Brieskorn manifolds $\Sigma(78k+1, 13, 6, 3, 3)$ give exotic but homotopically standard contact structures on S^7. However, it is not yet clear that they are pairwise non-contactomorphic, which we will show in Section 7.2.2.

7.2 Exotic contact structures on S^7

7.2.1 Application to $\Sigma(13, 11, 7, 4, 3)$

Before turning to the main example in Section 7.2.2, we briefly show that, if one is willing to use connected sums, there are even easier examples. They are based on the formula for the mean Euler characteristic for a connected sum [41, Theorem 5.19]:

Proposition 7.2.1. *Let Σ_1, Σ_2 be contact manifolds of dimension $2n-1$ that come along with Liouville fillings for which the mean Euler characteristic is defined. Then*

$$\chi_m(\Sigma_1 \# \Sigma_2) = \chi_m(\Sigma_1) + \chi_m(\Sigma_2) + (-1)^n \frac{1}{2}.$$

Here, we will use $\Sigma = \Sigma(13, 11, 7, 4, 3)$. Note that it is a homotopy sphere by Theorem 2.2.2. Further, application of Theorem 2.3.1 shows that the signature of its filling is 1344. Hence, it is diffeomorphic to S^7 and its almost contact structure is zero.

As for the mean Euler characteristic, note that all exponents are pairwise relatively prime. This makes the computation somewhat easier, as [32, Proposition 4.6] gives a simplified formula for such Brieskorn manifolds. Plugging in the numbers gives

$$\chi_m(\Sigma(13, 11, 7, 4, 3)) = -\frac{3047}{2546}.$$

Of course, it can also be worked out directly from Proposition 3.7.2, with a computation similar to the one we do in Section 7.2.2.

As $\chi_m(S^7) = -1/2$ for the standard contact structure, this shows that the contact structure on $\Sigma(13, 11, 7, 4, 3)$ is exotic. In order to generate infinitely many exotic contact structures, take the connected sum of k copies of $\Sigma(13, 11, 7, 4, 3)$ and use Proposition 7.2.1 to get

$$\chi_m(\#_k \Sigma(13, 11, 7, 4, 3)) = -k \cdot \frac{3047}{2546} + (k-1) \cdot \frac{1}{2} = -\frac{1}{2} - k \cdot \frac{887}{1273},$$

which is strictly monotone decreasing in k. Hence, the manifolds $\#_k \Sigma(13, 11, 7, 4, 3)$ are pairwise non-contactomorphic, and we get infinitely many exotic contact structures in S^7.

7.2.2 Application to $\Sigma(78k + 1, 13, 6, 3, 3)$

The example $\Sigma(78k + 1, 13, 6, 3, 3)$ is particularly nice because it does not need the connected sum construction. By Theorem 7.1.2, we already know that these manifolds are diffeomorphic to S^7 and have trivial almost contact structure. Now, we compute their mean Euler characteristic.

First, according to (4.4), the Robbin–Salamon index of the principal orbit is

$$\mu_P = 2 \cdot \operatorname{lcm}(a_j) \cdot \left(\sum_{j=0}^{4} \frac{1}{a_j} - 1 \right) = 156 - 14a_0 = 142 - 1092k.$$

Computing all the terms appearing in (3.17), we get Table 7.1.

Orbit space	period $/\frac{\pi}{2}$	χ^{S^1}	frequency
$\Sigma(a_0, 13, 6, 3, 3)$	$78a_0$	4	1
$\Sigma(13, 6, 3, 3)$	78	3	$a_0 - 1 = 78k$
$\Sigma(a_0, 6, 3, 3)$	$6a_0$	3	12
$\Sigma(6, 3, 3)$	6	0	$12(a_0 - 1) = 12 \cdot 78k$
$\Sigma(a_0, 13, 3, 3)$	$39a_0$	3	1
$\Sigma(13, 3, 3)$	39	2	$a_0 - 1 = 78k$
$\Sigma(a_0, 3, 3)$	$3a_0$	2	12
$\Sigma(3, 3)$	3	3	$12(a_0 - 1) = 12 \cdot 78k$
$\Sigma(a_0, 13)$	$13a_0$	1	4

Table 7.1: The contributions to $\chi_m(\Sigma(78k + 1, 13, 6, 3, 3))$

Hence, we can compute $\chi_m(\Sigma)$ in terms of k:

$$\begin{aligned}\chi_m(\Sigma) &= -\frac{4 + 3 \cdot 78k + 36 + 3 + 2 \cdot 78k + 24 + 3 \cdot 12 \cdot 78k + 4}{|142 - 1092k|} \\ &= \frac{71 + 3198k}{142 - 1092k}\end{aligned}$$

By a simple computation, the function

$$x \longmapsto \frac{71 + 3198x}{142 - 1092x}$$

is strictly monotone increasing. Hence, $\chi_m(\Sigma)$ can distinguish the different values of k.

Theorem 7.2.2. *The canonical contact structures on the Brieskorn manifolds* $\Sigma(78k + 1, 13, 6, 3, 3)$ *are all different. Hence, in combination with Theorem 7.1.2, we get infinitely many exotic but homotopically trivial contact structures on* S^7*.*

7.3 Application to other 7-manifolds

Having established the existence of infinitely many contact structures in the standard formal homotopy class on S^7, one can ask a similar question for other contact manifolds. In some cases, the answer is just a corollary of Theorem 7.2.2:

Theorem 7.3.1. *Let $(M, \xi = \ker(\alpha))$ be a contact 7-manifold that admits a Liouville filling for which the mean Euler characteristic is well-defined. Then, there exist infinitely many contact structures on M in the formal homotopy class of ξ.*

Proof. Take the connected sum of M with the manifolds from Theorem 7.2.2. These manifolds have trivial almost contact structure, corresponding to the zero element in $\pi_7(SO(8)/U(4))$. Hence, the lift of the classifying map $M \to BSO(7)$ to $U(3) \times \mathrm{id}$ does not change under the connected sum, so the formal homotopy class stays the same. However, the contact structures can be distinguished by the mean Euler characteristic, using Proposition 7.2.1. □

A similar theorem holds in dimensions $4m + 1$, where the Ustilovsky spheres take the place of the manifolds from Theorem 7.2.2. See e.g. [27] for the mean Euler characteristic of the Ustilovsky spheres.

Remark 7.3.2. There is also a version of the mean Euler characteristic using contact homology. For this purpose, the examples of Section 7.2.1 can be useful: All Reeb orbits in $\Sigma(13, 11, 7, 4, 3)$ have Conley–Zehnder index ≤ -3, so cylindrical contact homology is (conjecturally) well-defined. Hence, one can use these manifolds to prove a variant of Theorem 7.3.1 in which the assumption of a Liouville-filling is replaced by the assumption that cylindrical contact homology (and its mean Euler characteristic) is well-defined. Besides Brieskorn manifolds, e.g. the prequantization bundles from [27, Example 8.2] satisfy this assumption.

One may also ask whether there are infinitely many contact structures in other formal homotopy classes on S^7. The next proposition gives a partial answer to this question.

Proposition 7.3.3. *In any almost contact structure of the form $(2k, 0) \in \mathbb{Z} \oplus \mathbb{Z}_2$ on S^7, there are infinitely many contact structures.*

Proof. We use certain Brieskorn manifolds to construct a manifold diffeomorphic to S^7 with almost contact structure $(\pm 2, 0)$. Taking connected sums and applying Theorem 7.3.1 then finishes the proof.

We choose the manifolds $M_1 = \Sigma(11, 9, 9, 5, 3), M_2 = \Sigma(13, 10, 9, 3, 3)$ and $M_3 = \Sigma(167, 3, 2, 2, 2)$. It is straightforward to verify that they are diffeomorphic to S^7 and that their almost contact structures are -40, 72 and 194, respectively. Hence,

$$M_4 := 2M_1 \# M_2 \cong S^7$$

has almost contact structure -8. Further,

$$M_5 := 24 M_4 \# M_3 \cong S^7$$

has almost contact structure $+2$, and

$$M_6 := M_4 \# 3M_5 \cong S^7$$

has almost contact structure -2. $\square$

By contrast, the following lemma implies that the remaining almost contact structures on S^7 cannot be realized as connected sums of Brieskorn manifolds diffeomorphic to S^7.

Lemma 7.3.4. *Any Brieskorn manifold diffeomorphic to S^{4m-1}, $m \geq 2$, has almost contact structure of the form $(2k, 0) \in \mathbb{Z} \oplus \mathbb{Z}_2$ (resp. of the form $2k \in \mathbb{Z}$ if m is odd).*

Proof. By Morita's formula (7.1), we have

$$ac(\Sigma(a), \xi_a) = \begin{cases} \left(\frac{1}{4S_m}\sigma(W_a) - \frac{1}{2}\mu(a), 0\right) \in \mathbb{Z} \times \mathbb{Z}_2 & \text{if } m \text{ is even} \\ -\frac{1}{4S_m}\sigma(W_a) - \frac{1}{2}\mu(a) \in \mathbb{Z} & \text{if } m \text{ is odd.} \end{cases}$$

We see immediately that, if m is even, the second factor of the almost contact structure always vanishes. It remains to show that the first factor is an even integer.

By the assumption that $\Sigma(a)$ is diffeomorphic to S^{4m-1}, we know from (2.4) and (2.5) that $\sigma(W_a)$ is a multiple of σ_m. We have

$$\frac{\sigma_m}{4S_m} = \frac{\text{numerator}\left(\frac{4B_m}{m}\right) \cdot (2m)!}{B_m} \in 2\mathbb{Z},$$

so $\frac{\sigma(W_a)}{4S_m}$ is certainly an even integer.

As for $\mu(a) = \prod_{j=0}^{n}(a_j - 1)$, we use Theorem 2.2.2 to infer its divisibility by 4. First of all, there exists an exponent, say a_0, which is relatively prime to all other exponents. We assume that a_0 is odd, since otherwise, all other exponents are odd and $\mu(a)$ is divisible by 2^n. So we already get a factor of two in $\mu(a)$.

If item (i) of Theorem 2.2.2 applies, we get another factor of two for the same reason, so we are done. So assume that item (ii) holds with the set $\{a_1, \ldots, a_r\}$. In particular, $a_1, \ldots, a_r$ are even, while $a_0, a_{r+1}, \ldots, a_n$ are odd. Since r is odd and n is even, we have at least two odd exponents. Hence $\mu(a)$ is divisible by four. $\square$

One might ask further whether a result analogous to Theorem 7.2.2 holds for exotic 7-spheres. One problem here is that Morita's calculation of the almost contact structure in [53] is only valid for standard smooth spheres. Besides, it is not even clear which almost contact structure should be viewed as standard. Therefore, the best we can do is the following:

Corollary 7.3.5. *On any boundary parallelizable homotopy 7-sphere $M \in bP_8$, there exists an almost contact structure containing infinitely many contact structures.*

Proof. All elements of the group bP_8 are represented by $\Sigma(6k-1,3,2,2,2)$ ([14, p. 13]). Thus, M is diffeomorphic to a Brieskorn manifold, and we can apply Theorem 7.3.1 again. □

7.4 How this example was found

In the previous sections, the numbers $(78k+1,13,6,3,3)$ (and $(13,11,7,4,3)$ in Section 7.2.1) seemed to appear out of nowhere. In this section, we describe the strategy to find them.

Let $\Sigma = \Sigma(a_0,\dots,a_4)$ be any Brieskorn manifold with its standard contact structure ξ. Denote, as before, its filling by W_a, the middle dimension of its homology by $\mu = \operatorname{rank} H_4(W_a) = \prod_{i=0}^{4}(a_i-1)$ and its signature by σ. We are looking for examples that fulfill the following three conditions:

(i) Σ is a topological sphere, i.e. $H_{n-1}(\Sigma)=0$. This can be checked by Randell's algorithm.

(ii) Σ has the standard smooth structure. By (2.4), assuming (i) is satisfied, this is the case if and only if
$$\sigma \equiv 0 \mod 224.$$

(iii) (Σ,ξ) has the standard almost contact structure. By (7.1) and assuming (i) and (ii), this is equivalent to the condition
$$\frac{45}{28}\sigma - \frac{1}{2}\mu = 0.$$

To reformulate these conditions, let σ_a^+ (resp. σ_a^-) denote, as before, the dimension of the positive (resp. negative) eigenspace of $H_n(W_a)$. Then $\sigma = \sigma_a^+ - \sigma_a^-$ and $\mu = \sigma_a^+ + \sigma_a^-$, so condition (iii) becomes
$$31\sigma_a^+ = 59\sigma_a^-.$$

This gives $\sigma_a^+ = 59k$ and $\sigma_a^- = 31k$ for some positive integer k. Assuming this, condition (ii) is
$$\sigma = \sigma_a^+ - \sigma_a^- = 28k \overset{!}{=} 224s$$
for another positive integer s. Hence, $k = 8s$. Putting everything together, conditions (ii) and (iii) are satisfied (under the assumption of (i)) if and only if
$$\sigma_a^+ = 472s \qquad (7.2)$$
$$\sigma_a^- = 248s. \qquad (7.3)$$

In particular, $\mu = \sigma_a^+ + \sigma_a^- = 720s$.

With these preparations, it seems sensible to search for examples with the help of a computer. The algorithm does the following steps:

- Iterate over the integer s in some range, e.g. for $1 \leq s \leq 60$.
- Iterate over all tuples $(b_0, \ldots, b_4)$, $b_j \geq 1$ such that $\prod_{i=0}^{4} b_i = 720s$.
- Each such tuple gives a candidate $\Sigma(a)$ with $a_j = b_j + 1$. Compute the signature of its filling with (2.6) and (2.7).
- If (7.2) is fulfilled, use Randell's algorithm to check if Σ is also a topological sphere. Otherwise, discard it.

With this algorithm, the following list of examples was found (values of s without examples are skipped):

$s = 4$	$\Sigma(11, 7, 5, 5, 4)$			
$s = 5$	$\Sigma(11, 11, 7, 4, 3)$			
$s = 6$	$\Sigma(13, 11, 7, 4, 3)$			
$s = 7$	$\Sigma(11, 10, 9, 8, 2)$			
$s = 8$	$\Sigma(17, 16, 5, 4, 3)$	$\Sigma(21, 13, 5, 4, 3)$		
$s = 10$	$\Sigma(26, 13, 5, 4, 3)$	$\Sigma(41, 6, 5, 4, 4)$		
$s = 12$	$\Sigma(25, 11, 7, 7, 2)$	$\Sigma(28, 11, 9, 3, 3)$	$\Sigma(46, 7, 5, 5, 3)$	$\Sigma(37, 11, 5, 4, 3)$
$s = 14$	$\Sigma(22, 17, 7, 6, 2)$	$\Sigma(25, 13, 8, 6, 2)$	$\Sigma(29, 16, 7, 3, 3)$	$\Sigma(31, 13, 8, 3, 3)$
	$\Sigma(31, 15, 7, 3, 3)$	$\Sigma(43, 11, 5, 4, 3)$	$\Sigma(37, 11, 8, 3, 3)$	
$s = 15$	$\Sigma(25, 16, 7, 6, 2)$			
$s = 16$	$\Sigma(33, 13, 7, 6, 2)$			
$s = 18$	$\Sigma(37, 13, 7, 6, 2)$			
$s = 20$	$\Sigma(21, 17, 16, 4, 2)$			
$s = 21$	$\Sigma(43, 11, 10, 5, 2)$	$\Sigma(43, 19, 6, 3, 3)$		
$s = 22$	$\Sigma(25, 23, 11, 4, 2)$	$\Sigma(45, 13, 7, 6, 2)$		
$s = 23$	$\Sigma(31, 24, 7, 5, 2)$			
$s = 24$	$\Sigma(31, 25, 7, 5, 2)$	$\Sigma(31, 17, 13, 4, 2)$	$\Sigma(97, 7, 6, 4, 3)$	$\Sigma(91, 9, 5, 4, 3)$
$s = 25$	$\Sigma(31, 21, 11, 4, 2)$			
$s = 26$	$\Sigma(79, 13, 6, 3, 3)$			
$s = 27$	$\Sigma(37, 16, 13, 4, 2)$	$\Sigma(37, 19, 11, 4, 2)$	$\Sigma(46, 19, 7, 5, 2)$	
$s = 28$	$\Sigma(71, 9, 7, 7, 2)$	$\Sigma(64, 11, 9, 5, 2)$		
$s = 30$	$\Sigma(41, 19, 11, 4, 2)$			
$s = 33$	$\Sigma(41, 23, 10, 4, 2)$			

$s = 34$	$\Sigma(35, 31, 9, 4, 2)$	$\Sigma(52, 17, 11, 4, 2)$	$\Sigma(103, 11, 7, 3, 3)$	$\Sigma(86, 17, 4, 4, 3)$
$s = 36$	$\Sigma(37, 31, 9, 4, 2)$	$\Sigma(91, 17, 4, 4, 3)$		
$s = 39$	$\Sigma(79, 16, 7, 5, 2)$			
$s = 40$	$\Sigma(101, 17, 4, 4, 3)$			
$s = 42$	$\Sigma(113, 16, 4, 4, 3)$			
$s = 43$	$\Sigma(44, 37, 6, 5, 2)$			
$s = 44$	$\Sigma(49, 34, 6, 5, 2)$	$\Sigma(89, 16, 7, 5, 2)$		
$s = 45$	$\Sigma(136, 11, 7, 3, 3)$			
$s = 46$	$\Sigma(93, 16, 7, 5, 2)$			
$s = 48$	$\Sigma(97, 16, 7, 5, 2)$			
$s = 49$	$\Sigma(148, 11, 7, 3, 3)$			
$s = 50$	$\Sigma(121, 13, 6, 6, 2)$			
$s = 52$	$\Sigma(157, 13, 6, 3, 3)$	$\Sigma(131, 10, 9, 5, 2)$		
$s = 54$	$\Sigma(73, 28, 6, 5, 2)$			
$s = 57$	$\Sigma(91, 20, 9, 4, 2)$			
$s = 60$	$\Sigma(91, 31, 5, 3, 3)$			

The example $\Sigma(13, 11, 7, 4, 3)$ appears near the top. It was chosen simply as the first example whose exponents are relatively prime.

Unfortunately, this list does not display a simple regular pattern. Therefore, instead of continuing this brute-force method, one can try to find numbers a_1, a_2, a_3, a_4 such that, when $a_0 \to \infty$, the ratio σ_a^+/σ_a^- approaches the value $59/31$. In this computation, one can assume that the contribution of j_0/a_0 to (2.6) and (2.7) is spread out evenly over an interval of length one.

Thus, with another brute-force search, the numbers $13, 6, 3, 3$ were found quickly. Then, one can check that the values $79, 157, 235, 313$, etc. actually work for a_0. With this information, trying the tuples $a = (78k + 1, 13, 6, 3, 3)$ seems like the obvious choice. The remaining work was to verify conditions (i), (ii) and (iii), as was done in Sections 2.3 and 7.1.

The next example that can be found in this way is $\Sigma(504k + 1, 36, 7, 4, 2)$. With the same methods, it can be shown that this example also produces an infinite family of exotic contact structures on S^7.

7.5 Higher dimensions

An immediate question is whether an analog of Theorem 7.2.2 holds in higher dimensions. We may formulate it like this:

> Do there exist infinitely many exotic but homotopically trivial contact structures on S^{4m-1} for $m \geq 2$?

Note that this is not the case on S^3, see [24]. A similar question for S^{4m+1} was answered affirmatively by Ustilovsky [66].

In general, it seems hopeless to get analogs of $\Sigma(78k+1, 13, 6, 3, 3)$ for general dimensions $4m-1$. The reason is that all terms involving the Bernoulli numbers (in particular σ_m and S_m) get very complicated.

There is a somewhat simpler approach, making heavy use of connected sums. The strategy is to find Brieskorn manifolds Σ_1 and Σ_2 such that:

- Both Σ_1 and Σ_2 are diffeomorphic to S^{4m-1}.
- Viewing the almost contact structure as an element of $\mathbb{Z}$ (ignoring the second factor if m is even), we have $ac_1 := ac(\Sigma_1) > 0$ and $ac_2 := ac(\Sigma_2) < 0$.
- $\Sigma := (|ac_2|\Sigma_1)\#(ac_1\Sigma_2)$ (the connected sum of $|ac_2|$ copies of Σ_1 with ac_1 copies of Σ_2) has mean Euler characteristic $\chi_m(\Sigma) \neq -\frac{1}{2}$.

Then, Σ is diffeomorphic to S^{4m-1} with trivial almost contact structure. By taking further connected sums of Σ with itself, we get infinitely many values for the mean Euler characteristic, hence infinitely many exotic but homotopically trivial contact structures.

Now, the problem is to find such examples for Σ_1 and Σ_2. Since we require them to be diffeomorphic to S^{4m-1}, their signature should satisfy (2.4). So it should be either zero or very large. Unfortunately, there seem to be no examples with signature zero. It would be interesting to see a conceptual reason for this, possibly from the intersection matrix given in [56]. So the signature needs to have a specific, large value. One way to produce such examples is by mimicking Proposition 2.3.2. Thus, we first choose numbers $a_1, \dots, a_n$ with, say, a_1 relatively prime to the rest. Then we set

$$a_0^{(k)} := k \cdot \prod_{i=1}^{n} a_i + 1$$

and $\Sigma^{(k)} := \Sigma(a_0^{(k)}, a_1, \dots, a_n)$. (We could also choose $a_0 = k \cdot \prod_{i=1}^{n} a_i - 1$, which would work similarly.) With the same proof as for Proposition 2.3.2, we get $\sigma(\Sigma^{(k)}) = k \cdot \sigma(\Sigma^{(0)})$. Hence, once we computed $\sigma(\Sigma^{(0)})$, we can choose k such that $\sigma(\Sigma^{(k)}) \equiv 0$ (e.g. $k = \sigma_m$) to ensure diffeomorphicity to S^{4m-1}.

For Σ_1, we can choose $\Sigma_1^{(k)} = \Sigma(6k+1, 3, 2, \dots, 2)$. Its signature is

$$\sigma = \sigma(\Sigma(6k+1, 3, 2, \dots, 2)) = k \cdot \sigma(\Sigma(7, 3, 2, \dots, 2)) = (-1)^m 8k.$$

So we can choose

$$k = \frac{\sigma_m}{8} = 2^{2m-2} \cdot (2^{2m-1} - 1) \cdot \text{numerator}\left(\frac{4B_m}{m}\right).$$

As $\mu = 12k = \frac{3}{2}\sigma_m$, we get for the almost contact structure

$$\begin{aligned} ac &= (-1)^m \frac{\sigma}{4S_m} - \frac{1}{2}\mu \\ &= \left(\frac{1}{4S_m} - \frac{3}{4}\right)\sigma_m. \end{aligned}$$

To see that this is positive, we use some estimates for S_m. First, a well-known identity for Bernoulli numbers states that

$$B_m = \frac{2(2m)!}{(2\pi)^{2m}} \cdot \zeta(2m),$$

where ζ is the Riemann zeta function (see e.g. [52, p. 286]). As $\zeta(2m)$ converges to 1 very fast, $B_m \approx 2(2m)!/(2\pi)^{2m}$ is a good approximation. Therefore,

$$\frac{1}{4S_m} \approx \frac{(2m)!}{2^{2m+2}(2^{2m-1}-1)} \cdot \frac{(2\pi)^{2m}}{2(2m)!} = \frac{\pi^{2m}}{8(2^{2m-1}-1)} \approx \frac{1}{4}\left(\frac{\pi}{2}\right)^{2m}.$$

It is not hard to make this estimate precise enough to show that $1/4S_m > 3/4$ for all $m > 2$. So Σ_1 does indeed fulfill $ac > 0$.

The choice of Σ_2 is more of a problem. In view of the second condition, it seems reasonable to choose

$$\Sigma_2^{(k)} = \Sigma(k \cdot d(d+1) + 1, d+1, d, \ldots, d),$$

where d is sufficiently large. Then we expect that $\mu = k \cdot d^2 \cdot (d-1)^n$ is sufficiently large to make ac_2 negative. Another plausible choice might be $\Sigma(2d \cdot k + 1, 2, d, \ldots, d)$ for $d \gg 1$ odd. However, the precise value of $\sigma(\Sigma_2^{(k)})$ seems extremely hard to compute. Without such a computation at hand, $ac_2 < 0$ cannot be known for certain, and even assuming this, we cannot verify the third condition $\chi_m \neq -1/2$, although it looks entirely plausible. In this text, we restrict ourselves to dimensions 11 and 15, leaving the general case as a conjecture.

In dimension 11, it turns out that $d = 8$ works. So we take

$$\Sigma_2^{(k)} = \Sigma(72 \cdot k + 1, 9, 8, 8, 8, 8, 8).$$

A computer calculation gives $\mu^{(k)} = 9680832k$ and $\sigma^{(k)} = -1060560k$, so $k = 496 = \sigma_m/16$. This gives the almost contact structure $ac_2 = -396387936$, which is indeed negative.

As for the mean Euler characteristics, it turns out that

$$\chi_m(\Sigma_1) = -\frac{77393}{130978} \approx -0.5909 \qquad \text{and} \qquad \chi_m(\Sigma_2) = \frac{85520029}{193850} \approx 441.1660.$$

So $\Sigma := (|ac_2|\Sigma_1)\#(ac_1\Sigma_2)$ has mean Euler characteristic

$$\chi_m = -\frac{3345510952696507}{12695042650} \approx -263528.9,$$

for which we just need that it is not equal to $-1/2$.

In dimension 15, it turns out that $d = 8$ is not enough (ac_2 would still be positive), but $d = 9$ works. Then, the numbers for

$$\Sigma_2^{(k)} = \Sigma(90 \cdot k + 1, 10, 9, 9, 9, 9, 9, 9, 9)$$

are $\mu^{(k)} = 1698693120k$ and $\sigma^{(k)} = 86754800$, so we can choose $k = 4064 = \sigma_m/16$, giving $ac_2 = -172412979840 < 0$. Then $\Sigma := (|ac_2|\Sigma_1)\#(ac_1\Sigma_2)$ has trivial almost contact structure and mean Euler characteristic

$$\chi_m = \frac{744637007679318226185}{6671235576398} \approx 111619054.5.$$

This finishes the proof of Theorem 1.3.4.

Conjecture 7.5.1. *This method to find Σ_2 works in any dimension. Hence, there exist infinitely many exotic but homotopically trivial contact structures on S^{4m+3} for all $m \geq 2$.*

The following consideration from stochastics makes it plausible that ac_2 will indeed be negative for d sufficiently large. Let $X_0, \ldots, X_n$ be independent random variables, where X_i is distributed uniformly on the discrete set

$$\left\{\frac{1}{a_i}, \ldots, \frac{a_i - 1}{a_i}\right\}.$$

Their sum $S_n = \sum_{i=0}^n X_i$ is a random variable on a discrete set inside $(0, n+1)$, and each outcome gives a contribution to the signature of $\Sigma(a_0, \ldots, a_n)$ as in (2.6), (2.7). We can try to estimate $\sigma(W_a)$ with the help of the central limit theorem.

First, all X_i have mean value $1/2$ and standard deviation $\varsigma_i = \sqrt{\frac{a_i - 2}{12a_i}}$. For a_i large enough, $\varsigma_i \approx \sqrt{\frac{1}{12}}$ becomes a good approximation. The central limit theorem says that

$$\frac{S_n - \frac{n+1}{2}}{\sqrt{\sum_{i=0}^n \varsigma_i^2}} \approx \sqrt{\frac{12}{n+1}} \cdot \left(S_n - \frac{n+1}{2}\right)$$

will converge in distribution to the standard normal distribution. This means that the cumulative density function can be approximated, for large n, by

$$F_n(x) := P(S_n \leq x) \approx \Phi_{0,1}\left(\frac{12}{n+1}\left(x - \frac{n+1}{2}\right)\right), \tag{7.4}$$

where $\Phi_{0,1}$ is the cumulative density function of the standard normal distribution. A numeric computation shows that, if we use the right hand side to compute the signature as in (2.6), (2.7), we get that the quotient σ_a^+/σ_a^- is very close to one. Hence, it can be expected that $\sigma(W_a)$ is much smaller that μ, so that ac_2 will be negative.

Of course, this argument is far from being precise. Most importantly, the approximation of F_n with $\Phi_{0,1}$ is only good for heuristic purposes, as it is never exact for finite n. The Berry–Esseen theorem (a quantitative version of the central limit theorem) says that the error in (7.4) can be of order at most $n^{-1/2}$. This is not good enough for our purposes, because one would need to do this approximation for all positive integers up to n, thereby possibly collecting a total error of order $n \cdot n^{-1/2} = \sqrt{n}$.

Besides, one needs an argument that the mean Euler characteristic of $\Sigma := (|ac_2|\Sigma_1)\#(ac_1\Sigma_2)$ cannot be $-1/2$. In the examples, it is far away from this value, but of course that requires a proof.

8 Further questions

8.1 Are Brieskorn manifolds always different?

In many parts of the previous chapters, we explained methods to distinguish contact structures on Brieskorn manifolds. By contrast, we did not yet come across any example where two different Brieskorn manifolds (i.e. the exponents differ by more than a permutation) are contactomorphic. This raises the question whether such examples exist at all.

In fact, they do - at least in dimension three, where an explicit example was given by Milnor:

Theorem 8.1.1 ([51]). *The Brieskorn manifolds* $\Sigma(2,9,18)$ *and* $\Sigma(3,5,15)$ *are strictly contactomorphic to each other.*

This is proven in [51, Example 2], where it is shown that both are prequantization bundles with Chern number -1 over a surface of genus 4.

On the other hand, the question is completely open in higher dimensions. The reason for the lack of examples might be that there are no established methods for showing that two contact structures are equal.

8.2 Volume considerations

A candidate for two higher-dimensional Brieskorn manifold that might be equal is $\Sigma(4,4,4,4)$ and $\Sigma(2,6,6,6)$. Randell's algorithm shows that their homology is the same, namely

$$H_2(\Sigma(4,4,4,4)) \cong H_2(\Sigma(2,6,6,6)) \cong \mathbb{Z}^{21}.$$

Further, their symplectic homology, as far as one can tell from the chain level, seems to match completely. For SH^{+,S^1}, there are no differentials by degree reasons, and the chain groups are equal.

Even more compelling is the fact that in both cases, the quotient Σ/S^1 is a K3-surface (see [13, p. 163]). Thus, both are prequantization bundles over K3-surfaces. This does not prove that the contact structures are equal, though, since there are many symplectic structures on a K3-surface. In fact, the indications might be misleading:

Proposition 8.2.1. $\Sigma(4,4,4,4)$ *and* $\Sigma(2,6,6,6)$ *are not strictly contactomorphic.*

The idea of the proof is to compare the volumes of the quotients Σ/S^1.

Definition 8.2.2 ([4]). The *systolic volume* of a contact manifold M of dimension $2n-1$ with contact form α is defined as

$$\mathfrak{S}(M,\alpha) := \frac{\mathrm{vol}(M,\alpha)}{T_{\min}(M,\alpha)^n},$$

where $T_{\min}(M,\alpha)$ denotes the minimal length of a closed Reeb orbit and the volume is defined with the convention

$$\mathrm{vol}(M,\alpha) := \frac{1}{(n-1)!}\int_M \alpha \wedge (d\alpha)^{n-1}.$$

(Note that [4] does not have the factor $\frac{1}{(n-1)!}$ in the definition.)

For the computation of systolic volumes, a slightly different model for Brieskorn manifolds is useful. First, denote

$$w_k := \frac{\mathrm{lcm}_j\{a_j\}}{a_k} \in \mathbb{Z},$$

which will be referred to as weights. Then, redefine the Brieskorn manifold as

$$\Sigma(a_0,\dots,a_n) = \left\{ z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots z_n^{a_n} = 0,\ w_0|z_0|^2 + \cdots + w_n|z_n|^2 = 1 \right\}.$$

The difference with the earlier definition is that the sphere that intersects the Brieskorn variety is distorted to an ellipsoid. The diffeomorphism type remains unchanged. Further, equip $\Sigma(a)$ with the contact form

$$\alpha = -\frac{1}{4}d^c\left(\|z\|^2\right) = \frac{i}{4}\sum_{j=0}^{n}\left(z_j d\bar{z}_j - \bar{z}_j dz_j\right),$$

giving the same contact structure as before (see e.g. [41, Proposition 2.5]). The Reeb vector field and its flow are given by

$$\begin{aligned} R_\alpha &= (2iw_0z_0,\dots,2iw_nz_n) \\ \phi_t(z) &= \left(e^{2iw_0}z_0,\dots,e^{2iw_n}z_n\right). \end{aligned}$$

In this setup, the period of the principal orbit is always π (as by construction, the weights do not have any common divisor).

The advantage of this setup is the following: The quotient $\Sigma(a)/S^1$ by the S^1-action of the Reeb flow is a (possibly singular) hypersurface in the weighted projective space $\mathbb{CP}(w_0,\dots,w_n)$, cut out in homogeneous coordinates by the equation

$$Z_0^{a_0} + \cdots + Z_n^{a_n} = 0.$$

$\mathbb{CP}(w_0,\dots,w_n)$ is a symplectic toric orbifold, with symplectic form given by the symplectic reduction

$$\mathbb{CP}(w_0,\dots,w_n) = \{w_0|z_0|^2 + \cdots + w_n|z_n|^2 = 1\}/S^1 \tag{8.1}$$

from ω_{std} on $\mathbb{C}^{n+1}$. In this setup, $\Sigma(a)/S^1$ is (by construction) a subset of $\mathbb{CP}(w_0, \dots, w_n)$, and since $d\alpha = \omega_{std}$, the inclusion preserves the symplectic structure.

Before turning to the examples from Proposition 8.2.1, let us check that the systolic volumes of the manifolds in Theorem 8.1.1 are actually equal.

For $\Sigma(2, 9, 18)$, its quotient by the S^1-action is the hypersurface

$$\{Z_0^2 + Z_1^9 + Z_2^{18} = 0\} \subset \mathbb{CP}(9, 2, 1).$$

The symplectic volume of such a hypersurface depends only on its homology class, hence

$$\mathrm{vol}\left(\Sigma(2, 9, 18)/S^1\right) = 2 \cdot \mathrm{vol}\left(\{Z_0 = 0\} \subset \mathbb{CP}(9, 2, 1)\right).$$

$\mathbb{CP}(9, 2, 1)$ is a symplectic toric orbifold, in the sense of [47], with associated moment polytope a triangle with vertices $(0, 0), (\frac{1}{18}, 0)$ and $(0, \frac{1}{4})$. The normalization is chosen in such a way that it matches (8.1).

By a well-known theorem of Duistermaat and Heckman, the measure on a symplectic toric orbifold induced from the volume form $\omega^n/n!$ is proportional to the Lebesgue measure on its moment polytope, with proportionality constant $(2\pi)^n$. (This is usually stated in the case of manifolds, but the standard derivation as in e.g. [37] also works for orbifolds like the weighted projective space.) As the measure on the hyperplane $\{Z_0 = 0\}$ is induced from the ambient space, this theorem can be used to compute its volume. Namely, this hypersurface is the preimage of the facet from $(0, 0)$ to $(0, \frac{1}{4})$ under the moment map. So its volume is

$$\mathrm{vol}\left(\{Z_0 = 0\} \subset \mathbb{CP}(9, 2, 1)\right) = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2},$$

giving

$$\mathrm{vol}\left(\Sigma(2, 9, 18)/S^1\right) = \pi.$$

This shows immediately that

$$\mathrm{vol}(\Sigma(2, 9, 18)) = \mathrm{vol}\left(\Sigma(2, 9, 18)/S^1\right) \cdot T_{\min} = \pi^2 \qquad \text{and} \qquad \mathfrak{S}(\Sigma(2, 9, 18)) = 1.$$

A similar calculation shows $\mathfrak{S}(\Sigma(3, 5, 15)) = 1$, in accordance with Theorem 8.1.1.

As noted by Otto van Koert, this result also follows from the Gysin sequence of S^1-equivariant cohomology. Indeed, let Σ be a three-dimensional Brieskorn manifold on which the action by the Reeb flow is free. Normalize this action to an S^1-action with period one. Then, the Gysin sequence reads

$$\begin{aligned} 0 \longrightarrow H^1(\Sigma/S^1) \longrightarrow H^1(\Sigma) \longrightarrow H^0(\Sigma/S^1) \longrightarrow \\ \longrightarrow H^2(\Sigma/S^1) \longrightarrow H^2(\Sigma) \longrightarrow H^1(\Sigma/S^1) \longrightarrow 0. \end{aligned}$$

Denoting $\kappa := \mathrm{rank}(H_1(\Sigma))$ and assuming that $H_1(\Sigma)$ has no torsion (which $\Sigma(3, 5, 15) \cong \Sigma(2, 9, 18)$ satisfies by Randell's algorithm, with $\kappa = 8$), this sequence becomes

$$0 \longrightarrow \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}^\kappa \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^\kappa \longrightarrow \mathbb{Z}^\kappa \longrightarrow 0.$$

The only way this sequence can be exact is for the map $\mathbb{Z} \to \mathbb{Z}$ to be an isomorphism. As for any Gysin sequence, this map is the cup product with the Euler class of the S^1-bundle $\Sigma \to \Sigma/S^1$. By the prequantization condition, the Euler class coincides with the symplectic form on Σ/S^1, which gives $\omega = \pm 1 \in H^2(\Sigma/S^1;\mathbb{Z})$. Hence, such a Σ has systolic volume $\mathfrak{S}(\Sigma) = 1$.

Proof of Proposition 8.2.1. The computations for $\Sigma(4,4,4,4)$ and $\Sigma(2,6,6,6)$ are also similar as above. $\Sigma(4,4,4,4)/S^1$ is a hypersurface of degree 4 in the standard projective space, so

$$\operatorname{vol}\left(\Sigma(4,4,4,4)/S^1\right) = 4 \cdot \operatorname{vol}\left(\{Z_0 = 0\} \subset \mathbb{CP}^3\right) = 4 \cdot (2\pi)^2 \cdot \frac{1}{8} = 2\pi^2$$

and

$$\mathfrak{S}(\Sigma(4,4,4,4)) = 2.$$

$\Sigma(2,6,6,6)/S^1$ is once again a hypersurface in the weighted projective space $\mathbb{CP}(3,1,1,1)$, whose moment polytope is the convex hull of the four vertices $(0,0,0), (\frac{1}{6},0,0), (0,\frac{1}{2},0)$ and $(0,0,\frac{1}{2})$. So the computation is

$$\operatorname{vol}\left(\Sigma(2,6,6,6)/S^1\right) = 2 \cdot \operatorname{vol}\left(\{Z_0 = 0\} \subset \mathbb{CP}(2,6,6,6)\right) = 2 \cdot (2\pi)^2 \cdot \frac{1}{8} = \pi^2$$

and

$$\mathfrak{S}(\Sigma(2,6,6,6)) = 1.$$

$\square$

Note also that in this dimension, the Gysin sequence does not tell the systolic volume: Its non-trivial parts are

$$0 \to \underbrace{H^0(\Sigma/S^1)}_{\mathbb{Z}} \xrightarrow{\cup e} \underbrace{H^2(\Sigma/S^1)}_{\mathbb{Z}^{\kappa+1}} \to \underbrace{H^2(\Sigma)}_{\mathbb{Z}^\kappa} \to 0$$

and

$$0 \to \underbrace{H^3(\Sigma)}_{\mathbb{Z}^\kappa} \to \underbrace{H^2(\Sigma/S^1)}_{\mathbb{Z}^{\kappa+1}} \xrightarrow{\cup e} \underbrace{H^4(\Sigma/S^1)}_{\mathbb{Z}} \to 0,$$

which allow many possibilities for the Euler class in terms of the generators of $H^2(\Sigma/S^1)$.

Remark 8.2.3. Proposition 8.2.1 is related to the following question from [4], which is referred to as a generalization of the *weak Blaschke conjecture* from the theory of Zoll manifolds:

Question 8.2.4. *Let (Σ_1, α_1) and $\Sigma_2, \alpha_2)$ be two contact manifolds with periodic Reeb flow such that all simple Reeb orbits have the same period. If $(\Sigma_1, \ker(\alpha_1))$ and $(\Sigma_2, \ker(\alpha_2))$ are contactomorphic, do their systolic volumes agree?*

Corollary 8.2.5. *If the answer to Question 8.2.4 turns out to be yes, Proposition 8.2.1 implies that $\Sigma(4,4,4,4)$ and $\Sigma(2,6,6,6)$ are not contactomorphic.*

8.3 Some open questions

Let us start with the questions that originated from the last three chapters:

Question 8.3.1. *Do the methods from Chapter 5 work for coefficient rings other than $\mathbb{Z}_2$, i.e. with orientations? Can these methods be adapted to cases where there exist Floer cylinders in the fixed point set?*

Question 8.3.2. *Are there examples where breaking of pairs-of-pants as in Figure 3.4 actually occurs?*

Question 8.3.3. *If $\Sigma(a)$ is not index-positive, is it still true that $\check{SH}_*(\Sigma)$ has the module structure from Theorem 1.3.2?*

Question 8.3.4. *Does S^{4m+3}, $m \geq 2$, admit infinitely many exotic but homotopically trivial contact structures? If so, can they be realized as Brieskorn manifolds?*

Note that the second part of Question 8.3.4 is open also in dimensions 11 and 15, since we answered the first part using connected sums of Brieskorn manifolds.

Another question in the spirit of Chapter 5 is:

Question 8.3.5. *Is there another way to systematically exclude certain differentials in (any variant of) symplectic homology?*

Of course, the formulation is a bit vague. At first, one might suggest that the differential of symplectic homology between different critical submanifolds always vanishes (assuming that one starts with the Morse–Bott formalism of the unperturbed contact form). However, by [26, Section 7.1], there are non-trivial differentials for the A_k-surface singularities $\Sigma(k+1,2,2)$, and there is no reason to believe those are the only examples.

For contact homology, it was suggested in [68] that there are no differentials between the generators on different critical submanifolds. However, the argument is not entirely rigorous. It uses an almost complex structure for the differential which is invariant under the Reeb flow. While such an almost complex structure exists, it is by no means clear that the corresponding moduli spaces would be cut out transversally. The same issue applies to positive S^1-equivariant symplectic homology.

Finally, the following question was already mentioned in Section 8.1:

Question 8.3.6. *Are there examples of Brieskorn manifolds Σ_1, Σ_2 of dimension at least five, whose exponents differ by more than a permutation, which are contactomorphic to each other?*

Besides the classification of Brieskorn manifolds, another motivation for Question 8.3.6 comes from the fact that so far, no example for a contact manifold of dimension at least five with two distinct Stein fillings is known. If $\Sigma(a)$ and $\Sigma(a')$ were contactomorphic, then the associated fillings W_a and $W_{a'}$ might be different. For instance, if $\mu_a = \prod_j a_j \neq \prod_j a'_j = \mu_{a'}$, then W_a and $W_{a'}$ would have different topology.

It is hard to guess which Brieskorn manifolds might be contactomorphic. By Proposition 8.2.1 and Corollary 8.2.5, $\Sigma(4,4,4,4)$ and $\Sigma(2,6,6,6)$ do not seem so likely any more. Computer searches show that there are still many examples that are not distinguished by any of the invariants discussed, even in dimensions five and seven.

The author's best guess for contactomorphic Brieskorn manifolds would be $\Sigma(p,6,3,2)$, $\Sigma(p,4,4,2)$ and $\Sigma(p,3,3,3)$ for some $p > 1$ relatively prime to 2 and 3. Randell's algorithm shows that they all have the homology $H_2(\Sigma) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. By the classification of simply-connected, spin 5-manifolds [64], this fixes their diffeomorphism type uniquely. Furthermore, a computation shows that their mean Euler characteristic is $\frac{1}{2}$. While the chain groups of contact and symplectic homology look different, there may be differentials so that the homology matches.

Question 8.3.7. *Let $p > 1$ a natural number with $\gcd(p,2) = \gcd(p,3) = 1$. Are the Brieskorn manifolds $\Sigma(p,6,3,2)$, $\Sigma(p,4,4,2)$ and $\Sigma(p,3,3,3)$ contactomorphic?*

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